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Inferences of Type-II Generalized Hybrid Censored Competing Risks Data from New Extended Weibull Populations

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Abstract. Common in reliability analysis or medical studies, multiple causes of failure can be recorded with an (item/individual) life, which is known as the competing risks model. Due to the new extended Weibull (NEW) population with increasing and upside-down bathtub-shaped hazard rate functions, it occupies an important position in modelling the complex lifetime of a system. In this paper, we consider the competing risks model of the NEW populations with respect to the machinery of the type-II generalized hybrid censored scheme. Regarding independent causes of failure, the maximum likelihood and Bayes estimators are formulated. Also, the approximate confidence intervals, two bootstrap confidence intervals, and Bayesian credible intervals are formulated. The simulation study is constructed to assess and compare different estimation methods. The mechanism of the type-II generalized hybrid censored scheme is applied on a real competing data set and analyzed to illustrate our objectives.

1. Introduction

Weibull distribution is commonly used in reliability engineering because it is a best-fit distribution that represents a broad range of distribution shapes. However, the hazard rate function does not exhibit a bathtub or upside-down bathtub-shaped function, which cannot help to be used for modeling the complex lifetime of a system. To overcome this shortage, a number of extensions of the Weibull distribution were introduced. Hence, for example, the exponentiated Weibull distribution was introduced by Mudholkar and Srivastava [1], the additive Weibull distribution was introduced by Xie and Lai [2], the Marshall–Olkin extended Weibull distribution by Marshall and Olkin [3]. Different bathtub-shaped failure rate functions of the modified Weibull distributions

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were discussed by Xie et al. [4] and [5]. Flexible Weibull extension was presented by Bebbington et al. [6] and the generalized form of modified Weibull distribution by Carrasco et al. [7]. Also, the beta-modified and Beta generalized Weibull distributions by Nadarajah et al. [8] and Singla et al. [9], respectively. In several applications of reliability analysis, such as data obtained from the dynamic component of the commercial vehicle engines, the guinea pigs in the Regimen, and the maximum flood levels data with upside-down bathtub-shaped hazard rates, see Keller et al. [10], Gupta et al. [11] and Maswadah [12], respectively. One scale and two shape parameters extended the Weibull distribution with the upside-down bathtub or increasing-shaped hazard rate functions is presented by Peng and Yan [13].

The random variable T is called NEW random variable if its cumulative distribution function (CDF) is given by

$$F(t) = 1 - \exp\left\{-at^b e^{-\frac{c}{t}}\right\}, \quad t >; \ a, b > 0, \ c \geq 0, \tag{1.1}$$

where b and c are shape parameters and a is scale parameter.

The corresponding probability density function (PDF), reliability function and the hazard rate function (HRF) are given as follows.

$$f(t) = a(bt - c)t^{b-2}e^{-\frac{c}{t}} \exp\left\{-at^b e^{-\frac{c}{t}}\right\}, \quad t >; \ a, b > 0, \ c \geq 0, \tag{1.2}$$

$$R(t) = 1 - F(x) = \exp\left\{-at^b e^{-\frac{c}{t}}\right\}, \quad t >; \ a, b > 0, \ c \geq 0, \tag{1.3}$$

and

$$h(t) = \frac{f(t)}{R(t)} = a(c + bt)t^{b-2}e^{-\frac{c}{t}}, \quad t >; \ a, b > 0, \ c \geq 0, \tag{1.4}$$

Figures 1 and 2 show the graph of the PDF and HRF for different values of the parameters.

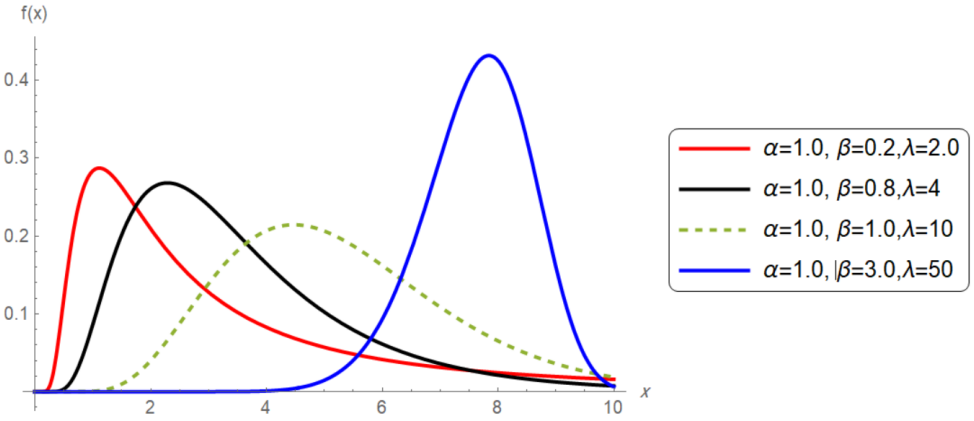

FIGURE 1. The graph of PDFs of NEW distribution.

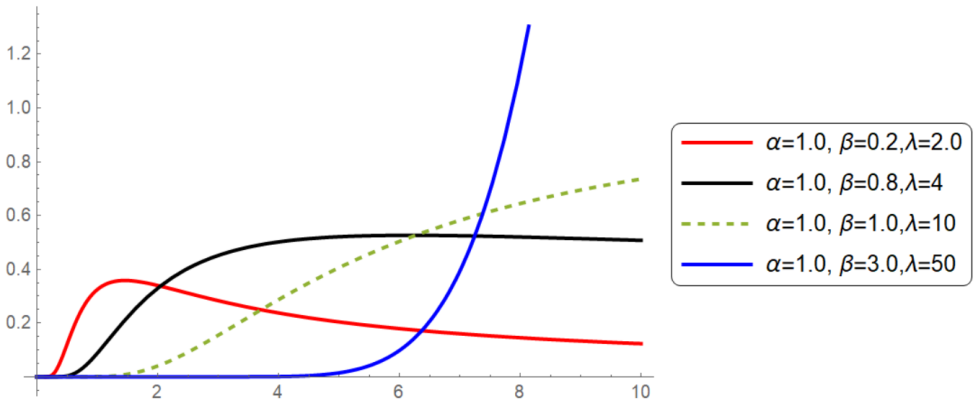

FIGURE 2. The graph of HFRs of Ishita distribution.

From Figure 1, the NEW distribution has unique mode, the hazard rate is increasing and decreasing based on the values of the parameters, see Figure 2.

In life testing experiments, censoring is a common phenomenon due to restrictions on data collection or time limitations. In literature, several types of censoring schemes, and the simplest ones are called type-I and type-II censoring schemes (CSs). Constant test time in a type-I censoring scheme and randomly failing times. On the contrary constant number of failures and random test time in the type-II censoring scheme. Collection between the fixed test time and fixed number of failures in the same scheme to formulate a hybrid censoring scheme (HCS). In the hybrid case the experiment takes prior test time (η) and the number of failures m. Therefore, the test is terminated at the $\min(\eta, T_m)$ in type-I HCS and $\max(\eta, T_m)$ in type-II HCS, where T_m is m-th failure time.

The lack of memory smaller number of failures may be zero in each of type-I CS or type-I HCS or a larger test time may be infinity in type-II CS or type-II HCS, see Gupta and Kundu [14], Kundu and Pradhan [15], Childs et al. [16] and [17]. The last schemes failure can be avoid by applied generalized hybrid censoring schemes (GHCSs), see [18]. In the problem at hand, type-II GHCS is adopted which is described as follows.

Suppose the ideal test times η_1 and η_2 are proposed to satisfy times $\eta_1 < \eta_2$. Also, from the n independent items put under test we need to read m failures from the test. Here, the test is terminated at η_1 if m failures observed before the time η_1 ($T_m < \eta_1$). But, the test is terminated at $\min(T_m, \eta_2)$ if T_m is larger than η_1. Figure 3 illustrates the schematic diagram of type-II GHCS, showing the failure times with test-terminated time under the type-II GHCS. If we denote to the observed data by $\mathbf{t} = (t_1 < t_2 < \cdots < t_k)$, then the integer number $k \geq m$ if $T_m < \eta_1$, and $k \leq m$ if $T_m > \eta_1$.

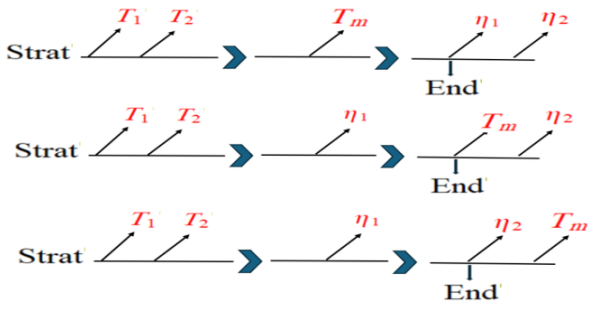

FIGURE 3. The schematic diagram of type-II GHCS.

