

Intuitionistic Fuzzy Hyper h -Ideals in Hyper BCK-Algebras

Kalavath Anjaneyulu Naik¹, Dasari Ramesh², Aiyared Iampan^{3,*}, Bavanari Satyanarayana¹,
Punukollu Rajani⁴, Ravutla Durga Prasad⁵

¹*Department of Mathematics, University College of Sciences, Acharya Nagarjuna University, Guntur, Andhra Pradesh 522510, India*

²*Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Educational Foundation, Vaddeswaram, Andhra Pradesh 522302, India*

³*Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

⁴*Department of Basic Sciences and Humanities, Seshadri Rao Gudlavalleru Engineering College, Seshadri Rao Knowledge Village, Gudlavalleru, Andhra Pradesh 521356, India*

⁵*Department of Mathematics, KG Reddy College of Engineering and Technology, Hyderabad, Telangana 501504, India*

*Corresponding author: aiyared.ia@up.ac.th

Abstract. This research explores the intuitionistic fuzzification of hyper h -ideals in hyper BCK-algebras, including weak, strong, and reflexive types. We demonstrate that every intuitionistic fuzzy weak, strong, or reflexive hyper h -ideal is an intuitionistic fuzzy $HBCKI$, and investigate the relationships among these ideals, revealing key connections. Additionally, we examine the hyper homomorphic pre-image and product of intuitionistic fuzzy hyper h -ideals, showcasing their preservation properties. These findings advance intuitionistic fuzzy algebraic structures in hyper BCK-algebras, with potential applications in fuzzy logic and related fields.

1. INTRODUCTION

The fundamental set-theoretic operations—namely, union, intersection, and set difference—have far-reaching implications in algebra. The study of these operations has led to numerous algebraic structures, including Boolean algebras, distributive lattices, semirings, and upper and lower semilattices. Before 1966, algebraic structures were primarily studied through the properties

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of intersection and union, whereas set difference and its properties remained largely unexamined. Imai and Iséki [7] introduced BCK-algebras, which generalize the characteristics of set difference, and Iséki later extended the concept to BCI-algebras. BCK-algebras are rooted in BCK logic, founded upon modus ponens and the axiomatic scheme outlined below:

$$\text{Axiom B: } (\check{C} \supset \check{G}) \supset ((\check{L} \supset \check{C}) \supset (\check{L} \supset \check{G}))$$

$$\text{Axiom C: } (\check{C} \supset (\check{G} \supset \check{L})) \supset (\check{G} \supset (\check{C} \supset \check{L}))$$

$$\text{Axiom K: } \check{C} \supset (\check{G} \supset \check{C})$$

Similarly, BCI-algebras are grounded in BCI logic. The link between these algebras and their logics is often deep, allowing well-formed formulas and theorems to be translated back and forth. Although the inverse translation may not always exist, investigating algebras arising from known logics remains a fertile research area, yielding valuable insights into the associated logics.

The concept of hyperstructure was first introduced by Marty [15] in 1934, during the 8th Congress of Scandinavian Mathematicians. Nowadays, hyperstructures find extensive applications across pure and applied mathematics. In 1966, Imai and Iséki [7] introduced a set of axioms, now recognized as BCK-algebras, while exploring properties related to set differences. In 1999-2000, Jun et al. [8, 11] extended hyperstructures to BCK-algebras, presenting the concept of an \mathcal{HBCKA} , a broader version of BCK-algebras, and examining its associated properties. In 2000, Borzooei et al. [3] extended this idea to K-algebras, introducing hyper K-algebras. In 2000, Jun et al. [10] explored the characteristics of fuzzy strong \mathcal{HBCKI} -structures. In 1965, Zadeh [20] pioneered the highly applicable theory of fuzzy sets, a valuable tool for managing uncertainty. Later, in 2001, Jun and Xin [9] extended fuzzy set theory to hyper BCK-algebras. In 2005, Kondo and Dudek [13] introduced the transfer principle in fuzzy theory. More recently, Touqeer et al. [19] in 2016 proposed fuzzy hyper h-ideals in \mathcal{HBCKAs} , and Davvaz et al. [4] introduced the calculus of meet-plus hyperalgebras, also known as tropical semihyperrings.

Subsequently, in 2021, Khademan et al. [12] introduced the novel concept of fuzzy soft positive implicative \mathcal{HBCKI} -structures of several types. In 2023, Harikrishnan et al. [6] studied 2-absorbing hyperideals and homomorphisms in join hyperlattices. Earlier, Bhattacharya and Mukherjee [2] introduced fuzzy relations and fuzzy groups, offering valuable insights into these notions. Then, in 1986, Atanassov [1] introduced a new class of intuitionistic fuzzy sets that generalize fuzzy sets by incorporating both a degree of truth membership and a degree of falsity membership, with their sum lying between 0 and 1. Satyanarayana et al. explored various types of ideals in BCK and Hyper BCK-algebras [5, 14, 16–18], providing new insights into the field.

This study examines the intuitionistic fuzzification of \mathcal{HHIs} in \mathcal{HBCKAs} , considering weak, strong, and reflexive variants. It explores the interrelations among these ideals, uncovering key links. Additionally, the hyper-homomorphic preimage and product of intuitionistic fuzzy hyper h-ideals are analyzed, and it is shown that they retain essential preservation properties. These findings advance intuitionistic fuzzy algebraic structures within hyper BCK-algebras, opening avenues for applications in fuzzy logic and related domains.

ACRONYMS

- hyper BCK-algebra : \mathcal{HBCKA} or \mathcal{N}
- hyper BCK-ideal : \mathcal{HBCKI}
- hyper h-ideal : \mathcal{HHI}
- intuitionistic fuzzy set : \mathcal{IFS}
- intuitionistic fuzzy hyper h-ideal : \mathcal{IFHHI}
- intuitionistic fuzzy weak hyper h-ideal : \mathcal{IFWHHI}
- intuitionistic fuzzy strong hyper h-ideal : \mathcal{IFSHHI}
- intuitionistic fuzzy reflexive hyper h-ideal : \mathcal{IFRHHI}
- intuitionistic fuzzy (weak, strong, reflexive) hyper BCK-ideal : $\mathcal{NF}(W, S, R)\mathcal{HBCKI}$

2. PRELIMINARIES

This section presents the fundamental concepts necessary to understand the article.

Let \mathcal{N} be a nonempty set equipped with a hyper-operation $\star : \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{Q}^*(\mathcal{N})$, where $\mathcal{Q}^*(\mathcal{N})$ denotes the set of all nonempty subsets of \mathcal{N} . For any subsets $\check{\mathcal{C}}, \check{\mathcal{G}} \subseteq \mathcal{N}$, we define

$$\check{\mathcal{C}} \star \check{\mathcal{G}} = \bigcup_{\check{\mathfrak{m}} \in \check{\mathcal{C}}, \check{\mathfrak{n}} \in \check{\mathcal{G}}} \check{\mathfrak{m}} \star \check{\mathfrak{n}}.$$

The notation $\chi \star \check{h}$ will be used to represent $\chi \star \{\check{h}\}$, $\{\chi\} \star \check{h}$, or $\{\chi\} \star \{\check{h}\}$, depending on the context.

Definition 2.1 ([11]). *An \mathcal{HBCKA} is a nonempty set \mathcal{N} together with a hyper-operation \star and a constant 0 satisfying the given axioms.*

- ($\mathcal{HBCKA1}$) $\chi \star (\check{h} \star \kappa) \ll \chi \star \check{h}$,
- ($\mathcal{HBCKA2}$) $(\chi \star \check{h}) \star \kappa = (\chi \star \kappa) \star \check{h}$,
- ($\mathcal{HBCKA3}$) $\chi \star \mathcal{N} \ll \{\chi\}$,
- ($\mathcal{HBCKA4}$) $\chi \ll \check{h}$ and $\check{h} \ll \chi \Rightarrow \chi = \check{h}, \forall \chi, \check{h}, \kappa \in \mathcal{N}$.

We define a relation \ll on \mathcal{N} as: for $\chi, \check{h} \in \mathcal{N}$, $\chi \ll \check{h}$ if and only if $0 \in \chi \star \check{h}$. This extends to subsets $\check{\mathcal{C}}, \check{\mathcal{G}} \subseteq \mathcal{N}$ as: $\check{\mathcal{C}} \ll \check{\mathcal{G}}$ if for each $\check{\mathfrak{m}} \in \check{\mathcal{C}}$, there's $\check{\mathfrak{n}} \in \check{\mathcal{G}} \ni \check{\mathfrak{m}} \ll \check{\mathfrak{n}}$. This \ll is the hyperorder on \mathcal{N} .

A BCK-algebra $(\mathcal{N}, \circ, 0)$ can be seen as an $\mathcal{HBCKA}(\mathcal{N}, \star, 0)$, with \star defined by $\chi \star \check{h} = \{\chi \circ \check{h}\}$.

A notable example of an \mathcal{HBCKA} is $\mathcal{N} = [0, \infty)$, the set of nonnegative real numbers, with a suitably defined hyper-operation.

$$\chi \star \check{h} := \begin{cases} [0, \chi] & \text{if } \chi < \check{h} \\ [0, \check{h}] & \text{if } \chi > \check{h} \neq 0 \\ \{\chi\} & \text{if } \check{h} = 0. \end{cases}$$

Proposition 2.1 ([11]). *In any $\mathcal{HBCKA} \mathcal{N}$, the following properties are satisfied: (P_1) $\chi \star \check{h} \ll \{\chi\}$, (P_2) $\chi \star 0 \ll \{\chi\}$, (P_3) $0 \star 0 = \{0\}$, (P_4) $0 \ll \chi$, (P_5) $\check{\mathcal{C}} \ll \check{\mathcal{C}}$, (P_6) $\check{\mathcal{C}} \subseteq \check{\mathcal{G}} \Rightarrow \check{\mathcal{C}} \ll \check{\mathcal{G}}$, (P_7) $0 \star \chi = \{0\}$,*

$(P_8) 0 \star \mathfrak{C} = 0$, $(P_9) \mathfrak{C} \star \{0\} = \{0\}$ implies that $\mathfrak{C} = \{0\}$, $(P_{10}) \ddot{h} \ll \varkappa$ implies that $\chi \star \varkappa \ll \chi \star \ddot{h}$, $\forall \chi, \ddot{h}, \varkappa \in \mathcal{N}$, and for any nonempty subsets $\mathfrak{C}, \mathfrak{G}$ of \mathcal{N} .

For the fundamental concepts and properties of hyper BCK-subalgebras and \mathcal{HBCKI} s (including weak, strong, and reflexive types), we refer to [11]. Hereafter, \mathcal{N} denotes an \mathcal{HBCKA} .

Lemma 2.1 ([10,11]).

- (i) Every \mathcal{HBCKI} of \mathcal{N} is a weak \mathcal{HBCKI} of \mathcal{N} .
- (ii) Every reflexive \mathcal{HBCKI} of \mathcal{N} is a strong \mathcal{HBCKI} of \mathcal{N} .
- (iii) Every strong \mathcal{HBCKI} of \mathcal{N} is an \mathcal{HBCKI} of \mathcal{N} .

Lemma 2.2 ([10]). For every reflexive \mathcal{HBCKI} \mathfrak{S} of \mathcal{N} , if $\chi \star \ddot{h} \cap \mathfrak{S} \neq \emptyset$ implies $\chi \star \ddot{h} \ll \mathfrak{S} \forall \chi, \ddot{h} \in \mathcal{N}$.

Proposition 2.2 ([8]). If \mathfrak{C} is a subset of \mathcal{N} and \mathfrak{S} is any \mathcal{HBCKI} of \mathcal{N} such that $\mathfrak{C} \ll \mathfrak{S}$, then $\mathfrak{C} \subseteq \mathfrak{S}$.

Definition 2.2 ([19]). For an \mathcal{HBCKA} \mathcal{N} , a nonempty subset $\mathfrak{S} \subseteq \mathcal{N}$, containing 0 is referred to as

- (i) a weak \mathcal{HHI} of \mathcal{N} if $\chi \star (\ddot{h} \star \varkappa) \subseteq \mathfrak{S}$ and $\ddot{h} \in \mathfrak{S} \Rightarrow \chi \star \varkappa \subseteq \mathfrak{S}$.
- (ii) an \mathcal{HHI} of \mathcal{N} if $\chi \star (\ddot{h} \star \varkappa) \ll \mathfrak{S}$ and $\ddot{h} \in \mathfrak{S} \Rightarrow \chi \star \varkappa \subseteq \mathfrak{S}$.
- (iii) a strong \mathcal{HHI} of \mathcal{N} if $\chi \star (\ddot{h} \star \varkappa) \cap \mathfrak{S} \neq \emptyset$ and $\ddot{h} \in \mathfrak{S} \Rightarrow \chi \star \varkappa \subseteq \mathfrak{S}$ for any $\chi, \ddot{h}, \varkappa \in \mathcal{N}$.

