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Nonexistence of Positive Solutions for System of Hadamard Fractional BVPs with P-Laplacian Operator

Sabbavarapu Nageswara Rao*

Department of Mathematics, College of Science, Jazan University, P.O. Box 114, Jazan, 45142, Kingdom of Saudi Arabia

**Corresponding author:* snrao@jazanu.edu.sa

Abstract. In this paper, we investigate a system of nonlinear Hadamard fractional differential equations involving (p_1, p_2, p_3)-Laplacian operators subject to three-point boundary conditions. By constructing appropriate Green's functions and establishing their qualitative properties, the given boundary value problem is transformed into an equivalent system of nonlinear integral equations in a cone of a Banach space. Using fixed point arguments and sharp integral inequalities, we derive sufficient conditions ensuring the nonexistence of positive solutions. In particular, explicit bounds on the involved parameters are obtained under which the associated operator fails to admit fixed points. The results cover both sublinear-type growth conditions and superlinear-type lower bounds on the nonlinearities. These findings extend existing nonexistence results for fractional boundary value problems to the setting of coupled systems with Hadamard fractional derivatives and p-Laplacian operators.

1. Introduction

In recent years, fractional differential equations have attracted considerable attention due to their wide applicability in various areas such as mathematics, physics, chemistry, biology, and engineering sciences. Consequently, a substantial number of monographs and research articles on fractional calculus have appeared; see [3,6,13,16] and the references therein. It is well known that fractional differential equations provide more accurate models for many real-world phenomena, including memory and hereditary properties, which arise naturally in fields such as biophysics, control theory, signal and image processing, aerodynamics, photoelasticity, and electromagnetics [10,12,17].

Boundary value problems (BVPs) for fractional differential equations play a fundamental role in both theoretical analysis and practical applications. In particular, the study of existence and

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nonexistence of solutions for such problems has received significant attention in recent years, leading to numerous important contributions; see [2, 7, 8, 14, 18, 19]. Furthermore, systems of fractional differential equations, especially those involving p-Laplacian operators and subject to multi-point or Riemann–Stieltjes integral boundary conditions, have been extensively investigated due to their richer structure and broader applicability [1, 4, 9, 11, 15].

Among the various types of fractional derivatives, the Hadamard fractional derivative, introduced by Hadamard in 1892 [5], has gained increasing interest. Unlike the classical Riemann–Liouville and Caputo derivatives, the Hadamard derivative is characterized by a logarithmic kernel, which leads to distinct analytical properties and makes it suitable for problems defined on domains with multiplicative structure. This difference significantly affects the qualitative behavior of solutions and requires the development of specialized analytical techniques.

Motivated by these developments, it is natural to investigate qualitative properties of solutions for systems of Hadamard fractional differential equations involving nonlinear operators such as the p-Laplacian. In particular, establishing conditions for the nonexistence of positive solutions is important for understanding the structure of the solution set and the influence of system parameters. Such results complement existence theories and provide sharper insight into the behavior of nonlinear fractional systems.

In this paper, we investigate the existence of no positive solutions for the following nonlinear Hadamard fractional differential equation with p-Laplacian operator

$$\begin{cases} -\mathcal{D}_{1^+}^{\flat_1}\Big(\phi_{\sigma_1}(\mathcal{D}_{1^+}^{\nu_1}\varpi_1(r))\Big) = \phi_{\sigma_1}(\lambda)\mathsf{f}(r,\varpi_1(r),\varpi_2(r),\varpi_3(r)),\ r\in(1,e), \\ -\mathcal{D}_{1^+}^{\flat_2}\Big(\phi_{\sigma_2}(\mathcal{D}_{1^+}^{\nu_2}\varpi_2(r))\Big) = \phi_{\sigma_2}(\mu)\mathsf{g}(r,\varpi_1(r),\varpi_2(r),\varpi_3(r)),\ r\in(1,e), \\ -\mathcal{D}_{1^+}^{\flat_3}\Big(\phi_{\sigma_3}(\mathcal{D}_{1^+}^{\nu_3}\varpi_3(r))\Big) = \phi_{\sigma_3}(\upsilon)\mathsf{h}(r,\varpi_1(r),\varpi_2(r),\varpi_3(r)),\ r\in(1,e), \end{cases} \tag{1.1}$$

subject to the three-point boundary conditions

$$\begin{cases} \varpi_1^{(i)}(1)=0,\ 0\le i\le 2,\ \phi_{\sigma_1}\Big(\mathcal{D}_{1^+}^{\nu_1}\varpi_1(1)\Big)=0, \\ \mathcal{D}_{1^+}^{\gamma_1}\varpi_1(e)-\kappa_1\mathcal{D}_{1^+}^{\gamma_1}\varpi_1(\eta_1)=\delta_1\mathcal{D}_{1^+}^{\gamma_1}\varpi_1(\zeta_1), \\ \varpi_2^{(j)}(1)=0,\ 0\le j\le 2,\ \phi_{\sigma_2}\Big(\mathcal{D}_{1^+}^{\nu_2}\varpi_2(1)\Big)=0, \\ \mathcal{D}_{1^+}^{\gamma_2}\varpi_2(e)-\kappa_2\mathcal{D}_{1^+}^{\gamma_2}\varpi_2(\eta_2)=\delta_2\mathcal{D}_{1^+}^{\gamma_2}\varpi_2(\zeta_2), \\ \varpi_3^{(l)}(1)=0,\ 0\le l\le 2,\ \phi_{\sigma_3}\Big(\mathcal{D}_{1^+}^{\nu_3}\varpi_3(1)\Big)=0, \\ \mathcal{D}_{1^+}^{\gamma_3}\varpi_3(e)-\kappa_3\mathcal{D}_{1^+}^{\gamma_3}\varpi_3(\eta_3)=\delta_3\mathcal{D}_{1^+}^{\gamma_3}\varpi_3(\zeta_3), \end{cases} \tag{1.2}$$

where $3<\nu_i\le 4, 0<\flat_i\le 1, 1<\gamma_i\le 2\ \mathcal{D}_{1^+}^{\nu_i}, \mathcal{D}_{1^+}^{\flat_i}, \mathcal{D}_{1^+}^{\gamma_i}$ denotes the Hadamard fractional derivatives, $\phi_{\sigma_i}(\nu)=\mid \nu \mid^{\sigma_i-2} \nu, \sigma_i>1, \phi_{\sigma_i}^{-1}=\phi_{\rho_i}, \frac{1}{\sigma_i}+\frac{1}{\rho_i}=1,\ \kappa_i,\delta_i\ge 0, 1<\eta_i<\zeta_i<e$ for $i=1,2,3.$ and $\mathsf{f},\mathsf{g},\mathsf{h}\in \mathsf{C}([1,e]\times[0,+\infty)^3,[0,+\infty))$.

