

## On Neutrosophic Crisp Axial Set Theory

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**ABSTRACT.** In this paper, we presented new concepts of neutrosophic crisp sets, which are known as neutrosophic crisp axial sets of type (1) and type (2), and studied the relationships between them. We also examine their fundamental and algebraic properties, as well as the studying of neutrosophic crisp axial intersections and unions. To conclude, we presented the concept of topology and some related ideas, laying the groundwork for future in-depth studies of these spaces.

### 1. Introduction

Salama et al. [1,2] examined a concept of neutrosophic crisp sets (NC-sets), and neutrosophic crisp topological spaces (NCT-spaces). Al-Nafee et al. [3] presented the n-valued Refined Neutrosophic crisp sets. Finally, the senses of new types of weakly neutrosophic crisp open mappings and new types of weakly neutrosophic crisp closed functions were informed by Al-Obaidi et al. [4,5]. Tomma and Al-Swidi [6] provided the ideas of stable interior and stable exterior via neutrosophic crisp sets. Hadi et al. [7] was introduced to the concept of neutrosophic axial set. Abdulkadhim et al. [8,9] established the interpretation of neutrosophic crisp generalized alpha generalized closed sets, and neutrosophic crisp generalized alpha generalized closed sets. Imran et al. [10-13] presented the view of new types of weakly neutrosophic crisp continuity, some new concepts of weakly neutrosophic crisp separation

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axioms, new concepts of neutrosophic crisp open sets, and neutrosophic crisp generalized sg-closed sets and their continuity. Al-Tamimi et al. [14] proposed the idea of partner sets for generalizations of MultiNeutrosophic sets. Abdulsada and Al-Swidi [15-17] presented the compatibility of center ideals with center topology, separation axioms of center topological space, and center set theory of proximity space. Al-Swidi and Al-Ethary [18] introduced the innovative concept of compactness with Gem-set. Altalkany and Al-Swidi [19] explored the class of focal function in i-topological spaces via proximity spaces. The main objective of the paper is to study new concepts of neutrosophic crisp sets called neutrosophic crisp axial sets of type (1) and (2). And study their basic and algebraic properties and relationships with each other. Moreover, we studied neutrosophic crisp axial intersections and unions. Finally, we presented the concept of the topology and some of the ideas entrusted to it, laying the groundwork for future in-depth studies of these spaces.

## 2. Preliminaries

**Definition 2.1.** [1] Let  $\mathcal{X}$  be a non-empty fixed sample space. A neutrosophic crisp set (NC-set)  $NC_{\mathcal{A}}$  is an object having the form  $NC_{\mathcal{A}} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle$ , where  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  are subsets of  $\mathcal{X}$ .

**Definition 2.2.** [2] The object having the form  $NC_{\mathcal{A}} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle$  is called:

- (i) A neutrosophic crisp set of type (1) (NC-set of type (1)) if satisfying  $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$ ,  $\mathcal{A}_1 \cap \mathcal{A}_3 = \phi$  and  $\mathcal{A}_2 \cap \mathcal{A}_3 = \phi$ .
- (ii) A neutrosophic crisp set of type (2) (NC-set of type (2)) if satisfying  $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$ ,  $\mathcal{A}_1 \cap \mathcal{A}_3 = \phi$  and  $\mathcal{A}_2 \cap \mathcal{A}_3 = \phi$ ,  $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 = \mathcal{X}$ .
- (iii) A neutrosophic crisp set of type (3) (NC-set of type (3)) if satisfying  $\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 = \phi$  and  $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 = \mathcal{X}$ .

**Definition 2.3.** [2] A family  $\tau_{NC}$  of NC-sets in a non-empty particular fixed space  $\mathcal{X}$  is called a neutrosophic crisp topology (in short, NCT-space) on  $\mathcal{X}$  satisfying the following conditions below:

- (i)  $NC_{\phi}, NC_{\mathcal{X}} \in \tau_{NC}$ .
- (ii)  $NC_{\mathcal{A}} \cap NC_{\mathcal{B}} \in \tau_{NC}$  where  $NC_{\mathcal{A}}, NC_{\mathcal{B}} \in \tau_{NC}$ .
- (iii)  $\sqcup NC_{\mathcal{A}_{\lambda}} \in \tau_{NC}$  for all collections  $\{NC_{\mathcal{A}_{\lambda}} | \lambda \in \Lambda\} \subseteq \tau_{NC}$ .

In this circumstance, the name of the ordered pair  $(\mathcal{X}, \tau_{NC})$  is NCT-space and every NC-set in  $\tau_{NC}$  is titled as neutrosophic crisp open set (in short, NCO-set).

## 3. Types of Neutrosophic Crisp Axial Sets

**Definition 3.1.** Let  $\mathcal{X}$  be a non-empty fixed sample space. A neutrosophic crisp axial set of type (1) (NCA-set of type (1))  $NCA_{\mathcal{A}} = \{\langle \mathcal{A}, \mathcal{H}_1, \mathcal{H}_2 \rangle : \mathcal{A} \cap \mathcal{H}_i = \phi, i = 1, 2\}$ , where  $\mathcal{A}, \mathcal{H}_1, \mathcal{H}_2$  are subsets of  $\mathcal{X}$ .

**Definition 3.2.**

- (i) The neutrosophic crisp axial empty set (NCA-empty set) of type (1) is denoted by:  $NCA_{\phi}$ .

(ii) The neutrosophic crisp axial universal set (NCA-universal set) of type (1) is denoted by:  $NCA_X$ .

**Example 3.3.** Let  $X = \{a, b, c\}$ , where

$$NCA_\phi = \{(\phi, \phi, \phi), (\phi, \phi, \{a\}), (\phi, \phi, \{b\}), (\phi, \phi, \{c\}), (\phi, \phi, \{a, b\}), (\phi, \phi, \{a, c\}), (\phi, \phi, \{b, c\}), (\phi, \phi, X), (\phi, \{a\}, \phi), (\phi, \{a\}, \{a\}), (\phi, \{a\}, \{b\}), (\phi, \{a\}, \{c\}), (\phi, \{a\}, \{a, b\}), (\phi, \{a\}, \{a, c\}), (\phi, \{a\}, \{b, c\}), (\phi, \{a\}, X), (\phi, \{b\}, \phi), (\phi, \{b\}, \{a\}), (\phi, \{b\}, \{b\}), (\phi, \{b\}, \{c\}), (\phi, \{b\}, \{a, b\}), (\phi, \{b\}, \{a, c\}), (\phi, \{b\}, \{b, c\}), (\phi, \{b\}, X), (\phi, \{c\}, \phi), (\phi, \{c\}, \{a\}), (\phi, \{c\}, \{b\}), (\phi, \{c\}, \{c\}), (\phi, \{c\}, \{a, b\}), (\phi, \{c\}, \{a, c\}), (\phi, \{c\}, \{b, c\}), (\phi, \{c\}, X), (\phi, \{a, b\}, \phi), (\phi, \{a, b\}, \{a\}), (\phi, \{a, b\}, \{b\}), (\phi, \{a, b\}, \{c\}), (\phi, \{a, b\}, \{a, b\}), (\phi, \{a, b\}, \{a, c\}), (\phi, \{a, b\}, \{b, c\}), (\phi, \{a, b\}, X), (\phi, \{a, c\}, \phi), (\phi, \{a, c\}, \{a\}), (\phi, \{a, c\}, \{b\}), (\phi, \{a, c\}, \{c\}), (\phi, \{a, c\}, \{a, b\}), (\phi, \{a, c\}, \{a, c\}), (\phi, \{a, c\}, \{b, c\}), (\phi, \{a, c\}, X), (\phi, \{b, c\}, \phi), (\phi, \{b, c\}, \{a\}), (\phi, \{b, c\}, \{b\}), (\phi, \{b, c\}, \{c\}), (\phi, \{b, c\}, \{a, b\}), (\phi, \{b, c\}, \{a, c\}), (\phi, \{b, c\}, \{b, c\}), (\phi, \{b, c\}, X), (\phi, X, \phi), (\phi, X, \{a\}), (\phi, X, \{b\}), (\phi, X, \{c\}), (\phi, X, \{a, b\}), (\phi, X, \{a, c\}), (\phi, X, \{b, c\}), (\phi, X, X)\};$$

$$NCA_{\{a\}} = \{(\{a\}, \phi, \phi), (\{a\}, \phi, \{b\}), (\{a\}, \phi, \{c\}), (\{a\}, \phi, \{b, c\}), (\{a\}, \{b\}, \phi), (\{a\}, \{b\}, \{b\}), (\{a\}, \{b\}, \{c\}), (\{a\}, \{b\}, \{b, c\}), (\{a\}, \{c\}, \phi), (\{a\}, \{c\}, \{b\}), (\{a\}, \{c\}, \{c\}), (\{a\}, \{c\}, \{b, c\}), (\{a\}, \{b, c\}, \phi), (\{a\}, \{b, c\}, \{b\}), (\{a\}, \{b, c\}, \{c\}), (\{a\}, \{b, c\}, \{b, c\})\};$$

$$NCA_{\{b\}} = \{(\{b\}, \phi, \phi), (\{b\}, \phi, \{a\}), (\{b\}, \phi, \{c\}), (\{b\}, \phi, \{a, c\}), (\{b\}, \{a\}, \phi), (\{b\}, \{a\}, \{a\}), (\{b\}, \{a\}, \{c\}), (\{b\}, \{a\}, \{a, c\}), (\{b\}, \{c\}, \phi), (\{b\}, \{c\}, \{a\}), (\{b\}, \{c\}, \{c\}), (\{b\}, \{c\}, \{a, c\}), (\{b\}, \{a, c\}, \phi), (\{b\}, \{a, c\}, \{a\}), (\{b\}, \{a, c\}, \{c\}), (\{b\}, \{a, c\}, \{a, c\})\};$$

$$NCA_{\{c\}} = \{(\{c\}, \phi, \phi), (\{c\}, \phi, \{a\}), (\{c\}, \phi, \{b\}), (\{c\}, \phi, \{a, b\}), (\{c\}, \{a\}, \phi), (\{c\}, \{a\}, \{a\}), (\{c\}, \{a\}, \{b\}), (\{c\}, \{a\}, \{a, b\}), (\{c\}, \{b\}, \phi), (\{c\}, \{b\}, \{a\}), (\{c\}, \{b\}, \{b\}), (\{c\}, \{b\}, \{a, b\}), (\{c\}, \{a, b\}, \phi), (\{c\}, \{a, b\}, \{a\}), (\{c\}, \{a, b\}, \{b\}), (\{c\}, \{a, b\}, \{a, b\})\};$$

$$NCA_{\{a,b\}} = \{(\{a, b\}, \phi, \phi), (\{a, b\}, \phi, \{c\}), (\{a, b\}, \{c\}, \phi), (\{a, b\}, \{c\}, \{c\})\};$$

$$NCA_{\{a,c\}} = \{(\{a, c\}, \phi, \phi), (\{a, c\}, \phi, \{b\}), (\{a, c\}, \{b\}, \phi), (\{a, c\}, \{b\}, \{b\})\};$$

$$NCA_{\{b,c\}} = \{(\{b, c\}, \phi, \phi), (\{b, c\}, \phi, \{a\}), (\{b, c\}, \{a\}, \phi), (\{b, c\}, \{a\}, \{a\})\};$$

$NCA_X = \{(X, \phi, \phi)\}$ . Then  $NCA_\phi, NCA_{\{a\}}, NCA_{\{b\}}, NCA_{\{c\}}, NCA_{\{a,b\}}, NCA_{\{a,c\}}, NCA_{\{b,c\}}$  and  $NCA_X$  are NCA-sets of type (1).

**Definition 3.4.** Let  $X$  be a non-empty fixed sample space. A neutrosophic crisp axial set of type (2) (NCA-set of type (2))  $NCA_B^A = \{(\mathcal{A}, \mathcal{H}_1, \mathcal{H}_2) : \mathcal{B} \cap \mathcal{H}_i = \phi, i = 1, 2\}$ , where  $\mathcal{A}, \mathcal{B}, \mathcal{H}_1, \mathcal{H}_2$  are subsets of  $X$ .

**Definition 3.5.**

(i) The neutrosophic crisp axial empty set (NCA-empty set) of type (2) is denoted by:  $NCA_X^\phi$  or  $NCA_\phi^\phi$ .

(ii) The neutrosophic crisp axial universal set (NCA-universal set) of type (2) is denoted by:  $NCA_X^X$  or  $NCA_X^X$ .