A common phenomenon in several fields of science such as engineering and biological science items is failing under several fatal risks, which is known as the competing risks model. The objective in these models is to assess the effect of each risk factor while accounting for the presence of other risk factors. Early, this problem was discussed by [19], [20], [21], [22], Bakoban and Abd-Elmougod [23] and Ganguly and Kundu [24]. Under consideration, independent causes of failure, several authors such as Algarni et al. [25], Tahani el al. [26], Alghamdi [27], Almarashi and Abd-Elmougod [28], and Alghamdia et al. [[29], [30]].

The type-II GHC data under competing risks model taken from n independent random items is denoted as $\mathbf{T} = ((T_1, \delta_1), (T_2, \delta_2), ..., (T_k, \delta_k))$. Suppose that, $\delta_i \in \{1, 2\}$ and $i = 1, 2, \cdots, k$. Therefore, the joint likelihood function of observed type-II GHC sample $\mathbf{t} = \{(t_1, \delta_1), (t_2, \delta_2), \cdots, (t_k, \delta_k)\}$ is given by

$$L(\mathbf{t}|\grave{}) = \frac{n!}{(n-k)!}\left(\prod_{i=1}^{k}[f_1(t_i)R_2(t_i)]^{\rho(\delta_i=1)}[f_2(t_i)R_1(t_i)]^{\rho(\delta_i=2)}\right)[R_1(t_k)R_2(t_k)]^{n-k}, \tag{1.5}$$

where, $R_j(.) = 1 - F_j(.)$, and $\rho(\delta_i = j) = \begin{cases} 0, & \delta_i \neq j \\ 1, & \delta_i = j \end{cases}$.

The statistical inference of competing NEW populations when the data are collected under the mechanism of type-II GHCS is our aim in this paper. The model and its assumptions are built, and hence, the point and interval estimates of the model parameters are formulated. The results of point estimators are proposed with maximum likelihood (ML) and Bayesian methods. Also, interval estimation is developed for approximate confidence intervals, bootstrap confidence, and Bayesian credible intervals. Assessment and comparison of the results through the discussion of two real data sets, as well as a Monte Carlo simulation study.

The plan of the main parts of this paper is described as follows. Abbreviations and model formulation are given in Section 2. The point estimates with ML and the Bayes method are developed in Section 3. The interval estimators of model parameters are formulated in Section 4 with resected to ML, bootstrap, and Bayes Methods. In Section 5, formulate the numerical discussion in the form of a real data analysis and a Monte Carlo simulation study. Finally, report some comments and future works about this model in Section 6.

2. Abbreviations and Model Formulation

In this section, we define the abbreviations used in the proposed model, which considers only two causes of failure. Additionally, formulate the model assumptions.

2.1. **Abbreviations.**

T_i:	The i-th lifetime of item.
T_{ij}:	The i-th lifetime of item under cause j, $j = 1, 2$
$F(.)$:	CDF of T_i
$f(.)$:	PDF of T_i
$F_j(.)$:	CDF of T_{ij}
$f_j(.)$:	PDF of T_{ij}
$R_j(.)$:	Relability function of T_{ij}
$h_j(.)$:	Hazard rate function of T_{ij}
δ_i:	The indicator i-th failure cause

2.2. **Model assumptions.**

(1) The i-th item lifetime is denoted by $T_i, i = 1, 2, \cdots, k$ and $T_i = \min(T_{i1}, T_{i2})$, where $T_{ij}, j = 1, 2$ is the i-th item lifetime under cause j.

(2) The random variable T_{ij} is distributed as NEW distribution with common shape parameters b and c and different scale parameter a. Hence, the CDF can be written as

$$F_j(t) = 1 - \exp\left\{-a_j t^b e^{-\frac{c}{t}}\right\}, \quad t >; \; a_j, b > 0, c \geq 0, \quad j = 1, 2. \tag{2.1}$$

The corresponding PDF, reliability and failure rate functions are given respectively by

$$f_j(t) = a_j(c + bt)t^{b-2}e^{-\frac{c}{t}} \exp\left\{-a_j t^b e^{-\frac{c}{t}}\right\}, \tag{2.2}$$

$$R_j(t) = \exp\left\{-a_j t^b e^{-\frac{c}{t}}\right\}, \tag{2.3}$$

and

$$h_j(t) = a_j(c + bt)t^{b-2}e^{-\frac{c}{t}}. \tag{2.4}$$

(3) The random variable $T_i = \min(T_{i1}, T_{i2})$ has the NEW distribution with shape parameters b and c and scale parameter $a_1 + a_2$.

(4) Each of observed integer numbers k_1 and k_2 under first and second causes of failure are generated from binomial distribution with probability $\frac{a_1}{a_1+a_2}$ and $\frac{a_2}{a_1+a_2}$, respectively.

(5) The joint likelihood function (1.5), respected type-II GHC sample $\boldsymbol{t} = \{(t_1, \delta_1), (t_2, \delta_2), ..., (t_k, \delta_k)\}$ and the distributions (2.1) to (2.4) is given by

$$\begin{aligned} L(a_1, a_2, b, c|\mathbf{t}) \propto\; & a_1^{k_1} a_2^{k_2} \exp\left\{\sum_{i=1}^{k} \log(c + bt_i) - c\sum_{i=1}^{k} \frac{1}{t_i} - (a_1 + a_2)\sum_{i=1}^{k} t_i^b e^{-\frac{c}{t_i}}\right. \\ & \left. + (b-2)\sum_{i=1}^{k} \log(t_i) - (n-k)(a_1 + a_2)t_k^b e^{-\frac{c}{t_k}}\right\}, \end{aligned} \tag{2.5}$$

3. Point Estimation

In this section, the point estimate of the model parameters is formulated with respect to ML estimation. Also, under consideration independent Gamma prior distributions with a squared error loss function, the Bayes point estimate is formulated.

3.1. **Maximum likelihood estimation.** From the joint likelihood function (2.5), the natural log-likelihood function is given by

$$\begin{aligned}\ell\left(a_1,a_2,b,c|\mathbf{t}\right) \quad \propto \quad & k_1\log(a_1)+k_2\log(a_2)+\sum_{i=1}^{k}\log(c+bt_i)-c\sum_{i=1}^{k}\frac{1}{t_i}-(a_1+a_2)\sum_{i=1}^{k}t_i^b e^{-\frac{c}{t_i}} \\ & +(b-2)\sum_{i=1}^{k}\log(t_i)-(n-k)(a_1+a_2)t_k^b e^{-\frac{c}{t_k}}. \end{aligned} \tag{3.1}$$

The point ML estimate of model parameters vector, $\boldsymbol{\theta}=(a_1,a_2,b,c)$ is obtained by maximize the logarithms function (3.1). The first partial derivatives of (3.1) to a_1 and a_2 is reduced to

$$a_l(b,c)=\frac{k_l}{\sum_{i=1}^{k}t_i^b e^{-\frac{c}{t_i}}+(n-k)t_k^b e^{-\frac{c}{t_k}}},\ l=1,2. \tag{3.2}$$

Also, the first partial derivatives of (3.1) to b and c is reduced to

$$\sum_{i=1}^{k}\frac{t_i}{c+bt_i}-(a_1+a_2)\sum_{i=1}^{k}t_i^b e^{-\frac{c}{t_i}}\log(t_i)+\sum_{i=1}^{k}\log(t_i)-(n-k)(a_1+a_2)t_k^b e^{-\frac{c}{t_k}}\log t_k=0, \tag{3.3}$$

and

$$\sum_{i=1}^{k}\frac{1}{c+bt_i}-\sum_{i=1}^{k}\frac{1}{t_i}+(a_1+a_2)\sum_{i=1}^{k}t_i^{b-1}e^{-\frac{c}{t_i}}+(n-k)(a_1+a_2)t_k^{b-1}e^{-\frac{c}{t_k}}=0. \tag{3.4}$$

Which are reduced to

$$\sum_{i=1}^{k}\frac{t_i}{c+bt_i}+\sum_{i=1}^{k}\log(t_i)-k\frac{\sum_{i=1}^{k}t_i^b e^{-\frac{c}{t_i}}\log(t_i)+(n-k)t_k^b e^{-\frac{c}{t_k}}\log(t_k)}{\sum_{i=1}^{k}t_i^b e^{-\frac{c}{t_i}}+(n-k)t_k^b e^{-\frac{c}{t_k}}}=0, \tag{3.5}$$

and

$$\sum_{i=1}^{k}\frac{1}{c+bt_i}-\sum_{i=1}^{k}\frac{1}{t_i}+k\frac{\sum_{i=1}^{k}t_i^{b-1}e^{-\frac{c}{t_i}}+(n-k)t_k^{b-1}e^{-\frac{c}{t_k}}}{\sum_{i=1}^{k}t_i^b e^{-\frac{c}{t_i}}+(n-k)t_k^b e^{-\frac{c}{t_k}}}=0. \tag{3.6}$$

The ML estimate of the parameters $\hat{} = (\hat{a}_1,\hat{a}_2,\hat{b},\hat{c})$ is obtained by solve two non-linear equations (3.5) and (3.6) with any iteration method such as Newton Raphson iteration. And then, substitute in (3.2).