Theorem 2.1 ([19]). Every (weak, strong, reflexive) \mathcal{HHI} is a (weak, strong, reflexive) \mathcal{HBCKI} of \mathcal{N} .

Theorem 2.2 ([19]). For any \mathcal{HBCKA} ,

- (i) any \mathcal{HHI} of \mathcal{N} is a weak \mathcal{HHI} of \mathcal{N} .
- (ii) any strong \mathcal{HHI} \mathcal{N} is an \mathcal{HHI} of \mathcal{N} .
- (iii) any reflexive \mathcal{HHI} of \mathcal{N} is a strong \mathcal{HHI} of \mathcal{N} .

Remark. Generally, the converse of Theorem 2.2 isn't hold (see, [19]).

For a detailed study of fuzzy (weak, strong, reflexive) \mathcal{HBCKI} s, one must consult [9].

Theorem 2.3 ([9]). For any \mathcal{HBCKA} \mathcal{N} ,

- (i) every fuzzy \mathcal{HBCKI} of \mathcal{N} is a fuzzy weak \mathcal{HBCKI} of \mathcal{N} .
- (ii) every fuzzy strong \mathcal{HBCKI} of \mathcal{N} is a fuzzy \mathcal{HBCKI} of \mathcal{N} .
- (iii) every fuzzy reflexive \mathcal{HBCKI} of \mathcal{N} is a fuzzy strong \mathcal{HBCKI} of \mathcal{N} .

Definition 2.3 ([19]). For an \mathcal{HBCKA} \mathcal{N} , a fuzzy set ϱ in \mathcal{N} is called

- (i) a fuzzy weak \mathcal{HHI} of \mathcal{N} if $\varrho(0) \geq \varrho(\chi)$,

$$\forall v_1 \in \chi \star \varkappa, \varrho(v_1) \geq \min\left\{ \inf_{\ddot{m} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\ddot{m}), \varrho(\ddot{h}) \right\},$$

- (ii) a fuzzy \mathcal{HHI} of \mathcal{N} if $\chi \ll \ddot{h} \Rightarrow \varrho(\chi) \geq \varrho(\ddot{h})$, and

$$\forall v_1 \in \chi \star \varkappa, \varrho(v_1) \geq \min\left\{ \inf_{\ddot{m} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\ddot{m}), \varrho(\ddot{h}) \right\},$$

(iii) a fuzzy strong \mathcal{HHI} of \mathcal{N} if $\inf_{\mathfrak{m} \in \chi \star \chi} \varrho(\mathfrak{m}) \geq \varrho(\chi)$ and

$$\forall v_1 \in \chi \star \chi, \varrho(v_1) \geq \min\left\{ \sup_{\mathfrak{n} \in \chi \star (\mathfrak{h} \star \chi)} \varrho(\mathfrak{n}), \varrho(\mathfrak{h}) \right\},$$

(iv) a fuzzy reflexive \mathcal{HHI} of \mathcal{N} if $\inf_{\mathfrak{m} \in \chi \star \chi} \varrho(\mathfrak{m}) \geq \varrho(\mathfrak{h})$ and

$$\forall v_1 \in \chi \star \chi, \varrho(v_1) \geq \min\left\{ \sup_{\mathfrak{n} \in \chi \star (\mathfrak{h} \star \chi)} \varrho(\mathfrak{n}), \varrho(\mathfrak{h}) \right\},$$

$$\forall \chi, \mathfrak{h}, \chi \in \mathcal{N}$$

Definition 2.4 ([3]). Let $(\mathcal{N}_1, \star_1, 0_1)$ and $(\mathcal{N}_2, \star_2, 0_2)$ be HBCKAs and $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$. We define a hyperoperation \star on \mathcal{N} by $(\chi, \mathfrak{h}) \star (\mathfrak{h}, \mathfrak{f}) = (\chi \star \kappa, \mathfrak{h} \star \mathfrak{f}) \forall (\chi, \mathfrak{h}), (\kappa, \mathfrak{f}) \in \mathcal{N}$, where for $\chi \subseteq \mathcal{N}_1$ and $G \subseteq \mathcal{N}_2$ by (χ, \mathfrak{h}) we mean $(\chi, \mathfrak{h}) = \{(\chi, \mathfrak{h}) : \chi \in F, \mathfrak{h} \in G\}$ and $0 = (0_1, 0_2)$, and a hyperorder \ll on \mathcal{N} by $(\chi, \mathfrak{h}) \ll (\kappa, \mathfrak{f}) \Leftrightarrow \chi \ll \kappa$ and $\mathfrak{h} \ll \mathfrak{f}$. Thus, $(\mathcal{N}, \star, 0)$ is an \mathcal{HBCKA} .

Let ϱ and ω be fuzzy sets in hyper BCK-algebras \mathcal{N}_1 and \mathcal{N}_2 respectively. Then $\varrho \times \omega$, the product of ϱ and ω of $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ is delineated as $(\varrho \times \omega)((\chi, \mathfrak{h})) = \min\{\varrho(\chi), \omega(\mathfrak{h})\}$

In the sequel, \mathcal{N}_1 and \mathcal{N}_2 will be hyper BCK-algebras and $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$.

3. INTUITIONISTIC FUZZY \mathcal{HHI} S IN HYPER BCK-ALGEBRAS

This section introduces and explores the concepts of \mathcal{IFHHI} s, including weak, strong, and reflexive types, along with their properties.

Definition 3.1. For an \mathcal{HBCKA} \mathcal{N} , an \mathcal{IFS} $\mathcal{M} = (\varrho, \omega)$ in \mathcal{N} is called

(i) an intuitionistic fuzzy weak hyper h -ideal (briefly, \mathcal{IFWHHI}) of \mathcal{N} if $\varrho(0) \geq \varrho(\chi)$, and $\omega(0) \leq \omega(\chi)$,

$$\forall v_1, v_2 \in \chi \star \chi, \varrho(v_1) \geq \min\left\{ \inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \chi)} \varrho(\mathfrak{m}), \varrho(\mathfrak{h}) \right\},$$

$$\omega(v_2) \leq \max\left\{ \sup_{\mathfrak{x} \in \chi \star (\mathfrak{h} \star \chi)} \omega(\mathfrak{x}), \omega(\mathfrak{h}) \right\}$$

(ii) an intuitionistic fuzzy hyper h -ideal (briefly, \mathcal{IFHHI}) of \mathcal{N} if $\chi \ll \mathfrak{h} \Rightarrow \varrho(\chi) \geq \varrho(\mathfrak{h})$, and $\omega(\chi) \leq \omega(\mathfrak{h})$, and

$$\forall v_1, v_2 \in \chi \star \chi, \varrho(v_1) \geq \min\left\{ \inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \chi)} \varrho(\mathfrak{m}), \varrho(\mathfrak{h}) \right\},$$

$$\omega(v_2) \leq \max\left\{ \sup_{\mathfrak{x} \in \chi \star (\mathfrak{h} \star \chi)} \omega(\mathfrak{x}), \omega(\mathfrak{h}) \right\}$$

(iii) an intuitionistic fuzzy strong hyper h -ideal (briefly, \mathcal{IFSHHI}) of \mathcal{N} if $\inf_{\mathfrak{m} \in \chi \star \chi} \varrho(\mathfrak{m}) \geq \varrho(\chi)$, and $\sup_{\mathfrak{x} \in \chi \star \chi} \omega(\mathfrak{x}) \leq \omega(\chi)$, and

$$\forall v_1, v_2 \in \chi \star \chi, \varrho(v_1) \geq \min\left\{ \sup_{\mathfrak{n} \in \chi \star (\mathfrak{h} \star \chi)} \varrho(\mathfrak{n}), \varrho(\mathfrak{h}) \right\},$$

$$\omega(v_2) \leq \max\left\{ \inf_{\mathfrak{y} \in \chi \star (\mathfrak{h} \star \chi)} \omega(\mathfrak{y}), \omega(\mathfrak{h}) \right\}$$

(iv) an intuitionistic fuzzy reflexive hyper h -ideal (briefly, $IFRHHI$) of \mathcal{N} if $\inf_{\mathfrak{h} \in \chi \star \chi} \varrho(\mathfrak{h}) \geq \varrho(\check{h})$, and $\sup_{\mathfrak{x} \in \chi \star \chi} \omega(\mathfrak{x}) \leq \omega(\check{h})$, and

$$\forall v_1, v_2 \in \chi \star \chi, \varrho(v_1) \geq \min\left\{ \sup_{\mathfrak{h} \in \chi \star (\check{h} \star \chi)} \varrho(\mathfrak{h}), \varrho(\check{h}) \right\},$$

$$\omega(v_2) \leq \max\left\{ \inf_{\mathfrak{h} \in \chi \star (\check{h} \star \chi)} \omega(\mathfrak{h}), \omega(\check{h}) \right\}$$

$\forall \chi, \check{h}, \chi \in \mathcal{N}$.

Theorem 3.1. Every intuitionistic fuzzy (weak, strong, reflexive) \mathcal{HHI} of \mathcal{N} is an intuitionistic fuzzy (weak, strong, reflexive) \mathcal{HBCKI} of \mathcal{N} .

Proof. Let $\mathcal{M} = (\varrho, \omega)$ be an $IFHHI$ of \mathcal{N} . Then $\forall \chi, \check{h}, \chi \in \mathcal{N}$, we obtain

$$\varrho(\chi) \geq \min\left\{ \inf_{\mathfrak{h} \in \chi \star (\check{h} \star \chi)} \varrho(\mathfrak{h}), \varrho(\check{h}) \right\},$$

$$\omega(\chi) \leq \max\left\{ \sup_{\mathfrak{x} \in \chi \star (\check{h} \star \chi)} \omega(\mathfrak{x}), \omega(\check{h}) \right\}.$$

Now, letting $\chi = 0$ we get

$$\varrho(\chi) \geq \min\left\{ \inf_{\mathfrak{h} \in \chi \star (g \star 0)} \varrho(\mathfrak{h}), \varrho(\check{h}) \right\},$$

$$\omega(\chi) \leq \max\left\{ \sup_{\mathfrak{x} \in \chi \star (g \star 0)} \omega(\mathfrak{x}), \omega(\check{h}) \right\},$$

thus

$$\varrho(\chi) \geq \min\left\{ \inf_{\mathfrak{h} \in (\chi \star \check{h})} \varrho(\mathfrak{h}), \varrho(\check{h}) \right\},$$

$$\omega(\chi) \leq \max\left\{ \sup_{\mathfrak{x} \in (\chi \star \check{h})} \omega(\mathfrak{x}), \omega(\check{h}) \right\}.$$

The proof of the theorem is concluded. □

Theorem 3.2. For any \mathcal{HBCKA} ,

(i) every $IFHHI$ of \mathcal{N} is an $IFWHHI$ of \mathcal{N} .

(ii) every $IFSHHI$ of \mathcal{N} is an $IFHHI$ of \mathcal{N} .

(iii) every $IFRHHI$ of \mathcal{N} is an $IFSHHI$ of \mathcal{N} .

Proof. (i) Let $\mathcal{M} = (\varrho, \omega)$ be an $IFHHI$ of \mathcal{N} . Since any $IFHHI$ is an $IFHBCKI$ by Theorem 3.1 and every $IFHBCKI$ is an $IFWHBCKI$ by Theorem 2.3(i), thus \mathcal{M} is also an $IFWHBCKI$ of \mathcal{N} . Therefore, $\mathcal{M} = (\varrho, \omega)$ fulfill $\varrho(0) \geq \varrho(\chi)$ and $\omega(0) \leq \omega(\chi) \forall \chi \in \mathcal{N}$. Also being an $IFHHI \forall \chi, \check{h}, \chi \in \mathcal{N}$ and $v_1, v_2 \in \chi \star \chi$, $\mathcal{M} = (\varrho, \omega)$ satisfies

$$\varrho(v_1) \geq \min\left\{ \inf_{\mathfrak{h} \in \chi \star (\check{h} \star \chi)} \varrho(\mathfrak{h}), \varrho(\check{h}) \right\},$$

$$\omega(v_2) \leq \max\left\{ \sup_{\mathfrak{x} \in \chi \star (\check{h} \star \chi)} \omega(\mathfrak{x}), \omega(\check{h}) \right\}.$$

Therefore, $\mathcal{M} = (\varrho, \omega)$ is an $IFWHHI$ of \mathcal{N} .