The following are the assumptions that will be employed in the subsequent section:

- (S1) The functions $\mathsf{f},\mathsf{g},\mathsf{h}:[1,e]\times[0,+\infty)^3\to[0,+\infty)$ are continuous.
- (S2) $\kappa_i,\delta_i\ge 0,\ 1<\eta_i<\zeta_i<e$ satisfy that
  $\Lambda_i=\Gamma(\nu_i)\Big[1-\kappa_i(\ln\eta_i)^{\nu_i-\gamma_i-1}-\delta_i(\ln\zeta_i)^{\nu_i-\gamma_i-1}\Big]>0$ for $i=1,2,3.$

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2) and estimate bounds for the Green's function. In Section 3, we consider the conditions of the nonexistence of a positive solution of given problem (1.1)-(1.2).

2. Preliminary Results

Some essential definitions and lemmas are taken from [10] for the reader's convenience. Consider the system of homogeneous Hadamard FDEs:

$$-\mathcal{D}_{1^+}^{\nu_1}\varpi_1(r) = 0;\ r \in (1,e), \tag{2.1}$$

$$\varpi_1^{(i)}(1) = 0,\ 0 \le i \le 2,\ \mathcal{D}_{1^+}^{\gamma_1}\varpi_1(e) - \kappa_1\mathcal{D}_{1^+}^{\gamma_1}\varpi_1(\eta_1) = \delta_1\mathcal{D}_{1^+}^{\gamma_1}\varpi_1(\zeta_1). \tag{2.2}$$

Lemma 2.1. *Let* $\Lambda_1 = \Gamma(\nu_1)\vartheta \neq 0; \vartheta = 1 - \kappa_1(\ln\eta_1)^{\nu_1-\gamma_1-1} - \delta_1(\ln\zeta_1)^{\nu_1-\gamma_1-1}$. *If* $x(r) \in \mathsf{C}[1,e]$ *and* $3 < \nu_1 \le 4$, *then the boundary value problem:*

$$\mathcal{D}_{1^+}^{\nu_1}\varpi_1(r) + x(r) = 0,\ 1 < r < e, \tag{2.3}$$

satisfying (2.2)*, has a unique solution*

$$\varpi_1(r) = \int_1^e \aleph_1(r,\wp)x(\wp)\frac{d\wp}{\wp},\ r \in [1,e],$$

where $\aleph_1(r,\wp)$ *is the Green's function for* (2.3)-(2.2) *and is given by*

$$\aleph_1(r,\wp) = \begin{cases} \aleph_{11}(r,\wp);\ 1 \le \wp \le \min\{r,\eta_1\} \le e, \\ \aleph_{12}(r,\wp);\ 1 \le r \le \wp \le \eta_1 \le \zeta_1 < e, \\ \aleph_{13}(r,\wp);\ 1 \le \eta_1 \le \wp \le \min\{\eta_1, r\} < e, \\ \aleph_{14}(r,\wp);\ 1 < \max\{r,\eta_1\} \le \wp \le \zeta_1 \le e, \\ \aleph_{15}(r,\wp);\ 1 \le \eta_1 \le \zeta_1 \le \wp \le r < e, \\ \aleph_{16}(r,\wp);\ 1 < \max\{\zeta_1, r\} \le \wp \le e, \end{cases} \tag{2.4}$$

where

$$\aleph_{11}(r,\wp) = \frac{1}{\Lambda_1}\Big[\Big((1-\ln\wp)^{\nu_1-\gamma_1-1} - \kappa_1\Big(\ln\frac{\eta_1}{\wp}\Big)^{\nu_1-\gamma_1-1} - \delta_1\Big(\ln\frac{\zeta_1}{\wp}\Big)^{\nu_1-\gamma_1-1}\Big)(\ln r)^{\nu_1-1} - \vartheta(\ln\frac{r}{\wp})^{\nu_1-1}\Big],$$

$$\aleph_{12}(r,\wp) = \frac{1}{\Lambda_1}\Big[(1-\ln\wp)^{\nu_1-\gamma_1-1} - \kappa_1\Big(\ln\frac{\eta_1}{\wp}\Big)^{\nu_1-\gamma_1-1} - \delta_1\Big(\ln\frac{\zeta_1}{\wp}\Big)^{\nu_1-\gamma_1-1}\Big](\ln r)^{\nu_1-1},$$

$$\aleph_{13}(r,\wp) = \frac{1}{\Lambda_1}\Big[\Big((1-\ln\wp)^{\nu_1-\gamma_1-1} - \delta_1\Big(\ln\frac{\zeta_1}{\wp}\Big)^{\nu_1-\gamma_1-1}\Big)(\ln r)^{\nu_1-1} - \vartheta(\ln\frac{r}{\wp})^{\nu_1-1}\Big],$$

$$\aleph_{14}(r,\wp) = \frac{1}{\Lambda_1}\Big[(1-\ln\wp)^{\nu_1-\gamma_1-1} - \delta_1\Big(\ln\frac{\zeta_1}{\wp}\Big)^{\nu_1-\gamma_1-1}\Big](\ln r)^{\nu_1-1},$$

$$\aleph_{15}(r,\wp) = \frac{1}{\Lambda_1}\Big[(1-\ln\wp)^{\nu_1-\gamma_1-1}(\ln r)^{\nu_1-1} - \vartheta(\ln\frac{r}{\wp})^{\nu_1-1}\Big],$$

$$\aleph_{16}(r,\wp) = \frac{1}{\Lambda_1}\Big[(1-\ln\wp)^{\nu_1-\gamma_1-1}(\ln r)^{\nu_1-1}\Big],$$

Lemma 2.2. *Let* $3 < \nu_1 \leq 4$, $0 < \flat_1 \leq 1$ *and* $\kappa \in \mathsf{C}[1,e]$ *Then the BVP:*

$$\begin{cases} -\mathcal{D}_{1^+}^{\flat_1}\Big(\phi_{\sigma_1}(\mathcal{D}_{1^+}^{\nu_1}\varpi_1(r))\Big) = \kappa(r),\ r \in (1,e), \\ \varpi_1^{(i)}(1) = 0,\ 0 \leq i \leq 2,\ \phi_{\sigma_1}\Big(\mathcal{D}_{1^+}^{\nu_1}\varpi_1(1)\Big) = 0, \\ \mathcal{D}_{1^+}^{\gamma_1}\varpi_1(e) - \kappa_1\mathcal{D}_{1^+}^{\gamma_1}\varpi_1(\eta_1) = \delta_1\mathcal{D}_{1^+}^{\gamma_1}\varpi_1(\zeta_1), \end{cases} \tag{2.5}$$

has a unique solution,

$$\varpi_1(r) = \int_1^e \aleph_1(r,\wp)\phi_{\rho_1}\Big(\int_1^\wp \frac{(\ln\frac{\wp}{\iota})^{\flat_1-1}}{\Gamma(\flat_1)}\kappa(\iota)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},\ r \in [1,e].$$

Lemma 2.3. *Let* $\Lambda_1 > 0$. *Then the Green's function* $\aleph_1(r,\wp)$ *is given by* (2.4) *satisfies the following inequalities:*

(i) $\aleph_1(r,\wp) \geq 0$, *for all* $(r,\wp) \in [1,e] \times [1,e]$,
(ii) $\aleph_1(r,\wp) \leq \aleph_1(e,\wp)$, *for all* $(r,\wp) \in [1,e] \times [1,e]$,
(iii) $\aleph_1(r,\wp) \geq \left(\frac{1}{4}\right)^{\nu_1-1}\aleph_1(e,\wp)$, *for all* $(r,\wp) \in [e^{1/4}, e^{3/4}] \times (1,e)$.

