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Analytical and Numerical Solution to the Generalized Logarithmic Regularized Boussinesq Equation with Coefficients Estimation

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Abstract. The logarithmic regularized Boussinesq equation (Log-RBE) is a partial differential equation that is used to describe the propagation of long ocean waves, like tsunamis, in coastal regions. In this paper, we first generalize the Log-RBE by considering model coefficients. Then, an exact Gaussian solitary wave solution is derived. Additionally, the implicit Crank-Nicolson finite difference (CNFD) scheme is employed to obtain a numerical solution that approximates the exact one. Furthermore, we study the identifiability of the model's coefficients, which leads to two estimation problems. Then, the modulating function method (MFM) is applied to address these estimation problems. All the theoretical findings were complemented by numerical experiments. The CNFD scheme's efficiency goes beyond its accuracy to the ease of its implementation and the short CPU time consumed to find the approximate solution. In addition, the MFM solutions to the coefficient estimation problems show satisfactory results even in the presence of noisy measurements.

1. Introduction

This research is interested first, in generalizing the logarithmic regularized Boussinesq equation (Log-RBE) given in [22] to be in the form:

$$w_{tt} + \lambda_1 w_{xx} + \lambda_2 (w \log |w|^m)_{xx} + \lambda_3 w_{xxtt} = 0, \qquad m \geq 1, \tag{1.1}$$

where the coefficients λ_1, λ_2, and λ_3 are associated with the properties of the medium being studied. Secondly, it is interested in deriving an exact and numerical Gaussian solitary wave solution to (1.1).

Although there have been extensive theoretical studies on the existence of Gaussian soliton solutions of the Log-RBE (see, for example, [7–9, 11, 12, 17, 23]), to the best of our knowledge, no author has approached a solution to equation (1.1) neither an exact solution nor a numerical one.

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In this paper, we propose finite difference approximation methods [5,6,13–15,21], aiming to secure accurate Gaussian soliton approximate solutions of model (1.1) on a finite spatial domain $[b,d]$. In particular, we present the implicit Crank-Nicolson finite difference (CNFD) scheme. The scheme is second-order in time and space. It utilizes the second-order centered approximation matrices to discrtize the spatial derivatives. Also, it employs the fixed-point iterative technique to solve the nonlinear system of equations that arises at each time step during the time integration process.

Furthermore, we conduct estimations for the coefficients λ_1, λ_2, and λ_3. Accurate estimation of these coefficients is vital to confirm the alignment between the utilized model and the physical environment under study [16]. Various methods have been proposed to address estimation problems. The idea in these methods is to minimize a suitable cost function that compares observed data with corresponding calculated values. However, the complexity of computations and the challenge of solving estimation issues increase with nonlinear partial differential equations. Consequently, there is a demand for a straightforward, effective, and robust approach to address such complexities. Hence, we propose to apply a method called the modulating function method (MFM) [2,3]. This method overcomes numerical instabilities resulting from differentiating noisy measurements [18]. Additionally, the estimation problem can be redefined as solving an algebraic system of equations. Notably, MFM provides a practical solution and demonstrates strong robustness characteristics. Moreover, it operates independently of the initial and boundary conditions, which are often unknown in real-world scenarios.

The rest of the paper is organized as follows. In Section 2, we solve (1.1) analytically and numerically. Estimating Log-RBE Coefficients is addressed in Section 3. Section 4 provides numerical results for both solution and estimation. Finally, some concluding remarks are given in Section 5.

2. Solution for the General Logarithmic Regularized Boussinesq Equation

In this section, we find analytical and numerical solutions to the generalized Log-RBE (1.1).