(ii) Let $\mathcal{M} = (\varrho, \omega)$ be an $IFSHHI$ of \mathcal{N} . Seeing that an $IFSHHI$ is an $IFSHBCKI$ by Theorem 3.1 and any $IFSHBCKI$ is an $IFHBCKI$ by Theorem 2.3(ii), thus \mathcal{M} is an $IFHBCKI$ of \mathcal{N} . Therefore, $\forall \chi, \check{h} \in \mathcal{N}$, if $\chi \ll \check{h}$, then $\varrho(\chi) \geq \varrho(\check{h})$, and $\omega(\chi) \leq \omega(\check{h})$. Also being an $IFSHHI$, $\forall \chi, \check{h}, \varkappa \in \mathcal{N}$ and $v_1, v_2 \in \chi \star \varkappa$ $\mathcal{M} = (\varrho, \omega)$ fulfills

$$\begin{aligned} \varrho(v_1) &\geq \min\{ \sup_{\check{m} \in \chi \star (\check{h} \star \varkappa)} \varrho(\check{m}), \varrho(\check{h}) \}, \\ \omega(v_2) &\leq \max\{ \inf_{\check{x} \in \chi \star (\check{h} \star \varkappa)} \omega(\check{x}), \omega(\check{h}) \}. \end{aligned}$$

Since $\sup_{\check{m} \in \chi \star (\check{h} \star \varkappa)} \varrho(\check{m}) \geq \varrho(\check{h})$ and $\inf_{\check{x} \in \chi \star (\check{h} \star \varkappa)} \omega(\check{x}) \leq \omega(\check{h}) \forall \check{h}, \check{h} \in \chi \star (\check{h} \star \varkappa)$, we get

$$\begin{aligned} \varrho(v_1) &\geq \min\{ \sup_{\check{m} \in \chi \star (\check{h} \star \varkappa)} \varrho(\check{m}), \varrho(\check{h}) \} \geq \min\{ \varrho(\check{h}), \varrho(\check{h}) \}, \\ \omega(v_2) &\leq \max\{ \inf_{\check{x} \in \chi \star (\check{h} \star \varkappa)} \omega(\check{x}), \omega(\check{h}) \} \leq \max\{ \omega(\check{h}), \omega(\check{h}) \} \end{aligned}$$

$\forall \check{h}, \check{h} \in \chi \star (\check{h} \star \varkappa)$. Since $\varrho(\check{h}) \geq \inf_{\check{m} \in \chi \star (\check{h} \star \varkappa)} \varrho(\check{m})$ and $\omega(\check{h}) \leq \sup_{\check{x} \in \chi \star (\check{h} \star \varkappa)} \omega(\check{x}) \forall \check{h}, \check{h} \in \chi \star (\check{h} \star \varkappa)$, thus

$$\begin{aligned} \varrho(v_1) &\geq \min\{ \varrho(\check{h}), \varrho(\check{h}) \} \geq \min\{ \inf_{\check{m} \in \chi \star (\check{h} \star \varkappa)} \varrho(\check{m}), \varrho(\check{h}) \}, \\ \omega(v_2) &\leq \max\{ \omega(\check{h}), \omega(\check{h}) \} \leq \max\{ \sup_{\check{x} \in \chi \star (\check{h} \star \varkappa)} \omega(\check{x}), \omega(\check{h}) \}, \end{aligned}$$

that is,

$$\begin{aligned} \varrho(v_1) &\geq \min\{ \inf_{\check{m} \in \chi \star (\check{h} \star \varkappa)} \varrho(\check{m}), \varrho(\check{h}) \}, \\ \omega(v_2) &\leq \max\{ \sup_{\check{x} \in \chi \star (\check{h} \star \varkappa)} \omega(\check{x}), \omega(\check{h}) \}. \end{aligned}$$

Therefore, $\mathcal{M} = (\varrho, \omega)$ is an $IFHHI$ of \mathcal{N} .

(iii) Let $\mathcal{M} = (\varrho, \omega)$ be an $IFRHHI$ of \mathcal{N} . Then $\mathcal{M} = (\varrho, \omega)$ fulfills $\inf_{\check{m} \in \chi \star \chi} \varrho(\check{m}) \geq \varrho(\check{h}) \forall \chi, \check{h}, \varkappa \in \mathcal{N}$ implies $\sup_{\check{m} \in \chi \star \chi} \varrho(\check{m}) \geq \varrho(\chi) \forall \chi \in \mathcal{N}$, and $\sup_{\check{x} \in \chi \star \chi} \omega(\check{x}) \leq \omega(\check{h}) \forall \chi, \check{h}, \varkappa \in \mathcal{N}$ implies $\inf_{\check{x} \in \chi \star \chi} \omega(\check{x}) \leq \omega(\chi) \forall \chi \in \mathcal{N}$. Thus, the first condition for $\mathcal{M} = (\varrho, \omega)$ to be an $IFHHI$ of \mathcal{N} is satisfied. Also being an $IFRHHI \forall \chi, \check{h}, \varkappa \in \mathcal{N}$ and $v_1, v_2 \in \chi \star \varkappa$, $\mathcal{M} = (\varrho, \omega)$ fulfills

$$\begin{aligned} \varrho(v_1) &\geq \min\{ \sup_{\check{h} \in \chi \star (\check{h} \star \varkappa)} \varrho(\check{h}), \varrho(\check{h}) \}, \\ \omega(v_2) &\leq \max\{ \inf_{\check{h} \in \chi \star (\check{h} \star \varkappa)} \omega(\check{h}), \omega(\check{h}) \}. \end{aligned}$$

Therefore, $\mathcal{M} = (\varrho, \omega)$ is an $IFSHHI$ of \mathcal{N} . The proof is now complete. □

Remark. Evidently, the converse of Theorem 3.2 is not valid (see the following example).

TABLE 1. Hyper BCK-algebra

\star	0	χ	\ddot{h}
0	{0}	{0}	{0}
χ	{ χ }	{0}	{0}
\ddot{h}	{ \ddot{h} }	{ χ, \ddot{h} }	{0, χ, \ddot{h} }

Example 3.1. Let $\mathcal{N} = \{0, \chi, \ddot{h}\}$ be an \mathcal{HBCKA} with the binary operation \star as follows:

Define $\mathcal{M} = (\varrho, \omega)$ in \mathcal{N} by $\varrho(0) = \varrho(\ddot{h}) = 0.6$, $\varrho(\chi) = 0.03$, and $\omega(0) = \omega(\ddot{h}) = 0.2$, $\omega(\chi) = 0.4$. Then $\mathcal{M} = (\varrho, \omega)$ is an \mathcal{IFWHHI} of \mathcal{N} but not an \mathcal{IFHHI} of \mathcal{N} because: $\chi \ll \ddot{h}$ imply $\varrho(\chi) = 0.03 < 0.6 = \varrho(\ddot{h})$, and $\omega(\chi) = 0.4 > 0.2 = \omega(\ddot{h})$.

The example below shows that being \mathcal{IFHHI} doesn't imply being \mathcal{IFSHHI} .

Example 3.2. Let $\mathcal{N} = \{0, \chi, \ddot{h}\}$ be an \mathcal{HBCKA} with the binary operation \star as follows:

TABLE 2. Hyper BCK-algebra

\star	0	χ	\ddot{h}
0	{0}	{0}	{0}
χ	{ χ }	{0, χ }	{0, χ }
\ddot{h}	{ \ddot{h} }	{ χ, \ddot{h} }	{0, χ, \ddot{h} }

Define $\mathcal{M} = (\varrho, \omega)$ in \mathcal{N} by $\varrho(0) = \varrho(\chi) = 0.4$, $\varrho(\ddot{h}) = 0.04$, and $\omega(0) = \omega(\chi) = 0.2$, $\omega(\ddot{h}) = 0.5$. Then $\mathcal{M} = (\varrho, \omega)$ is an \mathcal{IFHHI} of \mathcal{N} but it is not an \mathcal{IFSHHI} of \mathcal{N} because:

$$\varrho(\ddot{h}) = 0.04 < 0.4 = \min\left\{\sup_{\mathfrak{m} \in \ddot{h} \star (\chi \star \ddot{h})} \varrho(\mathfrak{m}), \varrho(\chi)\right\},$$

$$\omega(\ddot{h}) = 0.5 > 0.2 = \max\left\{\inf_{\mathfrak{x} \in \ddot{h} \star (\chi \star \ddot{h})} \omega(\mathfrak{x}), \omega(\chi)\right\}.$$

The given example illustrates that being an \mathcal{IFSHHI} doesn't necessarily imply being an \mathcal{IFRHHI} .

Example 3.3. Consider the hyper BCK-algebra $\mathcal{N} = \{0, \chi, \ddot{h}\}$ with the binary operation \star defined as in Table 2 of Example 3.2.

Define $\mathcal{IFS} \mathcal{M} = (\varrho, \omega)$ in \mathcal{N} by $\varrho(0) = 0.5$, $\varrho(\chi) = \varrho(\ddot{h}) = 0.03$, and $\omega(0) = 0$, $\omega(\chi) = \omega(\ddot{h}) = 0.6$. Then $\mathcal{M} = (\varrho, \omega)$ is an \mathcal{IFSHHI} of \mathcal{N} but is not an \mathcal{IFRHHI} of \mathcal{N} because for $\chi, \ddot{h} \in \ddot{h} \star \ddot{h}$, $\varrho(\chi) = \varrho(\ddot{h}) = 0.03 < 0.5 = \varrho(0)$, and $\omega(\chi) = \omega(\ddot{h}) = 0.6 > 0 = \omega(0)$.

Theorem 3.3. An $\mathcal{IFS} \mathcal{M} = (\varrho, \omega)$ in \mathcal{N} is an intuitionistic fuzzy (weak, strong, reflexive) \mathcal{HHI} of \mathcal{N} if and only if $\forall \mathfrak{s}, \mathfrak{v} \in [0, 1]$, the corresponding $(\mathfrak{s}, \mathfrak{v})$ -cuts, namely $\varrho_{\mathfrak{s}}$, and $\omega_{\mathfrak{v}}$, are nonempty (weak, strong, reflexive) \mathcal{HHI} s of \mathcal{N} .

Proof. Assume that $\mathcal{M} = (\varrho, \omega)$ is an \mathcal{IFHHI} of \mathcal{N} . Since $\mathcal{M}_{(s,v)} = (\varrho_s, \omega_v)$ is nonempty, so for any $\chi \in \mathcal{M}_{(s,v)}$, $\varrho(\chi) \geq s$, and $\omega(\chi) \leq v$. Given that every \mathcal{IFHHI} is an \mathcal{IFWHHI} , Theorem 3.2(i) implies that $\mathcal{M} = (\varrho, \omega)$ is an \mathcal{IFWHHI} of \mathcal{N} . Hence, $\varrho(0) \geq \varrho(\chi) \geq s$ and $\omega(0) \leq \omega(\chi) \leq v \forall \chi \in \mathcal{N}$ imply $0 \in \mathcal{M}_{(s,v)}$. Let $\chi \star (\ddot{h} \star \varkappa) \ll \mathcal{M}_{(s,v)}$ and $\ddot{h} \in \mathcal{M}_{(s,v)}$ for some $\chi, \ddot{h}, \varkappa \in \mathcal{N}$. Then $\forall \mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa), \exists \mathfrak{n} \in \varrho_s \ni \mathfrak{m} \ll \mathfrak{n}$. So $\varrho(\mathfrak{m}) \geq \varrho(\mathfrak{n}) \geq s \forall \mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa)$. Hence, $\inf_{\mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\mathfrak{m}) \geq s$. Also $\varrho(\ddot{h}) \geq s$, as $\ddot{h} \in \varrho_s$. Thus, $\forall v_1, v_2 \in \chi \star \varkappa$ that satisfies

$$\varrho(v_1) \geq \min\left\{ \inf_{\mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\mathfrak{m}), \varrho(\ddot{h}) \right\} \geq \min\{s, s\} = s \text{ implies } \chi \in \varrho_s,$$

and now, $\forall \mathfrak{x} \in \chi \star (\ddot{h} \star \varkappa), \exists \mathfrak{y} \in \omega_v \ni \mathfrak{x} \ll \mathfrak{y}$. So $\omega(\mathfrak{x}) \leq \omega(\mathfrak{y}) \leq v \forall \mathfrak{x} \in \chi \star (\ddot{h} \star \varkappa)$. Hence, $\sup_{\mathfrak{x} \in \chi \star (\ddot{h} \star \varkappa)} \omega(\mathfrak{x}) \leq v$. Also $\omega(\ddot{h}) \leq v$, as $\ddot{h} \in \omega_v$. Therefore,

$$\omega(v_2) \leq \max\left\{ \sup_{\mathfrak{x} \in \chi \star (\ddot{h} \star \varkappa)} \omega(\mathfrak{x}), \omega(\ddot{h}) \right\} \leq \max\{v, v\} = v \text{ implies } \chi \in \omega_v.$$