Remarks (1):

(1) When replace a_1 and a_2 from (3.2) in (2.5) obtain the joint profile log-likelihood function given by

$$g(b,c|\mathbf{t}) = k_1 \log\left[\frac{k_1}{\sum_{i=1}^{k} t_i^b e^{-\frac{c}{t_i}} + (n-k)t_k^b e^{-\frac{c}{t_k}}}\right] + k_2 \log\left[\frac{k_2}{\sum_{i=1}^{k} t_i^b e^{-\frac{c}{t_i}} + (n-k)t_k^b e^{-\frac{c}{t_k}}}\right] + \sum_{i=1}^{k} \log(c+bt_i) + (b-2)\sum_{i=1}^{k} \log(t_i) - c\sum_{i=1}^{k} \frac{1}{t_i} - k. \tag{3.7}$$

The initial values of iteration can be obtained from (3.7).

(2) When $\eta_1 = \eta_2 = \eta$ then, competing type-II GHCS reduce to ordinary competing type-II HCS.

(3) When $k_1 = 0$ or $k_2 = 0$ then, competing type-II GHCS reduce to ordinary type-II GHCS.

3.2. **Bayesian estimation.** When we have prior information about the model parameters, the Bayesian approach presents an attractive inferential method. This approach works even for a censored sample or small sample sizes. In the Bayesian approach, the prior information and the corresponding posterior information are discussed. As given by Peng and Yan [13] an independent prior distribution is considered. Gamma distributions are adopted for parameters a_1 and a_2, and a uniform prior distribution of b and c. Therefore, the prior information of the model parameters is given by

$$\pi_l^*(a_l) \propto a_l^{\alpha_l - 1} \exp\{-\beta_l a_l\},\ a_l;\ \alpha_l,\ \beta_l > 0 \text{ and } l = 1,2, \tag{3.8}$$

$$\pi_3^*(b) \propto 1 \text{ and } \pi_4^*(c) \propto 1. \tag{3.9}$$

Hence, the joint prior distribution is given by

$$\pi_l^*(a_1,a_2,b,c) \propto a_1^{\alpha_1-1} a_2^{\alpha_2-1} \exp\{-(\beta_1 a_1 + \beta_2 a_2)\}. \tag{3.10}$$

The posterior distribution of the parameters vector $\boldsymbol{\theta} = (a_1,a_2,b,c)$ can be obtained from

$$\pi(a_1,a_2,b,c|\mathbf{t}) = \frac{\pi_l^*(a_1,a_2,b,c)L(a_1,a_2,b,c|\boldsymbol{t})}{\iiiint \pi_l^*(a_1,a_2,b,c)L(a_1,a_2,b,c|\mathbf{t})\, da_1 da_2 dbdc}. \tag{3.11}$$

The closed form of posterior distribution requires a multidimensional integration as well as the corresponding Bayesian estimate. Under consideration squared error loss function the Bayesian estimate of any function $z(a_1,a_2,b,c)$ is defined by

$$\hat{z}_{\boldsymbol{B}} = \frac{\iiiint z(a_1,a_2,b,c)\pi_l^*(a_1,a_2,b,c)L(a_1,a_2,b,c|\boldsymbol{t})\, da_1 da_2 dbdc}{\iiiint \pi_l^*(a_1,a_2,b,c)L(a_1,a_2,b,c\boldsymbol{t})\, da_1 da_2 dbdc}. \tag{3.12}$$

The ratio of two integrals given by (3.12) can be computed with different methods, such as Lindley's approximation and numerical integration. In this paper, we applied the Markov chen Monte Carlo (MCMC) method. The posterior distribution defined by (3.11) can be written in the

form

$$\pi(a_1,a_2,b,c|\boldsymbol{t}) \propto a_1^{\alpha_1+k_1-1}a_2^{\alpha_2+k_2-1}\exp\left\{-(\beta_1a_1+\beta_2a_2)+\sum_{i=1}^{k}\log(c+bt_i)-c\sum_{i=1}^{k}\frac{1}{t_i}\right.$$
$$\left.-(a_1+a_2)\sum_{i=1}^{k}t_i^be^{-\frac{c}{t_i}}+(b-2)\sum_{i=1}^{k}\log(t_i)-(n-k)(a_1+a_2)t_k^be^{-\frac{c}{t_k}}\right\}, \tag{3.13}$$

MH under Gibbs sampling:

The Gibbs sampling technique was used to obtain the Bayesian estimate of the model parameters $\boldsymbol{\theta}=(a_1,a_2,b,c)$. From the joint posterior distribution (3.13), the full conditional posterior distribution can be describe by

$$\pi_1(a_1|a_2,b,c,\boldsymbol{t}) \propto a_1^{\alpha_1+k_1-1}\exp\left\{-a_1\left[\beta_1+\sum_{i=1}^{k}t_i^be^{-\frac{c}{t_i}}+(n-k)t_k^be^{-\frac{c}{t_k}}\right]\right\}, \tag{3.14}$$

and

$$\pi_2(a_2|a_1,b,c,\boldsymbol{t}) \propto a_2^{\alpha_2+k_2-1}\exp\left\{-a_2\left[\beta_2+\sum_{i=1}^{k}t_i^be^{-\frac{c}{t_i}}+(n-k)t_k^be^{-\frac{c}{t_k}}\right]\right\}. \tag{3.15}$$

The full conditional posterior distributions given by (3.14) and (3.15) are gamma distribution with parameters (α_l+k_l) and $\left(\beta_l+\sum_{i=1}^{k}t_i^be^{-\frac{c}{t_i}}+(n-k)t_k^be^{-\frac{c}{t_k}}\right)$, l =1,2. But, the full conditional posterior distributions of b and c can be written as

$$\pi_3(b|a_1,a_2,c,\boldsymbol{t}) \propto \exp\left\{\sum_{i=1}^{k}\log(c+bt_i)+b\sum_{i=1}^{k}\log(t_i)-(a_1+a_2)\left(\sum_{i=1}^{k}t_i^be^{-\frac{c}{t_i}}+(n-k)t_k^be^{-\frac{c}{t_k}}\right)\right\} \tag{3.16}$$

and

$$\pi_4(c|a_1,a_2,b,\boldsymbol{t}) \propto \exp\left\{\sum_{i=1}^{k}\log(c+bt_i)-c\sum_{i=1}^{k}\frac{1}{t_i}-(a_1+a_2)\left(\sum_{i=1}^{k}t_i^be^{-\frac{c}{t_i}}+(n-k)t_k^be^{-\frac{c}{t_k}}\right)\right\} \tag{3.17}$$

Now, we check the full conditional posterior distributions given by (3.16) and (3.17). Its clearly that each of $\frac{\partial^2\ln\pi_3((b|a_1,a_2,c,\boldsymbol{t}))}{\partial b^2}<0$ and $\frac{\partial^2\ln\pi_4((c|a_1,a_2,b,\boldsymbol{t}))}{\partial b^2}<0$, therefore the full conditional posterior distributions are log-concave functions and have only one maximum values. Hence, Metropolis-within-Gibbs is a suitable choice for generation from the posterior distribution. The MCMC algorithm to draw samples from the posterior density distribution and in turn compute the Bayes estimates and also, construct the corresponding credible intervals is given as follows.

MCMC Algorithms:

(1) Begin with initial gusses value $\boldsymbol{\theta}^{(0)}=(\hat{a}_1,\hat{a}_2,\hat{b},\hat{c})$.
(2) Put $s=1$
(3) From conditional gamma density (3.14) generate $a_1^{(s)}$.
(4) From conditional gamma density (3.15) generate $a_2^{(s)}$.

(5) With the normal proposal distribution generate $b_1^{(s)}$ from (3.16) using the MH algorithm.
(6) With the normal proposal distribution generate $c_1^{(s)}$ from (3.17) using the MH algorithm.
(7) Put $s = s + 1$.
(8) Repeat steps 3–7, MC times, we get $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \cdots, \boldsymbol{\theta}^{(MC)}$, where $\boldsymbol{\theta}^{(s)} = (a_1^{(s)}, a_2^{(s)}, b^{(s)}, c^{(s)})$ and $s = 1, 2, \cdots,$ MC.
(9) The Bayes estimate of the model parameters is given by

$$\hat{\boldsymbol{\theta}}_{\mathrm{B}} = \frac{1}{\mathrm{MC} - \mathrm{MC}^*} \sum_{i=\mathrm{MC}^*+1}^{\mathrm{MC}} \boldsymbol{\theta}^{(i)}, \tag{3.18}$$

where MC^* is the number of iteration need to reach the stationary distribution.

4. Interval Estimation

In this section, the approximate Fisher information matrix is estimated. The approximate confidence intervals of the model parameters are formulated. The approximate confidence interval with bootstrap-p and bootstrap-t techniques are discussed. Finally, we formulate equal two-sided Bayesian credible intervals.