**Remark 3.6.** If  $\mathcal{A} = \mathcal{B}$ , then  $NCA_B^A = NCA_A$ .

**Example 3.7.** Let  $X = \{a, b, c\}$ , then  $NCA_\phi^\phi = NCA_\phi, NCA_{\{a\}}^\phi, NCA_{\{b\}}^\phi, NCA_{\{c\}}^\phi, NCA_{\{a,b\}}^\phi, NCA_{\{a,c\}}^\phi, NCA_{\{b,c\}}^\phi, NCA_X^\phi, NCA_\phi^{\{a\}}, NCA_{\{a\}}^{\{a\}} = NCA_{\{a\}}, NCA_{\{b\}}^{\{a\}}, NCA_{\{c\}}^{\{a\}}, NCA_{\{a,b\}}^{\{a\}}, NCA_{\{a,c\}}^{\{a\}}, NCA_{\{b,c\}}^{\{a\}}, NCA_X^{\{a\}}$ ,

$NCA_{\phi}^{\{b\}}, NCA_{\{a\}}^{\{b\}}, NCA_{\{b\}}^{\{b\}} = NCA_{\{b\}}, NCA_{\{c\}}^{\{b\}}, NCA_{\{a,b\}}^{\{b\}}, NCA_{\{a,c\}}^{\{b\}}, NCA_{\{b,c\}}^{\{b\}}, NCA_X^{\{b\}}, NCA_{\phi}^{\{c\}}, NCA_{\{a\}}^{\{c\}},$   
 $NCA_{\{b\}}^{\{c\}}, NCA_{\{c\}}^{\{c\}} = NCA_{\{c\}}, NCA_{\{a,b\}}^{\{c\}}, NCA_{\{a,c\}}^{\{c\}}, NCA_{\{b,c\}}^{\{c\}}, NCA_X^{\{c\}}, NCA_{\phi}^{\{a,b\}}, NCA_{\{a\}}^{\{a,b\}}, NCA_{\{b\}}^{\{a,b\}},$   
 $NCA_{\{c\}}^{\{a,b\}}, NCA_{\{a,b\}}^{\{a,b\}} = NCA_{\{a,b\}}, NCA_{\{a,c\}}^{\{a,b\}}, NCA_{\{b,c\}}^{\{a,b\}}, NCA_X^{\{a,b\}}, NCA_{\phi}^{\{a,c\}}, NCA_{\{a\}}^{\{a,c\}}, NCA_{\{b\}}^{\{a,c\}}, NCA_{\{c\}}^{\{a,c\}},$   
 $NCA_{\{a,b\}}^{\{a,c\}}, NCA_{\{a,c\}}^{\{a,c\}} = NCA_{\{a,c\}}, NCA_{\{b,c\}}^{\{a,c\}}, NCA_X^{\{a,c\}}, NCA_{\phi}^{\{b,c\}}, NCA_{\{a\}}^{\{b,c\}}, NCA_{\{b\}}^{\{b,c\}}, NCA_{\{c\}}^{\{b,c\}}, NCA_{\{a,b\}}^{\{b,c\}},$   
 $NCA_{\{a,c\}}^{\{b,c\}}, NCA_{\{b,c\}}^{\{b,c\}} = NCA_{\{b,c\}}, NCA_X^{\{b,c\}}, NCA_{\phi}^X, NCA_{\{a\}}^X, NCA_{\{b\}}^X, NCA_{\{c\}}^X, NCA_{\{a,b\}}^X, NCA_{\{a,c\}}^X,$   
 $NCA_{\{b,c\}}^X$  and  $NCA_X^X = NCA_X$  are NCA-sets of type (2).

**Remark 3.8.**

- (i) Every family of NC-set of type (1), (2), (3) is a family of NCA-set of type (1), (2).
- (ii) Every NCA-set of type (1) is NCA-set of type (2).

The following example shows that the converse of the above remark not be true.

**Example 3.9.** Let  $X = \{a, b, c\}$ , then the set  $\{\{a, b\}, \{c\}, \{c\}\}$  is NCA-set of type (1), (2) but not NC-set of type (1), (2), (3). The set  $\{\{a\}, \{a, c\}, \{a, c\}\}$  is NCA-set of type (2) but not NCA-set of type (1).

**Remark 3.10.** The following diagram shows the relation between family of NC-sets of type (1), (2), (3) (family of NC-sets) and family of NCA-sets of type (1), (2).

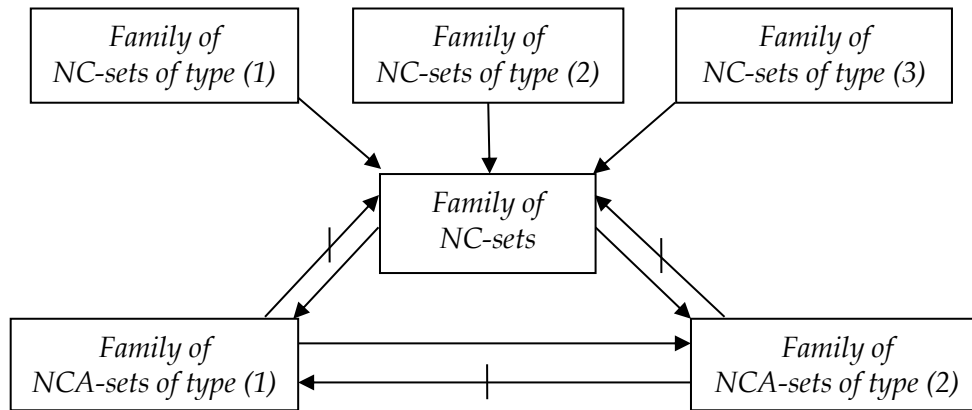


Fig.1. Diagram representing the relation between family of NC-sets and family of NCA-sets

**Definition 3.11.** Let  $NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{B}}$  be NCA-sets of type (1), where  $NCA_{\mathcal{A}} = \{\langle \mathcal{A}, \mathcal{H}_1, \mathcal{H}_2 \rangle : \mathcal{A} \cap \mathcal{H}_i = \phi, i = 1, 2\}$  and  $NCA_{\mathcal{B}} = \{\langle \mathcal{B}, \mathcal{K}_1, \mathcal{K}_2 \rangle : \mathcal{B} \cap \mathcal{K}_i = \phi, i = 1, 2\}$ . Then:

- (i)  $NCA_{\mathcal{A}} \sqsubseteq_1 NCA_{\mathcal{B}}$  iff  $\mathcal{A} \subseteq \mathcal{B}$ .
- (ii)  $NCA_{\mathcal{A}} =_1 NCA_{\mathcal{B}}$  iff  $NCA_{\mathcal{A}} \sqsubseteq_1 NCA_{\mathcal{B}}$  and  $NCA_{\mathcal{B}} \sqsubseteq_1 NCA_{\mathcal{A}}$ .

**Definition 3.12.** Let  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1}$  and  $NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$  be NCA-sets of type (2), where  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} = \{\langle \mathcal{A}_1, \mathcal{H}_1, \mathcal{H}_2 \rangle : \mathcal{B}_1 \cap \mathcal{H}_i = \phi, i = 1, 2\}$  and  $NCA_{\mathcal{B}_2}^{\mathcal{A}_2} = \{\langle \mathcal{A}_2, \mathcal{K}_1, \mathcal{K}_2 \rangle : \mathcal{B}_2 \cap \mathcal{K}_i = \phi, i = 1, 2\}$ . Then:

- (i)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqsubseteq_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$  iff  $\mathcal{A}_1 \subseteq \mathcal{A}_2$  and  $\mathcal{B}_2 \subseteq \mathcal{B}_1$ .
- (ii)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqsubseteq_3 NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$  iff  $\mathcal{A}_1 \subseteq \mathcal{A}_2$  and  $\mathcal{B}_1 \subseteq \mathcal{B}_2$ .

(iii)  $NCA_{B_1}^{\mathcal{A}_1} =_j NCA_{B_2}^{\mathcal{A}_2}$  iff  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_j NCA_{B_2}^{\mathcal{A}_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_j NCA_{B_1}^{\mathcal{A}_1}$ ,  $j = 2,3$ .

**Remark 3.13.**

- (i)  $NCA_{\mathcal{A}} \not\sqsubseteq_1 NCA_{\mathcal{B}}$  iff  $\mathcal{A} \not\subseteq \mathcal{B}$ .
- (ii)  $NCA_{\mathcal{A}} \neq_1 NCA_{\mathcal{B}}$  iff  $NCA_{\mathcal{A}} \not\sqsubseteq_1 NCA_{\mathcal{B}}$  or  $NCA_{\mathcal{B}} \not\sqsubseteq_1 NCA_{\mathcal{A}}$ .
- (iii)  $NCA_{B_1}^{\mathcal{A}_1} \not\sqsubseteq_2 NCA_{B_2}^{\mathcal{A}_2}$  iff  $\mathcal{A}_1 \not\subseteq \mathcal{A}_2$  or  $B_2 \not\subseteq B_1$ .
- (iv)  $NCA_{B_1}^{\mathcal{A}_1} \not\sqsubseteq_3 NCA_{B_2}^{\mathcal{A}_2}$  iff  $\mathcal{A}_1 \not\subseteq \mathcal{A}_2$  or  $B_1 \not\subseteq B_2$ .
- (v)  $NCA_{B_1}^{\mathcal{A}_1} \neq_j NCA_{B_2}^{\mathcal{A}_2}$  iff  $NCA_{B_1}^{\mathcal{A}_1} \not\sqsubseteq_j NCA_{B_2}^{\mathcal{A}_2}$  or  $NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_j NCA_{B_1}^{\mathcal{A}_1}$ ,  $j = 2,3$ .

**Remark 3.14.**

- (i) If  $NCA_{\mathcal{A}} \sqsubseteq_1 NCA_{\mathcal{B}}$  and  $NCA_{\mathcal{B}} \sqsubseteq_1 NCA_{\mathcal{C}}$ , then  $NCA_{\mathcal{A}} \sqsubseteq_1 NCA_{\mathcal{C}}$ .
- (ii) If  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_j NCA_{B_2}^{\mathcal{A}_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_j NCA_{B_3}^{\mathcal{A}_3}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_j NCA_{B_3}^{\mathcal{A}_3}$ ,  $j = 2,3$ .

**Definition 3.15.** Let  $\mathcal{X}$  be any non-empty set, the neutrosophic crisp axial points (NCA-points) is denoted by NCAP and define as follows:

- (i)  $NCAP_1 = \langle \{p_1\}, \{p_2\}, \{p_3\} \rangle$ ,  $p_1 \neq p_2, p_1 \neq p_3$  where  $p_1, p_2, p_3 \in \mathcal{X}$ .
- (ii)  $NCAP_2 = \langle \{p\}, \phi, \{p\}^c \rangle$ .
- (iii)  $NCAP_3 = \langle \phi, \{p\}, \{p\} \rangle$ .
- (iv)  $NCAP_4 = \langle \{p\}, \phi, \phi \rangle$ ,  $\{p\}$  is a singleton.
- (v)  $NCAP_5 = \langle \phi, \{p\}, \phi \rangle$ ,  $\{p\}$  is a singleton.
- (vi)  $NCAP_6 = \langle \phi, \phi, \{p\} \rangle$ ,  $\{p\}$  is a singleton.

**Remark 3.16.**  $NCAP_k$  are NCA-points but not NCA-sets,  $k = 1,2, \dots,6$ .

**Definition 3.17.**

- (i) The singular neutrosophic crisp axial point of NCA-set of type (1)  $NCA_{\mathcal{A}}$  is denoted by:  $SNCAP_{\mathcal{A}} = \langle \mathcal{A}, \mathcal{H}_1, \mathcal{H}_2 \rangle \ni \mathcal{A} \cap \mathcal{H}_i = \phi, i = 1,2$  and  $SNCAP_{\mathcal{A}} \in NCA_{\mathcal{A}}$ .
- (ii) The singular neutrosophic crisp axial point of NCA-set of type (2)  $NCA_{\mathcal{B}}^{\mathcal{A}}$  is denoted by:  $SNCAP_{\mathcal{B}}^{\mathcal{A}} = \langle \mathcal{A}, \mathcal{H}_1, \mathcal{H}_2 \rangle \ni \mathcal{B} \cap \mathcal{H}_i = \phi, i = 1,2$  and  $SNCAP_{\mathcal{B}}^{\mathcal{A}} \in NCA_{\mathcal{B}}^{\mathcal{A}}$ .

**Definition 3.18.** Let  $\mathcal{X}$  be non-empty set, and let  $NCA_{\mathcal{A}} = \{ \langle \mathcal{A}, \mathcal{H}_1, \mathcal{H}_2 \rangle : \mathcal{A} \cap \mathcal{H}_i = \phi, i = 1,2 \}$  be a NCA-set of type (1). Then:

- (i)  $NCAP_1 \in_1 NCA_{\mathcal{A}}$  iff  $p_1 \in \mathcal{A}, p_2 \in \mathcal{H}_1, p_3 \in \mathcal{H}_2$ .
- (ii)  $NCAP_2 \in_2 NCA_{\mathcal{A}}$  iff  $p \in \mathcal{A}$ .
- (iii)  $NCAP_3 \in_3 NCA_{\mathcal{A}}$  iff  $p \notin \mathcal{A}$ .
- (iv)  $NCAP_4 \in_4 NCA_{\mathcal{A}}$  iff  $p \in \mathcal{A}$ .
- (v)  $NCAP_5 \in_5 NCA_{\mathcal{A}}$  iff  $p \in \mathcal{H}_1$ .
- (vi)  $NCAP_6 \in_6 NCA_{\mathcal{A}}$  iff  $p \in \mathcal{H}_2$ .

**Definition 3.19.** Let  $\mathcal{X}$  be a non-empty set, and let  $NCA_{\mathcal{B}}^{\mathcal{A}} = \{ \langle \mathcal{A}, \mathcal{H}_1, \mathcal{H}_2 \rangle : \mathcal{B} \cap \mathcal{H}_i = \phi, i = 1,2 \}$  be a NCA-set of type (2). Then:

- (i)  $NCAP_1 \in_1 NCA_{\mathcal{B}}^{\mathcal{A}}$  iff  $p_1 \in \mathcal{A}, p_1 \in \mathcal{B}$  or  $p_1 \notin \mathcal{B}, p_2 \in \mathcal{H}_1, p_3 \in \mathcal{H}_2$ .

- (ii)  $NCAP_2 \in_2 NCA_B^A$  iff  $p \in \mathcal{A}$ ,  $p \in \mathcal{B}$  or  $p \notin \mathcal{B}$ .
- (iii)  $NCAP_3 \in_3 NCA_B^A$  iff  $p \notin \mathcal{A}$ ,  $p \notin \mathcal{B}$ .
- (iv)  $NCAP_4 \in_4 NCA_B^A$  iff  $p \in \mathcal{A}$ ,  $p \in \mathcal{B}$  or  $p \notin \mathcal{B}$ .
- (v)  $NCAP_5 \in_5 NCA_B^A$  iff  $p \in \mathcal{H}_1$ .
- (vi)  $NCAP_6 \in_6 NCA_B^A$  iff  $p \in \mathcal{H}_2$ .