2.1. **Analytical Solution.**

Theorem 2.1. *The function*

$$w(x,t) = e^{\mu_m - \nu_m (kx-ct)^2}, \tag{2.1}$$

represents an analytical solution to the generalized Log-RBE (1.1)*, where k, c, μ_m, and ν_m are parameters such that*

$$\mu_m = -\frac{-\lambda_2 k^2 m + 2\lambda_1 k^2 + 2c^2}{2\lambda_2 k^2 m}, \tag{2.2}$$

and

$$\nu_m = \frac{\lambda_2 m}{4c^2 \lambda_3}. \tag{2.3}$$

Proof. First, we assume that (1.1) exhibits a Gaussian solitary wave solution in the following form:

$$w(x,t) = e^{\mu_m - \nu_m (kx-ct)^2}, \tag{2.4}$$

which decays super-exponentially, with the parameters μ_m and ν_m yet to be determined. Subsequently, we utilize the transformation

$$u(x,t) = \ln w(x,t) = \mu_m - \nu_m (kx-ct)^2, \tag{2.5}$$

to find the exact μ_m and ν_m. Using (2.5), w can be written $w(x,t) = e^{u(x,t)}$, leading to:

$$\begin{aligned} &u_{tt} + u_t^2 + \lambda_1 \left[u_x^2 + u_{xx}\right] + \lambda_2\, m \left[u\, u_x^2 + u_{xx}\, u + 2\, u_x^2 + u_{xx}\right] + \\ &+\lambda_3 \left[u_x^2\, u_t^2 + u_{xx} u_{t+}^2\; u_x^2\, u_{tt} + 4\, u_x\, u_t\, u_{xt} + u_{xx}\, u_{tt} + 2\, u_x\, u_{ttx} + \right. \\ &\left. 2\, u_{txx}\, u_t + 2\, u_{tx}^2 + u_{ttxx}\right] = 0. \end{aligned} \tag{2.6}$$

Solving (2.6) for μ_m and ν_m completes the proof. □

Notice that if $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then values of μ_m and ν_m given in (2.2) and (2.3) are the same as those in Table 2 in [22].

2.2. **The Implicit Crank-Nicolson Finite Difference (CNFD) Scheme.** Applying the implicit CNFD scheme, which is a second-order in time and space multi-step scheme, produces an approximation solution U_q^n to the exact solution $w(x,t)$ of the generalized Log-RBE (1.1) on a uniform grid points (x_q, t_n), where $x \in [b,d]$, $t \in [T_0, T_1]$, the space grid points $x_q = b + qh,\ q = 1, \ldots, Q+1$, the space step $h = (d-b)/Q$, the time step $\Delta T = (T_1 - T_0)/n,\ n = 1, \ldots, N$, and the time levels $t_n = T_0 + n\Delta T$.

The formula of the scheme is given in the following matrix-vector form as

$$\begin{aligned} (\lambda_1 D_2 + 2(\lambda_3 D_2 + I)/\Delta T^2)\mathbf{U}^{n+1} + \lambda_2 D_2 \mathbf{M}^{n+1} &= -2\mathbf{U}^{n-1}/\Delta T^2 - \lambda_2 D_2 \mathbf{M}^n \\ &- (\lambda_1 D_2 - 4(\lambda_3 D_2 + I)/\Delta T^2)\mathbf{M}^n, \end{aligned} \tag{2.7}$$

where $\mathbf{M} = \mathbf{U} \log |\mathbf{U}|^m, I$ is the identity matrix and

$$D_2 = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & -2 \end{pmatrix}_{(Q-1)\times(Q-1)},$$

is the second-order centered finite difference approximation that represents the second derivative, for any given non-periodic boundary conditions.

To produce the solutions $\mathbf{U}^{n+1},\ n = 1, 2, \ldots, N-1$, the nonlinear system (2.7) is solved utilizing the fixed point iterative technique

$$\begin{aligned} \mathbf{U}^{(J+1)} &= -[\lambda_1 D_2 + 2(\lambda_3 D_2 + I)/\Delta T^2]^{-1}[2\mathbf{U}^{n-1}/\Delta T^2 + \lambda_2 D_2(\mathbf{n}^{(J)} + \mathbf{n}^n) \\ &+ (\lambda_1 D_2 - 4(\lambda_3 D_2 + I)/\Delta T^2)\mathbf{U}^n]. \end{aligned}$$

The iteration starts with an initial approximation $\mathbf{U}^{(0)}$ until accuracy is satisfied within δ in the l_∞ norm, i.e.,

$$l_\infty = \|\mathbf{U}^{(J+1)} - \mathbf{U}^{(J)}\|_\infty < \delta, \; J = 0, 1, 2, \ldots.$$

3. Coefficients Estimation

In this section, we first examine the ability to estimate the coefficients λ_1, λ_2, and λ_3 separately or simultaneously, known as identifiability. Based on this examination, we define our estimation problems and proceed to solve them using the MFM.