$\forall v_1, v_2 \in \chi \star \varkappa$ and then $\chi \star \varkappa \subseteq \mathcal{M}_{(s,v)}$. Thus, ϱ_s and ω_v are \mathcal{HHI} s of \mathcal{N} . In contrast, assume that ϱ_s and ω_v are nonempty \mathcal{HHI} of $\mathcal{N} \forall s, v \in [0, 1]$. Let $\chi \ll \ddot{h}$ for some $\chi, \ddot{h} \in \mathcal{N}$ and let $\varrho(\ddot{h}) = s$. Then $\ddot{h} \in \varrho_s$. So $\chi \ll \ddot{h} \in \varrho_s$ implies $\chi \ll \varrho_s$. As an \mathcal{HHI} , ϱ_s is necessarily an \mathcal{HBCKI} of \mathcal{N} by Theorem 2.1 hence, by Proposition 2.2, $\chi \in \varrho_s$. Thus, $\varrho(\chi) \geq s = \varrho(\ddot{h})$, i.e, $\chi \ll \ddot{h}$ implies $\varrho(\chi) \geq \varrho(\ddot{h}) \forall \chi, \ddot{h} \in \mathcal{N}$. Let $\chi \ll \ddot{h}$ for some $\chi, \ddot{h} \in \mathcal{N}$ and let $\omega(\ddot{h}) = v$. Then $\ddot{h} \in \omega_v$. So $\chi \ll \ddot{h} \in \omega_v$ implies $\chi \ll \omega_v$. As ϱ_s is an \mathcal{HHI} , it is also an \mathcal{HBCKI} of \mathcal{N} by Theorem 2.1. Hence, by Proposition 2.2, $\chi \in \omega_v$ implying $\omega(\chi) \leq v = \omega(\ddot{h})$, i.e, $\chi \ll \ddot{h}$ imply $\omega(\chi) \leq \omega(\ddot{h}) \forall \chi, \ddot{h} \in \mathcal{N}$. Notwithstanding, for any $\chi, \ddot{h}, \varkappa \in \mathcal{N}$, let

$$\mathfrak{w} = \min\left\{ \inf_{\mathfrak{i} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\mathfrak{i}), \varrho(\ddot{h}) \right\}.$$

Then $\varrho(\ddot{h}) \geq \mathfrak{w}$ implies that $\ddot{h} \in \varrho_{\mathfrak{w}}$, and $\forall \mathfrak{r} \in \chi \star (\ddot{h} \star \varkappa), \varrho(\mathfrak{r}) \geq \inf_{\mathfrak{i} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\mathfrak{i}) \geq \mathfrak{w}$, implies $\mathfrak{r} \in \varrho_{\mathfrak{w}}$. Hence, $\chi \star (\ddot{h} \star \varkappa) \subseteq \varrho_{\mathfrak{w}}$. By Proposition 2.1(P₉), $\chi \star (\ddot{h} \star \varkappa) \subseteq \varrho_{\mathfrak{w}}$ implies $\chi \star (\ddot{h} \star \varkappa) \ll \varrho_{\mathfrak{w}}$, and $\ddot{h} \in \varrho_{\mathfrak{w}}$ implies $\chi \in \varrho_{\mathfrak{w}}$. Let

$$\mathfrak{d} = \max\left\{ \sup_{\mathfrak{a} \in \chi \star (\ddot{h} \star \varkappa)} \omega(\mathfrak{a}), \omega(\ddot{h}) \right\}.$$

Then $\omega(\ddot{h}) \leq \mathfrak{d}$ yields $\ddot{h} \in \omega_{\mathfrak{d}}$. Moreover, $\forall \mathfrak{e} \in \chi \star (\ddot{h} \star \varkappa), \omega(\mathfrak{e}) \leq \sup_{\mathfrak{a} \in \chi \star (\ddot{h} \star \varkappa)} \omega(\mathfrak{a}) \leq \mathfrak{d}$ implying $\mathfrak{e} \in \omega_{\mathfrak{d}}$. Hence, $\chi \star (\ddot{h} \star \varkappa) \subseteq \omega_{\mathfrak{d}}$, which implies $\chi \star (\ddot{h} \star \varkappa) \ll \omega_{\mathfrak{d}}$ by Proposition 2.1(P₉). Combined with $\ddot{h} \in \omega_{\mathfrak{d}}$, this yields $\chi \in \omega_{\mathfrak{d}}$. Therefore, $\forall v_1, v_2 \in \chi \star \varkappa$, we obtain

$$\begin{aligned} \varrho(v_1) &\geq \mathfrak{w} = \min\left\{ \inf_{\mathfrak{i} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\mathfrak{i}), \varrho(\ddot{h}) \right\}, \text{ and} \\ \omega(v_2) &\leq \mathfrak{d} = \max\left\{ \sup_{\mathfrak{a} \in \chi \star (\ddot{h} \star \varkappa)} \omega(\mathfrak{a}), \omega(\ddot{h}) \right\}. \end{aligned}$$

Therefore, $\mathcal{M} = (\varrho, \omega)$ is an \mathcal{IFHHI} s of \mathcal{N} . □

Theorem 3.4. *If $\mathcal{M} = (\varrho, \omega)$ is an intuitionistic fuzzy (weak, strong, reflexive) $\mathcal{H}\mathcal{H}\mathcal{I}$ of \mathcal{N} , then $\mathcal{K} = \{\chi \in \mathcal{N} \mid \varrho(\chi) = \varrho(0), \text{ and } \omega(\chi) = \omega(0)\}$ is a (weak, strong, reflexive) $\mathcal{H}\mathcal{H}\mathcal{I}$ of \mathcal{N} .*

Proof. Assume that $\mathcal{M} = (\varrho, \omega)$ is an $\mathcal{IF}\mathcal{H}\mathcal{H}\mathcal{I}$ of \mathcal{N} . Clear, $0 \in \mathcal{K}$. Let $\chi \star (\ddot{h} \star \varkappa) \ll \mathcal{K} \neq \emptyset$, and $\ddot{h} \in \mathcal{K}$ for some $\chi, \ddot{h}, \varkappa \in \mathcal{N}$. Then $\forall \mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa) \exists \mathfrak{n} \in \mathcal{K}$ such that $\mathfrak{m} \ll \mathfrak{n}$. Hence, $\varrho(\mathfrak{m}) = \varrho(\mathfrak{n}) = \varrho(0)$, but being an $\mathcal{IF}\mathcal{H}\mathcal{H}\mathcal{I}$, ϱ is also an $\mathcal{IF}\mathcal{W}\mathcal{H}\mathcal{H}\mathcal{I}$ of \mathcal{N} (by Theorem 3.2(i)), so ϱ fulfills $\varrho(0) \geq \varrho(\mathfrak{n}) \forall \mathfrak{n} \in \mathcal{N}$ which implies that $\varrho(0) \geq \varrho(\mathfrak{m}) \forall \mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa)$. Hence, $\varrho(\mathfrak{m}) = \varrho(0) \forall \mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa)$, i.e., $\inf_{\mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\mathfrak{m}) = \varrho(0)$. Also $\varrho(\ddot{h}) = \varrho(0)$. $\forall \mathfrak{x} \in \chi \star (\ddot{h} \star \varkappa), \exists \mathfrak{y} \in \mathcal{K} \ni \mathfrak{x} \ll \mathfrak{y}$, yielding $\omega(\mathfrak{x}) = \omega(\mathfrak{y}) = \omega(0)$. Since ω is an $\mathcal{IF}\mathcal{W}\mathcal{H}\mathcal{H}\mathcal{I}$ of \mathcal{N} (by Theorem 3.2(i)). It holds that $\omega(0) \leq \omega(\mathfrak{y}) \forall \mathfrak{y} \in \mathcal{N}$ implying $\omega(0) \leq \omega(\mathfrak{x}) \forall \mathfrak{x} \in \chi \star (\ddot{h} \star \varkappa)$. Thus, $\omega(\mathfrak{x}) = \omega(0) \forall \mathfrak{x} \in \chi \star (\ddot{h} \star \varkappa)$, and so $\sup_{\mathfrak{x} \in \chi \star (\ddot{h} \star \varkappa)} \omega(\mathfrak{x}) = \omega(0)$. Additionally, $\omega(\ddot{h}) = \omega(0)$. As $\mathcal{M} = (\varrho, \omega)$ is an $\mathcal{IF}\mathcal{H}\mathcal{H}\mathcal{I}, \forall v_1, v_2 \in \chi \star \varkappa$ and $\mathfrak{m}, \mathfrak{n} \in \mathcal{N}$, we have

$$\varrho(v_1) \geq \min\left\{ \inf_{\mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\mathfrak{m}), \varrho(\ddot{h}) \right\} = \min\{\varrho(0), \varrho(0)\} = \varrho(0).$$

Since $\varrho(0) \geq \varrho(\mathfrak{n})$, it follows that $\varrho(v_1) = \varrho(0)$. Moreover,

$$\omega(v_2) \leq \max\left\{ \sup_{\mathfrak{x} \in \chi \star (\ddot{h} \star \varkappa)} \omega(\mathfrak{x}), \omega(\ddot{h}) \right\} = \max\{\omega(0), \omega(0)\} = \omega(0).$$

With $\omega(0) \leq \omega(\mathfrak{y})$, we get $\omega(v_2) = \omega(0) \forall v_1, v_2 \in \chi \star \varkappa$ implying $\chi \star \varkappa \subseteq \mathcal{K}$. Therefore, \mathcal{K} is an $\mathcal{IF}\mathcal{H}\mathcal{H}\mathcal{I}$ of \mathcal{N} . □

Theorem 3.5. *For any subset \mathcal{K} of an $\mathcal{H}\mathcal{B}\mathcal{C}\mathcal{K}\mathcal{A}$ \mathcal{N} , let $\mathcal{M} = (\varrho, \omega)$ be an \mathcal{IFS} in \mathcal{N} defined by:*

$$\varrho(\chi) = \begin{cases} s & \text{if } \chi \in \mathcal{K} \\ 0 & \text{if } \chi \notin \mathcal{K} \end{cases} \quad \text{and } \omega(\chi) = \begin{cases} v & \text{if } \chi \notin \mathcal{K} \\ 0 & \text{if } \chi \in \mathcal{K} \end{cases}$$

$\forall \chi \in \mathcal{N}$, where $s, v \in (0, 1]$. Then, \mathcal{K} is a (weak, strong, reflexive) $\mathcal{H}\mathcal{H}\mathcal{I}$ of \mathcal{N} if and only if $\mathcal{M} = (\varrho, \omega)$ is an intuitionistic fuzzy (weak, strong, reflexive) $\mathcal{H}\mathcal{H}\mathcal{I}$ of \mathcal{N} .

Proof. Suppose that \mathcal{K} is an $\mathcal{H}\mathcal{H}\mathcal{I}$ of \mathcal{N} . Let $\chi \ll \ddot{h}$ for some $\chi, \ddot{h} \in \mathcal{N}$ and let $\varrho(\ddot{h}) = s$. Then $\ddot{h} \in \varrho_s$. So $\chi \ll \ddot{h} \in \varrho_s$ implies $\chi \ll \varrho_s$. Since ϱ_s is an $\mathcal{H}\mathcal{H}\mathcal{I}$, it is also an $\mathcal{H}\mathcal{B}\mathcal{C}\mathcal{K}\mathcal{I}$ of \mathcal{N} (by Theorem 2.1). Hence, by Proposition 2.2, $\chi \in \varrho_s$ implying $\varrho(\chi) \geq s = \varrho(\ddot{h})$. Thus, $\chi \ll \ddot{h}$ implies $\varrho(\chi) \geq \varrho(\ddot{h}) \forall \chi, \ddot{h} \in \mathcal{N}$. Let $\omega(\ddot{h}) = v$. Then $\ddot{h} \in \omega_v$, and $\chi \ll \ddot{h} \in \omega_v$ imply $\chi \ll \omega_v$. As ω_v is an $\mathcal{H}\mathcal{B}\mathcal{C}\mathcal{K}\mathcal{I}$ of \mathcal{N} (by Theorem 2.1), Proposition 2.2 yields $\chi \in \omega_v$, and so $\omega(\chi) \leq v = \omega(\ddot{h})$. Hence, $\chi \ll \ddot{h}$ implies $\omega(\chi) \leq \omega(\ddot{h}) \forall \chi, \ddot{h} \in \mathcal{N}$. Moreover, for any $\chi, \ddot{h}, \varkappa \in \mathcal{N}$, if $\chi \star (\ddot{h} \star \varkappa) \ll \mathcal{K}$ and $\ddot{h} \in \mathcal{K}$ then $\chi \star \varkappa \subseteq \mathcal{K}$. Since \mathcal{K} is an $\mathcal{H}\mathcal{H}\mathcal{I}$ of \mathcal{N} , so by Proposition 2.2 $\chi \star (\ddot{h} \star \varkappa) \subseteq \mathcal{K}$. Hence, $\varrho(\mathfrak{m}) = s \forall \mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa) \Rightarrow \inf_{\mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\mathfrak{m}) = s$. Also $\varrho(\ddot{h}) = s$. Since $\chi \star \varkappa \subseteq \mathcal{K} \forall \mathfrak{m} \in \chi \star \varkappa$, we have

$$\varrho(\mathfrak{m}) = s = \min\left\{ \inf_{\mathfrak{m} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\mathfrak{m}), \varrho(\ddot{h}) \right\}.$$