**Remark 3.20.**

- (i)  $NCAP_1 \notin_1 NCA_{\mathcal{A}}$  iff  $p_1 \notin \mathcal{A}$  or  $p_2 \notin \mathcal{H}_1$  or  $p_3 \notin \mathcal{H}_2$ .
- (ii)  $NCAP_2 \notin_2 NCA_{\mathcal{A}}$  iff  $p \notin \mathcal{A}$ .
- (iii)  $NCAP_3 \notin_3 NCA_{\mathcal{A}}$  iff  $p \in \mathcal{A}$ .
- (iv)  $NCAP_4 \notin_4 NCA_{\mathcal{A}}$  iff  $p \notin \mathcal{A}$ .
- (v)  $NCAP_5 \notin_5 NCA_{\mathcal{A}}$  iff  $p \notin \mathcal{H}_1$ .
- (vi)  $NCAP_6 \notin_6 NCA_{\mathcal{A}}$  iff  $p \notin \mathcal{H}_2$ .
- (vii)  $NCAP_1 \notin_1 NCA_B^A$  iff  $p_1 \notin \mathcal{A}$  or  $p_1 \notin \mathcal{B}$  or  $p_1 \in \mathcal{B}$  or  $p_2 \notin \mathcal{H}_1$  or  $p_3 \notin \mathcal{H}_2$ .
- (viii)  $NCAP_2 \notin_2 NCA_B^A$  iff  $p \notin \mathcal{A}$  or  $p \notin \mathcal{B}$  or  $p \in \mathcal{B}$ .
- (ix)  $NCAP_3 \notin_3 NCA_B^A$  iff  $p \in \mathcal{A}$  or  $p \in \mathcal{B}$ .
- (x)  $NCAP_4 \notin_4 NCA_B^A$  iff  $p \notin \mathcal{A}$  or  $p \notin \mathcal{B}$  or  $p \in \mathcal{B}$ .
- (xi)  $NCAP_5 \notin_5 NCA_B^A$  iff  $p \notin \mathcal{H}_1$ .
- (xii)  $NCAP_6 \notin_6 NCA_B^A$  iff  $p \notin \mathcal{H}_2$ .

**Definition 3.21.** Let  $NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{B}}$  be two NCA-sets of type (1), the intersection between two NCA-sets of type (1) as follows:  $NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{B}} = NCA_{\mathcal{A} \cap \mathcal{B}} = \{(\mathcal{A} \cap \mathcal{B}, \mathcal{H}_1, \mathcal{H}_2): (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{H}_i = \phi, i = 1, 2\}$ .

**Definition 3.22.** Let  $NCA_{B_1}^{A_1}$  and  $NCA_{B_2}^{A_2}$  be two NCA-sets of type (2), the intersection between two NCA-sets of type (2) as follows:

- (i)  $NCA_{B_1}^{A_1} \sqcap_2 NCA_{B_2}^{A_2} = NCA_{B_1 \cup B_2}^{A_1 \cap A_2} = \{(\mathcal{A}_1 \cap \mathcal{A}_2, \mathcal{H}_1, \mathcal{H}_2): (B_1 \cup B_2) \cap \mathcal{H}_i = \phi, i = 1, 2\}$
- (ii)  $NCA_{B_1}^{A_1} \sqcap_3 NCA_{B_2}^{A_2} = NCA_{B_1 \cap B_2}^{A_1 \cap A_2} = \{(\mathcal{A}_1 \cap \mathcal{A}_2, \mathcal{H}_1, \mathcal{H}_2): (B_1 \cap B_2) \cap \mathcal{H}_i = \phi, i = 1, 2\}$ .

**Definition 3.23.** Let  $NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{B}}$  be two NCA-sets of type (1), the union between two NCA-sets of type (1) as follows:  $NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{B}} = NCA_{\mathcal{A} \cup \mathcal{B}} = \{(\mathcal{A} \cup \mathcal{B}, \mathcal{H}_1, \mathcal{H}_2): (\mathcal{A} \cup \mathcal{B}) \cap \mathcal{H}_i = \phi, i = 1, 2\}$ .

**Definition 3.24.** Let  $NCA_{B_1}^{A_1}$  and  $NCA_{B_2}^{A_2}$  be two NCA-sets of type (2), the union between two NCA-sets of type (2) as follows:

- (i)  $NCA_{B_1}^{A_1} \sqcup_2 NCA_{B_2}^{A_2} = NCA_{B_1 \cap B_2}^{A_1 \cup A_2} = \{(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{H}_1, \mathcal{H}_2): (B_1 \cap B_2) \cap \mathcal{H}_i = \phi, i = 1, 2\}$
- (ii)  $NCA_{B_1}^{A_1} \sqcup_3 NCA_{B_2}^{A_2} = NCA_{B_1 \cup B_2}^{A_1 \cup A_2} = \{(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{H}_1, \mathcal{H}_2): (B_1 \cup B_2) \cap \mathcal{H}_i = \phi, i = 1, 2\}$ .

**Remark 3.25.** Let  $NCA_{B_1}^{A_1}$  and  $NCA_{B_2}^{A_2}$  be two NCA-sets of type (2), then:

- (i)  $NCA_{B_1}^{A_1} \sqcap_2 NCA_{B_2}^{A_2} \sqsubseteq_2 NCA_{B_1}^{A_1} \sqcap_3 NCA_{B_2}^{A_2}$
- (ii)  $NCA_{B_1}^{A_1} \sqcap_3 NCA_{B_2}^{A_2} \sqsubseteq_3 NCA_{B_1}^{A_1} \sqcap_2 NCA_{B_2}^{A_2}$
- (iii)  $NCA_{B_1}^{A_1} \sqcup_3 NCA_{B_2}^{A_2} \sqsubseteq_2 NCA_{B_1}^{A_1} \sqcup_2 NCA_{B_2}^{A_2}$

$$(iv) NCA_{B_1}^{A_1} \sqcup_2 NCA_{B_2}^{A_2} \sqsubseteq_3 NCA_{B_1}^{A_1} \sqcup_3 NCA_{B_2}^{A_2}.$$

**Definition 3.26.**

(i) The NCA-sum between two NCA-sets of type (1),  $NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{B}}$  is  $NCA_{\mathcal{A}} \oplus_1 NCA_{\mathcal{B}} = NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{B}}$ .

(ii) The NCA-sum between two NCA-sets of type (2),  $NCA_{B_1}^{A_1}$  and  $NCA_{B_2}^{A_2}$  is  $NCA_{B_1}^{A_1} \oplus_j NCA_{B_2}^{A_2} = NCA_{B_1}^{A_1} \sqcup_j NCA_{B_2}^{A_2}$ ,  $j = 2, 3$ .

**Definition 3.27.** Let  $\mathcal{X}$  be a non-empty fixed set and  $NCA_{\mathcal{A}}$  be NCA-set of type (1), the complement of  $NCA_{\mathcal{A}}$  as follows:  $(NCA_{\mathcal{A}})^{C_1} = NCA_{\mathcal{A}^C} = \{(\mathcal{A}^C, \mathcal{H}_1, \mathcal{H}_2) : \mathcal{A}^C \cap \mathcal{H}_i = \phi, i = 1, 2\}$ .

**Definition 3.28.** Let  $\mathcal{X}$  be a non-empty fixed set and  $NCA_{\mathcal{B}}^{\mathcal{A}}$  be NCA-set of type (2), the complement of  $NCA_{\mathcal{B}}^{\mathcal{A}}$  as follows:

$$(i) (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_2} = NCA_{\mathcal{B}^C}^{\mathcal{A}^C} = \{(\mathcal{A}^C, \mathcal{H}_1, \mathcal{H}_2) : \mathcal{B}^C \cap \mathcal{H}_i = \phi, i = 1, 2\}$$

$$(ii) (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_3} = NCA_{\mathcal{B}}^{\mathcal{A}^C} = \{(\mathcal{A}^C, \mathcal{H}_1, \mathcal{H}_2) : \mathcal{B} \cap \mathcal{H}_i = \phi, i = 1, 2\}.$$

**Proposition 3.29.** Let  $NCA_{\mathcal{A}}$  be NCA-set of type (1), then:

$$(i) NCA_{\mathcal{A}} \sqcap_1 (NCA_{\mathcal{A}})^{C_1} = NCA_{\phi}$$

$$(ii) NCA_{\mathcal{A}} \sqcup_1 (NCA_{\mathcal{A}})^{C_1} = NCA_{\mathcal{X}}.$$

*Proof.*

$$(i) NCA_{\mathcal{A}} \sqcap_1 (NCA_{\mathcal{A}})^{C_1} = NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{A}^C} = NCA_{\mathcal{A} \cap \mathcal{A}^C} = NCA_{\phi}.$$

$$(ii) NCA_{\mathcal{A}} \sqcup_1 (NCA_{\mathcal{A}})^{C_1} = NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{A}^C} = NCA_{\mathcal{A} \cup \mathcal{A}^C} = NCA_{\mathcal{X}}. \blacksquare$$

**Proposition 3.30.** Let  $NCA_{\mathcal{B}}^{\mathcal{A}}$  be NCA-set of type (2), then:

$$(i) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcap_2 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_2} = NCA_{\mathcal{X}}^{\phi}$$

$$(ii) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcap_3 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_2} = NCA_{\phi}^{\phi}$$

$$(iii) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcup_2 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_2} = NCA_{\phi}^{\mathcal{X}}$$

$$(iv) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcup_3 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_2} = NCA_{\mathcal{X}}^{\mathcal{X}}.$$

*Proof.*

$$(i) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcap_2 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_2} = NCA_{\mathcal{B}}^{\mathcal{A}} \sqcap_2 NCA_{\mathcal{B}^C}^{\mathcal{A}^C} = NCA_{\mathcal{B} \cup \mathcal{B}^C}^{\mathcal{A} \cap \mathcal{A}^C} = NCA_{\mathcal{X}}^{\phi}.$$

$$(ii) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcap_3 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_2} = NCA_{\mathcal{B}}^{\mathcal{A}} \sqcap_3 NCA_{\mathcal{B}^C}^{\mathcal{A}^C} = NCA_{\mathcal{B} \cap \mathcal{B}^C}^{\mathcal{A} \cap \mathcal{A}^C} = NCA_{\phi}^{\phi}.$$

$$(iii) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcup_2 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_2} = NCA_{\mathcal{B}}^{\mathcal{A}} \sqcup_2 NCA_{\mathcal{B}^C}^{\mathcal{A}^C} = NCA_{\mathcal{B} \cap \mathcal{B}^C}^{\mathcal{A} \cup \mathcal{A}^C} = NCA_{\phi}^{\mathcal{X}}.$$

$$(iv) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcup_3 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_2} = NCA_{\mathcal{B}}^{\mathcal{A}} \sqcup_3 NCA_{\mathcal{B}^C}^{\mathcal{A}^C} = NCA_{\mathcal{B} \cup \mathcal{B}^C}^{\mathcal{A} \cup \mathcal{A}^C} = NCA_{\mathcal{X}}^{\mathcal{X}}. \blacksquare$$

**Proposition 3.31.** Let  $NCA_{\mathcal{B}}^{\mathcal{A}}$  be NCA-set of type (2), then:

$$(i) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcap_2 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_3} = NCA_{\mathcal{B}}^{\mathcal{A}} \sqcap_3 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_3}$$

$$(ii) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcup_2 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_3} = NCA_{\mathcal{B}}^{\mathcal{A}} \sqcup_3 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_3}$$

*Proof.* Let  $NCA_{\mathcal{B}}^{\mathcal{A}}$  be NCA-set of type (2).

$$(i) NCA_{\mathcal{B}}^{\mathcal{A}} \sqcap_2 (NCA_{\mathcal{B}}^{\mathcal{A}})^{C_3} = NCA_{\mathcal{B}}^{\mathcal{A}} \sqcap_2 NCA_{\mathcal{B}}^{\mathcal{A}^C} = NCA_{\mathcal{B} \cup \mathcal{B}}^{\mathcal{A} \cap \mathcal{A}^C} = NCA_{\mathcal{B}}^{\phi}.$$

And  $NCA_B^{\mathcal{A}} \sqcap_3 (NCA_B^{\mathcal{A}})^{C_3} = NCA_B^{\mathcal{A}} \sqcap_3 NCA_B^{\mathcal{A}^C} = NCA_{B \cap B}^{\mathcal{A} \cap \mathcal{A}^C} = NCA_B^{\phi}$ .

Hence,  $NCA_B^{\mathcal{A}} \sqcap_2 (NCA_B^{\mathcal{A}})^{C_3} = NCA_B^{\mathcal{A}} \sqcap_3 (NCA_B^{\mathcal{A}})^{C_3}$ .

(ii)  $NCA_B^{\mathcal{A}} \sqcup_2 (NCA_B^{\mathcal{A}})^{C_3} = NCA_B^{\mathcal{A}} \sqcup_2 NCA_B^{\mathcal{A}^C} = NCA_{B \cap B}^{\mathcal{A} \cup \mathcal{A}^C} = NCA_B^{\chi}$ .

And  $NCA_B^{\mathcal{A}} \sqcup_3 (NCA_B^{\mathcal{A}})^{C_3} = NCA_B^{\mathcal{A}} \sqcup_3 NCA_B^{\mathcal{A}^C} = NCA_{B \cup B}^{\mathcal{A} \cup \mathcal{A}^C} = NCA_B^{\chi}$ .