3.1. **Identifiability.**

In this subsection, we investigate the ability to uniquely determine unknowns from accessible measurements [19].

Let $\{w(x,t^*), w_{tt}(x,t^*)\}$ and $\{\bar{w}(x,t^*), \bar{w}_{tt}(x,t^*)\}$ be two sets of measurements corresponding to $\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3\}$, respectively, where t^* is any time instant. The error between measurements in these two sets can be defined as follows:

$$\begin{aligned} \Delta w(x) &= w(x,t^*) - \bar{w}(x,t^*), \\ \Delta w_{tt}(x) &= w_{tt}(x,t^*) - \bar{w}_{tt}(x,t^*), \\ \Delta \lambda_i &= \lambda_i - \bar{\lambda}_i, \qquad i = 1,2,3. \end{aligned} \tag{3.1}$$

The coefficients λ_1, λ_2, and λ_3 of the generalized Log-RBE (1.1) are simultaneously identifiable if and only if $\Delta w(x) = \Delta w_{tt}(x) = 0$ implies that $\Delta\lambda_i = 0$ for $i = 1,2,3$.

Proposition 3.1. *The coefficients λ_1, λ_2, and λ_3 of the Log-RBE* (1.1) *are not simultaneously identifiable.*

Proof. Writing equation (1.1) in terms of the two sets of measurements and their corresponding coefficients gives

$$w_{tt}(x,t^*) + \lambda_1 w_{xx}(x,t^*) + \lambda_2 (w(x,t^*) \log|w(x,t^*)|^m)_{xx} + \lambda_3 w_{xxtt}(x,t^*) = 0, \tag{3.2}$$

$$\bar{w}_{tt}(x,t^*) + \bar{\lambda}_1 \bar{w}_{xx}(x,t^*) + \bar{\lambda}_2 (\bar{w}(x,t^*) \log|\bar{w}(x,t^*)|^m)_{xx} + \bar{\lambda}_3 w_{xxtt}(x,t^*) = 0. \tag{3.3}$$

Subtracting (3.3) from (3.2) and assuming that $\Delta w(x) = \Delta w_{tt}(x) = 0$ lead to

$$\Delta\lambda_1 w_{xx}(x,t^*) + \Delta\lambda_2 (w(x,t^*) \log|w(x,t^*)|^m)_{xx} + \Delta\lambda_3 w_{xxtt}(x,t^*) = 0. \tag{3.4}$$

Hence, from (3.4), $\Delta\lambda_i = 0$, for $i = 1,2,3$, if the functions $w_{xx}(x,t^*)$, $w_{xxtt}(x,t^*)$, and $(w(x,t^*) \log|w(x,t^*)|^m)_{xx}$ are linearly independent, which is not satisfied. □

Remark 3.1. *The linear dependence of the functions $w_{xx}(x,t^*)$, $w_{xxtt}(x,t^*)$, and $(w(x,t^*) \log|w(x,t^*)|^m)_{xx}$ can be demonstrated by a counter-example. For example, if $\lambda_i = 1$ for $i = 1,2,3$, $m = k = 2$, and $c = 1$, then one can write:*

$$(w \log|w|^m)_{xx} = -\frac{5}{4} w_{xx} - w_{xxtt}. \tag{3.5}$$

Hence, the three coefficients can be estimated separately but not simultaneously.