We can also formulate similar results as Lemmas 2.1-2.2 for the following BVP:

$$\begin{cases} -\mathcal{D}_{1^+}^{\flat_2}\Big(\phi_{\sigma_2}(\mathcal{D}_{1^+}^{\nu_2}\varpi_2(r))\Big) = \phi_{\sigma_2}(\mu)\mathsf{g}(r,\varpi_1(r),\varpi_2(r),\varpi_3(r)),\ r \in (1,e), \\ \varpi_2^{(j)}(1) = 0,\ 0 \leq j \leq 2,\ \phi_{\sigma_2}\Big(\mathcal{D}_{1^+}^{\nu_2}\varpi_2(1)\Big) = 0, \\ \mathcal{D}_{1^+}^{\gamma_2}\varpi_2(e) - \kappa_2\mathcal{D}_{1^+}^{\gamma_2}\varpi_2(\eta_2) = \delta_2\mathcal{D}_{1^+}^{\gamma_2}\varpi_2(\zeta_2), \end{cases} \tag{2.6}$$

and

$$\begin{cases} -\mathcal{D}_{1^+}^{\flat_3}\Big(\phi_{\sigma_3}(\mathcal{D}_{1^+}^{\nu_3}\varpi_3(r))\Big) = \phi_{\sigma_3}(\upsilon)\mathsf{h}(r,\varpi_1(r),\varpi_2(r),\varpi_3(r)),\ r \in (1,e), \\ \varpi_3^{(l)}(1) = 0,\ 0 \leq l \leq 2,\ \phi_{\sigma_3}\Big(\mathcal{D}_{1^+}^{\nu_3}\varpi_3(1)\Big) = 0 \\ \mathcal{D}_{1^+}^{\gamma_3}\varpi_3(e) - \kappa_3\mathcal{D}_{1^+}^{\gamma_3}\varpi_3(\eta_3) = \delta_3\mathcal{D}_{1^+}^{\gamma_3}\varpi_3(\zeta_3), \end{cases} \tag{2.7}$$

Remark: Similar results are obtained from lemma (2.1)-(2.3) to the Hadamard FDEs (2.6) and (2.7) using the Green's functions $\aleph_2(r,\wp)$ and $\aleph_3(r,\wp)$.

Consider the following condition:

$$\aleph_p(r,\wp) \geq m\aleph_p(e,\wp), \text{ for all } (r,\wp) \in I \times [1,e],\ p = 1,2,3,$$

where $I = [e^{1/4}, e^{3/4}]$, $m = \min\left\{\left(\frac{1}{4}\right)^{\nu_1-1}, \left(\frac{1}{4}\right)^{\nu_2-1}, \left(\frac{1}{4}\right)^{\nu_3-1}\right\}$.

We consider the Banach space $\mathsf{X} = \mathsf{C}[1,e]$, with the norm $\|\varpi\| = \sup_{r\in[1,e]} |\varpi(r)|$, and the Banach space $\mathsf{Y} = \mathsf{X} \times \mathsf{X} \times \mathsf{X}$ with the norm $\|(\varpi_1,\varpi_2,\varpi_3)\|_{\mathsf{Y}} = \|\varpi_1\| + \|\varpi_2\| + \|\varpi_3\|$. Define a cone $\mathsf{P} \subset \mathsf{Y}$ by

$$\mathsf{P} = \Big\{(\varpi_1,\varpi_2,\varpi_3) \in \mathsf{Y} : \varpi_1(r) \geq 0,\ \varpi_2(r) \geq 0,\ \varpi_3(r) \geq 0,\ \forall r \in [1,e], \\ \min_{r\in I}\Big\{\varpi_1(r) + \varpi_2(r) + \varpi_3(r)\Big\} \geq m\|(\varpi_1,\varpi_2,\varpi_3)\|_{\mathsf{Y}}\Big\}.$$

Consider the coupled system of integral equations

$$\varpi_1(r) = \lambda \int_1^e \aleph_1(r,\wp)\phi_{\rho_1}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_1-1}}{\Gamma(\mathfrak{b}_1)} \mathsf{f}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},\ r\in[1,e],$$
$$\varpi_2(r) = \mu \int_1^e \aleph_2(r,\wp)\phi_{\rho_2}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_2-1}}{\Gamma(\mathfrak{b}_2)} \mathsf{g}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},\ r\in[1,e],$$
$$\varpi_3(r) = \upsilon \int_1^e \aleph_3(r,\wp)\phi_{\rho_3}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_3-1}}{\Gamma(\mathfrak{b}_3)} \mathsf{h}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},\ r\in[1,e].$$

By Lemma 2.2, $(\varpi_1,\varpi_2,\varpi_3)\in\mathsf{P}$ is a solution of BVPs (1.1)-(1.2) if and only if it is a solution of the system of integral equations.

3. Main Results

In this section, we give some sufficient conditions for the nonexistence of a positive solutions for the BVP (1.1)-(1.2). For $I=\left[e^{1/4},e^{3/4}\right]\subset(1,e)$, the subsequent extreme limits are introduced:

$$\mathsf{f}_0^s = \limsup_{\varpi_1+\varpi_2+\varpi_3\to 0}\max_{r\in[1,e]}\frac{\mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_1}(\varpi_1+\varpi_2+\varpi_3)},\ \mathsf{g}_0^s = \limsup_{\varpi_1+\varpi_2+\varpi_3\to 0}\max_{r\in[1,e]}\frac{\mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_2}(\varpi_1+\varpi_2+\varpi_3)},$$
$$\mathsf{h}_0^s = \limsup_{\varpi_1+\varpi_2+\varpi_3\to 0}\max_{r\in[1,e]}\frac{\mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_3}(\varpi_1+\varpi_2+\varpi_3)},\ \mathsf{f}_0^i = \liminf_{\varpi_1+\varpi_2+\varpi_3\to 0}\min_{r\in I}\frac{\mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_1}(\varpi_1+\varpi_2+\varpi_3)},$$
$$\mathsf{g}_0^i = \liminf_{\varpi_1+\varpi_2+\varpi_3\to 0}\min_{r\in I}\frac{\mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_2}(\varpi_1+\varpi_2+\varpi_3)},\ \mathsf{h}_0^i = \liminf_{\varpi_1+\varpi_2+\varpi_3\to 0}\min_{r\in I}\frac{\mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_3}(\varpi_1+\varpi_2+\varpi_3)},$$
$$\mathsf{f}_\infty^s = \limsup_{\varpi_1+\varpi_2+\varpi_3\to \infty}\max_{r\in[1,e]}\frac{\mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_1}(\varpi_1+\varpi_2+\varpi_3)},\ \mathsf{g}_\infty^s = \limsup_{\varpi_1+\varpi_2+\varpi_3\to \infty}\max_{r\in[1,e]}\frac{\mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_2}(\varpi_1+\varpi_2+\varpi_3)},$$
$$\mathsf{h}_\infty^s = \limsup_{\varpi_1+\varpi_2+\varpi_3\to \infty}\max_{r\in[1,e]}\frac{\mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_3}(\varpi_1+\varpi_2+\varpi_3)},\ \mathsf{f}_\infty^i = \liminf_{\varpi_1+\varpi_2+\varpi_3\to \infty}\min_{r\in I}\frac{\mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_1}(\varpi_1+\varpi_2+\varpi_3)},$$
$$\mathsf{g}_\infty^i = \liminf_{\varpi_1+\varpi_2+\varpi_3\to \infty}\max_{r\in I}\frac{\mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_2}(\varpi_1+\varpi_2+\varpi_3)},\ \mathsf{h}_\infty^i = \liminf_{\varpi_1+\varpi_2+\varpi_3\to \infty}\min_{t\in I}\frac{\mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3)}{\phi_{\sigma_3}(\varpi_1+\varpi_2+\varpi_3)}.$$