Hence,  $NCA_B^{\mathcal{A}} \sqcup_2 (NCA_B^{\mathcal{A}})^{C_3} = NCA_B^{\mathcal{A}} \sqcup_3 (NCA_B^{\mathcal{A}})^{C_3}$ . ■

**Proposition 3.32.**

(i)  $NCA_{\mathcal{A}} \sqsubseteq_1 NCA_B$  iff  $NCA_{\mathcal{A}} \sqcap_1 (NCA_B)^{C_1} = NCA_{\phi}$ .

(ii)  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 NCA_{B_2}^{\mathcal{A}_2}$  iff  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_2} = NCA_{\chi}^{\phi}$ .

(iii)  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 NCA_{B_2}^{\mathcal{A}_2}$  iff  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_2} = NCA_{\chi}^{\phi}$ .

(iv)  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 NCA_{B_2}^{\mathcal{A}_2}$  iff  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_2} = NCA_{\chi}^{\phi}$ .

**Remark 3.33.**

(i) If  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 NCA_{B_2}^{\mathcal{A}_2}$ , is not necessarily the  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_3} = NCA_{\phi}^{\phi}$ .

(ii) If  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_3} = NCA_{\phi}^{\phi}$ , is not necessarily the  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 NCA_{B_2}^{\mathcal{A}_2}$ .

(iii) If  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 NCA_{B_2}^{\mathcal{A}_2}$ , is not necessarily the  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_3} = NCA_{\chi}^{\phi}$ .

(iv) If  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_3} = NCA_{\chi}^{\phi}$ , is not necessarily the  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 NCA_{B_2}^{\mathcal{A}_2}$ .

(v) If  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 NCA_{B_2}^{\mathcal{A}_2}$ , is not necessarily the  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_3} = NCA_{\phi}^{\phi}$ .

(vi) If  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_3} = NCA_{\phi}^{\phi}$ , is not necessarily the  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 NCA_{B_2}^{\mathcal{A}_2}$ .

(vii) If  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 NCA_{B_2}^{\mathcal{A}_2}$ , is not necessarily the  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_3} = NCA_{\chi}^{\phi}$ .

(viii) If  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_3} = NCA_{\chi}^{\phi}$ , is not necessarily the  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 NCA_{B_2}^{\mathcal{A}_2}$ .

**Proposition 3.34.**  $NCA_{\mathcal{A}} \sqcap_1 NCA_B = NCA_{\phi}$  iff  $NCA_{\mathcal{A}} \sqsubseteq_1 (NCA_B)^{C_1}$  and  $NCA_B \sqsubseteq_1 (NCA_{\mathcal{A}})^{C_1}$ .

*Proof.* Let  $NCA_{\mathcal{A}}$  and  $NCA_B$  be two NCA-sets of type (1) and  $NCA_{\mathcal{A}} \sqcap_1 NCA_B = NCA_{\phi}$ .

Since  $NCA_{\mathcal{A}} \sqcap_1 NCA_B = NCA_{\mathcal{A} \cap B}$ , we have  $NCA_{\mathcal{A} \cap B} = NCA_{\phi}$  implies  $\mathcal{A} \cap B = \phi$ , then  $\mathcal{A} \subseteq B^C$  and  $B \subseteq \mathcal{A}^C$ . Therefore,  $NCA_{\mathcal{A}} \sqsubseteq_1 NCA_{B^C}$  and  $NCA_B \sqsubseteq_1 NCA_{\mathcal{A}^C}$ .

Hence,  $NCA_{\mathcal{A}} \sqsubseteq_1 (NCA_B)^{C_1}$  and  $NCA_B \sqsubseteq_1 (NCA_{\mathcal{A}})^{C_1}$ .

Conversely, let  $NCA_{\mathcal{A}} \sqsubseteq_1 (NCA_B)^{C_1}$  and  $NCA_B \sqsubseteq_1 (NCA_{\mathcal{A}})^{C_1}$ . Since  $NCA_{\mathcal{A}} \sqsubseteq_1 (NCA_B)^{C_1} = NCA_{B^C}$  implies  $NCA_{\mathcal{A}} \sqsubseteq_1 NCA_{B^C}$ , then  $\mathcal{A} \subseteq B^C$ .

And  $NCA_B \sqsubseteq_1 (NCA_{\mathcal{A}})^{C_1} = NCA_{\mathcal{A}^C}$  implies  $NCA_B \sqsubseteq_1 NCA_{\mathcal{A}^C}$ , then  $B \subseteq \mathcal{A}^C$ . Thus  $\mathcal{A} \cap B = \phi$ .

Hence,  $NCA_{\mathcal{A}} \sqcap_1 NCA_B = NCA_{\mathcal{A} \cap B} = NCA_{\phi}$ . ■

**Proposition 3.35.**

(i)  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\chi}^{\phi}$  iff  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_2 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ .

(ii)  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\phi}^{\phi}$  iff  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_3 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ .

(iii)  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\chi}^{\phi}$  iff  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_3}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_2 (NCA_{B_1}^{\mathcal{A}_1})^{C_3}$ .

(iv)  $NCA_{B_1}^{\mathcal{A}_1} \Pi_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\mathcal{X}}^{\phi}$  iff  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_3}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_3 (NCA_{B_1}^{\mathcal{A}_1})^{C_3}$

*Proof.* Let  $NCA_{B_1}^{\mathcal{A}_1}$  and  $NCA_{B_2}^{\mathcal{A}_2}$  be two NCA-sets of type (2).

(i) Suppose that  $NCA_{B_1}^{\mathcal{A}_1} \Pi_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\mathcal{X}}^{\phi}$ .

Since  $NCA_{B_1}^{\mathcal{A}_1} \Pi_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{B_1 \cup B_2}^{\mathcal{A}_1 \cap \mathcal{A}_2}$ , we have  $NCA_{B_1 \cup B_2}^{\mathcal{A}_1 \cap \mathcal{A}_2} = NCA_{\mathcal{X}}^{\phi}$  implies  $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$  and  $B_1 \cup B_2 = \mathcal{X}$  ( $B_1^c \cap B_2^c = \phi$ ).

If  $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$ , then  $\mathcal{A}_1 \subseteq \mathcal{A}_2^c$  and  $\mathcal{A}_2 \subseteq \mathcal{A}_1^c$ . And  $B_1 \cup B_2 = \mathcal{X}$ , we have  $B_1^c \subseteq B_2$  and  $B_2^c \subseteq B_1$ . Since  $\mathcal{A}_1 \subseteq \mathcal{A}_2^c$  and  $B_2^c \subseteq B_1$  implies  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 NCA_{B_2^c}^{\mathcal{A}_2^c} = (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and since  $\mathcal{A}_2 \subseteq \mathcal{A}_1^c$  and  $B_1^c \subseteq B_2$  implies  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_2 NCA_{B_1^c}^{\mathcal{A}_1^c} = (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ . Hence,  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_2 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ .

Conversely, let  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_2 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ .

Since  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_2} = NCA_{B_2^c}^{\mathcal{A}_2^c}$ , we have  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 NCA_{B_2^c}^{\mathcal{A}_2^c}$  implies  $\mathcal{A}_1 \subseteq \mathcal{A}_2^c$  and  $B_2^c \subseteq B_1$ . And  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_2 (NCA_{B_1}^{\mathcal{A}_1})^{C_2} = NCA_{B_1^c}^{\mathcal{A}_1^c}$ , we have  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_2 NCA_{B_1^c}^{\mathcal{A}_1^c}$  implies  $\mathcal{A}_2 \subseteq \mathcal{A}_1^c$  and  $B_1^c \subseteq B_2$ . Thus  $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$  and  $B_1 \cup B_2 = \mathcal{X}$ . Therefore  $NCA_{B_1}^{\mathcal{A}_1} \Pi_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{B_1 \cup B_2}^{\mathcal{A}_1 \cap \mathcal{A}_2} = NCA_{\mathcal{X}}^{\phi}$ . Hence,  $NCA_{B_1}^{\mathcal{A}_1} \Pi_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\mathcal{X}}^{\phi}$ .

(ii) Suppose that  $NCA_{B_1}^{\mathcal{A}_1} \Pi_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\phi}^{\phi}$ .

Since  $NCA_{B_1}^{\mathcal{A}_1} \Pi_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{B_1 \cap B_2}^{\mathcal{A}_1 \cap \mathcal{A}_2}$ , we have  $NCA_{B_1 \cap B_2}^{\mathcal{A}_1 \cap \mathcal{A}_2} = NCA_{\phi}^{\phi}$  implies  $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$  and  $B_1 \cap B_2 = \phi$ .

If  $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$ , then  $\mathcal{A}_1 \subseteq \mathcal{A}_2^c$  and  $\mathcal{A}_2 \subseteq \mathcal{A}_1^c$ . And  $B_1 \cap B_2 = \phi$ , we have  $B_1 \subseteq B_2^c$  and  $B_2 \subseteq B_1^c$ . Since  $\mathcal{A}_1 \subseteq \mathcal{A}_2^c$  and  $B_1 \subseteq B_2^c$  implies  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 NCA_{B_2^c}^{\mathcal{A}_2^c} = (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and since  $\mathcal{A}_2 \subseteq \mathcal{A}_1^c$  and  $B_2 \subseteq B_1^c$  implies  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_3 NCA_{B_1^c}^{\mathcal{A}_1^c} = (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ . Hence,  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_3 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ .

Conversely, let  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_3 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ .

Since  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_2} = NCA_{B_2^c}^{\mathcal{A}_2^c}$ , we have  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 NCA_{B_2^c}^{\mathcal{A}_2^c}$  implies  $\mathcal{A}_1 \subseteq \mathcal{A}_2^c$  and  $B_1 \subseteq B_2^c$ . And  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_3 (NCA_{B_1}^{\mathcal{A}_1})^{C_2} = NCA_{B_1^c}^{\mathcal{A}_1^c}$ , we have  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_3 NCA_{B_1^c}^{\mathcal{A}_1^c}$  implies  $\mathcal{A}_2 \subseteq \mathcal{A}_1^c$  and  $B_2 \subseteq B_1^c$ . Thus  $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$  and  $B_1 \cap B_2 = \phi$ . Therefore  $NCA_{B_1}^{\mathcal{A}_1} \Pi_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{B_1 \cap B_2}^{\mathcal{A}_1 \cap \mathcal{A}_2} = NCA_{\phi}^{\phi}$ . Hence,  $NCA_{B_1}^{\mathcal{A}_1} \Pi_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\phi}^{\phi}$ . The proof is clear for others. ■

**Remark 3.36.**

(i) If  $NCA_{B_1}^{\mathcal{A}_1} \Pi_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\mathcal{X}}^{\phi}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \not\sqsubseteq_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_3 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ .

- (ii) If  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_3 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 NCA_{B_2}^{\mathcal{A}_2} \neq NCA_{\mathcal{X}}^{\phi}$ .
- (iii) If  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\phi}^{\phi}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \not\sqsubseteq_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_3}$  and  $NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_3 (NCA_{B_1}^{\mathcal{A}_1})^{C_3}$ .
- (iv) If  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_3}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_3 (NCA_{B_1}^{\mathcal{A}_1})^{C_3}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} \neq NCA_{\phi}^{\phi}$ .
- (v) If  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\phi}^{\phi}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \not\sqsubseteq_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_3}$  and  $NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_2 (NCA_{B_1}^{\mathcal{A}_1})^{C_3}$ .
- (vi) If  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_3}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_2 (NCA_{B_1}^{\mathcal{A}_1})^{C_3}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} \neq NCA_{\phi}^{\phi}$ .
- (vii) If  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\phi}^{\phi}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \not\sqsubseteq_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_2 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ .
- (viii) If  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_2 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} \neq NCA_{\phi}^{\phi}$ .

**Theorem 3.37.** If  $NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{B}} = NCA_{\phi}$  and  $NCAP_k \in_k NCA_{\mathcal{A}}$ , then  $NCAP_k \notin_k NCA_{\mathcal{B}}$ ,  $k = 1, 2, \dots, 6$ .

*Proof.* Let  $NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{B}}$  be two NCA-sets of type (1) and let  $NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{B}} = NCA_{\phi}$  and  $NCAP_k \in_k NCA_{\mathcal{A}}$ ,  $k = 1, 2, \dots, 6$ . Since  $NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{B}} = NCA_{\phi}$ , then  $NCA_{\mathcal{A}} \sqsubseteq_1 (NCA_{\mathcal{B}})^{C_1}$  and  $NCA_{\mathcal{B}} \sqsubseteq_1 (NCA_{\mathcal{A}})^{C_1}$ .

$NCAP_1 \in_1 NCA_{\mathcal{A}}$ , implies  $NCAP_1 \in_1 (NCA_{\mathcal{B}})^{C_1}$ . Therefore  $NCAP_1 \notin_1 NCA_{\mathcal{B}}$ . Hence,  $NCAP_k \notin_k NCA_{\mathcal{B}}$ ,  $k = 1, 2, \dots, 6$ . ■

**Theorem 3.38.**

- (i) If  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\mathcal{X}}^{\phi}$  and  $NCAP_k \in_k NCA_{B_1}^{\mathcal{A}_1}$ , then  $NCAP_k \notin_k NCA_{B_2}^{\mathcal{A}_2}$ ,  $k = 1, 2, \dots, 6$ .
- (ii) If  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\phi}^{\phi}$  and  $NCAP_k \in_k NCA_{B_1}^{\mathcal{A}_1}$ , then  $NCAP_k \notin_k NCA_{B_2}^{\mathcal{A}_2}$ ,  $k = 1, 2, \dots, 6$ .

*Proof.* Let  $NCA_{B_1}^{\mathcal{A}_1}$  and  $NCA_{B_2}^{\mathcal{A}_2}$  be two NCA-sets of type (2).