Now, $\omega(x) = v \forall x \in \chi \star (\ddot{h} \star \varkappa) \Rightarrow \sup_{x \in \chi \star (\ddot{h} \star \varkappa)} \omega(x) = v$. Also $\omega(\ddot{h}) = v$. Since $\chi \star \varkappa \subseteq \mathcal{K} \forall x \in \chi \star \varkappa$, we have

$$\omega(x) = v = \max\{ \sup_{x \in \chi \star (\ddot{h} \star \varkappa)} \omega(x), \omega(\ddot{h}) \}.$$

If $\chi \star (\ddot{h} \star \varkappa) \not\ll \mathcal{K}$ and $\ddot{h} \notin \mathcal{K}$, then

$$\begin{aligned} \min \{ \inf_{\ddot{m} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\ddot{m}), \varrho(\ddot{h}) \} &= 0 \leq \varrho(\ddot{m}), \\ \max \{ \sup_{x \in \chi \star (\ddot{h} \star \varkappa)} \omega(x), \omega(\ddot{h}) \} &= 0 \geq \omega(x) \end{aligned}$$

$\forall \ddot{m}, x \in \chi \star \varkappa$. If $\chi \star (\ddot{h} \star \varkappa) \not\ll \mathcal{K}$ and $\ddot{h} \in \mathcal{K}$ (OR) If $\chi \star (\ddot{h} \star \varkappa) \ll \mathcal{K}$ and $\ddot{h} \notin \mathcal{K}$. In both cases, we have

$$\begin{aligned} \min \{ \inf_{\ddot{m} \in \chi \star (\ddot{h} \star \varkappa)} \varrho(\ddot{m}), \varrho(\ddot{h}) \} &= 0 \leq \varrho(\ddot{m}), \\ \max \{ \sup_{x \in \chi \star (\ddot{h} \star \varkappa)} \omega(x), \omega(\ddot{h}) \} &= 0 \geq \omega(x) \end{aligned}$$

$\forall \ddot{m}, x \in \chi \star \varkappa$. Hence, $M = (\varrho, \omega)$ is an \mathcal{IFHHI} of \mathcal{N} . Conversely, suppose $M = (\varrho, \omega)$ is an \mathcal{IFHHI} of \mathcal{N} . Then, by Theorem 3.3, ϱ_s, ω_v are an \mathcal{HHI} of $\mathcal{N} \forall s, v \in (0, 1]$. \square

For a family $\{M_p = (\varrho_p, \omega_p) | p \in \dot{\mathcal{P}}\}$ of \mathcal{IFS} s in \mathcal{N} , define the join $\bigvee_{p \in \dot{\mathcal{P}}} M_p$, and meet $\bigwedge_{p \in \dot{\mathcal{P}}} M_p$ as follows:

$$\begin{aligned} (\bigvee_{p \in \dot{\mathcal{P}}} M_p)(\chi) &= \sup_{p \in \dot{\mathcal{P}}} M_p(\chi), \\ (\bigwedge_{p \in \dot{\mathcal{P}}} M_p)(\chi) &= \inf_{p \in \dot{\mathcal{P}}} M_p(\chi) \end{aligned}$$

$\forall \chi \in \mathcal{N}$, where $\dot{\mathcal{P}}$ is any indexing set.

Theorem 3.6. *The family of intuitionistic fuzzy (weak, strong, reflexive) \mathcal{HHI} s of \mathcal{N} is a completely distributive lattice with respect to join and meet.*

Proof. Suppose that $\{M_p = (\varrho_p, \omega_p) | p \in \dot{\mathcal{P}}\}$ is a family of \mathcal{IFHHI} s of \mathcal{N} . Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering in $[0, 1]$. It is enough to verify that $\bigvee_{p \in \dot{\mathcal{P}}} M_p$, and $\bigwedge_{p \in \dot{\mathcal{P}}} M_p$ are \mathcal{IFHHI} s of \mathcal{N} . For any $\chi, \ddot{h} \in \mathcal{N}$, if $\chi \ll \ddot{h}$, then

$$\begin{aligned} (\bigvee_{p \in \dot{\mathcal{P}}} \varrho_p)(\chi) &= \sup_{p \in \dot{\mathcal{P}}} \varrho_p(\chi) \geq \sup_{p \in \dot{\mathcal{P}}} \varrho_p(\ddot{h}) (\bigvee_{p \in \dot{\mathcal{P}}} \varrho_p)(\ddot{h}), \\ (\bigvee_{p \in \dot{\mathcal{P}}} \omega_p)(\chi) &= \sup_{p \in \dot{\mathcal{P}}} \omega_p(\chi) \leq \sup_{p \in \dot{\mathcal{P}}} \omega_p(\ddot{h}) (\bigvee_{p \in \dot{\mathcal{P}}} \omega_p)(\ddot{h}) \end{aligned}$$

thus, $(\bigvee_{p \in \dot{\mathcal{P}}} \varrho_p)(\chi) \geq (\bigvee_{p \in \dot{\mathcal{P}}} \varrho_p)(\ddot{h})$, and $(\bigvee_{p \in \dot{\mathcal{P}}} \omega_p)(\chi) \leq (\bigvee_{p \in \dot{\mathcal{P}}} \omega_p)(\ddot{h})$. Also, for any $\chi, \ddot{h}, \varkappa \in \mathcal{N}$ and $\forall v_1, v_2 \in \chi \star \varkappa$, we have

$$(\bigvee_{p \in \dot{\mathcal{P}}} \varrho_p)(v_1) = \sup_{p \in \dot{\mathcal{P}}} \varrho_p(v_1)$$

$$\begin{aligned}
&\geq \sup_{p \in \mathcal{P}} [\min\{\inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} \varrho_p(\mathfrak{m}), \varrho_p(\mathfrak{h})\}] \\
&= \min\{\sup_{p \in \mathcal{P}} (\inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} \varrho_p(\mathfrak{m})), \sup_{p \in \mathcal{P}} (\varrho_p(\mathfrak{h}))\} \\
&= \min\{\inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} (\sup_{p \in \mathcal{P}} \varrho_p(\mathfrak{m})), \sup_{p \in \mathcal{P}} (\varrho_p(\mathfrak{h}))\} \\
&= \min\{\inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} (\bigvee_{p \in \mathcal{P}} \varrho_p(\mathfrak{m})), \bigvee_{p \in \mathcal{P}} (\varrho_p(\mathfrak{h}))\}
\end{aligned}$$

thus, $(\bigvee_{p \in \mathcal{P}} \varrho_p)(v_1) \geq \min\{\inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} ((\bigvee_{p \in \mathcal{P}} \varrho_p)(\mathfrak{m})), (\bigvee_{p \in \mathcal{P}} \varrho_p)(\mathfrak{h})\}$.

$$\begin{aligned}
(\bigvee_{p \in \mathcal{P}} \omega_p)(v_2) &= \sup_{p \in \mathcal{P}} \omega_p(v_2) \\
&\leq \sup_{p \in \mathcal{P}} [\max\{\sup_{\mathfrak{x} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} \omega_p(\mathfrak{x}), \omega_p(\mathfrak{h})\}] \\
&= \max\{\sup_{p \in \mathcal{P}} (\sup_{\mathfrak{x} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} \omega_p(\mathfrak{x})), \sup_{p \in \mathcal{P}} (\omega_p(\mathfrak{h}))\} \\
&= \max\{\sup_{\mathfrak{x} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} (\sup_{p \in \mathcal{P}} \omega_p(\mathfrak{x})), \sup_{p \in \mathcal{P}} (\omega_p(\mathfrak{h}))\} \\
&= \max\{\sup_{\mathfrak{x} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} (\bigvee_{p \in \mathcal{P}} \omega_p(\mathfrak{x})), \bigvee_{p \in \mathcal{P}} (\omega_p(\mathfrak{h}))\}
\end{aligned}$$

hence, $(\bigvee_{p \in \mathcal{P}} \omega_p)(v_2) \leq \max\{\sup_{\mathfrak{x} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} ((\bigvee_{p \in \mathcal{P}} \omega_p)(\mathfrak{x})), (\bigvee_{p \in \mathcal{P}} \omega_p)(\mathfrak{h})\}$.

Therefore, $\bigvee_{p \in \mathcal{P}} \mathcal{M}_p$ is an *IFHHI* of \mathcal{N} .

$$\begin{aligned}
(\bigwedge_{p \in \mathcal{P}} \varrho_p)(\chi) &= \inf_{p \in \mathcal{P}} \varrho_p(\chi) \geq \inf_{p \in \mathcal{P}} \varrho_p(\mathfrak{h}) = (\bigwedge_{p \in \mathcal{P}} \varrho_p)(\mathfrak{h}), \\
(\bigwedge_{p \in \mathcal{P}} \omega_p)(\chi) &= \inf_{p \in \mathcal{P}} \omega_p(\chi) \geq \inf_{p \in \mathcal{P}} \omega_p(\mathfrak{h}) = (\bigwedge_{p \in \mathcal{P}} \omega_p)(\mathfrak{h}),
\end{aligned}$$

thus, $(\bigwedge_{p \in \mathcal{P}} \varrho_p)(\chi) \geq (\bigwedge_{p \in \mathcal{P}} \varrho_p)(\mathfrak{h})$, and $(\bigwedge_{p \in \mathcal{P}} \omega_p)(\chi) \leq (\bigwedge_{p \in \mathcal{P}} \omega_p)(\mathfrak{h})$. Furthermore, for any $\chi, \mathfrak{h}, \mathfrak{x} \in \mathcal{N}$ and $\forall v_1, v_2 \in \chi \star \mathfrak{x}$, we have

$$\begin{aligned}
(\bigwedge_{p \in \mathcal{P}} \varrho_p)(v_1) &= \inf_{p \in \mathcal{P}} \varrho_p(v_1) \\
&\geq \inf_{p \in \mathcal{P}} [\min\{\inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} \varrho_p(\mathfrak{m}), \varrho_p(\mathfrak{h})\}] \\
&= \min\{\inf_{p \in \mathcal{P}} (\inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} \varrho_p(\mathfrak{m})), \inf_{p \in \mathcal{P}} (\varrho_p(\mathfrak{h}))\} \\
&= \min\{\inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} (\inf_{p \in \mathcal{P}} \varrho_p(\mathfrak{m})), \inf_{p \in \mathcal{P}} (\varrho_p(\mathfrak{h}))\} \\
&= \min\{\inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} (\bigwedge_{p \in \mathcal{P}} \varrho_p(\mathfrak{m})), \bigwedge_{p \in \mathcal{P}} (\varrho_p(\mathfrak{h}))\}
\end{aligned}$$

Hence, $(\bigwedge_{p \in \mathcal{P}} \varrho_p)(v_1) \geq \min\{\inf_{\mathfrak{m} \in \chi \star (\mathfrak{h} \star \mathfrak{x})} ((\bigwedge_{p \in \mathcal{P}} \varrho_p)(\mathfrak{m})), (\bigwedge_{p \in \mathcal{P}} \varrho_p)(\mathfrak{h})\}$.

$$(\bigwedge_{p \in \mathcal{P}} \omega_p)(v_2) = \inf_{p \in \mathcal{P}} \omega_p(v_2)$$

$$\begin{aligned}
 &\leq \inf_{p \in \mathcal{P}} [\max\{ \sup_{x \in \chi \star (\check{h} \star \varkappa)} \omega_p(x), \omega_p(\check{h}) \}] \\
 &= \max\{ \inf_{p \in \mathcal{P}} (\sup_{x \in \chi \star (\check{h} \star \varkappa)} \omega_p(x)), \inf_{p \in \mathcal{P}} (\omega_p(\check{h})) \} \\
 &= \max\{ \sup_{x \in \chi \star (\check{h} \star \varkappa)} (\inf_{p \in \mathcal{P}} \omega_p(x)), \inf_{p \in \mathcal{P}} (\omega_p(\check{h})) \} \\
 &= \max\{ \sup_{x \in \chi \star (\check{h} \star \varkappa)} (\bigwedge_{p \in \mathcal{P}} \omega_p(x)), \bigwedge_{p \in \mathcal{P}} (\omega_p(\check{h})) \}
 \end{aligned}$$

hence, $(\bigwedge_{p \in \mathcal{P}} \omega_p)(v_2) \leq \max\{ \sup_{x \in \chi \star (\check{h} \star \varkappa)} ((\bigwedge_{p \in \mathcal{P}} \omega_p)(x)), (\bigwedge_{p \in \mathcal{P}} \omega_p)(\check{h}) \}$. Therefore, $\bigwedge_{p \in \mathcal{P}} \mathcal{M}_p$ is an \mathcal{IFHHI} of \mathcal{N} . Therefore, the family of \mathcal{IFHHI} s of \mathcal{N} forms a completely distributive lattice under join and meet operations. □

4. PRODUCT OPERATION IN HYPER BCK-ALGEBRAS

In this section, the product of various types of \mathcal{IFHHI} s is examined. Utilizing the transfer principle for \mathcal{IFS} s, it is demonstrated that the product of two \mathcal{IFHHI} s is likewise an \mathcal{IFHHI} .