By using the Green's functions $\aleph_1,\aleph_2$ and $\aleph_3$ from Section 2, our problem (1.1)-(1.2) can be written equivalently as the following nonlinear system of integral equations

$$\varpi_1(r) = \lambda \int_1^e \aleph_1(r,\wp)\phi_{\rho_1}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_1-1}}{\Gamma(\mathfrak{b}_1)} \mathsf{f}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},\ r\in[1,e],$$
$$\varpi_2(r) = \mu \int_1^e \aleph_2(r,\wp)\phi_{\rho_2}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_2-1}}{\Gamma(\mathfrak{b}_2)} \mathsf{g}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},\ r\in[1,e],$$
$$\varpi_3(r) = \upsilon \int_1^e \aleph_3(r,\wp)\phi_{\rho_3}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_3-1}}{\Gamma(\mathfrak{b}_3)} \mathsf{h}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},\ r\in[1,e].$$

We consider the Banach space $\mathsf{X}=\mathsf{C}[1,e]$, with the norm $\|\varpi\|=\sup_{r\in[1,e]}|\varpi(r)|$, and the Banach space $\mathsf{Y}=\mathsf{X}\times\mathsf{X}\times\mathsf{X}$ with the norm $\|(\varpi_1,\varpi_2,\varpi_3)\|_{\mathsf{Y}}=\|\varpi_1\|+\|\varpi_2\|+\|\varpi_3\|$. Define a cone $\mathsf{P}\subset\mathsf{Y}$ by

$$\mathsf{P} = \Big\{(\varpi_1,\varpi_2,\varpi_3) \in \mathsf{Y} : \varpi_1(r) \geq 0,\ \varpi_2(r) \geq 0,\ \varpi_3(r) \geq 0,\ \forall r \in [1,e],$$
$$\min_{r\in I}\Big\{\varpi_1(r)+\varpi_2(r)+\varpi_3(r)\Big\} \geq m\|(\varpi_1,\varpi_2,\varpi_3)\|_{\mathsf{Y}}\Big\}.$$

Now we define the operators $\mho_1, \mho_2, \mho_3 : \mathsf{Y} \to \mathsf{X}$ and $\mho : \mathsf{P} \to \mathsf{Y}$ by

$$\mho(\varpi_1,\varpi_2,\varpi_3) = \Big(\mho_1(\varpi_1,\varpi_2,\varpi_3), \mho_2(\varpi_1,\varpi_2,\varpi_3), \mho_3(\varpi_1,\varpi_2,\varpi_3)\Big) \tag{3.1}$$

with

$$\mho_1(\varpi_1,\varpi_2,\varpi_3)(r) = \lambda\int_1^e \aleph_1(r,s)\phi_{\rho_1}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_1-1}}{\Gamma(\mathfrak{b}_1)}\mathsf{f}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},\ r\in[1,e],$$
$$\mho_2(\varpi_1,\varpi_2,\varpi_3)(r) = \mu\int_1^e \aleph_2(r,\wp)\phi_{\rho_2}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_2-1}}{\Gamma(\mathfrak{b}_2)}\mathsf{g}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},\ r\in[1,e],$$
$$\mho_3(\varpi_1,\varpi_2,\varpi_3)(r) = \upsilon\int_1^e \aleph_3(r,\wp)\phi_{\rho_3}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_3-1}}{\Gamma(\mathfrak{b}_3)}\mathsf{h}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},\ r\in[1,e].$$

It is clear that the existence of a positive solution to the system (1.1)-(1.2) is equivalent to the existence of fixed points of the operator $\mho$.

Lemma 3.1. *If* $(S1)-(S2)$ *hold, then* $\mho : \mathsf{P} \to \mathsf{P}$ *is a completely continuous operator.*

Proof. Let $(\varpi_1,\varpi_2,\varpi_3) \in \mathsf{P}$ be arbitrary. First, we show that $\mho(\mathsf{P}) \subset \mathsf{P}$. By Lemma 2.3, the Green's functions $\aleph_i(r,\wp)$ are nonnegative and satisfy the required upper and lower bounds. Using these properties together with the definition of the operators $\mho_1, \mho_2$ and $\mho_3$, we obtain the following estimates:

$$\|\mho_1(\varpi_1,\varpi_2,\varpi_3)\| \leq \lambda\int_1^e \aleph_1(e,\wp)\phi_{\rho_1}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_1-1}}{\Gamma(\mathfrak{b}_1)}\mathsf{f}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},$$
$$\|\mho_2(\varpi_1,\varpi_2,\varpi_3)\| \leq \mu\int_1^e \aleph_2(e,\wp)\phi_{\rho_2}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_2-1}}{\Gamma(\mathfrak{b}_2)}\mathsf{g}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp},$$
$$\|\mho_3(\varpi_1,\varpi_2,\varpi_3)\| \leq \upsilon\int_1^e \aleph_3(e,\wp)\phi_{\rho_3}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_3-1}}{\Gamma(\mathfrak{b}_3)}\mathsf{h}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp}.$$

Next, we estimate the minimum of the sum of the components over the interval I. By using the lower bound of the Green's functions given in Lemma 2.3, we have

$$\begin{aligned}
&\min_{r\in I}\{\mho_1(\varpi_1,\varpi_2,\varpi_3)(r)+\mho_2(\varpi_1,\varpi_2,\varpi_3)(r)+\mho_3(\varpi_1,\varpi_2,\varpi_3)(r)\}\\
&\quad= \min_{r\in I}\Big\{\lambda\int_1^e \aleph_1(r,\wp)\phi_{\rho_1}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_1-1}}{\Gamma(\mathfrak{b}_1)}\mathsf{f}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp}\\
&\qquad+\mu\int_1^e \aleph_2(r,\wp)\phi_{\rho_2}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_2-1}}{\Gamma(\mathfrak{b}_2)}\mathsf{g}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp}\\
&\qquad+\upsilon\int_1^e \aleph_3(r,\wp)\phi_{\rho_3}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_3-1}}{\Gamma(\mathfrak{b}_3)}\mathsf{h}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp}\Big\}\\
&\quad\geq m\Big\{\lambda\int_1^e \aleph_1(e,\wp)\phi_{\rho_1}\Big(\int_1^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_1-1}}{\Gamma(\mathfrak{b}_1)}\mathsf{f}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp}
\end{aligned}$$