- (i) Let  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\mathcal{X}}^{\phi}$  and  $NCAP_k \in_k NCA_{B_1}^{\mathcal{A}_1}$ ,  $k = 1, 2, \dots, 6$ .

Since  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\mathcal{X}}^{\phi}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_2 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_2 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ .

$NCAP_1 \in_1 NCA_{B_1}^{\mathcal{A}_1}$ , implies  $NCAP_1 \in_1 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$ . Therefore,  $NCAP_1 \notin_1 NCA_{B_2}^{\mathcal{A}_2}$ .

Hence,  $NCAP_k \notin_k NCA_{B_2}^{\mathcal{A}_2}$ ,  $k = 1, 2, \dots, 6$ .

- (ii) Let  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\phi}^{\phi}$  and  $NCAP_k \in_k NCA_{B_1}^{\mathcal{A}_1}$ ,  $k = 1, 2, \dots, 6$ .

Since  $NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{\phi}^{\phi}$ , then  $NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_3 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$  and  $NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_3 (NCA_{B_1}^{\mathcal{A}_1})^{C_2}$ .

$NCAP_1 \in_1 NCA_{B_1}^{\mathcal{A}_1}$ , implies  $NCAP_1 \in_1 (NCA_{B_2}^{\mathcal{A}_2})^{C_2}$ . Therefore,  $NCAP_1 \notin_1 NCA_{B_2}^{\mathcal{A}_2}$ .

Hence,  $NCAP_k \notin_k NCA_{B_2}^{\mathcal{A}_2}$ ,  $k = 1, 2, \dots, 6$ . ■

**Theorem 3.39.** Let  $NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{B}}$  be two NCA-sets of type (1), then:

- (i)  $(NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{B}})^{C_1} = (NCA_{\mathcal{A}})^{C_1} \sqcup_1 (NCA_{\mathcal{B}})^{C_1}$
- (ii)  $(NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{B}})^{C_1} = (NCA_{\mathcal{A}})^{C_1} \sqcap_1 (NCA_{\mathcal{B}})^{C_1}$ .

*Proof.* Let  $NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{B}}$  be two NCA-sets of type (1).

- (i) Since  $NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{B}} = NCA_{\mathcal{A} \cap \mathcal{B}} = \{(\mathcal{A} \cap \mathcal{B}, \mathcal{H}_1, \mathcal{H}_2): (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{H}_i = \phi, i = 1, 2\}$ .

$(NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{B}})^{C_1} = (NCA_{\mathcal{A} \cap \mathcal{B}})^{C_1} = NCA_{(\mathcal{A} \cap \mathcal{B})^c} = NCA_{\mathcal{A}^c \cup \mathcal{B}^c}$ .

And  $(NCA_{\mathcal{A}})^{C_1} \sqcup_1 (NCA_{\mathcal{B}})^{C_1} = NCA_{\mathcal{A}^c} \sqcup_1 NCA_{\mathcal{B}^c} = NCA_{\mathcal{A}^c \cup \mathcal{B}^c}$ .

Thus,  $(NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{B}})^{C_1} = (NCA_{\mathcal{A}})^{C_1} \sqcup_1 (NCA_{\mathcal{B}})^{C_1}$ .

(ii) Since  $NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{B}} = NCA_{\mathcal{A} \cup \mathcal{B}} = \{(\mathcal{A} \cup \mathcal{B}, \mathcal{H}_1, \mathcal{H}_2): (\mathcal{A} \cup \mathcal{B}) \cap \mathcal{H}_i = \phi, i = 1, 2\}$ .

$(NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{B}})^{C_1} = (NCA_{\mathcal{A} \cup \mathcal{B}})^{C_1} = NCA_{(\mathcal{A} \cup \mathcal{B})^c} = NCA_{\mathcal{A}^c \cap \mathcal{B}^c}$ .

And  $(NCA_{\mathcal{A}})^{C_1} \sqcup_1 (NCA_{\mathcal{B}})^{C_1} = NCA_{\mathcal{A}^c} \sqcup_1 NCA_{\mathcal{B}^c} = NCA_{\mathcal{A}^c \cap \mathcal{B}^c}$ .

Hence,  $(NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{B}})^{C_1} = (NCA_{\mathcal{A}})^{C_1} \sqcup_1 (NCA_{\mathcal{B}})^{C_1}$ . ■

**Theorem 3.40.** Let  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1}$  and  $NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$  be two NCA-sets of type (2), then:

(i)  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2} = (NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_2} \sqcup_2 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2}$

(ii)  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2} = (NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_2} \sqcup_2 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2}$

(iii)  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_3 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2} = (NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_2} \sqcup_3 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2}$

(iv)  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_3 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2} = (NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_2} \sqcup_3 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2}$ .

*Proof.* Let  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1}$  and  $NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$  be two NCA-sets of type (2), where  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} = \{(\mathcal{A}_1, \mathcal{H}_1, \mathcal{H}_2): \mathcal{B}_1 \cap \mathcal{H}_i = \phi, i = 1, 2\}$  and  $NCA_{\mathcal{B}_2}^{\mathcal{A}_2} = \{(\mathcal{A}_2, \mathcal{K}_1, \mathcal{K}_2): \mathcal{B}_2 \cap \mathcal{K}_i = \phi, i = 1, 2\}$ .

(i) Since  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2} = NCA_{\mathcal{B}_1 \cup \mathcal{B}_2}^{\mathcal{A}_1 \cap \mathcal{A}_2}$ .

$(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2} = (NCA_{\mathcal{B}_1 \cup \mathcal{B}_2}^{\mathcal{A}_1 \cap \mathcal{A}_2})^{C_2} = NCA_{(\mathcal{B}_1 \cup \mathcal{B}_2)^c}^{\mathcal{A}_1 \cap \mathcal{A}_2} = NCA_{\mathcal{B}_1^c \cap \mathcal{B}_2^c}^{\mathcal{A}_1 \cap \mathcal{A}_2}$ .

And  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_2} \sqcup_2 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2} = NCA_{\mathcal{B}_1^c}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2^c}^{\mathcal{A}_2} = NCA_{\mathcal{B}_1^c \cap \mathcal{B}_2^c}^{\mathcal{A}_1 \cap \mathcal{A}_2}$ .

Hence,  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2} = (NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_2} \sqcup_2 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2}$ .

(ii) Since  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2} = NCA_{\mathcal{B}_1 \cap \mathcal{B}_2}^{\mathcal{A}_1 \cup \mathcal{A}_2}$ .

$(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2} = (NCA_{\mathcal{B}_1 \cap \mathcal{B}_2}^{\mathcal{A}_1 \cup \mathcal{A}_2})^{C_2} = NCA_{(\mathcal{B}_1 \cap \mathcal{B}_2)^c}^{\mathcal{A}_1 \cup \mathcal{A}_2} = NCA_{\mathcal{B}_1^c \cup \mathcal{B}_2^c}^{\mathcal{A}_1 \cup \mathcal{A}_2}$ .

And  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_2} \sqcup_2 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2} = NCA_{\mathcal{B}_1^c}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2^c}^{\mathcal{A}_2} = NCA_{\mathcal{B}_1^c \cup \mathcal{B}_2^c}^{\mathcal{A}_1 \cup \mathcal{A}_2}$ .

Thus,  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2} = (NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_2} \sqcup_2 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_2}$ . The proof is clear for others. ■

**Remark 3.41.** Let  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1}$  and  $NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$  be two NCA-sets of type (2), then:

(i)  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3} \sqsubseteq_2 (NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_3} \sqcup_2 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3}$

(ii)  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_3} \sqcup_2 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3} \sqsubseteq_2 (NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3}$

(iii)  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_3} \sqcup_3 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3} \sqsubseteq_2 (NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_3 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3}$

(iv)  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_3 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3} \sqsubseteq_2 (NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_3} \sqcup_3 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3}$ .

**Remark 3.42.** Let  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1}$  and  $NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$  be two NCA-sets of type (2), then:

(i)  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_3} \sqcup_2 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3} \sqsubseteq_3 (NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3}$

(ii)  $(NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcup_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3} \sqsubseteq_3 (NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_3} \sqcup_2 (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_3}$

$$(iii) (NCA_{B_1}^{A_1} \sqcap_3 NCA_{B_2}^{A_2})^{C_3} \sqsubseteq_3 (NCA_{B_1}^{A_1})^{C_3} \sqcup_3 (NCA_{B_2}^{A_2})^{C_3}$$

$$(iv) (NCA_{B_1}^{A_1})^{C_3} \sqcap_3 (NCA_{B_2}^{A_2})^{C_3} \sqsubseteq_3 (NCA_{B_1}^{A_1} \sqcup_3 NCA_{B_2}^{A_2})^{C_3}.$$

**Lemma 3.43.** Let  $NCA_{B_1}^{A_1}$  and  $NCA_{B_2}^{A_2}$  be two NCA-sets of type (2), then:

$$(i) (NCA_{B_1}^{A_1} \sqcap_2 NCA_{B_2}^{A_2})^{C_3} = (NCA_{B_1}^{A_1})^{C_3} \sqcup_3 (NCA_{B_2}^{A_2})^{C_3}$$

$$(ii) (NCA_{B_1}^{A_1} \sqcup_2 NCA_{B_2}^{A_2})^{C_3} = (NCA_{B_1}^{A_1})^{C_3} \sqcap_3 (NCA_{B_2}^{A_2})^{C_3}$$

$$(iii) (NCA_{B_1}^{A_1} \sqcap_3 NCA_{B_2}^{A_2})^{C_3} = (NCA_{B_1}^{A_1})^{C_3} \sqcup_2 (NCA_{B_2}^{A_2})^{C_3}$$

$$(iv) (NCA_{B_1}^{A_1} \sqcup_3 NCA_{B_2}^{A_2})^{C_3} = (NCA_{B_1}^{A_1})^{C_3} \sqcap_2 (NCA_{B_2}^{A_2})^{C_3}$$

*Proof.* Let  $NCA_{B_1}^{A_1}$  and  $NCA_{B_2}^{A_2}$  be two NCA-sets of type (2).

$$(i) \text{ Since } NCA_{B_1}^{A_1} \sqcap_2 NCA_{B_2}^{A_2} = NCA_{B_1 \cup B_2}^{A_1 \cap A_2}.$$

$$(NCA_{B_1}^{A_1} \sqcap_2 NCA_{B_2}^{A_2})^{C_3} = (NCA_{B_1 \cup B_2}^{A_1 \cap A_2})^{C_3} = NCA_{B_1 \cup B_2}^{(A_1 \cap A_2)^C} = NCA_{B_1 \cup B_2}^{A_1^C \cup A_2^C}.$$

$$\text{And } (NCA_{B_1}^{A_1})^{C_3} \sqcup_3 (NCA_{B_2}^{A_2})^{C_3} = NCA_{B_1}^{A_1^C} \sqcup_3 NCA_{B_2}^{A_2^C} = NCA_{B_1 \cup B_2}^{A_1^C \cup A_2^C}.$$

$$\text{Hence, } (NCA_{B_1}^{A_1} \sqcap_2 NCA_{B_2}^{A_2})^{C_3} = (NCA_{B_1}^{A_1})^{C_3} \sqcup_3 (NCA_{B_2}^{A_2})^{C_3}.$$

$$(ii) \text{ Since } NCA_{B_1}^{A_1} \sqcup_2 NCA_{B_2}^{A_2} = NCA_{B_1 \cap B_2}^{A_1 \cup A_2}.$$

$$(NCA_{B_1}^{A_1} \sqcup_2 NCA_{B_2}^{A_2})^{C_3} = (NCA_{B_1 \cap B_2}^{A_1 \cup A_2})^{C_3} = NCA_{B_1 \cap B_2}^{(A_1 \cup A_2)^C} = NCA_{B_1 \cap B_2}^{A_1^C \cap A_2^C}.$$

$$\text{And } (NCA_{B_1}^{A_1})^{C_3} \sqcap_3 (NCA_{B_2}^{A_2})^{C_3} = NCA_{B_1}^{A_1^C} \sqcap_3 NCA_{B_2}^{A_2^C} = NCA_{B_1 \cap B_2}^{A_1^C \cap A_2^C}.$$

Thus,  $(NCA_{B_1}^{A_1} \sqcup_2 NCA_{B_2}^{A_2})^{C_3} = (NCA_{B_1}^{A_1})^{C_3} \sqcap_3 (NCA_{B_2}^{A_2})^{C_3}$ . The proof is clear for others. ■

**Remark 3.44.** Let  $NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{B}}$  be two NCA-sets of type (1), then:

$$(i) (NCA_{\mathcal{A}})^{C_1 C_1} = NCA_{\mathcal{A}}$$

$$(ii) (NCA_{\phi})^{C_1} = NCA_{\mathcal{X}}$$

$$(iii) (NCA_{\mathcal{X}})^{C_1} = NCA_{\phi}$$

$$(iv) \text{ If } NCA_{\mathcal{A}} \sqsubseteq_1 NCA_{\mathcal{B}}, \text{ then } (NCA_{\mathcal{B}})^{C_1} \sqsubseteq_1 (NCA_{\mathcal{A}})^{C_1}$$

**Remark 3.45.**

$$(i) (NCA_{\mathcal{X}}^{\phi})^{C_2} = NCA_{\mathcal{X}}^{\mathcal{X}}$$

$$(ii) (NCA_{\phi}^{\mathcal{X}})^{C_2} = NCA_{\mathcal{X}}^{\phi}$$

$$(iii) (NCA_{\phi}^{\phi})^{C_2} = NCA_{\mathcal{X}}^{\mathcal{X}}$$

$$(iv) (NCA_{\mathcal{X}}^{\mathcal{X}})^{C_2} = NCA_{\phi}^{\phi}$$

$$(v) (NCA_{\mathcal{X}}^{\phi})^{C_3} = NCA_{\mathcal{X}}^{\mathcal{X}}$$

$$(vi) (NCA_{\phi}^{\mathcal{X}})^{C_3} = NCA_{\phi}^{\phi}$$

$$(vii) (NCA_{\phi}^{\phi})^{C_3} = NCA_{\phi}^{\mathcal{X}}$$

$$(viii) (NCA_{\mathcal{X}}^{\mathcal{X}})^{C_3} = NCA_{\mathcal{X}}^{\phi}.$$

**Proposition 3.46.** Let  $NCA_{B_1}^{A_1}$  and  $NCA_{B_2}^{A_2}$  be two NCA-sets of type (2), then:

- (i)  $(NCA_{B_1}^{A_1})^{C_j} = NCA_{B_1}^{A_1}, j = 2,3$
- (ii) If  $NCA_{B_1}^{A_1} \sqsubseteq_2 NCA_{B_2}^{A_2}$ , then  $(NCA_{B_2}^{A_2})^{C_2} \sqsubseteq_2 (NCA_{B_1}^{A_1})^{C_2}$
- (iii) If  $NCA_{B_1}^{A_1} \sqsubseteq_3 NCA_{B_2}^{A_2}$ , then  $(NCA_{B_2}^{A_2})^{C_2} \sqsubseteq_3 (NCA_{B_1}^{A_1})^{C_2}$
- (iv) If  $NCA_{B_1}^{A_1} \sqsubseteq_2 NCA_{B_2}^{A_2}$ , then  $(NCA_{B_2}^{A_2})^{C_3} \sqsubseteq_3 (NCA_{B_1}^{A_1})^{C_3}$
- (v) If  $NCA_{B_1}^{A_1} \sqsubseteq_3 NCA_{B_2}^{A_2}$ , then  $(NCA_{B_2}^{A_2})^{C_3} \sqsubseteq_2 (NCA_{B_1}^{A_1})^{C_3}$ .