4.1. **Approximate confidence interval.** The problem of estimating the approximate confidence intervals need to construct the Fisher information matrix. This matrix was defined as the minus expectation of second partially derivative of the log-likelihood function resected to the model parameters. Generally, this expectation is more complicated to compute specially under high dimensional parameters model. Therefore, the natural alternative of this matrix is approximate Fisher information matrix which denoted by $\Psi\left(a_1, a_2, b, c\right)$.

$$\Psi\left(a_1, a_2, b, c\right) = \left(-\frac{\partial^2 \ell\left(a_1, a_2, b, c|\boldsymbol{t}\right)}{\partial \boldsymbol{\theta}_l^2}\right)\Big|_{\theta_l = \hat{\theta}_l},\ l = 1, 2, 3, 4, \tag{4.1}$$

where $\boldsymbol{\theta} = (a_1, a_2, b, c)$. Also, the second partial derivative of log-likelihood function are given by

$$\frac{\partial^2 \ell(a_1,a_2,b,c|\boldsymbol{t})}{\partial a_s^2} = -\frac{k_s}{a_s^2}, \quad s = 1, 2, \tag{4.2}$$

$$\begin{aligned}\frac{\partial^2 \ell(a_1,a_2,b,c|\boldsymbol{t})}{\partial b^2} &= -\sum_{i=1}^{k} \frac{t_i^2}{(c+bt_i)^2} - (a_1+a_2)\sum_{i=1}^{k} t_i^b \left(\log t_i\right)^2 e^{-\frac{c}{t_i}} \\ &\quad -(n-k)(a_1+a_2)t_k^b \left(\log t_k\right)^2 e^{-\frac{c}{t_k}},\end{aligned} \tag{4.3}$$

$$\frac{\partial^2 \ell(a_1,a_2,b,c|\mathbf{t})}{\partial c^2} = -\sum_{i=1}^{k} \frac{1}{(c+bt_i)^2} - (a_1+a_2)\sum_{i=1}^{k} t_i^{b-2} e^{-\frac{c}{t_i}} - (n-k)(a_1+a_2)t_k^{b-2} e^{-\frac{c}{t_k}}, \tag{4.4}$$

$$\frac{\partial^2 \ell(a_1,a_2,b,c|\boldsymbol{t})}{\partial a_1 \partial a_2} = \frac{\partial^2 \ell(a_1,a_2,b,c|\boldsymbol{t})}{\partial a_2 \partial a_1} = 0, \tag{4.5}$$

$$\frac{\partial^2 \ell(a_1,a_2,b,c|\boldsymbol{t})}{\partial a_s \partial b} = -\sum_{i=1}^{k} t_i^b \left(\log t_i\right) e^{-\frac{c}{t_i}} - (n-k) t_k^b \left(\log t_k\right) e^{-\frac{c}{t_k}}, \quad s = 1, 2, \tag{4.6}$$

$$\frac{\partial^2 \ell(a_1,a_2,b,c|\boldsymbol{t})}{\partial a_s \partial c} = \sum_{i=1}^{k} t_i^{b-1} e^{-\frac{c}{t_i}} + (n-k) t_k^{b-1} e^{-\frac{c}{t_k}}, \quad s = 1,2 \tag{4.7}$$

$$\begin{aligned}\frac{\partial^2 \ell(a_1,a_2,b,c|\boldsymbol{t})}{\partial b \partial c} &= -\sum_{i=1}^{k} \frac{t_i}{c+bt_i} + (a_1+a_2) \sum_{i=1}^{k} t_i^{b-1} (\log t_i) e^{-\frac{c}{t_i}} \\ &\quad +(n-k)(a_1+a_2) (\log t_k) t_k^{b-1} e^{-\frac{c}{t_k}}.\end{aligned} \tag{4.8}$$

The estimate value $\hat{\boldsymbol{\theta}} = (\hat{a}_1, \hat{a}_2, \hat{b}, \hat{c})$ is approximate bivariate normal distribution under some mild regularity conditions with mean (a_1, a_2, b, c) and variance covariance matrix $\Psi^{-1}\left(\hat{a}_1, \hat{a}_2, \hat{b}, \hat{c}\right)$ as follows

$$(\hat{a}_1, \hat{a}_2, \hat{b}, \hat{c}) \to \mathcal{N}\left((a_1, a_2, b, c), \Psi^{-1}(\hat{a}_1, \hat{a}_2, \hat{b}, \hat{c})\right), \tag{4.9}$$

Under normal property (4.9), $100(1-2\gamma)\%$ approximate confidence intervals of (a_1, a_2, b, c) are given by

$$\hat{\boldsymbol{\theta}}_l \mp z_\gamma \sqrt{\zeta_{ll}}, \qquad l = 1,2,3,4, \tag{4.10}$$

where $\hat{\boldsymbol{\theta}} = (\hat{a}_1, \hat{a}_2, \hat{b}, \hat{c})$, z_γ is the normal percentile value with confidence level 2γ and ζ_{ll} are the diagonal of variance covariance matrix $\Psi^{-1}\left(\hat{a}_1, \hat{a}_2, \hat{b}, \hat{c}\right)$.

4.2. **Bootstrap confidence interval.** Statistical inference under the resampling method is commonly used in the literature. The bootstrap techniques, as resampling methods not only used to estimate confidence intervals. It can be used to calibrate hypothesis tests or estimate the bias and variance of an estimator. The nonparametric and parametric bootstrap techniques can be applied for statistical inference of population parameters. In this paper, parametric bootstrap techniques are used to formulate the approximate confidence intervals. For more details about parametric bootstrap techniques, see [31], [32], and [33]. The percentile bootstrap-p (boot-p) and bootstrap t (boot-t) techniques are suggested to construct confidence intervals for the model parameters. The algorithms used to construct the bootstrap confidence interval are described as follows.

Bootstrap confidence intervals algorithm:

(1) Form original type-II GHC sample $\boldsymbol{t} = \{(t_1, \delta_1), (t_2, \delta_2), \cdots, (t_k, \delta_k)\}$, compute the estimate values $\hat{\boldsymbol{\theta}} = (\hat{a}_1, \hat{a}_2, \hat{b}, \hat{c})$.
(2) Put $\nu = 1$
(3) Under consideration η_1, η_2, n, m generate type-II GHC sample $\boldsymbol{t}^* = \{t_1^*, t_2^*, \cdots, t_k^*\}$ from NEW$(a_1 + a_1, b, c)$.
(4) The integer numbers k_1 and k_2 under first and second causes of failure are generated from binomial distribution with probability $\frac{a_1}{a_1+a_2}$ and $\frac{a_2}{a_1+a_2}$, respectively.
(5) The bootstrap sample estimate $\hat{\boldsymbol{\theta}}^* = (\hat{a}_1^*, \hat{a}_2^*, \hat{b}^*, \hat{c}^*)$ is computed under given $\boldsymbol{t}^* = \{t_1^*, t_2^*, \cdots, t_k^*\}$, k_1 and k_2.
(6) Put $\nu = \nu + 1$.
(7) The Steps from 3 to 6 are repeat $\boldsymbol{NB}$ times.
(8) Record the bootstrap sample estimate $(\hat{\boldsymbol{\theta}}_l^{*(1)}, \hat{\boldsymbol{\theta}}_l^{*(2)}, \cdots, \hat{\boldsymbol{\theta}}_l^{*(\boldsymbol{NB})})$, $l = 1,2,3,4$.

Boot-p confidence intervals (BPCIs)

Suppose that, the ordered bootstrap sample estimate $(\hat{\boldsymbol{\theta}}^*_{l(1)}, \hat{\boldsymbol{\theta}}^*_{l(2)}, \cdots, \hat{\boldsymbol{\theta}}^*_{l(\boldsymbol{NB})})$ are distributed as the empirical distribution $\Phi(x)$ where, $\Phi(x) = P(\hat{\boldsymbol{\theta}}^*_l \leq x)$. Therefore, $100(1-2\gamma)\%$ BPCIs computed from

$$(\hat{\boldsymbol{\theta}}^*_{l(\gamma \boldsymbol{NB})},\ \hat{\boldsymbol{\theta}}^*_{l((1-\gamma)\boldsymbol{NB})}), \tag{4.11}$$

where of $\hat{\boldsymbol{\theta}}^*_l = \Phi^{-1}(x)$.