$$
\begin{aligned}
&+\mu\int_1^e \aleph_2(e,\wp)\phi_{\rho_2}\Big(\int_1^{\wp}\frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_2-1}}{\Gamma(\mathfrak{b}_2)}\mathsf{g}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp}\\
&+\upsilon\int_1^e \aleph_3(e,\wp)\phi_{\rho_3}\Big(\int_1^{\wp}\frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_3-1}}{\Gamma(\mathfrak{b}_3)}\mathsf{h}\big(\iota,\varpi_1(\iota),\varpi_2(\iota),\varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp}\Big\}\\
\geq\ & m\Big(\|\mho_1(\varpi_1,\varpi_2,\varpi_3)\|+\|\mho_2(\varpi_1,\varpi_2,\varpi_3)\|+\|\mho_3(\varpi_1,\varpi_2,\varpi_3)\|\Big)\\
=\ & m\|\mho(\varpi_1,\varpi_2,\varpi_3)\|.
\end{aligned}
$$

Hence, $\mho(\mathsf{P})\subset\mathsf{P}$. Next, we prove that $\mho$ is completely continuous. From assumption $(S1)$ and the continuity of the kernel functions, it follows that the operators $\mho_1,\mho_2$ and $\mho_3$ are continuous. Moreover, for any bounded subset of P, the corresponding images under $\mho$ are uniformly bounded due to the above estimates. Furthermore, the integral operators involved are equicontinuous on $[1,e]$, since the kernels $\aleph_i(r,\wp)$ are continuous and the nonlinearities are continuous functions. Therefore, by the Arzela–Ascoli theorem, the operator $\mho$ maps bounded sets into relatively compact sets in Y. Consequently, $\mho:\mathsf{P}\to\mathsf{P}$ is completely continuous. □

Theorem 3.1. *If $(S1)-(S2)$ hold. If there exist ${}_1,{}_2,{}_3>0$ such that*

$$
\begin{aligned}
&\mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3)\leq\phi_{\sigma_1}\big[{}_1(\varpi_1+\varpi_2+\varpi_3)\big],\\
&\mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3)\leq\phi_{\sigma_2}\big[{}_2(\varpi_1+\varpi_2+\varpi_3)\big],\\
&\mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3)\leq\phi_{\sigma_3}\big[{}_3(\varpi_1+\varpi_2+\varpi_3)\big],\quad\forall r\in[1,e],\ \varpi_1,\varpi_2,\varpi_3\geq 0.
\end{aligned}\tag{3.2}
$$

then there exist positive constants $\lambda_0,\mu_0,\upsilon_0$ such that, for every $\lambda\in(0,\lambda_0),\mu\in(0,\mu_0),\upsilon\in(0,\upsilon_0)$, the BVP (1.1)-(1.2) *has no positive solution.*

Proof. Define

$$
\lambda_0=\frac{1}{3_1\mathsf{B}},\quad \mu_0=\frac{1}{3_2\mathsf{D}},\quad \upsilon_0=\frac{1}{3_3\mathsf{F}},
$$

where

$$
\begin{aligned}
\mathsf{B}&=\phi_{\rho_1}\Big(\frac{1}{\Gamma(\mathfrak{b}_1+1)}\Big)\int_1^e\aleph_1(e,\wp)\frac{d\wp}{\wp},\\
\mathsf{D}&=\phi_{\rho_2}\Big(\frac{1}{\Gamma(\mathfrak{b}_2+1)}\Big)\int_1^e\aleph_2(e,\wp)\frac{d\wp}{\wp},\\
\mathsf{F}&=\phi_{\rho_3}\Big(\frac{1}{\Gamma(\mathfrak{b}_3+1)}\Big)\int_1^e\aleph_3(e,\wp)\frac{d\wp}{\wp}.
\end{aligned}
$$

Let $\lambda\in(0,\lambda_0),\mu\in(0,\mu_0),\upsilon\in(0,\upsilon_0)$. Suppose, for contradiction, that the boundary value problem (1.1)-(1.2) admits a positive solution $(\varpi_1,\varpi_2,\varpi_3)\in\mathsf{P}$.

Using the definition of the operator $\mho$ together with condition (3.2) and the positivity of the Green's functions, we estimate each component. For $r\in[1,e]$, we obtain

$$
\begin{aligned}
\varpi_1(r)&\leq\lambda\int_1^e\aleph_1(e,\wp)\phi_{\rho_1}\Big(\int_1^{\wp}\frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_1-1}}{\Gamma(\mathfrak{b}_1)}\mathsf{f}\big(\iota,\varpi_1,\varpi_2,\varpi_3\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp}\\
&\leq\lambda_1\phi_{\rho_1}\Big(\frac{1}{\Gamma(\mathfrak{b}_1+1)}\Big)\int_1^e\aleph_1(e,\wp)(\varpi_1(\wp)+\varpi_2(\wp)+\varpi_3(\wp))\frac{d\wp}{\wp}
\end{aligned}
$$

$$\leq \lambda_1 \mathsf{B} \|(\varpi_1, \varpi_2, \varpi_3)\|.$$

Taking supremum over $r \in [1, e]$, we obtain

$$\|\varpi_1\| \leq \lambda_1 \mathsf{B} \|(\varpi_1, \varpi_2, \varpi_3)\|.$$

Similarly,

$$\|\varpi_2\| \leq \mu_2 \mathsf{D} \|(\varpi_1, \varpi_2, \varpi_3)\|, \quad \|\varpi_3\| \leq \upsilon_3 \mathsf{F} \|(\varpi_1, \varpi_2, \varpi_3)\|.$$

Since $\lambda < \lambda_0, \mu < \mu_0, \upsilon < \upsilon_0$, it follows that

$$\|\varpi_1\| < \frac{1}{3}\|(\varpi_1, \varpi_2, \varpi_3)\|, \quad \|\varpi_2\| < \frac{1}{3}\|(\varpi_1, \varpi_2, \varpi_3)\|, \quad \|\varpi_3\| < \frac{1}{3}\|(\varpi_1, \varpi_2, \varpi_3)\|.$$

Adding these inequalities, we obtain

$$\|(\varpi_1, \varpi_2, \varpi_3)\| = \|\varpi_1\| + \|\varpi_2\| + \|\varpi_3\| < \|(\varpi_1, \varpi_2, \varpi_3)\|,$$

which is a contradiction. Hence, the boundary value problem (1.1)-(1.2) admits no positive solution. □

Remark 3.1. *In the proof of Theorem 3.1, one may alternatively define*

$$\lambda_0 = \frac{\nu_1}{3K_1 \mathsf{B}}, \quad \mu_0 = \frac{\nu_2}{3K_2 \mathsf{D}}, \quad \upsilon_0 = \frac{\nu_3}{3K_3 \mathsf{F}},$$

where $\nu_1, \nu_2, \nu_3 > 0$ *satisfy* $\nu_1 + \nu_2 + \nu_3 = 1$.