Similarly, we can study the identifiability of simultaneously estimating two of the coefficients: $\{\lambda_1, \lambda_2\}$, $\{\lambda_1, \lambda_3\}$, or $\{\lambda_2, \lambda_3\}$. In these cases, the functions in the sets $\{w_{xx}(x,t^*), (w(x,t^*)\log|w(x,t^*)|^m)_{xx}\}$, $\{w_{xx}(x,t^*), w_{xxtt}(x,t^*)\}$, or $\{(w(x,t^*)\log|w(x,t^*)|^m)_{xx}, w_{xxtt}(x,t^*)\}$, respectively, should be linearly independent, a condition that is met.

Building upon the prior study, the following estimation problems are posed:

EP1:: Given $w(x,t^*)$ and $w_{tt}(x,t^*)$, find separately λ_1, λ_2, or λ_3.

EP2:: Given $w(x,t^*)$ and $w_{tt}(x,t^*)$, find simultaneously $\{\lambda_1, \lambda_2\}$, $\{\lambda_1, \lambda_3\}$, or $\{\lambda_2, \lambda_3\}$.

The MFM solutions to **EP1** and **EP2** are given in the next subsection.

3.2. **Modulating Function Method (MFM).** First, we recall the definition of a modulating function.

Definition 3.1 ([20]). *Let $l \in \mathbb{N}^*$ and $\psi \neq 0$ be a function defined on $[b,d] \subset \mathbb{R}$. If ψ satisfies the properties: $\forall\, \kappa \in \{0,\dots,l-1\}$,*

(P_1) :: $\psi \in C^l([b,d])$,

(P_2) :: $\psi^{(\kappa)}(b) = \psi^{(\kappa)}(d) = 0$.

Then, ψ is called a modulating function of l^{th} order.

The MFM solution to **EP1** is given in the next proposition.

Proposition 3.2. *Let λ_1, λ_2, or λ_3 be an unknown coefficient in (1.1). Let $\{\psi_r(x)\}_{r=1}^{r=R}$ be a class of at least second-order modulating functions with $R \geq 1$. Then, the unknown coefficient can be estimated by solving the linear system*

$$\mathcal{B}\Lambda = K, \tag{3.6}$$

where the r^{th} element of $\mathcal{B}$ and K, and the unknown Λ are given by:

• *If λ_1 is the unknown:*

$$\mathcal{B}_r = \int_b^d w(x,t^*)\psi_r''(x)\,\mathrm{d}x, \tag{3.7}$$

$$\begin{aligned} K_r = &-\int_b^d w_{tt}(x,t^*)\psi_r(x)\,\mathrm{d}x - \lambda_2\int_b^d w(x,t^*)\ \log|w(x,t^*)|^m\ \psi_r''(x)\,\mathrm{d}x \\ &- \lambda_3\int_b^d w_{tt}(x,t^*)\psi_r''(x)\,\mathrm{d}x, \end{aligned} \tag{3.8}$$

and $\Lambda = \lambda_1$.

• *If λ_2 is the unknown:*

$$\mathcal{B}_r = \int_b^d w(x,t^*)\ \log|w(x,t^*)|^m\ \psi_r''(x)\,\mathrm{d}x, \tag{3.9}$$

$$K_r = -\int_b^d w_{tt}(x,t^*)\psi_r(x)\,\mathrm{d}x - \lambda_1\int_b^d w(x,t^*)\psi_r''(x)\,\mathrm{d}x - \lambda_3\int_b^d w_{tt}(x,t^*)\psi_r''(x)\,\mathrm{d}x, \tag{3.10}$$

and $\Lambda = \lambda_2$.

• *If* λ_3 *is the unknown:*

$$\mathcal{B}_r = \int_b^d w_{tt}(x,t^*)\psi_r''(x)\,\mathrm{d}x, \tag{3.11}$$

$$\begin{aligned} K_r = &- \int_b^d w_{tt}(x,t^*)\psi_r(x)\,\mathrm{d}x - \lambda_1 \int_b^d w(x,t^*)\psi_r''(x)\,\mathrm{d}x \\ &- \lambda_2 \int_b^d w(x,t^*)\ \log|w(x,t^*)|^m\, \psi_r''(x)\,\mathrm{d}x, \end{aligned} \tag{3.12}$$

and $\Lambda = \lambda_3$.