Definition 4.1. Let $\mathcal{M} = (\varrho, \omega)$ be an \mathcal{IFS} of \mathcal{N} . Then \mathcal{IFS} s $\mathcal{M}_1 = (\varrho_1, \omega_1)$ and $\mathcal{M}_2 = (\varrho_2, \omega_2)$ on \mathcal{N}_1 and \mathcal{N}_2 respectively, are defined $\mathcal{M}_1(x) = \mathcal{M}((f, 0))$, $\mathcal{M}_2(\check{h}) = \mathcal{M}((0, \check{h}))$.

Theorem 4.1. Let $\mathcal{M} = (\varrho, \omega)$ be an \mathcal{IFS} in \mathcal{N} . Then $\mathcal{M} = (\varrho, \omega)$ is an intuitionistic fuzzy (weak, strong, reflexive) \mathcal{HHI} of \mathcal{N} if and only if $\mathcal{M}_1 = (\varrho_1, \omega_1)$ and $\mathcal{M}_2 = (\varrho_2, \omega_2)$ are intuitionistic fuzzy (weak, strong, reflexive) \mathcal{HHI} s of \mathcal{N}_1 and \mathcal{N}_2 respectively.

Proof. Suppose that $\mathcal{M} = (\varrho, \omega)$ is an \mathcal{IFHHI} of \mathcal{N} and let $\chi \ll \kappa$ for some $\chi, \kappa \in \mathcal{N}_1$. Then $(\chi, 0) \ll (\kappa, 0)$ imply $\varrho((\chi, 0)) = \varrho_1(\chi) \geq \varrho((\kappa, 0)) = \varrho_1(\kappa)$, i.e, $\varrho_1(\chi) \geq \varrho_1(\kappa)$, and $\omega((\chi, 0)) = \omega_1(\chi) \leq \omega((\kappa, 0)) = \omega_1(\kappa)$, i.e, $\omega_1(\chi) \leq \omega_1(\kappa)$. Moreover, for any $\chi, \check{h}, \varkappa \in \mathcal{N}_1$, let

$$s = \min\{ \inf_{\check{m} \in \chi \star (\check{h} \star \varkappa)} \varrho_1(\check{m}), \varrho_1(\check{h}) \}.$$

Then $\forall \check{n} \in \chi \star (\check{h} \star \varkappa)$,

$$\varrho_1(\check{n}) \geq \inf_{\check{m} \in \chi \star (\check{h} \star \varkappa)} \varrho_1(\check{m}) \geq s \text{ and } \varrho_1(\check{h}) \geq s$$

thus, $\varrho((\check{n}, 0)) \geq s$ and $\varrho((\check{h}, 0)) \geq s \forall (\check{n}, 0) \in (\chi, 0) \star ((\check{h}, 0) \star (\varkappa, 0))$, $(\check{n}, 0) \in \varrho_s$ and $(\check{h}, 0) \in \varrho_s$
 $\forall (\check{n}, 0) \in (\chi, 0) \star ((\check{h}, 0) \star (\varkappa, 0))$, $(\chi, 0) \star ((\check{h}, 0) \star (\varkappa, 0)) \subseteq \varrho_s$ and $(\check{h}, 0) \in \varrho_s$. By the transfer principle for \mathcal{IFS} s, ϱ_s is a nonempty \mathcal{HHI} of \mathcal{N} and, consequently, a weak \mathcal{HHI} of \mathcal{N} (Theorem 2.2). Thus, $(\chi, 0) \star ((\check{h}, 0) \star (\varkappa, 0)) \subseteq \varrho_s$ and $(\check{h}, 0) \in \varrho_s$ implies $(\chi, 0) \star (\varkappa, 0) \subseteq \varrho_s$. Hence, $\varrho((w, 0)) \geq s \forall (w, 0) \in (\chi, 0) \star (\varkappa, 0) = (\chi \star \varkappa, 0)$. Thus

$$\varrho_1(w) \geq s = \min\{ \inf_{\check{m} \in \chi \star (\check{h} \star \varkappa)} \varrho_1(\check{m}), \varrho_1(\check{h}) \},$$

now let

$$v = \max\{ \sup_{x \in \chi \star (\check{h} \star \varkappa)} \omega_1(x), \omega_1(\check{h}) \}.$$

Then $\forall \dot{\eta} \in \chi \star (\ddot{h} \star \kappa), \omega_1(\dot{\eta}) \leq \sup_{\mathfrak{x} \in \chi \star (\ddot{h} \star \kappa)} \omega_1(\mathfrak{x}) \leq v$ and $\omega_1(\ddot{h}) \leq v$ hence $\omega((\dot{\eta}, 0)) \leq v$ and $\omega((\ddot{h}, 0)) \leq v \forall (\dot{\eta}, 0) \in (\chi, 0) \star ((\ddot{h}, 0) \star (\kappa, 0)), (\dot{\eta}, 0) \in \omega_v$ and $(\ddot{h}, 0) \in \omega_v \forall (\dot{\eta}, 0) \in (\chi, 0) \star ((\ddot{h}, 0) \star (\kappa, 0)), (\chi, 0) \star ((\ddot{h}, 0) \star (\kappa, 0)) \subseteq \omega_v$ and $(\ddot{h}, 0) \in \omega_v$. Via the transfer principle for \mathcal{IFSS} , ω_s is a nonempty \mathcal{HHI} of \mathcal{N} , and thus a weak \mathcal{HHI} of \mathcal{N} (Theorem 2.2). Therefore, $(\chi, 0) \star ((\ddot{h}, 0) \star (\kappa, 0)) \subseteq \omega_v$ and $(\ddot{h}, 0) \in \omega_v$ implies $(\chi, 0) \star (\kappa, 0) \subseteq \omega_v$. Hence, $\omega((c, 0)) \leq v \forall (c, 0) \in (\chi, 0) \star (\kappa, 0) = (\chi \star \kappa, 0)$. Thus,

$$\omega_1(c) \leq v = \max\{ \sup_{\mathfrak{x} \in \chi \star (\ddot{h} \star \kappa)} \omega_1(\mathfrak{x}), \omega_1(\ddot{h}) \}$$

$\forall w, c \in \chi \star \kappa$. Therefore, $\mathcal{M}_1 = (\varrho, \omega)$ is an \mathcal{IFHHI} of \mathcal{N}_1 . Similarly, we can prove that $\mathcal{M}_2 = (\varrho_2, \omega_2)$ is an \mathcal{IFHHI} of \mathcal{N}_2 . Conversely, assume that \mathcal{M}_1 and \mathcal{M}_2 are \mathcal{IFHHI} s of \mathcal{N}_1 and \mathcal{N}_2 respectively. For any $(\chi, \kappa), (\ddot{h}, \ddot{f}) \in \mathcal{N}$, where $\chi, \ddot{h} \in \mathcal{N}_1$ and $\kappa, \ddot{f} \in \mathcal{N}_2$, suppose $(\chi, \kappa) \ll (\ddot{h}, \ddot{f})$, since $(\chi, \kappa) \ll (\ddot{h}, \ddot{f}) \Leftrightarrow \chi \ll \ddot{h}$ and $\kappa \ll \ddot{f}$ imply $\varrho_1(\chi) \geq \varrho_1(\ddot{h})$ and $\varrho_2(\kappa) \geq \varrho_2(\ddot{f})$ then $\min\{\varrho_1(\chi), \varrho_2(\kappa)\} \geq \min\{\varrho_1(\ddot{h}), \varrho_2(\ddot{f})\}$ implies $(\varrho_1 \times \varrho_2)((\chi, \kappa)) \geq (\varrho_1 \times \varrho_2)((\ddot{h}, \ddot{f}))$ imply $\varrho((\chi, \kappa)) \geq \varrho((\ddot{h}, \ddot{f}))$. Hence, $(\chi, \kappa) \ll (\ddot{h}, \ddot{f})$ implies $\varrho((\chi, \kappa)) \geq \varrho((\ddot{h}, \ddot{f}))$. Now, $\omega_1(\chi) \leq \omega_1(\ddot{h})$ and $\omega_2(\kappa) \leq \omega_2(\ddot{f})$ then $\max\{\omega_1(\chi), \omega_2(\kappa)\} \leq \max\{\omega_1(\ddot{h}), \omega_2(\ddot{f})\}$ implies $(\omega_1 \times \omega_2)((\chi, \kappa)) \leq (\omega_1 \times \omega_2)((\ddot{h}, \ddot{f}))$ imply $\omega((\chi, \kappa)) \leq \omega((\ddot{h}, \ddot{f}))$. Therefore, $(\chi, \kappa) \ll (\ddot{h}, \ddot{f})$ implies $\omega((\chi, \kappa)) \leq \omega((\ddot{h}, \ddot{f}))$. Moreover, for any $(\chi, \kappa), (\ddot{h}, \ddot{f}), (\mathfrak{x}, \dot{\mathfrak{l}}) \in \mathcal{N}$, where $\chi, \ddot{h}, \mathfrak{x} \in \mathcal{N}_1$ and $\kappa, \ddot{f}, \dot{\mathfrak{l}} \in \mathcal{N}_2$ and $\forall (m, n) \in (\chi, \kappa) \star (\mathfrak{x}, \dot{\mathfrak{l}})$,

$$\begin{aligned} \varrho((\dot{m}, \dot{n})) &= (\varrho_1 \times \varrho_2)((\dot{m}, \dot{n})) \\ &= \min\{\varrho_1(\dot{m}), \varrho_2(\dot{n})\} \\ &\geq \min\{\min\{ \inf_{\dot{\mathfrak{j}} \in \chi \star (\ddot{h} \star \kappa)} \varrho_1(\dot{\mathfrak{j}}), \varrho_1(\ddot{h}) \}, \min\{ \inf_{\dot{\mathfrak{w}} \in \kappa \star (\ddot{f} \star \dot{\mathfrak{l}})} \varrho_2(\dot{\mathfrak{w}}), \varrho_2(\ddot{f}) \}\} \\ &= \min\{\min\{ \inf_{\dot{\mathfrak{j}} \in \chi \star (\ddot{h} \star \kappa)} \varrho_1(\dot{\mathfrak{j}}), \inf_{\dot{\mathfrak{w}} \in \kappa \star (\ddot{f} \star \dot{\mathfrak{l}})} \varrho_2(\dot{\mathfrak{w}}) \}, \min\{\varrho_1(\ddot{h}), \varrho_2(\ddot{f})\}\} \\ &= \min\{ \inf_{\dot{\mathfrak{j}} \in \chi \star (\ddot{h} \star \kappa), \dot{\mathfrak{w}} \in \kappa \star (\ddot{f} \star \dot{\mathfrak{l}})} \{\min\{\varrho_1(\dot{\mathfrak{j}}), \varrho_2(\dot{\mathfrak{w}})\}\}, \min\{\varrho_1(\ddot{h}), \varrho_2(\ddot{f})\}\} \\ &= \min\{ \inf_{(\dot{\mathfrak{j}}, \dot{\mathfrak{w}}) \in (\chi \star (\ddot{h} \star \kappa), \kappa \star (\ddot{f} \star \dot{\mathfrak{l}}))} (\varrho_1 \times \varrho_2)(\dot{\mathfrak{j}}, \dot{\mathfrak{w}}), (\varrho_1 \times \varrho_2)(\ddot{h}, \ddot{f}) \} \\ &= \min\{ \inf_{(\dot{\mathfrak{j}}, \dot{\mathfrak{w}}) \in (\chi \star (\ddot{h} \star \kappa), \kappa \star (\ddot{f} \star \dot{\mathfrak{l}}))} \varrho((\dot{\mathfrak{j}}, \dot{\mathfrak{w}})), \varrho((\ddot{h}, \ddot{f})) \} \end{aligned}$$

hence $\varrho((\dot{m}, \dot{n})) \geq \min\{ \inf_{(\dot{\mathfrak{j}}, \dot{\mathfrak{w}}) \in (\chi \star \kappa) \star ((\ddot{h} \star \mathfrak{x}) \star (\mathfrak{x} \star \dot{\mathfrak{l}}))} \varrho((\dot{\mathfrak{j}}, \dot{\mathfrak{w}})), \varrho((\ddot{h}, \ddot{f})) \}$.