*Proof.* The proof is clear for others. ■

#### 4. Algebraic Properties of Neutrosophic Crisp Axial Sets

**Proposition 4.1.** Let  $NCA_{\mathcal{A}}$  be NCA-set of type (1), then:

- (i)  $NCA_{\mathcal{A}} \sqcap_1 NCA_{\phi} = NCA_{\phi}$
- (ii)  $NCA_{\mathcal{A}} \sqcup_1 NCA_{\phi} = NCA_{\mathcal{A}}$
- (iii)  $NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{X}} = NCA_{\mathcal{A}}$
- (iv)  $NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{X}} = NCA_{\mathcal{X}}$

*Proof.* The proof is clear for others. ■

**Remark 4.2.** Let  $NCA_B^A$  be NCA-set of type (2), then:

- (i)  $NCA_B^A \sqcap_2 NCA_{\mathcal{X}}^{\phi} = NCA_{\mathcal{X}}^{\phi}$
- (ii)  $NCA_B^A \sqcup_2 NCA_{\mathcal{X}}^{\phi} = NCA_B^A$
- (iii)  $NCA_B^A \sqcap_2 NCA_{\phi}^{\mathcal{X}} = NCA_B^A$
- (iv)  $NCA_B^A \sqcup_2 NCA_{\phi}^{\mathcal{X}} = NCA_{\phi}^{\mathcal{X}}$
- (v)  $NCA_B^A \sqcap_3 NCA_{\phi}^{\phi} = NCA_{\phi}^{\phi}$
- (vi)  $NCA_B^A \sqcup_3 NCA_{\phi}^{\phi} = NCA_B^A$
- (vii)  $NCA_B^A \sqcap_3 NCA_{\mathcal{X}}^{\mathcal{X}} = NCA_B^A$
- (viii)  $NCA_B^A \sqcup_3 NCA_{\mathcal{X}}^{\mathcal{X}} = NCA_{\mathcal{X}}^{\mathcal{X}}$

**Theorem 4.3.** Let  $NCA_{\mathcal{A}}, NCA_B$  and  $NCA_C$  be NCA-sets of type (1), then:

- (i)  $NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{A}} = NCA_{\mathcal{A}}$
- (ii)  $NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{A}} = NCA_{\mathcal{A}}$
- (iii)  $NCA_{\mathcal{A}} \sqcap_1 NCA_B = NCA_B \sqcap_1 NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{A}} \sqcup_1 NCA_B = NCA_B \sqcup_1 NCA_{\mathcal{A}}$
- (iv)  $NCA_{\mathcal{A}} \sqcap_1 NCA_B \sqsubseteq_1 NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{A}} \sqcap_1 NCA_B \sqsubseteq_1 NCA_B$
- (v)  $NCA_{\mathcal{A}} \sqsubseteq_1 NCA_{\mathcal{A}} \sqcup_1 NCA_B$  and  $NCA_B \sqsubseteq_1 NCA_{\mathcal{A}} \sqcup_1 NCA_B$
- (vi)  $(NCA_{\mathcal{A}} \sqcap_1 NCA_B) \sqcap_1 NCA_C = NCA_{\mathcal{A}} \sqcap_1 (NCA_B \sqcap_1 NCA_C)$
- (vii)  $(NCA_{\mathcal{A}} \sqcup_1 NCA_B) \sqcup_1 NCA_C = NCA_{\mathcal{A}} \sqcup_1 (NCA_B \sqcup_1 NCA_C)$
- (viii)  $(NCA_{\mathcal{A}} \sqcup_1 NCA_B) \sqcap_1 NCA_C = (NCA_{\mathcal{A}} \sqcap_1 NCA_C) \sqcup_1 (NCA_B \sqcap_1 NCA_C)$

*Proof.* (i), (ii) are clear.

$$(iii) NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{B}} = NCA_{\mathcal{A} \cap \mathcal{B}} = NCA_{\mathcal{B} \cap \mathcal{A}} = NCA_{\mathcal{B}} \sqcap_1 NCA_{\mathcal{A}}.$$

$$NCA_{\mathcal{A}} \sqcup_1 NCA_{\mathcal{B}} = NCA_{\mathcal{A} \cup \mathcal{B}} = NCA_{\mathcal{B} \cup \mathcal{A}} = NCA_{\mathcal{B}} \sqcup_1 NCA_{\mathcal{A}}. \text{ The proof is clear for others. } \blacksquare$$

**Theorem 4.4.** Let  $NCA_{B_1}^{\mathcal{A}_1}$ ,  $NCA_{B_2}^{\mathcal{A}_2}$  and  $NCA_{B_3}^{\mathcal{A}_3}$  be NCA-sets of type (2), then:

$$(i) NCA_{B_1}^{\mathcal{A}_1} \sqcap_j NCA_{B_1}^{\mathcal{A}_1} = NCA_{B_1}^{\mathcal{A}_1}, j = 2, 3$$

$$(ii) NCA_{B_1}^{\mathcal{A}_1} \sqcup_j NCA_{B_1}^{\mathcal{A}_1} = NCA_{B_1}^{\mathcal{A}_1}, j = 2, 3$$

$$(iii) NCA_{B_1}^{\mathcal{A}_1} \sqcap_j NCA_{B_2}^{\mathcal{A}_2} = NCA_{B_2}^{\mathcal{A}_2} \sqcap_j NCA_{B_1}^{\mathcal{A}_1}, j = 2, 3$$

$$(iv) NCA_{B_1}^{\mathcal{A}_1} \sqcup_j NCA_{B_2}^{\mathcal{A}_2} = NCA_{B_2}^{\mathcal{A}_2} \sqcup_j NCA_{B_1}^{\mathcal{A}_1}, j = 2, 3$$

$$(v) NCA_{B_1}^{\mathcal{A}_1} \sqcap_j NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_j NCA_{B_1}^{\mathcal{A}_1} \text{ and } NCA_{B_1}^{\mathcal{A}_1} \sqcap_j NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_j NCA_{B_2}^{\mathcal{A}_2}, j = 2, 3$$

$$(vi) NCA_{B_1}^{\mathcal{A}_1} \sqsubseteq_j NCA_{B_1}^{\mathcal{A}_1} \sqcup_j NCA_{B_1}^{\mathcal{A}_1} \text{ and } NCA_{B_2}^{\mathcal{A}_2} \sqsubseteq_j NCA_{B_1}^{\mathcal{A}_1} \sqcup_j NCA_{B_1}^{\mathcal{A}_1}, j = 2, 3$$

$$(vii) (NCA_{B_1}^{\mathcal{A}_1} \sqcap_j NCA_{B_2}^{\mathcal{A}_2}) \sqcap_j NCA_{B_3}^{\mathcal{A}_3} = NCA_{B_1}^{\mathcal{A}_1} \sqcap_j (NCA_{B_2}^{\mathcal{A}_2} \sqcap_j NCA_{B_3}^{\mathcal{A}_3}), j = 2, 3$$

$$(viii) (NCA_{B_1}^{\mathcal{A}_1} \sqcup_j NCA_{B_2}^{\mathcal{A}_2}) \sqcup_j NCA_{B_3}^{\mathcal{A}_3} = NCA_{B_1}^{\mathcal{A}_1} \sqcup_j (NCA_{B_2}^{\mathcal{A}_2} \sqcup_j NCA_{B_3}^{\mathcal{A}_3}), j = 2, 3$$

$$(ix) (NCA_{B_1}^{\mathcal{A}_1} \sqcup_j NCA_{B_2}^{\mathcal{A}_2}) \sqcap_j NCA_{B_3}^{\mathcal{A}_3} = (NCA_{B_1}^{\mathcal{A}_1} \sqcap_j NCA_{B_3}^{\mathcal{A}_3}) \sqcup_j (NCA_{B_2}^{\mathcal{A}_2} \sqcap_j NCA_{B_3}^{\mathcal{A}_3}), j = 2, 3$$

*Proof.* (i), (ii) are clear.

$$(iii) NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{B_1 \cap B_2}^{\mathcal{A}_1 \cap \mathcal{A}_2} = NCA_{B_2 \cap B_1}^{\mathcal{A}_2 \cap \mathcal{A}_1} = NCA_{B_2}^{\mathcal{A}_2} \sqcap_2 NCA_{B_1}^{\mathcal{A}_1}.$$

$$NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{B_1 \cap B_2}^{\mathcal{A}_1 \cap \mathcal{A}_2} = NCA_{B_2 \cap B_1}^{\mathcal{A}_2 \cap \mathcal{A}_1} = NCA_{B_2}^{\mathcal{A}_2} \sqcap_3 NCA_{B_1}^{\mathcal{A}_1}.$$

$$(iv) NCA_{B_1}^{\mathcal{A}_1} \sqcup_2 NCA_{B_2}^{\mathcal{A}_2} = NCA_{B_1 \cup B_2}^{\mathcal{A}_1 \cup \mathcal{A}_2} = NCA_{B_2 \cup B_1}^{\mathcal{A}_2 \cup \mathcal{A}_1} = NCA_{B_2}^{\mathcal{A}_2} \sqcup_2 NCA_{B_1}^{\mathcal{A}_1}.$$

$$NCA_{B_1}^{\mathcal{A}_1} \sqcup_3 NCA_{B_2}^{\mathcal{A}_2} = NCA_{B_1 \cup B_2}^{\mathcal{A}_1 \cup \mathcal{A}_2} = NCA_{B_2 \cup B_1}^{\mathcal{A}_2 \cup \mathcal{A}_1} = NCA_{B_2}^{\mathcal{A}_2} \sqcup_3 NCA_{B_1}^{\mathcal{A}_1}.$$

The proof is clear for others.  $\blacksquare$

**Remark 4.5.**

$$(i) NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_3 NCA_{B_1}^{\mathcal{A}_1} \text{ and } NCA_{B_1}^{\mathcal{A}_1} \sqcap_2 NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_3 NCA_{B_2}^{\mathcal{A}_2}.$$

$$(ii) NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_2 NCA_{B_1}^{\mathcal{A}_1} \text{ and } NCA_{B_1}^{\mathcal{A}_1} \sqcap_3 NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_2 NCA_{B_2}^{\mathcal{A}_2}.$$

$$(iii) NCA_{B_1}^{\mathcal{A}_1} \not\sqsubseteq_3 NCA_{B_1}^{\mathcal{A}_1} \sqcup_2 NCA_{B_1}^{\mathcal{A}_1} \text{ and } NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_3 NCA_{B_1}^{\mathcal{A}_1} \sqcup_2 NCA_{B_1}^{\mathcal{A}_1}.$$

$$(iv) NCA_{B_1}^{\mathcal{A}_1} \not\sqsubseteq_2 NCA_{B_1}^{\mathcal{A}_1} \sqcup_3 NCA_{B_1}^{\mathcal{A}_1} \text{ and } NCA_{B_2}^{\mathcal{A}_2} \not\sqsubseteq_2 NCA_{B_1}^{\mathcal{A}_1} \sqcup_3 NCA_{B_1}^{\mathcal{A}_1}.$$

**Example 4.6.** Let  $\mathcal{X} = \{a, b, c\}$  and  $NCA_{\{b\}}^{\{a\}}$  and  $NCA_{\{c\}}^{\{b\}}$  be two NCA-sets of type (2).