Remark 3.2. *If* $\mathsf{f}_0^s, \mathsf{g}_0^s, \mathsf{h}_0^s, \mathsf{f}_\infty^s, \mathsf{g}_\infty^s, \mathsf{h}_\infty^s < \infty$, *then there exist positive constants* k_1, k_2, k_3 *such that condition* (3.2) *holds. Consequently, the conclusion of Theorem 3.1 follows.*

Theorem 3.2. *If* $(S1) - (S2)$ *hold and there exists a positive constant* $m_1 > 0$ *such that*

$$\mathsf{f}(r, \varpi_1, \varpi_2, \varpi_3) \geq \phi_{\sigma_1}\Big[m_1(\varpi_1 + \varpi_2 + \varpi_3)\Big], \quad \forall r \in I, ; \varpi_1, \varpi_2, \varpi_3 \geq 0, \tag{3.3}$$

then there exists a positive constant $\overline{\lambda_0}$ *such that, for any* $\lambda > \overline{\lambda_0}, \mu > 0$ *and* $\upsilon > 0$, *the BVP* (1.1)-(1.2) *has no positive solution.*

Proof. Define

$$\overline{\lambda_0} = \frac{1}{m^2 m_1 \mathsf{A}},$$

where

$$\mathsf{A} = \phi_{\rho_1}\Big(\frac{1}{\Gamma(\mathfrak{b}_1 + 1)}\Big) \int_{e^{1/4}}^{e^{3/4}} \aleph_1(e, \wp), (\ln \wp - 1/4)^{\mathfrak{b}_1(\rho_1 - 1)} \frac{d\wp}{\wp}.$$

Let $\lambda > \overline{\lambda_0}, \mu > 0$ and $\upsilon > 0$. Suppose, for contradiction, that the boundary value problem (1.1)-(1.2) admits a positive solution $(\varpi_1, \varpi_2, \varpi_3) \in \mathsf{P}$. From the definition of the operator $\mho_1$, we have for $r \in [1, e]$

$$\varpi_1(r) = \lambda \int_1^e \aleph_1(r, \wp) \phi_{\rho_1}\Big(\int_1^{\wp} \frac{(\ln \frac{\wp}{\iota})^{\mathfrak{b}_1 - 1}}{\Gamma(\mathfrak{b}_1)} \mathsf{f}\big(\iota, \varpi_1(\iota), \varpi_2(\iota), \varpi_3(\iota)\big) \frac{d\iota}{\iota}\Big) \frac{d\wp}{\wp}.$$

Using the lower bound of the Green's function given in Lemma 2.3 and restricting the integration to the interval $I = [e^{1/4}, e^{3/4}]$, we obtain

$$\varpi_1(r) \geq \lambda m \int_{e^{1/4}}^{e^{3/4}} \aleph_1(e,\wp)\phi_{\rho_1}\Big(\int_{e^{1/4}}^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_1-1}}{\Gamma(\mathfrak{b}_1)} \mathsf{f}\big(\iota, \varpi_1(\iota), \varpi_2(\iota), \varpi_3(\iota)\big)\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp}.$$

Applying condition (3.3), we obtain

$$\varpi_1(r) \geq \lambda m \int_{e^{1/4}}^{e^{3/4}} \aleph_1(e,\wp)\phi_{\rho_1}\Big(\int_{e^{1/4}}^{\wp} \frac{(\ln\frac{\wp}{\iota})^{\mathfrak{b}_1-1}}{\Gamma(\mathfrak{b}_1)} \phi_{\sigma_1}\Big[m_1(\varpi_1(\iota)+\varpi_2(\iota)+\varpi_3(\iota))\Big]\frac{d\iota}{\iota}\Big)\frac{d\wp}{\wp}.$$

Since $(\varpi_1, \varpi_2, \varpi_3) \in \mathsf{P}$, we have

$$\varpi_1(\iota) + \varpi_2(\iota) + \varpi_3(\iota) \geq m\|(\varpi_1, \varpi_2, \varpi_3)\|_{\mathsf{Y}}, \quad \forall \iota \in I.$$

Hence,

$$\begin{aligned}\varpi_1(r) &\geq \lambda m^2 m_1 \|(\varpi_1, \varpi_2, \varpi_3)\|_{\mathsf{Y}} \phi_{\rho_1}\Big(\frac{1}{\Gamma(\mathfrak{b}_1+1)}\Big)\int_{e^{1/4}}^{e^{3/4}} \aleph_1(e,\wp)(\ln\wp - 1/4)^{\mathfrak{b}_1(\rho_1-1)}\frac{d\wp}{\wp} \\ &= \lambda m^2 m_1 \mathsf{A}\|(\varpi_1, \varpi_2, \varpi_3)\|_{\mathsf{Y}}.\end{aligned}$$

Taking supremum over $r \in [1, e]$, we obtain

$$\|\varpi_1\| \geq \lambda m^2 m_1 \mathsf{A}\|(\varpi_1, \varpi_2, \varpi_3)\|_{\mathsf{Y}}.$$

Since $\lambda > \overline{\lambda_0}$, it follows that

$$\|\varpi_1\| > \|(\varpi_1, \varpi_2, \varpi_3)\|_{\mathsf{Y}}.$$

Consequently,

$$\|(\varpi_1, \varpi_2, \varpi_3)\| = \|\varpi_1\| + \|\varpi_2\| + \|\varpi_3\| > \|(\varpi_1, \varpi_2, \varpi_3)\|_{\mathsf{Y}},$$

which is a contradiction. Therefore, the boundary value problem (1.1)-(1.2) admits no positive solution. □

Similarly, we obtain the theorems listed below.

Theorem 3.3. *If $(S1)-(S2)$ hold and there exist positive number $m_2 > 0$ such that*

$$\mathsf{g}(r, \varpi_1, \varpi_2, \varpi_3) \geq \phi_{\sigma_2}\Big[m_2(\varpi_1 + \varpi_2 + \varpi_3)\Big], \forall r \in I, \varpi_1, \varpi_2, \varpi_3 \geq 0 \tag{3.4}$$

then there exists a positive solution $\overline{\mu_0}$ such that, for any $\lambda > 0$, $\mu > \overline{\mu_0}$, and $\upsilon > 0$, the BVP (1.1)-(1.2) *has no positive solution.*

Where $\overline{\mu_0} = \frac{1}{m^2 m_2 \mathsf{C}}$ and $\mathsf{C} = \phi_{\rho_2}\Big(\frac{1}{\Gamma(\mathfrak{b}_2+1)}\Big)\int_{e^{1/4}}^{e^{3/4}} \aleph_2(e,\wp)(\ln\wp - 1/4)^{\mathfrak{b}_2(\rho_2-1)}\frac{d\wp}{\wp}$.