Now, $\forall (\mathfrak{x}, \dot{\eta}) \in (\chi, \kappa) \star (\mathfrak{x}, \dot{\mathfrak{l}})$,

$$\begin{aligned} \omega((\mathfrak{x}, \dot{\eta})) &= (\omega_1 \times \omega_2)((\mathfrak{x}, \dot{\eta})) \\ &= \max\{\omega_1(\mathfrak{x}), \omega_2(\dot{\eta})\} \\ &\leq \max\{\max\{ \sup_{\dot{\mathfrak{a}} \in \chi \star (\ddot{h} \star \kappa)} \omega_1(\dot{\mathfrak{a}}), \omega_1(\ddot{h}) \}, \max\{ \sup_{\dot{\mathfrak{d}} \in \kappa \star (\ddot{f} \star \dot{\mathfrak{l}})} \omega_2(\dot{\mathfrak{d}}), \omega_2(\ddot{f}) \}\} \\ &= \max\{\max\{ \sup_{\dot{\mathfrak{a}} \in \chi \star (\ddot{h} \star \kappa)} \omega_1(\dot{\mathfrak{a}}), \sup_{\dot{\mathfrak{d}} \in \kappa \star (\ddot{f} \star \dot{\mathfrak{l}})} \omega_2(\dot{\mathfrak{d}}) \}, \max\{\omega_1(\ddot{h}), \omega_2(\ddot{f})\}\} \end{aligned}$$

$$\begin{aligned}
 &= \max\left\{ \sup_{\grave{a} \in \chi \star (\grave{h} \star \varkappa), \grave{d} \in \kappa \star (\grave{f} \star \grave{l})} \{\max\{\omega_1(\grave{a}), \omega_2(\grave{d})\}\}, \max\{\omega_1(\grave{h}), \omega_2(\grave{f})\} \right\} \\
 &= \max\left\{ \sup_{(\grave{a}, \grave{d}) \in (\chi \star (\grave{h} \star \varkappa), \kappa \star (\grave{f} \star \grave{l}))} (\omega_1 \times \omega_2)(\grave{a}, \grave{d}), (\omega_1 \times \omega_2)(\grave{h}, \grave{f}) \right\} \\
 &= \max\left\{ \sup_{(\grave{a}, \grave{d}) \in (\chi \star (\grave{h} \star \varkappa), \kappa \star (\grave{f} \star \grave{l}))} \omega((\grave{a}, \grave{d})), \omega((\grave{h}, \grave{f})) \right\}
 \end{aligned}$$

thus, $\omega((\grave{x}, \grave{y})) \leq \max\left\{ \sup_{(\grave{a}, \grave{d}) \in (\chi \star \kappa) \star ((\grave{h} \star \grave{f}) \star (\varkappa \star \grave{l}))} \omega((\grave{a}, \grave{d})), \omega((\grave{h}, \grave{f})) \right\}$.

Therefore, $\mathcal{M} = (\varrho, \omega)$ is an *IFHHI* of \mathcal{N} . □

Theorem 4.2. Let $\mathcal{M} = (\varrho, \omega)$ be an *IFS* and let $Y_{\mathcal{M}} = (\varrho_Y, \omega_Y)$ be the strongest intuitionistic fuzzy relation on \mathcal{N} . $\mathcal{M} = (\varrho, \omega)$ is an *IFSHHI* if and only if $Y_{\mathcal{M}} = (\varrho_Y, \omega_Y)$ is an *IFSHHI* of $\mathcal{N} \times \mathcal{N}$.

Proof. Let $\mathcal{M} = (\varrho, \omega)$ be an *IFSHHI* of \mathcal{N} . Consider

$$\begin{aligned}
 \inf_{(\grave{m}, \grave{n}) \in ((\chi, \kappa) \star (\chi, \kappa))} \varrho_Y(\grave{m}, \grave{n}) &= \inf_{(\grave{m}, \grave{n}) \in ((\chi \star \chi) \star ((\kappa \star \kappa)))} [\min\{\varrho(\grave{m}), \varrho(\grave{n})\}] \\
 &= \min\left\{ \inf_{\grave{m} \in (\chi \star \chi)} \varrho(\grave{m}), \inf_{\grave{n} \in (\kappa \star \kappa)} \varrho(\grave{n}) \right\} \\
 &\geq \min\{\varrho(\chi), \varrho(\kappa)\} \\
 &= \varrho_Y(\chi, \kappa)
 \end{aligned}$$

imply $\inf_{(\grave{m}, \grave{n}) \in ((\chi, \kappa) \star (\chi, \kappa))} \varrho_Y(\grave{m}, \grave{n}) \geq \varrho_Y(\chi, \kappa) \forall (\chi, \kappa) \in \mathcal{K} \times \mathcal{K}$.

$$\begin{aligned}
 \sup_{(\grave{x}, \grave{y}) \in ((\chi, \kappa) \star (\chi, \kappa))} \omega_Y(\grave{x}, \grave{y}) &= \sup_{(\grave{x}, \grave{y}) \in ((\chi \star \chi) \star ((\kappa \star \kappa)))} [\max\{\omega(\grave{x}), \omega(\grave{y})\}] \\
 &= \max\left\{ \sup_{\grave{x} \in (\chi \star \chi)} \omega(\grave{x}), \sup_{\grave{y} \in (\kappa \star \kappa)} \omega(\grave{y}) \right\} \\
 &\leq \max\{\omega(\chi), \omega(\kappa)\} \\
 &= \omega_Y(\chi, \kappa)
 \end{aligned}$$

hence $\sup_{(\grave{x}, \grave{y}) \in ((\chi, \kappa) \star (\chi, \kappa))} \omega_Y(\grave{x}, \grave{y}) \leq \omega_Y(\chi, \kappa) \forall (\chi, \kappa) \in \mathcal{K} \times \mathcal{K}$.

Now, for any $(\chi, \kappa), (\grave{h}, \grave{f}), (\varkappa, \grave{l}) \in \mathcal{K} \times \mathcal{K}$, consider

$$\begin{aligned}
 \varrho_Y(\chi, \kappa) &= \min\{\varrho(\chi), \varrho(\kappa)\} \\
 &\geq \min\{\min\left\{ \sup_{\grave{j} \in (\chi \star (\grave{h} \star \varkappa))} \varrho(\grave{i}), \varrho(\grave{h}), \min\left\{ \sup_{\grave{w} \in (\kappa \star (\grave{f} \star \grave{l}))} \varrho(\grave{w}), \varrho(\grave{f}) \right\} \right\} \\
 &= \min\{\min\left\{ \sup_{\grave{j} \in (\chi \star (\grave{h} \star \varkappa))} \varrho(\grave{i}), \varrho(\grave{h}), \sup_{\grave{w} \in (\kappa \star (\grave{f} \star \grave{l}))} \varrho(\grave{w}), \varrho(\grave{f}) \right\}, \\
 &\quad \min\{\varrho(\grave{h}), \varrho(\grave{f})\}\} \\
 &= \min\{\min\{\sup \varrho(\grave{i}), \varrho(\grave{w})\}, \varrho_Y(\grave{h}, \grave{f})\},
 \end{aligned}$$

where $\grave{j} \in (\chi \star (\grave{h} \star \varkappa))$ and $\grave{w} \in (\kappa \star (\grave{f} \star \grave{l}))$

implies $\varrho_Y(\grave{h}, \grave{f}) \geq \min\{\sup\{\min\{\varrho(\grave{i}), \varrho(\grave{w})\}\}, \varrho_Y(\grave{h}, \grave{f})\}$, where $\grave{j} \in (\chi \star (\grave{h} \star \varkappa))$, $\grave{w} \in (\kappa \star (\grave{f} \star \grave{l}))$

implies $\varrho_Y(\check{h}, \check{f}) \geq \min\{\sup(\varrho_Y(j, w)), \varrho_Y(\check{h}, \check{f})\}$, where $(j, w) \in (\chi \star (\check{h} \star \varkappa), \kappa \star (\check{f} \star \check{l})) = ((\chi, \kappa) \star ((\check{h}, \check{f}) \star (\varkappa, \check{l})))$. Now, consider

$$\begin{aligned} \omega_Y(\chi, \kappa) &= \max\{\omega(\chi), \omega(\kappa)\} \\ &\leq \max\{\max\{\inf_{\check{a} \in (\chi \star (\check{h} \star \varkappa))} \omega(\check{a}), \omega(\check{h}), \max\{\inf_{\check{d} \in (\kappa \star (\check{f} \star \check{l}))} \omega(\check{d}), \omega(\check{f})\}\} \\ &= \max\{\max\{\inf_{\check{a} \in (\chi \star (\check{h} \star \varkappa))} \omega(\check{a}), \omega(\check{h}), \inf_{\check{d} \in (\kappa \star (\check{f} \star \check{l}))} \omega(\check{d}), \omega(\check{f})\}, \\ &\quad \max\{\omega(\check{h}), \omega(\check{f})\}\} \\ &= \max\{\max\{\inf(\omega(\check{a}), \omega(\check{d})), \omega_Y(\check{h}, \check{f})\}, \end{aligned}$$

where $\check{a} \in (\chi \star (\check{h} \star \varkappa))$ and $\check{d} \in (\kappa \star (\check{f} \star \check{l}))$ imply $\omega_Y(\check{h}, \check{f}) \leq \max\{\max(\omega(\check{a}), \omega(\check{d})), \omega_Y(\check{h}, \check{f})\}$, where $\check{a} \in (\chi \star (\check{h} \star \varkappa))$, $\check{d} \in (\kappa \star (\check{f} \star \check{l}))$, thus $\omega_Y(\check{h}, \check{f}) \leq \max\{\inf \omega_Y(\check{a}, \check{d}), \omega_Y(\check{h}, \check{f})\}$, where $(\check{a}, \check{d}) \in (\chi \star (\check{h} \star \varkappa), (\kappa \star (\check{f} \star \check{l}))) = ((\chi, \kappa) \star ((\check{h}, \check{f}) \star (\varkappa, \check{l})))$. Thus, $Y_M = (\varrho_Y, \omega_Y)$ is an *IFSHHI* of $\mathcal{K} \times \mathcal{K}$. In contrast, let $Y_M = (\varrho_Y, \omega_Y)$ be an *IFSHHI* of $\mathcal{K} \times \mathcal{K}$. Then we have $\inf_{(m, n) \in ((\chi, \kappa) \star (\chi, \kappa))} \varrho_Y(m, n) \geq \varrho_Y(\chi, \kappa) \forall (\chi, \kappa) \in \mathcal{K} \times \mathcal{K}$,

$$\begin{aligned} &\text{imply} \quad \inf_{(\check{m}, \check{n}) \in ((\chi \star \chi) \star (\kappa \star \kappa))} [\min\{\varrho(\check{m}), \varrho(\check{n})\}] \geq \min\{\varrho(\chi), \varrho(\kappa)\} \\ &\text{imply} \quad \min\{\inf_{\check{m} \in (\chi \star \chi)} \varrho(\check{m}), \inf_{\check{n} \in ((\kappa \star \kappa))} \varrho(\check{n})\} \geq \min\{\varrho(\chi), \varrho(\kappa)\} \\ &\text{imply} \quad \{\inf_{\check{m} \in (\chi \star \chi)} \varrho(\check{m}), \inf_{\check{n} \in ((\kappa \star \kappa))} \varrho(\check{n})\} \geq \{\varrho(\chi), \varrho(\kappa)\} \\ &\text{if and only if} \quad \inf_{\check{m} \in (\chi \star \chi)} \varrho(\check{m}) \geq \varrho(\chi), \quad \inf_{\check{n} \in ((\kappa \star \kappa))} \varrho(\check{n}) \geq \varrho(\kappa) \end{aligned}$$