$$(i) \text{ Since } NCA_{\{b\}}^{\{a\}} \sqcap_2 NCA_{\{c\}}^{\{b\}} = NCA_{\{b\} \cap \{c\}}^{\{a\} \cap \{b\}} = NCA_{\{b,c\}}^{\emptyset}. \text{ Then}$$

$$NCA_{\{b\}}^{\{a\}} \sqcap_2 NCA_{\{c\}}^{\{b\}} \not\sqsubseteq_3 NCA_{\{b\}}^{\{a\}} \text{ and } NCA_{\{b\}}^{\{a\}} \sqcap_2 NCA_{\{c\}}^{\{b\}} \not\sqsubseteq_3 NCA_{\{c\}}^{\{b\}}$$

$$(ii) \text{ Since } NCA_{\{b\}}^{\{a\}} \sqcap_3 NCA_{\{c\}}^{\{b\}} = NCA_{\{b\} \cap \{c\}}^{\{a\} \cap \{b\}} = NCA_{\emptyset}^{\emptyset}. \text{ Then}$$

$$NCA_{\{b\}}^{\{a\}} \sqcap_3 NCA_{\{c\}}^{\{b\}} \not\sqsubseteq_2 NCA_{\{b\}}^{\{a\}} \text{ and } NCA_{\{b\}}^{\{a\}} \sqcap_3 NCA_{\{c\}}^{\{b\}} \not\sqsubseteq_2 NCA_{\{c\}}^{\{b\}}.$$

$$(iii) \text{ Since } NCA_{\{b\}}^{\{a\}} \sqcup_2 NCA_{\{c\}}^{\{b\}} = NCA_{\{b\} \cup \{c\}}^{\{a\} \cup \{b\}} = NCA_{\emptyset}^{\{a,b\}}. \text{ Then}$$

$$NCA_{\{b\}}^{\{a\}} \not\sqsubseteq_3 NCA_{\{b\}}^{\{a\}} \sqcup_2 NCA_{\{c\}}^{\{b\}} \text{ and } NCA_{\{c\}}^{\{b\}} \not\sqsubseteq_3 NCA_{\{b\}}^{\{a\}} \sqcup_2 NCA_{\{c\}}^{\{b\}}.$$

(iv) Since  $NCA_{\{b\}}^{\{a\}} \sqcup_3 NCA_{\{c\}}^{\{b\}} = NCA_{\{b\} \cup \{c\}}^{\{a\} \cup \{b\}} = NCA_{\{b,c\}}^{\{a,b\}}$ . Then

$$NCA_{\{b\}}^{\{a\}} \not\sqsubseteq_2 NCA_{\{b\}}^{\{a\}} \sqcup_3 NCA_{\{c\}}^{\{b\}} \text{ and } NCA_{\{c\}}^{\{b\}} \not\sqsubseteq_2 NCA_{\{b\}}^{\{a\}} \sqcup_3 NCA_{\{c\}}^{\{b\}}.$$

**Definition 4.7.** Let  $NCA_{\mathcal{A}}$  and  $NCA_{\mathcal{B}}$  be two NCA-sets of type (1), then:

- (i)  $NCA_{\mathcal{A}-1}NCA_{\mathcal{B}} = NCA_{\mathcal{A}} \sqcap_1 (NCA_{\mathcal{B}})^{C_1}$
- (ii)  $NCA_{\mathcal{A}\Delta_1}NCA_{\mathcal{B}} = (NCA_{\mathcal{A}} \sqcap_1 (NCA_{\mathcal{B}})^{C_1}) \sqcup_1 (NCA_{\mathcal{B}} \sqcap_1 (NCA_{\mathcal{A}})^{C_1})$ .

**Definition 4.8.** Let  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1}$  and  $NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$  be two NCA-sets of type (2), then:

- (i)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} -_j NCA_{\mathcal{B}_2}^{\mathcal{A}_2} = NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcap_j (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_j}, j = 2, 3$
- (ii)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \Delta_j NCA_{\mathcal{B}_2}^{\mathcal{A}_2} = (NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcap_j (NCA_{\mathcal{B}_2}^{\mathcal{A}_2})^{C_j}) \sqcup_j (NCA_{\mathcal{B}_2}^{\mathcal{A}_2} \sqcap_j (NCA_{\mathcal{B}_1}^{\mathcal{A}_1})^{C_j}), j = 2, 3$ .

**Proposition 4.9.** Let  $NCA_{\mathcal{A}}, NCA_{\mathcal{B}}$  and  $NCA_{\mathcal{C}}$  be NCA-sets of type (1), then:

- (i)  $NCA_{\mathcal{A}\Delta_1}NCA_{\mathcal{A}} = NCA_{\phi}$
- (ii)  $NCA_{\mathcal{A}\Delta_1}NCA_{\phi} = NCA_{\mathcal{A}}$
- (iii)  $NCA_{\mathcal{A}\Delta_1}NCA_{\mathcal{B}} = NCA_{\phi}$  iff  $NCA_{\mathcal{A}} =_1 NCA_{\mathcal{B}}$
- (iv)  $NCA_{\mathcal{A}\Delta_1}NCA_{\mathcal{B}} = NCA_{\mathcal{B}\Delta_1}NCA_{\mathcal{A}}$
- (v)  $NCA_{\mathcal{A}\Delta_1}(NCA_{\mathcal{B}\Delta_1}NCA_{\mathcal{C}}) = (NCA_{\mathcal{A}\Delta_1}NCA_{\mathcal{B}})\Delta_1 NCA_{\mathcal{C}}$
- (vi)  $NCA_{\mathcal{A}\Delta_1}(NCA_{\mathcal{B}\Delta_1}NCA_{\mathcal{C}}) = (NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{B}})\Delta_1 (NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{C}})$ .

*Proof.* The proof is clear for others. ■

**Theorem 4.10.** Let  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1}, NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$  and  $NCA_{\mathcal{B}_3}^{\mathcal{A}_3}$  be NCA-sets of type (2), then:

- (i)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \Delta_2 NCA_{\mathcal{B}_1}^{\mathcal{A}_1} = NCA_{\mathcal{X}}^{\phi}$  and  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \Delta_3 NCA_{\mathcal{B}_1}^{\mathcal{A}_1} = NCA_{\phi}^{\phi}$
- (ii)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \Delta_2 NCA_{\mathcal{X}}^{\phi} = NCA_{\mathcal{B}_1}^{\mathcal{A}_1}$  and  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \Delta_3 NCA_{\phi}^{\phi} = NCA_{\mathcal{B}_1}^{\mathcal{A}_1}$
- (iii)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \Delta_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2} = NCA_{\mathcal{X}}^{\phi}$  iff  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} =_2 NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$
- (iv)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \Delta_3 NCA_{\mathcal{B}_2}^{\mathcal{A}_2} = NCA_{\phi}^{\phi}$  iff  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} =_3 NCA_{\mathcal{B}_2}^{\mathcal{A}_2}$
- (v)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \Delta_j NCA_{\mathcal{B}_2}^{\mathcal{A}_2} = NCA_{\mathcal{B}_2}^{\mathcal{A}_2} \Delta_j NCA_{\mathcal{B}_1}^{\mathcal{A}_1}, j = 2, 3$
- (vi)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \Delta_j (NCA_{\mathcal{B}_2}^{\mathcal{A}_2} \Delta_j NCA_{\mathcal{B}_3}^{\mathcal{A}_3}) = (NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \Delta_j NCA_{\mathcal{B}_2}^{\mathcal{A}_2}) \Delta_j NCA_{\mathcal{B}_3}^{\mathcal{A}_3}, j = 2, 3$
- (vii)  $NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcap_j (NCA_{\mathcal{B}_2}^{\mathcal{A}_2} \Delta_j NCA_{\mathcal{B}_3}^{\mathcal{A}_3}) = (NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcap_j NCA_{\mathcal{B}_2}^{\mathcal{A}_2}) \Delta_j (NCA_{\mathcal{B}_1}^{\mathcal{A}_1} \sqcap_j NCA_{\mathcal{B}_3}^{\mathcal{A}_3}), j = 2, 3$

*Proof.* The proof is clear for others. ■

### 5. Neutrosophic Crisp Axial Topological Spaces

**Definition 5.1.** Let  $\mathcal{X}$  be a fixed set that is not empty, a neutrosophic crisp axial topological space of type

(1) (in short, NCAT-space of type (1)) is a family  $\tau_{NCA}^1$  satisfies the following conditions:

- (i)  $NCA_{\phi}, NCA_{\mathcal{X}} \in \tau_{NCA}^1$ .
- (ii) If  $NCA_{\mathcal{A}}, NCA_{\mathcal{B}} \in \tau_{NCA}^1$ , then  $NCA_{\mathcal{A}} \sqcap_1 NCA_{\mathcal{B}} \in \tau_{NCA}^1$ .
- (iii) Let  $NCA_{\mathcal{A}_{\lambda}} \in \tau_{NCA}^1, \forall \lambda \in \Lambda$ , then  $\sqcup_{\lambda \in \Lambda} NCA_{\mathcal{A}_{\lambda}} \in \tau_{NCA}^1$ .

Then  $(X, \tau_{NCA}^1)$  is a NCAT-space of type (1). The elements in  $\tau_{NCA}^1$  are called neutrosophic crisp axial open sets of type (1) and it is denoted by (NCAO-sets of type (1)), the complement of type 1 for (NCAO-sets of type (1)) are neutrosophic crisp axial closed sets of type (1) and it is denoted by (NCAC-sets of type (1)).

**Example 5.2.** Let  $X = \{a, b, c\}$ , then  $\tau_{NCA}^1 = \{NCA_\phi, NCA_X, NCA_{\{a\}}, NCA_{\{b\}}, NCA_{\{a,b\}}\}$  is a NCAT-space of type (1).

**Definition 5.3.** Let  $X$  be a fixed set that is not empty, a neutrosophic crisp axial topological space of type (2) (in short, NCAT-space of type (2)) is a family  $\tau_{NCA}^2$  satisfies the following conditions:

- (i)  $NCA_X^\phi, NCA_X^x \in \tau_{NCA}^2$ .
- (ii) If  $NCA_{B_1}^{A_1}, NCA_{B_2}^{A_2} \in \tau_{NCA}^2$ , then  $NCA_{B_1}^{A_1} \cap_2 NCA_{B_2}^{A_2} \in \tau_{NCA}^2$ .
- (iii) Let  $NCA_{B_\lambda}^{A_\lambda} \in \tau_{NCA}^2, \forall \lambda \in \Lambda$ , then  $\cup_{2 \lambda \in \Lambda} NCA_{B_\lambda}^{A_\lambda} \in \tau_{NCA}^2$ .

Then  $(X, \tau_{NCA}^2)$  is a NCAT-space of type (2). The elements in  $\tau_{NCA}^2$  are called neutrosophic crisp axial open sets of type (2) and it is denoted by (NCAO-sets of type (2)), the complement of type 2 for (NCAO-sets of type (2)) are neutrosophic crisp axial closed sets of type (2) and it is denoted by (NCAC-sets of type (2)).

**Definition 5.4.** Let  $X$  be a fixed set that is not empty, a neutrosophic crisp axial topological space of type (3) (in short, NCAT-space of type (3)) is a family  $\tau_{NCA}^3$  satisfies the following conditions:

- (i)  $NCA_\phi^\phi, NCA_X^x \in \tau_{NCA}^3$ .
- (ii) If  $NCA_{B_1}^{A_1}, NCA_{B_2}^{A_2} \in \tau_{NCA}^3$ , then  $NCA_{B_1}^{A_1} \cap_3 NCA_{B_2}^{A_2} \in \tau_{NCA}^3$ .
- (iii) Let  $NCA_{B_\lambda}^{A_\lambda} \in \tau_{NCA}^3, \forall \lambda \in \Lambda$ , then  $\cup_{3 \lambda \in \Lambda} NCA_{B_\lambda}^{A_\lambda} \in \tau_{NCA}^3$ .

Then  $(X, \tau_{NCA}^3)$  is a NCAT-space of type (3). The elements in  $\tau_{NCA}^3$  are called neutrosophic crisp axial open sets of type (2) and it is denoted by (NCAO-sets of type (2)), the complement of type 2 for (NCAO-sets of type (2)) are neutrosophic crisp axial closed sets of type (2) and it is denoted by (NCAC-sets of type (2)).

**Example 5.5.** Let  $X = \{a, b, c\}$ , then

$\tau_{NCA}^2 = \{NCA_X^\phi, NCA_X^x, NCA_{\{a\}}, NCA_{\{b\}}, NCA_{\{a,b\}}, NCA_{\{b,c\}}^\phi\}$  is a NCAT-space of type (2) and  $\tau_{NCA}^3 = \{NCA_\phi^\phi, NCA_X^x, NCA_{\{a\}}, NCA_{\{b\}}, NCA_{\{a,b\}}\}$  is a NCAT-space of type (3).

**Remark 5.6.**

- (i) The intersection of two NCAT-spaces of type (1) is a NCAT-space of type (1).
- (ii) The intersection of two NCAT-spaces of type (2) is a NCAT-space of type (2).
- (iii) The intersection of two NCAT-spaces of type (3) is a NCAT-space of type (3)

**Remark 5.7.**

- (i) The union of two NCAT-spaces of type (1) does not necessarily have to be NCAT-space of type (1).
- (ii) The union of two NCAT-spaces of type (2) does not necessarily have to be NCAT-space of type (2).
- (iii) The union of two NCAT-spaces of type (3) does not necessarily have to be NCAT-space of type (3).

**Example 5.8.** Let  $\mathcal{X} = \{a, b, c\}$ .

Consider two NCAT-spaces of type (1)  $\tau_{1NCA}^1$  and  $\tau_{2NCA}^1$  defined as follows:

$\tau_{1NCA}^1 = \{NCA_\phi, NCA_{\mathcal{X}}, NCA_{\{a\}}\}$  and  $\tau_{2NCA}^1 = \{NCA_\phi, NCA_{\mathcal{X}}, NCA_{\{b\}}\}$ . Then

$\tau_{1NCA}^1 \sqcup \tau_{2NCA}^1 = \{NCA_\phi, NCA_{\mathcal{X}}, NCA_{\{a\}}, NCA_{\{b\}}\}$  which is not a NCAT-space of type (1).