Boot-t confidence intervals (BTCIs)

For the empirical distribution $\Phi(x)$ on the ordered bootstrap sample estimate define a new statistic Ω as follows

$$\Omega^*_l = \sqrt{k}\frac{\hat{\boldsymbol{\theta}}^*_l - \hat{\boldsymbol{\theta}}_l}{\sqrt{\text{var}\left(\theta^*_l\right)}}, \quad l = 1, 2, 3, 4. \tag{4.12}$$

The values Ω^*_l are computed for all ordered bootstrap sample estimate to be $\Omega l(1)^* < \Omega^*_{l(2)} < \cdots < \Omega^*_{l(\mathbf{NB})}$. These values are distributed as the empirical distribution $\Phi(x)$ where, $\Phi(x) = P(\Omega^*_l \leq x)$. Hence, we define

$$\boldsymbol{\theta}^*_{l\text{boot-t}} = \hat{\boldsymbol{\theta}}_l + \sqrt{r\text{Var}(\hat{\boldsymbol{\theta}}_l)}\Phi^{-1}(x), \tag{4.13}$$

The corresponding $(1-2\gamma)100\%$ BTCIs are define by

$$\left(\boldsymbol{\theta}^*_{l\text{boot-t}(\nu)}, \boldsymbol{\theta}^*_{l\text{boot-t}(1-\nu)}\right). \tag{4.14}$$

4.3. **Equal two side credible interval.** From the estimated values $\boldsymbol{\theta}^{(\text{MC}^*+1)}, \boldsymbol{\theta}^{(\text{MC}^*+2)}, \cdots, \boldsymbol{\theta}^{(MC)}$, where $\boldsymbol{\theta}^{(i)} = (a_1^{(i)}, a_2^{(i)}, b^{(i)}, c^{(i)})$, $i = MC^* + 1$, $\text{MC}^* + 2$, $\cdots$, MC, we get the ordered values define by

$$\boldsymbol{\theta}_{(1)}, \boldsymbol{\theta}_{(2)}, \cdots, \boldsymbol{\theta}_{(\text{MC-MC}^*)}. \tag{4.15}$$

From (4.15), the $(1-2\gamma)100\%$ equal two side credible intervals are defined by

$$\left(\hat{\boldsymbol{\theta}}_{\text{B}(\gamma(\text{MC-MC}^*))},\ \hat{\boldsymbol{\theta}}_{\text{B}((1-\gamma)(\text{MC-MC}^*))}\right). \tag{4.16}$$

5. Numerical Studies

In this section, The performance of the proposed statistical inference procedures is discussed and assessed in this section for the competing risks model. Firstly, to determine the efficiency and reliability of the estimators, simulation studies are conducted under various choices of model parameters and censoring schemes, aiming to demonstrate the robustness and accuracy of the estimation methods. Secondly, apply the results to real data sets to illustrate the theoretical results. At the end, report the important results obtained from the numerical results.

5.1. **Simulation studies.** To obtain a reliable statistical inference, it must be evaluated how well the development of estimation methods performs in this paper. The Monte Carlo simulation study is conducted using Mathematica version 12 to assess the accuracy and efficiency of the estimators. In this section, point estimates are evaluated based on mean squared error, while interval estimates are assessed through coverage probability and average interval length as key metrics. These measures under repeated sampling conditions offer a comprehensive understanding of the estimators' behavior. Therefore, two sets of parameter values are used with non-informative and informative prior information. The true parameter values are selected under given prior information by generating 10 values and choosing their mean. Specifically, when the prior information is set as (α, β)={(2.0, 2.5), (3.0, 1.0)}. The true parameter values are selected accordingly based on this prior information (a, b, c)={(0.7, 1.0, 0.8, 4.0), (1.5, 2.0, 3.0, 10.0)}. In this study, test the effect of varying the total sample size, $k = k_1 + k_2$, as well as the impact of different combinations of the tested times (η_1, η_2). To do so, we generate from NEW distribution 1000 random samples. The following steps outline the algorithm used to perform the Monte Carlo simulation study.

Monte Carlo simulation study

Step 1: Generate a sample of size n from NEW distribution with shape parameters b and c and scale parameter $(a_1 + a_2)$, $\mathbf{t} = (t_1, t_2, ..., t_n)$.

Step 2: For given m and (η_1, η_2) determine the type-II GHC sample $\mathbf{t} = (t_1, t_2, ..., t_k)$.

Step 3: The integer values k_1 and k_2 are generated from binomial distribution with probability $\frac{a_1}{a_1+a_2}$ and $\frac{a_2}{a_1+a_2}$, respectively.

Step 4: The numbers m_{sj}, $s, j = 1, 2$ are generated from binomial distributions under size J_s and the success of probability $\frac{\beta_{s1}}{\beta_{s1}+\beta_{s2}}$ and $\frac{\beta_{s2}}{\beta_{s1}+\beta_{s2}}$, respectively.

Step 5: Compute the point ML and Bayes estimates.

Step 6: Compute the interval estimate with 95% asymptotic ML, 95% two bootstrap and 95% Bayes equal two side credible interval.

Step 7: Steps from 1 to 6 are repeated 1000 times.

Step 8: The results of mean squared error (MSE) are computed with respected to non-informative and informative prior information (π^0 and π^1), respectively in Tables 1 and 2.

Step 9: The coverage percentage (CP) and average interval length (AIL) are computed and reported in Tables 3 and 4.

TABLE 1. The estimate values of ME and MSE $\theta = \{0.7, 1.0, 0.8, 4.0\}$

		MLE				Bayes(π^0)				Bayes(π^1)			
(n, m, η_1, η_2)		a_1	a_2	b	c	a_1	a_2	b	c	a_1	a_2	b	c
(35,15,2.0,3.5)	ME	0.887	1.275	0.974	5.472	0.852	1.261	0.949	5.441	0.801	1.174	0.903	5.201
	MSE	0.143	0.237	0.164	0.456	0.133	0.229	0.165	0.438	0.098	0.189	0.101	0.379
(35,25,2.0,3.5)	ME	0.830	1.236	0.931	5.407	0.814	1.222	0.902	5.411	0.777	1.125	0.876	4.875
	MSE	0.113	0.211	0.141	0.425	0.110	0.201	0.137	0.415	0.074	0.101	0.092	0.297
(50,25,2.0,3.5)	ME	0.841	1.238	0.925	5.401	0.817	1.228	0.914	5.405	0.781	1.117	0.877	4.871
	MSE	0.117	0.214	0.139	0.421	0.113	0.206	0.131	0.417	0.072	0.105	0.095	0.294
(50,40,2.0,3.5)	ME	0.789	1.183	0.879	4.782	0.774	1.172	0.875	4.764	0.735	1.081	0.835	4.689
	MSE	0.082	0.172	0.104	0.389	0.075	0.166	0.102	0.378	0.055	0.084	0.064	0.188
(70,40,2.0,3.5)	ME	0.791	1.181	0.875	4.787	0.776	1.169	0.871	4.768	0.741	1.083	0.829	4.684
	MSE	0.084	0.175	0.107	0.386	0.078	0.169	0.099	0.381	0.052	0.081	0.067	0.191
(70,60,2.0,3.5)	ME	0.755	1.142	0.848	4.730	0.742	1.139	0.845	4.702	0.728	1.051	0.828	4.622
	MSE	0.059	0.149	0.087	0.322	0.054	0.141	0.081	0.350	0.035	0.061	0.043	0.139
(35,15,2.0,5.0)	ME	0.875	1.261	0.962	5.459	0.843	1.252	0.936	5.439	0.789	1.157	0.900	5.192
	MSE	0.136	0.224	0.151	0.447	0.122	0.220	0.158	0.431	0.092	0.183	0.097	0.371
(35,25,2.0,5.0)	ME	0.822	1.231	0.933	5.392	0.804	1.211	0.900	5.403	0.765	1.118	0.870	4.869
	MSE	0.099	0.201	0.129	0.416	0.095	0.187	0.130	0.401	0.066	0.094	0.085	0.290
(50,25,2.0,5.0)	ME	0.833	1.231	0.915	5.403	0.807	1.221	0.900	5.394	0.777	1.109	0.872	4.864
	MSE	0.104	0.203	0.131	0.417	0.104	0.201	0.119	0.409	0.059	0.100	0.088	0.283
(50,40,2.0,5.0)	ME	0.781	1.177	0.871	4.774	0.770	1.165	0.871	4.750	0.719	1.071	0.824	4.681
	MSE	0.077	0.165	0.101	0.382	0.069	0.157	0.097	0.371	0.047	0.080	0.061	0.182
(70,40,2.0,5.0)	ME	0.779	1.171	0.868	4.781	0.764	1.160	0.859	4.760	0.733	1.074	0.814	4.680
	MSE	0.071	0.171	0.097	0.369	0.071	0.158	0.092	0.374	0.047	0.073	0.060	0.185
(70,60,2.0,5.0)	ME	0.751	1.133	0.829	4.717	0.733	1.131	0.840	4.689	0.721	1.053	0.818	4.611
	MSE	0.047	0.142	0.069	0.303	0.051	0.137	0.075	0.341	0.028	0.045	0.041	0.132
(90,70,3.0,6.0)	ME	0.725	1.087	0.811	4.521	0.717	1.066	0.825	4.498	0.711	1.001	0.804	4.362
	MSE	0.028	0.099	0.051	0.245	0.017	0.087	0.049	0.236	0.012	0.031	0.028	0.114