Proof. **STEP 1:** Multiply (1.1) by $\psi_r(x)$, for $r = 1, 2, \ldots, R$:

$$\begin{aligned} w_{tt}(x,t^*)\psi_r(x) + \lambda_1 w_{xx}(x,t^*)\psi_r(x) + \lambda_2 \Big[v(x,t^*)\ \log|w(x,t^*)|^m\Big]_{xx}\psi_r(x) \\ + \lambda_3 w_{xxtt}(x,t^*)\psi_r(x) = 0, \end{aligned} \tag{3.13}$$

where t^* is a fixed time instance.

STEP 2: Integrate over the space interval $[b,d]$:

$$\begin{aligned} \int_b^d w_{tt}(x,t^*)\psi_r(x)\,\mathrm{d}x + \lambda_1 \int_b^d w_{xx}(x,t^*)\psi_r(x)\,\mathrm{d}x \\ + \lambda_2 \int_b^d \Big[w(x,t^*)\ \log|w(x,t^*)|^m\Big]_{xx}\psi_r(x)\,\mathrm{d}x + \lambda_3 \int_b^d w_{xxtt}(x,t^*)\psi_r(x)\,\mathrm{d}x = 0. \end{aligned} \tag{3.14}$$

STEP 3: Apply integration by parts w.r.t. x twice to the second, third, and fourth terms in (3.14) leads to:

$$\begin{aligned} \int_b^d w_{tt}(x,t^*)\psi_r(x)\,\mathrm{d}x + \lambda_1 \int_b^d w(x,t^*)\psi_r''(x)\,\mathrm{d}x \\ + \lambda_2 \int_b^d w(x,t^*)\ \log|w(x,t^*)|^m\, \psi_r''(x)\,\mathrm{d}x + \lambda_3 \int_b^d w_{tt}(x,t^*)\psi_r''(x)\,\mathrm{d}x = 0, \end{aligned} \tag{3.15}$$

where property (P_2) in Definition 3.1 is used to eliminate all boundary terms resulting from the integration by parts. If λ_1 is the unknown, then the second integral in (3.15) corresponds to the r^{th} row of $\mathcal{B}$, while the remaining integrals represent the r^{th} component of $-K$. Similarly for λ_2 and λ_3. □

It is worth noting that in **STEP 3**, we have successfully transferred all spatial derivative dependencies of the solution to the modulating function, which is both known and deterministic. This approach overcomes the need for differentiation of noisy measurements, a well-known ill-posed problem [18]. Furthermore, the MFM incorporates integrals that act as noise-reducing filters for combating the effects of noisy measurements [10]. These factors underlie the robustness of the MFM.

The MFM solution of **EP2** is given in the next proposition.

Proposition 3.3. *Let $\{\lambda_1, \lambda_2\}$, $\{\lambda_1, \lambda_3\}$, or $\{\lambda_2, \lambda_3\}$ be unknown coefficients in (1.1). Let $\{\psi_r(x)\}_{r=1}^{r=R}$ be a class of at least second-order modulating functions with $R \geq 2$. Then, the unknown coefficients can be estimated by solving the linear system*

$$\mathcal{B}\Lambda = K, \tag{3.16}$$

where the rows of the $R \times 2$ matrix $\mathcal{B}$, the elements of the vector K, and the vector Λ are given by:

• *If λ_1 and λ_2 are the unknowns:*

$$\mathcal{B}_r = \begin{bmatrix} \int_b^d w(x,t^*)\psi_r''(x)\,\mathrm{d}x \\ \int_b^d w(x,t^*)\ \log|w(x,t^*)|^m\,\psi_r''(x)\,\mathrm{d}x \end{bmatrix}^\top, \tag{3.17}$$

$$K_r = -\int_b^d w_{tt}(x,t^*)\psi_r(x)\,\mathrm{d}x - \lambda_3\int_b^d w_{tt}(x,t^*)\psi_r''(x)\,\mathrm{d}x, \tag{3.18}$$

and

$$\Lambda = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}^\top. \tag{3.19}$$