Theorem 3.4. *Assume that $(S1)-(S2)$ hold. If there exist positive number $m_3 > 0$ such that*

$$\mathsf{h}(r, \varpi_1, \varpi_2, \varpi_3) \geq \phi_{\sigma_3}\Big[m_3(\varpi_1 + \varpi_2 + \varpi_3)\Big], \forall r \in I, \varpi_1, \varpi_2, \varpi_3 \geq 0 \tag{3.5}$$

then there exists a positive solution $\overline{\upsilon_0}$ such that, for any $\lambda > 0$, $\mu > 0$, and $\upsilon > \overline{\upsilon_0}$, the BVP (1.1)-(1.2) *has no positive solution.*

Where $\overline{\upsilon_0} = \frac{1}{m^2 m_3 \mathsf{E}}$ and $\mathsf{E} = \phi_{\rho_3}\Big(\frac{1}{\Gamma(\mathfrak{b}_3+1)}\Big)\int_{e^{1/4}}^{e^{3/4}} \aleph_3(e,\wp)(\ln\wp - 1/4)^{\mathfrak{b}_3(\rho_3-1)}\frac{d\wp}{\wp}$.

Remark 3.3:

(i) If $\mathsf{f}_0^i, \mathsf{f}_\infty^i > 0$ and $\mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3) > 0$ for all $r \in I$ and $\varpi_1,\varpi_2,\varpi_3 \geq 0$ with $\varpi_1+\varpi_2+\varpi_3 > 0$ then relation (3.3) holds, and we obtain the conclusion of theorem 3.2

(ii) If $\mathsf{g}_0^i, \mathsf{g}_\infty^i > 0$ and $\mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3) > 0$ for all $r \in I$ $\varpi_1,\varpi_2,\varpi_3 \geq 0$ with $\varpi_1+\varpi_2+\varpi_3 > 0$. If relation (3.4) holds, we have the conclusion of theorem 3.3

(iii) If $\mathsf{h}_0^i, \mathsf{h}_\infty^i > 0$ and $\mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3) > 0$ for all $r \in I$ $\varpi_1,\varpi_2,\varpi_3 \geq 0$ with $\varpi_1+\varpi_2+\varpi_3 > 0$. If relation (3.5) holds, we have the conclusion of theorem 3.4.

Theorem 3.5. *If $(S1)-(S2)$ hold. If there exist $m_1, m_2 > 0$ such that*

$$\begin{aligned} &\mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3) \geq \phi_{\sigma_1}\Big[m_1(\varpi_1+\varpi_2+\varpi_3)\Big] \\ &\mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3) \geq \phi_{\sigma_2}\Big[m_2(\varpi_1+\varpi_2+\varpi_3)\Big],\ r \in I, \varpi_1,\varpi_2,\varpi_3 \geq 0, \end{aligned} \tag{3.6}$$

then there exist positive constants $\lambda \geq \overline{\overline{\lambda_0}}, \mu \geq \overline{\overline{\mu_0}}$ and $\upsilon > 0$ the BVP (1.1)-(1.2) *has no positive solution*

Proof. We define $\overline{\overline{\lambda_0}} = \frac{1}{2m^2 m_1 \mathsf{A}} = (\frac{\overline{\lambda_0}}{2})$ and $\overline{\overline{\mu_0}} = \frac{1}{2m^2 m_2 \mathsf{C}} = (\frac{\overline{\mu_0}}{2})$. Then, for every $\lambda > \overline{\overline{\lambda_0}}, \mu > \overline{\overline{\mu_0}}$ and $\upsilon > 0$, the problem (1.1)-(1.2) has positive solution $(\varpi_1(r),\varpi_2(r),\varpi_3(r)), r \in [1,e]$. Then, analogous to the proof of theorem 3.2, we deduce

$$\begin{aligned} \|\varpi_1\| &\geq \lambda m^2 m_1 \mathsf{A}\|(\varpi_1,\varpi_2,\varpi_3)\| \\ \|\varpi_2\| &\geq \mu m^2 m_2 \mathsf{C}\|(\varpi_1,\varpi_2,\varpi_3)\| \end{aligned}$$

and so

$$\begin{aligned} \|(\varpi_1,\varpi_2,\varpi_3)\| &= \|\varpi_1\|+\|\varpi_2\|+\|\varpi_3\| \geq \|\varpi_1\|+\|\varpi_2\| \\ &\geq \Big[\lambda m^2 m_1 \mathsf{A} + \mu m^2 m_2 \mathsf{C}\Big]\|(\varpi_1,\varpi_2,\varpi_3)\| \\ &> \Big[\overline{\overline{\lambda_0}} m^2 m_1 \mathsf{A} + \overline{\overline{\mu_0}} m^2 m_2 \mathsf{C}\Big]\|(\varpi_1,\varpi_2,\varpi_3)\| \\ &= \Big(\frac{1}{2}+\frac{1}{2}\Big)\|(\varpi_1,\varpi_2,\varpi_3)\| = \|(\varpi_1,\varpi_2,\varpi_3)\| \end{aligned}$$

which is a contradiction. □

Remark: In the proof of Theorem 3.5, we can also define $\overline{\overline{\lambda_0}} = \frac{\overline{\nu_1}}{m^2 m_1 \mathsf{A}}$; $\overline{\overline{\mu_0}} = \frac{\overline{\nu_2}}{m^2 m_2 \mathsf{C}}$ with $\overline{\nu_1}, \overline{\nu_2} > 0$ with $\overline{\nu_1}+\overline{\nu_2} = 1$. Similarly, we establish the following theorems

Theorem 3.6. *If $(S1)-(S2)$ hold. If there exist $m_1, m_3 > 0$ such that*

$$\begin{aligned} &\mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3) \geq \phi_{\sigma_1}\Big[m_1(\varpi_1+\varpi_2+\varpi_3)\Big] \\ &\mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3) \geq \phi_{\sigma_3}\Big[m_3(\varpi_1+\varpi_2+\varpi_3)\Big],\ r \in I, \varpi_1,\varpi_2,\varpi_3 \geq 0, \end{aligned} \tag{3.7}$$

then there exist positive constants $\lambda \geq \overline{\overline{\lambda_0}}'$, $\mu > 0$ and $\upsilon \geq \overline{\overline{\upsilon_0}}'$ the BVP (1.1)-(1.2) *has no positive solution*

Theorem 3.7. *Assume that* $(S1)-(S2)$ *hold. If there exist* $m_2, m_3 > 0$ *such that*

$$\begin{aligned} \mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3) &\geq \phi_{\sigma_2}\Big[m_2(\varpi_1+\varpi_2+\varpi_3)\Big] \\ \mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3) &\geq \phi_{\sigma_3}\Big[m_3(\varpi_1+\varpi_2+\varpi_3)\Big],\ r\in I, \varpi_1,\varpi_2,\varpi_3\geq 0, \end{aligned} \tag{3.8}$$

then there exist positive constants $\lambda > 0, \mu \geq \overline{\overline{\mu_0}}''$ *and* $\upsilon \geq \overline{\overline{\upsilon_0}}''$ *the BVP* (1.1)-(1.2) *has no positive solution*

Remark 3.4:

(i) If $\mathsf{f}_0^i, \mathsf{f}_\infty^i, \mathsf{g}_0^i, \mathsf{g}_\infty^i > 0$ and $\mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3) > 0, \mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3) > 0$ for all $r \in I$ and $\varpi_1,\varpi_2,\varpi_3 > 0$, thus, relation (3.6) is satisfied, leading to the conclusion of theorem 3.5.