TABLE 2. The estimate values of CP and MIL when $\theta = \{0.7, 1.0, 0.8, 4.0\}$

		ACI				Boot-p				Boot-t				CI			
(n, m, η_1, η_2)		a_1	a_2	b	c	a_1	a_2	b	c	a_1	a_2	b	c	a_1	a_2	b	c
(35,15,2.0,3.5)	CP	0.89	0.88	0.90	0.88	0.87	0.90	0.90	0.89	0.90	0.91	0.90	0.91	0.90	0.92	0.92	0.91
	MIL	1.74	2.45	1.94	7.66	1.85	2.94	2.10	7.98	1.65	2.40	1.91	7.28	1.51	2.24	1.83	7.42
(35,25,2.0,3.5)	CP	0.90	0.89	0.91	0.90	0.93	0.90	0.92	0.91	0.93	0.91	0.92	0.91	0.92	0.92	0.93	0.96
	MIL	1.66	2.37	1.90	7.55	1.74	2.88	2.02	7.84	1.51	2.29	1.84	7.20	1.45	2.14	1.72	7.40
(50,25,2.0,3.5)	CP	0.90	0.91	0.91	0.92	0.93	0.89	0.92	0.95	0.93	0.93	0.94	0.92	0.92	0.94	0.91	0.92
	MIL	1.67	2.39	1.93	7.54	1.78	2.85	2.05	7.81	1.55	2.24	1.87	7.18	1.43	2.16	1.74	7.39
(50,40,2.0,3.5)	CP	0.93	0.91	0.91	0.92	0.90	0.93	0.92	0.92	0.94	0.93	0.92	0.93	0.92	0.94	0.93	0.92
	MIL	1.51	2.18	1.81	7.25	1.64	2.79	1.92	7.49	1.40	2.17	1.72	7.00	1.32	2.07	1.61	7.15
(70,40,2.0,3.5)	CP	0.92	0.91	0.93	0.92	0.92	0.93	0.92	0.92	0.93	0.93	0.93	0.93	0.94	0.95	0.92	0.93
	MIL	1.53	2.17	1.84	7.14	1.66	2.74	1.93	7.51	1.42	2.14	1.70	7.02	1.28	2.03	1.63	7.17
(70,60,2.0,3.5)	CP	0.93	0.93	0.91	0.94	0.90	0.93	0.94	0.92	0.94	0.94	0.92	0.95	0.92	0.92	0.93	0.96
	MIL	1.42	2.07	1.69	7.04	1.52	2.60	1.83	7.18	1.28	2.04	1.63	6.82	1.17	2.00	1.45	6.88
(35,15,2.0,5.0)	CP	0.90	0.89	0.91	0.90	0.89	0.90	0.91	0.89	0.91	0.91	0.92	0.91	0.92	0.91	0.92	0.93
	MIL	1.70	2.41	1.89	7.61	1.83	2.91	2.08	7.93	1.59	2.41	1.87	7.28	1.44	2.19	1.81	7.35
(35,25,2.0,5.0)	CP	0.91	0.90	0.90	0.92	0.93	0.94	0.92	0.93	0.92	0.93	0.92	0.94	0.91	0.92	0.94	0.92
	MIL	1.57	2.25	1.82	7.25	1.71	2.78	1.89	7.65	1.14	2.14	1.69	7.00	1.33	2.05	1.64	7.29
(50,25,2.0,5.0)	CP	0.92	0.91	0.93	0.92	0.90	0.91	0.92	0.92	0.94	0.93	0.91	0.92	0.94	0.94	0.95	0.93
	MIL	1.57	2.30	1.84	7.31	1.70	2.74	2.01	7.69	1.45	2.12	1.72	7.04	1.31	2.04	1.70	7.18
(50,40,2.0,5.0)	CP	0.92	0.93	0.93	0.92	0.94	0.91	0.92	0.93	0.94	0.91	0.95	0.96	0.94	0.93	0.93	0.96
	MIL	1.40	2.03	1.67	7.04	1.50	2.62	1.81	7.29	1.26	2.04	1.61	6.89	1.14	2.00	1.45	7.01
(70,40,2.0,5.0)	CP	0.90	0.91	0.92	0.93	0.92	0.94	0.92	0.90	0.93	0.94	0.93	0.96	0.93	0.91	0.92	0.94
	MIL	1.42	2.10	1.77	7.01	1.53	2.49	1.81	7.36	1.29	2.03	1.58	6.88	1.19	2.00	1.47	7.23
(70,60,2.0,5.0)	CP	0.92	0.93	0.94	0.90	0.92	0.93	0.92	0.92	0.92	0.94	0.93	0.95	0.93	0.92	0.93	0.92
	MIL	1.29	1.88	1.56	6.79	1.43	2.41	1.70	7.09	1.14	1.92	1.51	6.71	1.11	1.79	1.33	6.73
(90,70,3,6)	CP	0.93	0.90	0.94	0.92	0.92	0.94	0.92	0.95	0.92	0.95	0.93	0.94	0.93	0.92	0.94	0.96
	MIL	1.14	1.70	1.42	6.49	1.01	2.28	1.51	6.86	1.00	1.79	1.33	6.55	1.00	1.65	1.18	6.20

TABLE 3. The estimate values of ME and MSE $\theta = \{1.5, 2.0, 3.0, 10.0\}$

		MLE				Bayes(P^0)				Bayes(P^0)			
(n, m, η_1, η_2)		a_1	a_2	b	c	a_1	a_2	b	c	a_1	a_2	b	c
(35,15,1.8,3.5)	ME	1.875	2.456	3.345	12.452	1.854	2.448	3.318	12.400	1.755	2.380	3.233	12.280
	MSE	0.240	0.287	0.351	1.423	0.233	0.279	0.345	1.411	0.187	0.203	0.245	1.221
(35,25,1.8,3.5)	ME	1.855	2.431	3.324	12.430	1.834	2.432	3.307	12.375	1.725	2.364	3.211	12.256
	MSE	0.228	0.274	0.340	1.408	0.219	0.270	0.331	1.400	0.181	0.192	0.237	1.214
(50,25,1.8,3.5)	ME	1.857	2.428	3.321	12.432	1.829	2.435	3.304	12.371	1.719	2.367	3.213	12.260
	MSE	0.223	0.271	0.342	1.413	0.216	0.272	0.333	1.402	0.179	0.191	0.234	1.215
(50,40,1.8,3.5)	ME	1.814	2.389	3.300	12.380	1.801	2.401	3.281	12.342	1.690	2.337	3.187	12.204
	MSE	0.213	0.262	0.327	1.394	0.203	0.255	0.314	1.382	0.167	0.179	0.225	1.199
(70,40,1.8,3.5)	ME	1.816	2.384	3.303	12.383	1.807	2.400	3.277	12.339	1.688	2.334	3.189	12.207
	MSE	0.215	0.260	0.324	1.392	0.205	0.257	0.312	1.379	0.164	0.176	0.227	1.196
(70,60,1.8,3.5)	ME	1.788	2.357	3.272	12.349	1.781	2.371	3.255	12.317	1.666	2.309	3.170	12.181
	MSE	0.201	0.251	0.319	1.379	0.189	0.242	0.303	1.369	0.160	0.170	0.213	1.183
(35,15,1.8,5.5)	ME	1.871	2.453	3.341	12.449	1.850	2.442	3.314	12.395	1.751	2.377	3.230	12.277
	MSE	0.232	0.281	0.345	1.420	0.228	0.273	0.341	1.408	0.182	0.201	0.240	1.217
(35,25,1.8,5.5)	ME	1.847	2.424	3.317	12.422	1.830	2.424	3.301	12.370	1.716	2.355	3.203	12.251
	MSE	0.221	0.269	0.332	1.401	0.211	0.260	0.324	1.392	0.175	0.179	0.233	1.205
(50,25,1.8,5.5)	ME	1.851	2.419	3.312	12.421	1.821	2.427	3.297	12.366	1.713	2.362	3.214	12.254
	MSE	0.218	0.272	0.335	1.407	0.212	0.266	0.331	1.395	0.172	0.188	0.230	1.210
(50,40,1.8,5.5)	ME	1.801	2.377	3.288	12.370	1.792	2.389	3.280	12.336	1.691	2.330	3.182	12.193
	MSE	0.211	0.255	0.322	1.391	0.197	0.247	0.310	1.379	0.161	0.173	0.221	1.196
(70,40,1.8,5.5)	ME	1.808	2.380	3.300	12.377	1.801	2.392	3.272	12.334	1.680	2.328	3.180	12.202
	MSE	0.211	0.254	0.319	1.390	0.201	0.251	0.307	1.373	0.160	0.170	0.222	1.188
(70,60,1.8,5.5)	ME	1.783	2.354	3.269	12.343	1.770	2.370	3.245	12.311	1.663	2.302	3.162	12.174
	MSE	0.192	0.244	0.309	1.367	0.174	0.235	0.300	1.362	0.149	0.164	0.210	1.174
(80,60,3,7)	ME	1.702	2.314	3.224	12.303	1.725	2.324	3.213	12.259	1.617	2.259	3.124	12.142
	MSE	0.174	0.221	0.291	1.341	0.149	0.214	0.282	1.341	0.132	0.150	0.200	1.161