• *If λ_1 and λ_3 are the unknowns:*

$$\mathcal{B}_r = \begin{bmatrix} \int_b^d w(x,t^*)\psi_r''(x)\,\mathrm{d}x \\ \int_b^d w_{tt}(x,t^*)\psi_r''(x)\,\mathrm{d}x \end{bmatrix}^\top, \tag{3.20}$$

$$K_r = -\int_b^d w_{tt}(x,t^*)\psi_r(x)\,\mathrm{d}x - \lambda_2\int_b^d w(x,t^*)\ \log|w(x,t^*)|^m\,\psi_r''(x)\,\mathrm{d}x, \tag{3.21}$$

and

$$\Lambda = \begin{bmatrix} \lambda_1 & \lambda_3 \end{bmatrix}^\top. \tag{3.22}$$

• *If λ_2 and λ_3 are the unknowns:*

$$\mathcal{B}_r = \begin{bmatrix} \int_b^d w(x,t^*)\ \log|w(x,t^*)|^m\,\psi_r''(x)\,\mathrm{d}x \\ \int_b^d w_{tt}(x,t^*)\psi_r''(x)\,\mathrm{d}x \end{bmatrix}^\top, \tag{3.23}$$

$$K_r = -\int_b^d w_{tt}(x,t^*)\psi_r(x)\,\mathrm{d}x - \lambda_1\int_b^d w(x,t^*)\psi_r''(x)\,\mathrm{d}x, \tag{3.24}$$

and

$$\Lambda = \begin{bmatrix} \lambda_2 & \lambda_3 \end{bmatrix}^\top. \tag{3.25}$$

Proof. The proof follows the same steps of Proposition 3.2's proof; hence, it is omitted here. □

4. Numerical Simulations

To illustrate the performance of the proposed methods, the implicit CNFD scheme for the numerical solution and the MFM for the estimation problems, some numerical results, with the aid of Matlab codes, are presented in this section.

Table 1. Error norm $L_2(w(x,t))$ (4.1) obtained by applying the CNFD scheme (2.7) to solve the generalized Log-RBE (1.1) for $m = 1,\ k = 2,\ c = 1.75,\ h = 0.05$ and $\Delta T = 10^{-2}$.

t	$L_2(w(x,t))$
-1	0
-0.6	$3.702 * 10^{-4}$
-0.2	$7.4856 * 10^{-4}$
0.2	$1.7412 * 10^{-3}$
0.6	$3.4522 * 10^{-3}$
1	$8.2943 * 10^{-3}$

4.1. **Numerical Solution of the Direct Problem.** This section exhibits the performance of the CNFD scheme (2.7) for approximating the numerical solution of the generalized Log-RBE (1.1) in one dimension, where $t \in [-1, 1],\ \lambda_1 = \lambda_2 = \lambda_3 = 1$, and the step size $h = 0.05$. The time integration starts with the initial condition, obtained by (2.1),

$$w(x, -1) = e^{\mu_m - \nu_m (kx+c)^2},\ x \in [-8, 8],$$

with time-step $\Delta T = 10^{-2}$.

Table 1 exhibits the results of our first test of the CNFD scheme (2.7). We assign some values for the parameters, $m = 1,\ k = 2$ and $c = 1.75$, and compute the relative error evaluated in the integrated error norm

$$L_2 = \frac{\left(\sum[\text{numerical solution} - \text{exact solution}]^2\right)^{\frac{1}{2}}}{\left(\sum[\text{exact solution}]^2\right)^{\frac{1}{2}}}, \tag{4.1}$$

to measure the accuracy of the scheme. The CNFD scheme is an implicit scheme implying that a nonlinear system of equations needs to be solved at each time-step, however, the errors shown in Table 1 are satisfying and justify the efficiency of the scheme.