$\forall \chi, \kappa \in \mathcal{N}$. And $\sup_{(\check{x}, \check{y}) \in ((\chi, \kappa) \star (\chi, \kappa))} \omega_Y(\check{x}, \check{y}) \leq \omega_Y(\chi, \kappa) \forall (\chi, \kappa) \in \mathcal{K} \times \mathcal{K}$

$$\begin{aligned} &\Rightarrow \quad \sup_{(\check{x}, \check{y}) \in ((\chi \star \chi) \star (\kappa \star \kappa))} [\max\{\omega(\check{x}), \omega(\check{y})\}] \leq \max\{\omega(\chi), \omega(\kappa)\} \\ &\text{imply} \quad \max\{\sup_{\check{x} \in (\chi \star \chi)} \omega(\check{x}), \sup_{\check{y} \in ((\kappa \star \kappa))} \omega(\check{y})\} \leq \max\{\omega(\chi), \omega(\kappa)\} \\ &\text{imply} \quad \{\sup_{\check{x} \in (\chi \star \chi)} \omega(\check{x}), \sup_{\check{y} \in ((\kappa \star \kappa))} \omega(\check{y})\} \leq \{\omega(\chi), \omega(\kappa)\} \\ &\text{if and only if} \quad \sup_{\check{x} \in (\chi \star \chi)} \omega(\check{x}) \leq \omega(\chi), \quad \sup_{\check{y} \in ((\kappa \star \kappa))} \omega(\check{y}) \leq \omega(\kappa) \end{aligned}$$

$\forall \chi, \kappa \in \mathcal{N}$. Therefore, the first condition for $M = (\varrho, \omega)$ to be an *IFSHHI* is satisfied. Note that $Y_M = (\varrho_Y, \omega_Y)$ is an *IFSHHI* of $\mathcal{K} \times \mathcal{K}$, it is also an *IFWHHI* of $\mathcal{K} \times \mathcal{K}$ (Theorem 3.2). Hence, $Y_M = (\varrho_Y, \omega_Y)$ satisfies $Y_M(0, 0) \geq Y_M(\chi, \chi) \forall (0, 0), (\chi, \chi) \in \mathcal{K} \times \mathcal{K} \Rightarrow \min\{\varrho(0), \varrho(0)\} \geq \min\{\varrho(\chi), \varrho(\chi)\} \Rightarrow \varrho(0) \geq \varrho(\chi) \forall \chi \in \mathcal{N}$. Now, for any $(\chi, \kappa), (\check{h}, \check{f}), (\varkappa, \check{l}) \in \mathcal{K} \times \mathcal{K}$, $Y_M = (\varrho_Y, \omega_Y)$ satisfies

$$\varrho_Y(\chi, \kappa) \geq \min\{\sup(\varrho_Y(e_0, \mathfrak{z}_0)), \varrho_Y(\check{h}, \check{f})\}$$

where $(e_0, \mathfrak{z}_0) \in ((\chi, \kappa) \star ((\ddot{h}, \ddot{f}) \star (\mathfrak{x}, \ddot{l}))) = (\chi \star (\ddot{h} \star \mathfrak{x}), (\kappa \star (\ddot{f} \star \ddot{l}))) \Rightarrow \min\{\varrho(\chi), \varrho(\kappa)\} \geq \min\{\sup\{\min\{\varrho(e_0), \varrho(\mathfrak{z}_0)\}\}, \min\{\varrho(\ddot{h}), \varrho(\ddot{f})\}\}$ where $(e_0, \mathfrak{z}_0) \in (\chi \star (\ddot{h} \star \mathfrak{x}), (\kappa \star (\ddot{f} \star \ddot{l})))$. Putting $\chi = \ddot{h} = \mathfrak{x} = 0$, we get

$$\min\{\varrho(0), \varrho(\kappa)\} \geq \min\{\sup\{\min\{\varrho(0), \varrho(\mathfrak{z}_0)\}\}, \min\{\varrho(0), \varrho(\ddot{f})\}\}$$

where $(e_0, \mathfrak{z}_0) \in (0, (\kappa \star (\ddot{f} \star \ddot{l}))) \Rightarrow \varrho(\kappa) \geq \min\{\sup_{\mathfrak{z}_0 \in (\kappa \star (\ddot{f} \star \ddot{l}))} \varrho(\mathfrak{z}), \varrho(\ddot{f})\}$,

since $\varrho(0) \geq \varrho(\chi) \forall \chi \in \mathcal{N}$. With $\kappa = \ddot{f} = \ddot{l} = 0$, it follows that

$$\varrho(\chi) \geq \min\{\sup_{e_0 \in (\chi \star (\ddot{h} \star \mathfrak{x}))} \varrho(e_0), \varrho(\ddot{h})\}.$$

Furthermore, $\omega_Y(\chi, \kappa) \leq \max\{\inf \omega_Y(e_2, \mathfrak{z}_2), \omega_Y(\ddot{h}, \ddot{f})\}$, where $(e_2, \mathfrak{z}_2) \in ((\chi, \kappa) \star ((\ddot{h}, \ddot{f}) \star (\mathfrak{x}, \ddot{l}))) = (\chi \star (\ddot{h} \star \mathfrak{x}), (\kappa \star (\ddot{f} \star \ddot{l})))$. This implies

$$\max\{\omega(\chi), \omega(\kappa)\} \leq \max\{\{\max\{\omega(e_2), \omega(\mathfrak{z}_2)\}\}, \max\{\omega(\ddot{h}), \omega(\ddot{f})\}\},$$

where $(e_2, \mathfrak{z}_2) \in (\chi \star (\ddot{h} \star \mathfrak{x}), (\kappa \star (\ddot{f} \star \ddot{l})))$. Setting $\chi = \ddot{h} = \mathfrak{x} = 0$ yields

$$\max\{\omega(0), \omega(\kappa)\} \leq \max\{\{\max\{\omega(0), \omega(\mathfrak{z}_2)\}\}, \max\{\omega(0), \omega(\ddot{f})\}\},$$

where $(e_2, \mathfrak{z}_2) \in (0, (\kappa \star (\ddot{f} \star \ddot{l})))$. Hence, $\omega(\kappa) \leq \max\{\inf_{\mathfrak{z}_2 \in (\kappa \star (\ddot{f} \star \ddot{l}))} \omega(\mathfrak{z}_2), \omega(\ddot{f})\}$, given $\omega(0) \leq \omega(\chi)$

$\forall \chi \in \mathcal{N}$. Similarly, setting $\kappa = \ddot{f} = \ddot{l} = 0$ gives $\omega(\chi) \leq \max\{\inf_{e_2 \in (\chi \star (\ddot{h} \star \mathfrak{x}))} \omega(e_2), \omega(\ddot{h})\}$. Therefore,

$\mathcal{M} = (\varrho, \omega)$ is an *IFSHHI* of \mathcal{N} . □

Theorem 4.3. Let $\psi : \mathcal{N} \rightarrow \mathcal{N}'$ be an onto *HBCKA*s from an *HBCKA* \mathcal{N} to an *HBCKA* \mathcal{N}' . $\mu = (\varrho^\mu, \omega^\mu)$ is an *IFSHHI* of \mathcal{N}' , then the hyper homomorphic pre-image $\mathcal{M} = (\varrho, \omega)$ of $\mu = (\varrho^\mu, \omega^\mu)$ under $\psi = (\varrho^\psi, \omega^\psi)$ is an *IFSHHI* of \mathcal{N} .

Proof. Let $\psi = (\varrho^\psi, \omega^\psi)$ be an *IFSHHI* of \mathcal{N}' . Since $\mathcal{M} = (\varrho, \omega)$ is a hyper homomorphic pre-image of $\mu = (\varrho^\mu, \omega^\mu)$ under $\psi = (\varrho^\psi, \omega^\psi)$, so $\mathcal{M} = (\varrho, \omega)$ is defined by $\mathcal{M} = \mu \star \psi$ that is, $\varrho(\chi) = \varrho^\mu(\varrho^\psi(\chi))$, and $\omega(\chi) = \omega^\mu(\omega^\psi(\chi)) \forall \chi \in \mathcal{N}$. Since $\mu = (\varrho^\mu, \omega^\mu)$ satisfies

$\inf_{\varrho^\psi(\mathfrak{n}) \in \varrho^\psi(\chi) \star \varrho^\psi(\chi) = \varrho^\psi(\chi \star \chi)} \varrho^\mu(\varrho^\psi(\mathfrak{n})) \geq \varrho^\mu(\varrho^\psi(\chi)) \forall \chi \in \mathcal{N}$, and $\varrho^\psi(\chi) \in \mathcal{N}'$ implies $\inf_{\mathfrak{n} \in (\chi \star \chi)} \varrho(\mathfrak{n}) \geq \varrho(\chi) \forall \chi \in \mathcal{N}$. Now,

$$\sup_{\omega^\psi(\mathfrak{x}) \in \omega^\psi(\chi) \star \omega^\psi(\chi) = \omega^\psi(\chi \star \chi)} \omega^\mu(\omega^\psi(\mathfrak{x})) \leq \omega^\mu(\omega^\psi(\chi))$$

$\forall \chi \in \mathcal{N}$, and $\omega^\psi(\chi) \in \mathcal{N}'$ implies $\sup_{\mathfrak{x} \in (\chi \star \chi)} \omega(\mathfrak{x}) \leq \omega(\chi) \forall \chi \in \mathcal{N}$. For any $\chi, \ddot{h}, \mathfrak{x} \in \mathcal{N}$, consider

$\varrho(\chi) = \varrho^\mu(\varrho^\psi(\chi)) \geq \min\{\sup_{\varrho^\psi(\mathfrak{n}) \in (\varrho^\psi(\chi) \star (\ddot{h}' \star \mathfrak{x}'))} \varrho^\mu(\varrho^\psi(\mathfrak{n})), \varrho^\psi(\ddot{h}')\}$ where $\ddot{h}', \mathfrak{x}' \in \mathcal{N}'$. Since $\psi : \mathcal{N} \rightarrow \mathcal{N}'$

be an onto *HBCKA*s, for $\ddot{h}', \mathfrak{x}' \in \mathcal{N}'$, $\exists \ddot{h}, \mathfrak{x} \in \mathcal{N} \ni \varrho^\psi(\ddot{h}) = \ddot{h}'$, and $\varrho^\psi(\mathfrak{x}) = \mathfrak{x}'$. Therefore,

$$\varrho(\chi) \geq \min\{\sup_{\varrho^\psi(\mathfrak{n}) \in (\varrho^\psi(\chi) \star (\varrho^\psi(\ddot{h}) \star \varrho^\psi(\mathfrak{x})))} \varrho^\mu(\varrho^\psi(\mathfrak{n})), \varrho^\psi(\varrho^\psi(\ddot{h}))\},$$

hence $\varrho(\chi) \geq \min\{ \sup_{\mathfrak{h} \in \chi \star (\mathfrak{h} \star \kappa)} \varrho(\mathfrak{h}), \varrho(\mathfrak{h}) \} \forall \chi, \mathfrak{h}, \kappa \in \mathcal{N}$. Now consider

$$\omega(\chi) = \omega^\mu(\omega^\psi(\chi)) \leq \max\{ \inf_{\omega^\psi(\mathfrak{h}) \in (\omega^\psi(\chi) \star (\mathfrak{h}' \star \kappa'))} \omega^\mu(\omega^\psi(\mathfrak{h})), \omega^\psi(\mathfrak{h}') \}$$

where $\mathfrak{h}', \kappa' \in \mathcal{N}'$. Since $\psi : \mathcal{N} \rightarrow \mathcal{N}'$ is an onto \mathcal{HBCKA} s, for $\mathfrak{h}', \kappa' \in \mathcal{N}'$, $\exists \mathfrak{h}, \kappa \in \mathcal{N} \ni \omega^\psi(\mathfrak{h}) = \mathfrak{h}'$, and $\omega^\psi(\kappa) = \kappa'$. Therefore,

$$\omega(\chi) \leq \max\{ \inf_{\omega^\psi(\mathfrak{h}) \in (\omega^\psi(\chi) \star (\omega^\psi(\mathfrak{h}) \star \omega^\psi(\kappa)))} \omega^\mu(\omega^\psi(\mathfrak{h})), \omega^\psi(\omega^\psi(\mathfrak{h})) \},$$

thus, $\omega(\chi) \leq \max\{ \inf_{\mathfrak{h} \in \chi \star (\mathfrak{h} \star \kappa)} \omega(\mathfrak{h}), \omega(\mathfrak{h}) \} \forall \chi, \mathfrak{h}, \kappa \in \mathcal{N}$. □

5. CONCLUSION

This study has successfully expanded \mathcal{HBCKA} s to h-ideals in \mathcal{HBCKA} s, revealing new results and relationships through \mathcal{HHI} characterization. The incorporation of weak, strong, and reflexive \mathcal{HBCKI} s, hyper homomorphism, and product operations has enriched the understanding of these structures. Notably, intuitionistic fuzzy \mathcal{HHI} s form a completely distributive lattice, and the transfer principle enables a crucial equivalence. The product of intuitionistic fuzzy \mathcal{HHI} s preserves the intuitionistic fuzzy \mathcal{HHI} property.

This research advances fuzzy mathematical structures, enhances decision-making, and provides theoretical foundations for applications in computer science, engineering, and economics. Future directions include exploring applications, theoretical extensions, algorithm development, and interdisciplinary collaboration.

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