TABLE 4. The estimate values of CP and MIL when $\theta = \{1.5, 2.0, 3.0, 10.0\}$

		ACI				Boot-p				Boot-t				CI			
(n, m, η_1, η_2)		a_1	a_2	b	c	a_1	a_2	b	c	a_1	a_2	b	c	a_1	a_2	b	c
(35,15,1.8,3.5)	CP	0.90	0.88	0.90	0.91	0.91	0.87	0.91	0.90	0.91	0.92	0.90	0.90	0.92	0.90	0.91	0.92
	MIL	3.54	4.52	6.45	19.45	3.75	4.73	6.69	19.72	3.39	4.40	6.32	19.20	3.14	4.17	6.08	18.75
(35,25,1.8,3.5)	CP	0.91	0.91	0.92	0.90	0.91	0.93	0.92	0.91	0.92	0.92	0.93	0.90	0.91	0.93	0.90	0.94
	MIL	3.38	4.35	6.01	19.43	3.58	4.60	6.51	19.54	3.21	4.19	6.11	19.03	3.00	4.02	5.92	18.59
(50,25,1.8,3.5)	CP	0.92	0.89	0.92	0.91	0.91	0.92	0.93	0.91	0.94	0.92	0.91	0.92	0.92	0.92	0.94	0.92
	MIL	3.41	4.31	6.04	19.42	3.61	4.62	6.53	19.49	3.17	4.16	6.14	19.07	3.02	4.00	5.89	18.54
(50,40,1.8,3.5)	CP	0.93	0.93	0.92	0.92	0.94	0.93	0.97	0.92	0.93	0.92	0.94	0.94	0.96	0.93	0.91	0.93
	MIL	3.24	4.22	5.84	19.39	3.41	4.43	6.33	19.41	3.02	4.07	6.00	18.70	2.87	3.82	5.79	18.47
(70,40,1.8,3.5)	CP	0.92	0.92	0.92	0.89	0.93	0.92	0.94	0.94	0.94	0.92	0.92	0.94	0.93	0.93	0.94	0.96
	MIL	3.27	4.25	5.81	19.40	3.43	4.46	6.29	19.37	3.05	4.03	6.02	18.64	2.85	3.81	5.74	18.49
(70,60,1.8,3.5)	CP	0.94	0.93	0.94	0.93	0.94	0.90	0.93	0.95	0.93	0.95	0.94	0.93	0.96	0.95	0.91	0.97
	MIL	3.15	4.09	5.71	19.38	3.24	4.25	6.22	19.29	3.91	4.00	5.87	18.40	2.55	3.61	5.61	18.29
(35,15,1.8,5.5)	CP	0.89	0.90	0.90	0.90	0.91	0.90	0.91	0.91	0.91	0.92	0.91	0.92	0.92	0.90	0.91	0.91
	MIL	3.49	4.50	6.41	19.42	3.72	4.71	6.66	19.68	3.38	4.41	6.28	19.18	3.15	4.15	6.05	18.71
(35,25,1.8,5.5)	CP	0.92	0.91	0.93	0.91	0.91	0.93	0.90	0.91	0.93	0.92	0.96	0.92	0.91	0.92	0.91	0.96
	MIL	3.36	4.32	6.00	19.39	3.53	4.60	6.49	19.49	3.22	4.17	6.08	19.01	2.98	4.00	5.87	18.56
(50,25,1.8,5.5)	CP	0.91	0.92	0.92	0.93	0.93	0.92	0.94	0.93	0.94	0.95	0.94	0.92	0.95	0.92	0.93	0.95
	MIL	3.37	4.28	6.01	19.40	3.58	4.60	6.49	19.46	3.13	4.13	6.11	19.02	2.99	3.94	5.84	18.51
(50,40,1.8,5.5)	CP	0.92	0.93	0.94	0.91	0.93	0.94	0.94	0.92	0.94	0.92	0.93	0.91	0.94	0.97	0.92	0.93
	MIL	3.21	4.18	5.81	19.40	3.42	4.39	6.30	19.36	3.00	4.02	5.97	18.66	2.82	3.80	5.74	18.43
(70,40,1.8,5.5)	CP	0.93	0.92	0.91	0.92	0.92	0.94	0.94	0.94	0.92	0.92	0.93	0.94	0.95	0.91	0.94	0.95
	MIL	3.23	4.23	5.77	19.38	3.41	4.42	6.26	19.33	3.01	4.00	5.97	18.60	2.84	3.82	5.71	18.47
(70,60,1.8,5.5)	CP	0.92	0.92	0.96	0.94	0.94	0.92	0.94	0.93	0.92	0.94	0.92	0.93	0.95	0.93	0.91	0.96
	MIL	3.11	4.04	5.69	19.36	3.19	4.20	6.20	19.26	3.87	3.97	5.84	18.38	2.50	3.60	5.57	18.23
(80,60,3,7)	CP	0.93	0.95	0.94	0.93	0.92	0.91	0.96	0.94	0.92	0.93	0.94	0.96	0.95	0.94	0.93	0.94
	MIL	3.01	3.97	5.61	19.28	3.08	4.07	6.06	19.12	3.75	3.82	5.7	18.30	2.38	3.48	5.42	18.08

TABLE 5. The real lifetime data obtained from germ-free male mice with two primary causes of failure

Thymic lymphoma									
1.58	1.92	1.93	1.94	1.95	2.02	2.12	2.15	2.29	2.3
2.37	2.40	2.44	2.47	2.59	3.00	3.01	3.21	3.37	4.15
4.34	4.44	4.85	4.96	5.29	5.37	6.24	7.07	8.00	
Other causes									
1.36	2.46	2.55	3.76	4.21	5.65	6.16	6.17	l 6.52	6.55
6.58	6.6	6.62	6.75	l 6.81	7.34	7.36	7.37	7.57	7.69
7.77	8.00	8.07	8.25	8.55	8.57	8.64	8.68	8.70	8.70
8.73	8.82	8.95	9.10	9.34	9.42	10.15	10.19		

TABLE 6. Type-II GHC competing risks data.

$\mathbf{t}_i$	1.36	1.58	1.92	1.93	1.94	1.95	2.02	2.12	2.15	2.29	2.3	2.37
δ_i	2	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{t}_i$	2.4	2.44	2.46	2.47	2.55	2.59	3.0	3.01	3.21	3.37	3.76	4.15
δ_i	1	1	2	1	2	1	1	1	1	1	2	1
$\mathbf{t}_i$	4.21	4.34	4.44	4.85	4.96	5.29	5.37	5.65	6.16	6.17	6.24	6.52
δ_i	2	1	1	1	1	1	1	2	2	2	1	2
$\mathbf{t}_i$	6.55	6.58	6.6	6.62								
δ_i	2	2	2	2								

5.2. **Data Analysis.** In this study, we adopt a real data sets. The dataset was presented by Hoel [34], who conducted it over a period of 5–6 weeks as part of a laboratory experiment. The experiment involved male mice living in a germ-free environment, and observing the lifetimes of male mice that received a radiation dose of 300 roentgens. These data considered by different authors, Sarhan et al. [35], Cramer and Schmiedt [36], Bakoban and Abd-Elmougod [23], AL-Wageh et al. [37] and Al-Essa et al. [38]. In our study, two specific causes of failure are considered, thymic lymphoma and Other causes. The data after divided by 100 for simplicity is presented by Table 5. Figure 4 shows a reasonably good fit of the data in Table 5 to the NEW distribution with the Kolmogorov-Smirnov (K-S) distance equal {0.1678, 0.1358}, respectively. These figures are used before applying statistical inference to validate distribution assumptions. For given m =40, τ_1 =4 and τ_1 =7 the results of k_1 =27 and k_2 =13 and the corresponding type-II GHC competing risks data are given by Table 6. Based on the type-II GHC data given in Table 6, we report the MLEs along with their approximate 95% confidence intervals in Table 7.

For the interest model parameter, two bootstrap confidence intervals were constructed. 95% percentile bootstrap confidence intervals and 95% bootstrap-t confidence intervals using 500 generated samples in Table 7. Also, non-informative prior information is adopted in the Bayesian approach. In this example, run Chen 21000 iterations, and delete the first 1000 iterations as burn-in. Figures from 5 to 8 describe convergence in MCMC.