In Figure (1), we plot the numerical Gaussian solitary wave solution $w(x,t)$ utilizing CNFD scheme (2.7) with $m = 1,\ k = 1,\ c = 1$. The scheme produces the numerical solution with acceptable global error $L_2(w(x,t)) = 9.8003 * 10^{-3}$.

The strength of the scheme is extended to show in Figure (2), at time $t = 0$, one Gaussian soliton solution utilizing the CNFD scheme (2.7) with $m = 2,\ k = 1,\ \lambda_3 = 4,\ \Delta T = 10^{-3}$ and various values of $c = 0.25,\ 0.5,\ 0.75$. In addition, the strength of the scheme is compared to the exact solution (2.1) with $c = 0.75$. The accuracy is measured with a global error $L_2(w(x,t)) = 2.107 * 10^{-4}$.

4.2. **Coefficients Estimation Using MFM.** The efficiency and robustness of the MFM is numerically demonstrated in noise-free and noisy cases. In the noisy case, 5% of a white Gaussian random noise with zero mean has been added to the data.

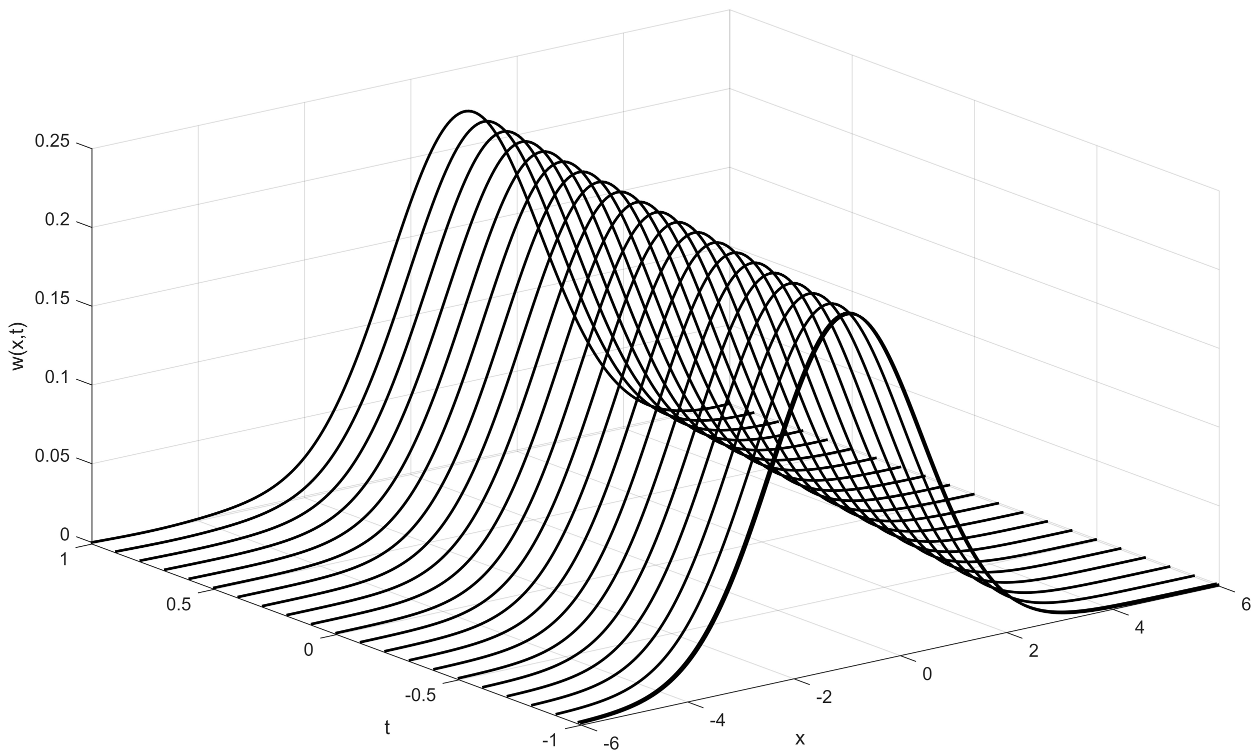

FIGURE 1. Gaussian solitary wave solution $w(x,t)$ produced by the CNFD scheme (2.7) with $\Delta T = 10^{-2}$ for $m = 1,\ k = 1,\ c = 1$.