**Example 5.9.** Let  $\mathcal{X} = \{a, b, c\}$ .

Consider two NCAT-spaces of type (2)  $\tau_{1NCA}^2$  and  $\tau_{2NCA}^2$  defined as follows:

$\tau_{1NCA}^2 = \{NCA_{\mathcal{X}}^\phi, NCA_{\mathcal{X}}^x, NCA_\phi^{\{a\}}\}$  and  $\tau_{2NCA}^2 = \{NCA_{\mathcal{X}}^\phi, NCA_{\mathcal{X}}^x, NCA_\phi^{\{b\}}\}$ . Then  $\tau_{1NCA}^2 \sqcup \tau_{2NCA}^2 = \{NCA_{\mathcal{X}}^\phi, NCA_{\mathcal{X}}^x, NCA_\phi^{\{a\}}, NCA_\phi^{\{b\}}\}$  which is not a NCAT-space of type (2).

**Example 5.10.** Let  $\mathcal{X} = \{a, b, c\}$ .

Consider two NCAT-spaces of type (3)  $\tau_{1NCA}^3$  and  $\tau_{2NCA}^3$  defined as follows:

$\tau_{1NCA}^3 = \{NCA_\phi^\phi, NCA_{\mathcal{X}}^x, NCA_\phi^{\{b\}}\}$  and  $\tau_{2NCA}^3 = \{NCA_\phi^\phi, NCA_{\mathcal{X}}^x, NCA_\phi^{\{c\}}\}$ . Then  $\tau_{1NCA}^3 \sqcup \tau_{2NCA}^3 = \{NCA_\phi^\phi, NCA_{\mathcal{X}}^x, NCA_\phi^{\{b\}}, NCA_\phi^{\{c\}}\}$  which is not a NCAT-space of type (3).

**The first idea:** is to divide the message into words, where each word represents a set. The family formed from these sets is then examined to determine whether it constitutes a sub-base, a base, or a topology after adding both the empty set ( $\phi$ ) and the universal set ( $\mathcal{X}$ ).

In parallel, the message is encrypted, then segmented in the same manner as the clear (unencrypted) message, and a topology is constructed from it as done for the clear message. Consequently, we obtain a topology for the clear message and a topology for the encrypted message, and the topological properties between them are studied.

In general, the family composed of the clear and encrypted messages initially forms a sub-base; after adding all intersections, it becomes a base, and after including all unions, it turns into a topology. Here,  $\mathcal{X}$  denotes the set of letters of that message.

**Example 5.11.** *Ali is a good man.*

Let  $\mathcal{X} = \{\text{Ali is, a good, man}\} = \{1,2,3\}$ .

The plain message:

The  $\phi, \{\text{Ali is}\}, \{\text{a good}\}, \{\text{man}\}, \{\text{Ali is, a good}\}, \{\text{Ali is, man}\}, \{\text{a good, man}\}, \mathcal{X}$  are subsets of  $\mathcal{X}$ .

Then  $\tau_{NCA}^1 = \{NCA_\phi, NCA_{\mathcal{X}}, NCA_{\{\text{Ali is}\}}, NCA_{\{\text{a good}\}}, NCA_{\{\text{man}\}}, NCA_{\{\text{Ali is, a good}\}},$

$NCA_{\{\text{Ali is, man}\}}, NCA_{\{\text{a good, man}\}}\}$  is a NCAT-space of type (1).

$\tau_{NCA}^2 = \{NCA_{\mathcal{X}}^\phi, NCA_\phi^x, NCA_\phi^{\{\text{Ali is}\}}, NCA_\phi^{\{\text{a good}\}}, NCA_\phi^{\{\text{man}\}}, NCA_\phi^{\{\text{Ali is, a good}\}},$

$NCA_\phi^{\{\text{Ali is, man}\}}, NCA_\phi^{\{\text{a good, man}\}}\}$  is a NCAT-space of type (2).

$\tau_{NCA}^3 = \{NCA_\phi^\phi, NCA_{\mathcal{X}}^x, NCA_\phi^{\{\text{Ali is}\}}, NCA_\phi^{\{\text{a good}\}}, NCA_\phi^{\{\text{man}\}}, NCA_\phi^{\{\text{Ali is, a good}\}},$

$NCA_\phi^{\{\text{Ali is, man}\}}, NCA_\phi^{\{\text{a good, man}\}}\}$  is a NCAT-space of type (3).

The encrypted message:

The  $\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \mathcal{X}$  are subsets of  $\mathcal{X}$ .

Then  $\tau_{NCA}^1 = \{NCA_\phi, NCA_X, NCA_{\{1\}}, NCA_{\{2\}}, NCA_{\{3\}}, NCA_{\{1,2\}}, NCA_{\{1,3\}}, NCA_{\{2,3\}}\}$  is a NCAT-space of type (1).

$\tau_{NCA}^2 = \{NCA_\phi^\phi, NCA_X^\chi, NCA_\phi^{\{1\}}, NCA_\phi^{\{2\}}, NCA_\phi^{\{3\}}, NCA_\phi^{\{1,2\}}, NCA_\phi^{\{1,3\}}, NCA_\phi^{\{2,3\}}\}$  is a NCAT-space of type (2).

$\tau_{NCA}^3 = \{NCA_\phi^\phi, NCA_X^\chi, NCA_\phi^{\{1\}}, NCA_\phi^{\{2\}}, NCA_\phi^{\{3\}}, NCA_\phi^{\{1,2\}}, NCA_\phi^{\{1,3\}}, NCA_\phi^{\{2,3\}}\}$  is a NCAT-space of type (3).

**The second idea:** is to transform the message into a topological structure within a neutrosophic crisp axial space. First, the number of characters in the message is calculated and then divided by three. If the result is a natural number, the message is partitioned from left to right into groups of three characters, forming neutrosophic crisp axial sets. These sets constitute a sub-base. By adding the neutrosophic crisp axial intersections, the structure becomes a base; and by further adding neutrosophic crisp axial unions, it evolves into a topology.

Using the same procedure, the message characters are encrypted using the RSA algorithm, and then partitioned again into neutrosophic crisp axial sets, forming another sub-base, from which we derive the topology of the encrypted message.

At this stage, we obtain both a neutrosophic crisp axial topology for the encrypted message and another for the plain message. The topological properties of these two spaces can then be studied and compared.

**Example 5.12.** *Ahmed is a good man.*

So that  $A = 00, H = 07, M = 12, E = 04, D = 03, I = 08, S = 18, G = 06, O = 14, N = 13$ .

The plain message:

Let  $X = \{a, h, m, e, d, i, s, a, g, o, o, d, m, a, n\}$   
 $= \{00, 07, 12, 04, 03, 08, 18, 00, 06, 14, 14, 03, 12, 00, 13\}$ ,

such that  $\mathcal{M}_1 = \{00, 07, 12, 04, 03\}$ ,  $\mathcal{M}_2 = \{08, 18, 00, 06, 14\}$ ,  $\mathcal{M}_3 = \{14, 03, 12, 00, 13\}$ .

We find the intersections:  $\mathcal{M}_4 = \{00\}$ ,  $\mathcal{M}_5 = \{00, 03, 12\}$ ,  $\mathcal{M}_6 = \{00, 14\}$ .

We find the unions:  $\mathcal{M}_7 = \{00, 07, 12, 04, 03, 08, 18, 06, 14\}$ ,  $\mathcal{M}_8 = \{00, 07, 12, 04, 03, 14, 13\}$ ,

$\mathcal{M}_9 = \{08, 18, 00, 06, 14, 03, 12, 13\}$ ,  $\mathcal{M}_{10} = \{00, 07, 12, 04, 03, 14\}$ ,

$\mathcal{M}_{11} = \{08, 18, 00, 06, 14, 03, 12\}$ ,  $\mathcal{M}_{12} = \{00, 03, 12, 14\}$ .

Then  $\tau_{NCA}^1 = \{NCA_\phi, NCA_X, NCA_{\mathcal{M}_1}, NCA_{\mathcal{M}_2}, NCA_{\mathcal{M}_3}, NCA_{\mathcal{M}_4}, NCA_{\mathcal{M}_5}, NCA_{\mathcal{M}_6}, NCA_{\mathcal{M}_7}, NCA_{\mathcal{M}_8}, NCA_{\mathcal{M}_9}, NCA_{\mathcal{M}_{10}}, NCA_{\mathcal{M}_{11}}, NCA_{\mathcal{M}_{12}}\}$  is a NCAT-space of type (1).

$\tau_{NCA}^2 = \{NCA_\phi^\phi, NCA_X^\chi, NCA_\phi^{\mathcal{M}_1}, NCA_\phi^{\mathcal{M}_2}, NCA_\phi^{\mathcal{M}_3}, NCA_\phi^{\mathcal{M}_4}, NCA_\phi^{\mathcal{M}_5}, NCA_\phi^{\mathcal{M}_6}, NCA_\phi^{\mathcal{M}_7}, NCA_\phi^{\mathcal{M}_8}, NCA_\phi^{\mathcal{M}_9}, NCA_\phi^{\mathcal{M}_{10}}, NCA_\phi^{\mathcal{M}_{11}}, NCA_\phi^{\mathcal{M}_{12}}\}$  is a NCAT-space of type (2).

$\tau_{NCA}^3 = \{NCA_\phi^\phi, NCA_X^\chi, NCA_\phi^{\mathcal{M}_1}, NCA_\phi^{\mathcal{M}_2}, NCA_\phi^{\mathcal{M}_3}, NCA_\phi^{\mathcal{M}_4}, NCA_\phi^{\mathcal{M}_5}, NCA_\phi^{\mathcal{M}_6}, NCA_\phi^{\mathcal{M}_7}, NCA_\phi^{\mathcal{M}_8}, NCA_\phi^{\mathcal{M}_9}, NCA_\phi^{\mathcal{M}_{10}}, NCA_\phi^{\mathcal{M}_{11}}, NCA_\phi^{\mathcal{M}_{12}}\}$  is a NCAT-space of type (3).

The encrypted message using the RSA algorithm:

Let  $\mathcal{X} = \{00,63,38,23,61,08,44,00,41,14,14,61,38,00,13\}$ , such that

$\mathcal{K}_1 = \{00,63,38,23,61\}$ ,  $\mathcal{K}_2 = \{08,44,00,41,14\}$ ,  $\mathcal{K}_3 = \{14,61,38,00,13\}$ .

We find the intersections:  $\mathcal{K}_4 = \{00\}$ ,  $\mathcal{K}_5 = \{00,61,38\}$ ,  $\mathcal{K}_6 = \{00,14\}$ .

We find the unions:  $\mathcal{K}_7 = \{00,63,38,23,61,08,44,41,14\}$ ,  $\mathcal{K}_8 = \{00,63,38,23,61,14,13\}$ ,

$\mathcal{K}_9 = \{08,44,00,41,14,61,38,13\}$ ,  $\mathcal{K}_{10} = \{00,63,38,23,61,14\}$ ,

$\mathcal{K}_{11} = \{08,44,00,41,14,38,61\}$ ,  $\mathcal{K}_{12} = \{00,61,38,14\}$ .

Then  $\tau_{NCA}^1 = \{NCA_\phi, NCA_{\mathcal{X}}, NCA_{\mathcal{K}_1}, NCA_{\mathcal{K}_2}, NCA_{\mathcal{K}_3}, NCA_{\mathcal{K}_4}, NCA_{\mathcal{K}_5}, NCA_{\mathcal{K}_6},$

$NCA_{\mathcal{K}_7}, NCA_{\mathcal{K}_8}, NCA_{\mathcal{K}_9}, NCA_{\mathcal{K}_{10}}, NCA_{\mathcal{K}_{11}}, NCA_{\mathcal{K}_{12}}\}$  is a NCAT-space of type (1).

$\tau_{NCA}^2 = \{NCA_\phi^\phi, NCA_\phi^{\mathcal{X}}, NCA_\phi^{\mathcal{K}_1}, NCA_\phi^{\mathcal{K}_2}, NCA_\phi^{\mathcal{K}_3}, NCA_\phi^{\mathcal{K}_4}, NCA_\phi^{\mathcal{K}_5}, NCA_\phi^{\mathcal{K}_6}, NCA_\phi^{\mathcal{K}_7},$

$NCA_\phi^{\mathcal{K}_8}, NCA_\phi^{\mathcal{K}_9}, NCA_\phi^{\mathcal{K}_{10}}, NCA_\phi^{\mathcal{K}_{11}}, NCA_\phi^{\mathcal{K}_{12}}\}$  is a NCAT-space of type (2).

$\tau_{NCA}^3 = \{NCA_\phi^\phi, NCA_\phi^{\mathcal{X}}, NCA_\phi^{\mathcal{K}_1}, NCA_\phi^{\mathcal{K}_2}, NCA_\phi^{\mathcal{K}_3}, NCA_\phi^{\mathcal{K}_4}, NCA_\phi^{\mathcal{K}_5}, NCA_\phi^{\mathcal{K}_6}, NCA_\phi^{\mathcal{K}_7},$

$NCA_\phi^{\mathcal{K}_8}, NCA_\phi^{\mathcal{K}_9}, NCA_\phi^{\mathcal{K}_{10}}, NCA_\phi^{\mathcal{K}_{11}}, NCA_\phi^{\mathcal{K}_{12}}\}$  is a NCAT-space of type (3).

## 6. Conclusion

In this paper we studied new concepts of neutrosophic crisp sets (NC-sets), specifically neutrosophic crisp axial sets (NCA-sets) of type (1) and (2). Along with looking at a number of basic and algebraic characteristics and providing examples. Additionally, a comparison between neutrosophic crisp axial sets of type (1) and (2). Finally, we presented the concept of the topology and some of the ideas entrusted to it, laying the groundwork for future in-depth studies of these spaces.

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