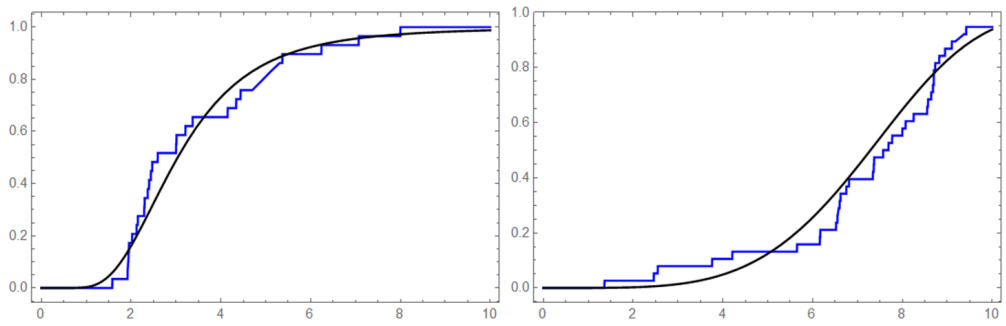

FIGURE 4. The empirical and theoretical CDF of two causes in Table 5.

TABLE 7. The point ML and Bayes with corresponding 95% interval estimates.

Parameter	$(.)_{ML}$	$(.)_{Bayes}$	95% ACI	95% Boot-p	95% Boot-t	95% CI
a_1	0.897003	0.849884	(0.001,3.536)	(0.113, 1.999)	(0.145, 1.884)	(0.166, 1.875)
a_2	0.43189	0.425872	(0.001,1.714)	(0.0247, 1.652)	(0.069, 1.002)	(0.077, 1.006)
b	0.247518	0.354854	(0.001, 1.451)	(0.100, 1.231)	(0.005, 0.989)	(0.014,0.972)
c	5.73104	5.612930	(1.037, 10.425)	(2.9431, 8.654)	(2.912, 7.952)	(2.943, 7.911)

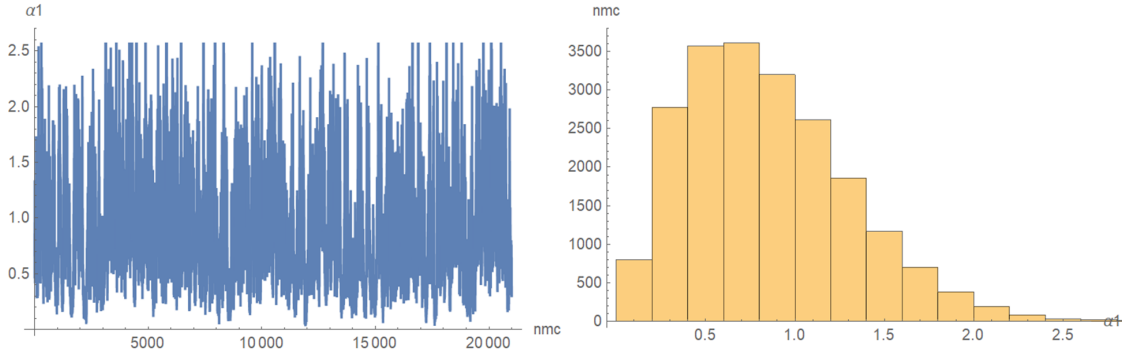

FIGURE 5. Simulation number and corresponding histogram of a_1 generated by MCMC.

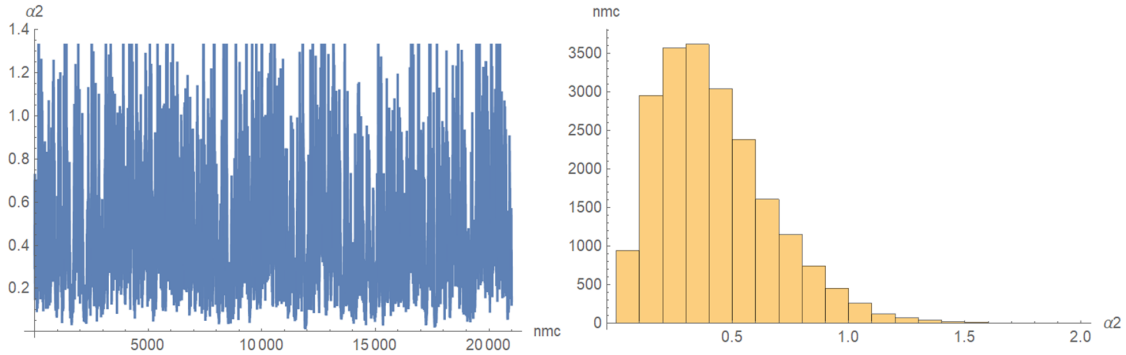

FIGURE 6. Simulation number and corresponding histogram of a_2 generated by MCMC.

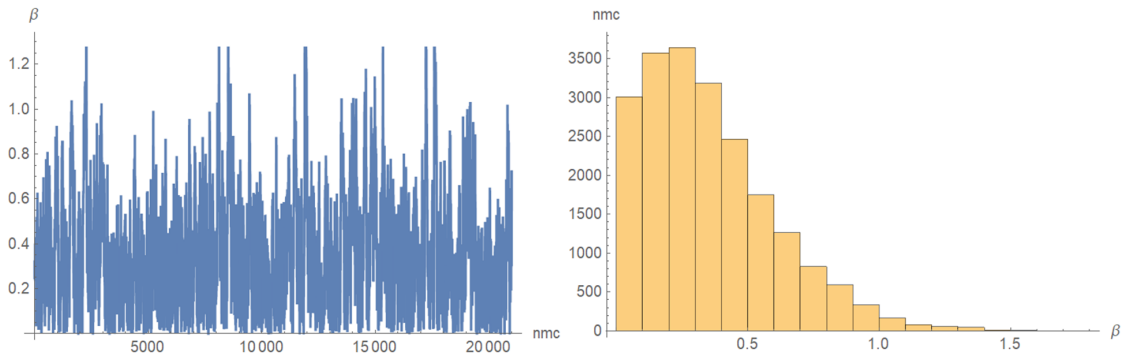

FIGURE 7. Simulation number and corresponding histogram of b generated by MCMC.

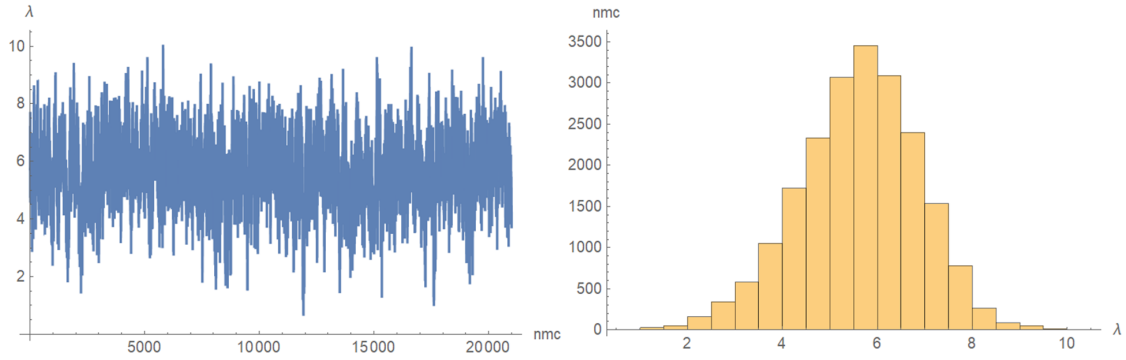

FIGURE 8. Simulation number and corresponding histogram of c generated by MCMC.

5.3. **Numerical discussion.** The numerical discussion presented here offers a detailed analysis of the statistical inference procedures derived from the Monte Carlo simulation study and real data analysis applied to the competing risks model. The results have shown that the performance of classical and Bayesian estimation methods work well in all cases. Here some point are observed from the numerical results.

(1) The type-II GHCS is a practical and flexible tool for several life-testing experiments, especially biomedical and reliability studies.
(2) When the multiple causes of failure are appeared the competing risks model essential for accurate analysis.
(3) The two censoring times τ_1 and τ_2 have more effect on the numerical results. the larger value of $\frac{m}{n}$ gives the best results in all cases.
(4) The results are closed for the ML and non-informative Bayes estimates.
(5) The results under the informative Bayes estimate are the best choice among other estimation methods.
(6) The bootstrap-t CIs are better than both the non-informative symmetric credible intervals and asymptotic CIs and for all choices.

(7) The mean squared error (MSE) generally decreases when the proportion (m/n) increases. Suggesting that higher numbers of observed failures are relative to the sample size to improve estimation accuracy.
(8) Also, the mean interval length (MIL) generally decreases when the proportion (m/n) increases.
(9) As the proportion (m/n) increases, the coverage percentage approaches the nominal level (1-α)%, indicating better reliability of the interval estimates.

6. Conclusions and Some Comments

In this paper, we examine products with lifetimes following the NEW distribution. In life testing experiments, we address a significant issue known as the competing risks problem. When the shape parameters are common, but the scale parameters differ, the lifetimes are modeled using the NEW distribution. Under a type-II GHCS, the model parameters are estimated with ML, bootstrap, and Bayesian methods. The behavior of the proposed estimators is examined through a simulation study and a real data example. The results show that the outperform of Bayesian estimators in point estimate and both bootstrap-t and Bayesian estimators in interval estimates.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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