(ii) If $\mathsf{f}_0^i, \mathsf{f}_\infty^i, \mathsf{h}_0^i, \mathsf{h}_\infty^i > 0$ and $\mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3) > 0, \mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3) > 0$ for all $r \in I$ and $\varpi_1,\varpi_2,\varpi_3 > 0$, then relation (3.7) is true, and we get the result of theorem 3.6.

(iii) If $\mathsf{g}_0^i, \mathsf{g}_\infty^i, \mathsf{h}_0^i, \mathsf{h}_\infty^i > 0$ and $\mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3) > 0, \mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3) > 0$ for all $r \in I$ and $\varpi_1,\varpi_2,\varpi_3 > 0$, then the relation (3.8) is valid, and the conclusion of theorem 3.7 is reached.

Theorem 3.8. *If* $(S1)-(S2)$ *hold. If there exist* $m_1, m_2, m_3 > 0$ *such that*

$$\begin{aligned} \mathsf{f}(r,\varpi_1,\varpi_2,\varpi_3) &\geq \phi_{\sigma_1}\Big[m_1(\varpi_1+\varpi_2+\varpi_3)\Big], \\ \mathsf{g}(r,\varpi_1,\varpi_2,\varpi_3) &\geq \phi_{\sigma_2}\Big[m_2(\varpi_1+\varpi_2+\varpi_3)\Big], \\ \mathsf{h}(r,\varpi_1,\varpi_2,\varpi_3) &\geq \phi_{\sigma_3}\Big[m_3(\varpi_1+\varpi_2+\varpi_3)\Big],\ \forall\, r\in I,\ \varpi_1,\varpi_2,\varpi_3\geq 0, \end{aligned} \tag{3.9}$$

then there exist positive constants $\overline{\lambda_0}, \overline{\mu_0}$ *and* $\overline{\upsilon_0}$ *such that, for every* $\lambda > \overline{\lambda_0}, \mu > \overline{\mu_0}$ *and* $\upsilon > \overline{\upsilon_0}$*, the BVP* (1.1)-(1.2) *has no positive solution.*

Proof. Define

$$\overline{\lambda_0} = \frac{1}{3m^2 m_1 \mathsf{A}}, \quad \overline{\mu_0} = \frac{1}{3m^2 m_2 \mathsf{C}}, \quad \overline{\upsilon_0} = \frac{1}{3m^2 m_3 \mathsf{E}}.$$

Let $\lambda > \overline{\lambda_0}, \mu > \overline{\mu_0}$ and $\upsilon > \overline{\upsilon_0}$. Suppose, for contradiction, that the problem (1.1)-(1.2) admits a positive solution $(\varpi_1,\varpi_2,\varpi_3) \in \mathsf{P}$. Using the lower bounds of the Green's functions given in Lemma 2.3, together with condition (3.9), and arguing as in the proofs of Theorems 3.2-3.4, we obtain the following estimates:

$$\begin{aligned} \|\varpi_1\| &\geq \lambda m^2 m_1 \mathsf{A}\|(\varpi_1,\varpi_2,\varpi_3)\|, \\ \|\varpi_2\| &\geq \mu m^2 m_2 \mathsf{C}\|(\varpi_1,\varpi_2,\varpi_3)\|, \\ \|\varpi_3\| &\geq \upsilon m^2 m_3 \mathsf{E}\|(\varpi_1,\varpi_2,\varpi_3)\|. \end{aligned}$$

Summing these inequalities, we obtain

$$\begin{aligned} \|(\varpi_1,\varpi_2,\varpi_3)\| &= \|\varpi_1\| + \|\varpi_2\| + \|\varpi_3\| \\ &\geq \Big[\lambda m^2 m_1 \mathsf{A} + \mu m^2 m_2 \mathsf{C} + \upsilon m^2 m_3 \mathsf{E}\Big]\|(\varpi_1,\varpi_2,\varpi_3)\| \\ &\geq \Big[\overline{\lambda_0} m^2 m_1 \mathsf{A} + \overline{\mu_0} m^2 m_2 \mathsf{C} + \overline{\upsilon_0} m^2 m_3 \mathsf{E}\Big]\|(\varpi_1,\varpi_2,\varpi_3)\| \\ &= \Big(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\Big)\|(\varpi_1,\varpi_2,\varpi_3)\| \\ &= \|(\varpi_1,\varpi_2,\varpi_3)\|. \end{aligned}$$

Since $\lambda > \overline{\lambda_0}, \mu > \overline{\mu_0}$ and $\upsilon > \overline{\upsilon_0}$, the above inequality is in fact strict, which yields

$$\|(\varpi_1, \varpi_2, \varpi_3)\| > \|(\varpi_1, \varpi_2, \varpi_3)\|,$$

a contradiction. Therefore, the boundary value problem (1.1)-(1.2) admits no positive solution. □

Remark 3.5: In the proof of Theorem 3.8, one may alternatively define

$$\overline{\lambda_0} = \frac{\nu_1'}{m^2 m_1 \mathsf{A}}, \quad \overline{\mu_0} = \frac{\nu_2'}{m^2 m_2 \mathsf{C}}, \quad \overline{\upsilon_0} = \frac{\nu_3'}{m^2 m_3 \mathsf{E}},$$

where $\nu_1', \nu_2', \nu_3' > 0$ with $\nu_1' + \nu_2' + \nu_3' = 1$.

4. Conclusion and Future Work

In this paper, we investigated a system of nonlinear Hadamard fractional differential equations involving (p_1, p_2, p_3)-Laplacian operators subject to three-point boundary conditions. By constructing suitable Green's functions and establishing their fundamental properties, the boundary value problem was reduced to an equivalent system of nonlinear integral equations in a cone of a Banach space. Using fixed point arguments together with appropriate growth conditions on the nonlinear terms, we derived several sufficient criteria ensuring the nonexistence of positive solutions. In particular, explicit parameter thresholds were obtained that guarantee the absence of fixed points of the associated operator. The results cover both upper growth conditions leading to nonexistence for sufficiently small parameters, and lower growth conditions leading to nonexistence for sufficiently large parameters. The analysis presented here contributes to the qualitative theory of fractional boundary value problems, especially for systems involving Hadamard fractional derivatives, where the logarithmic kernel introduces additional analytical complexity.

Future Work. The present study can be extended in several directions:

(i) Investigating the existence and multiplicity of positive solutions under complementary growth conditions, in order to obtain sharper bifurcation-type results.

(ii) Extending the analysis to more general boundary conditions, such as multi-point, integral, or nonlocal boundary conditions, which frequently arise in applications.

(iii) Studying similar systems involving other types of fractional derivatives, such as Caputo or Riemann–Liouville operators, and comparing the qualitative behavior of solutions.

(iv) Considering sign-changing solutions or weakening the cone conditions, which may lead to a broader class of admissible solutions.

(v) Developing numerical methods and approximation schemes for Hadamard fractional systems, which remain relatively less explored compared to other fractional models.

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