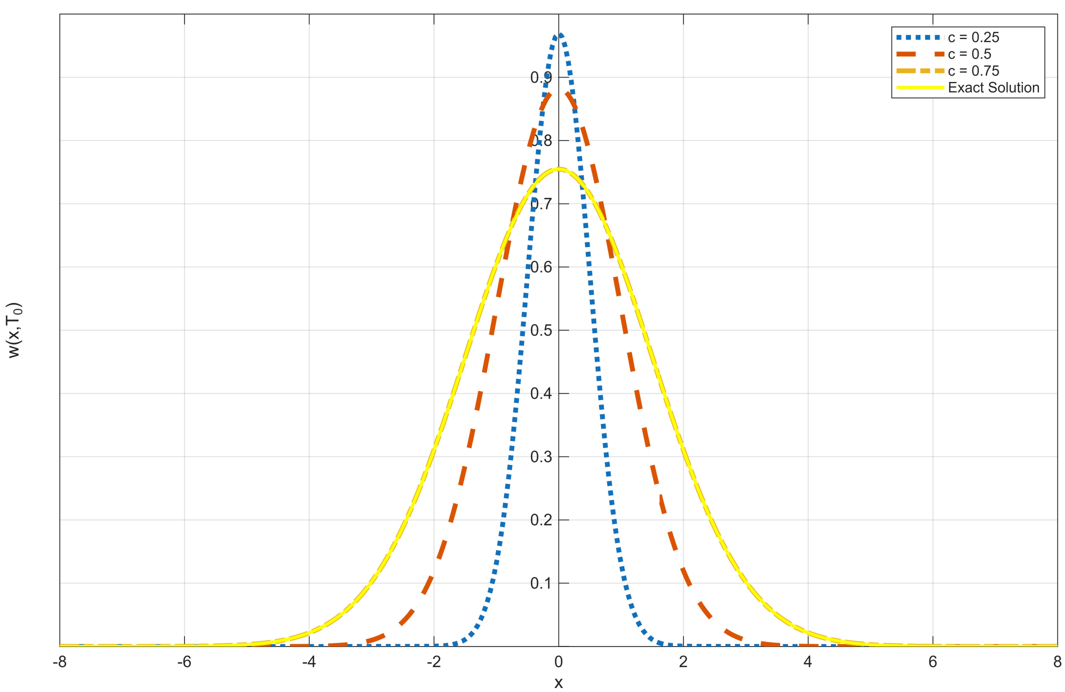

FIGURE 2. CNFD scheme (2.7) produces at $t = 0$ one Gaussian soliton solution $w(x,t)$ for $m = 2,\ k = 1,\ \lambda_3 = 4$ and $\Delta T = 10^{-3}$.

TABLE 2. Estimated values of λ_1 (EP1) in noise-free and noisy cases, where $\lambda_2 = \lambda_3 = 1$.

λ_1	$\hat{\lambda}_1$ (noise-free)	Relative Error	$\hat{\lambda}_1$ (noisy)	Relative Error
-1	-1.0000	2.2204e-16	-1.00003	3.4717e-05
0.5	0.5000	2.2204e-16	0.49997	6.9434e-05
1	1.0000	4.4409e-16	0.99997	3.4717e-05

TABLE 3. Estimated values for λ_2 (EP1) in noise-free and noisy cases, where $\lambda_1 = \lambda_3 = 1$.

λ_2	$\hat{\lambda}_2$ (noise-free)	Relative Error	$\hat{\lambda}_2$ (noisy)	Relative Error
2	2.0000	2.2204e-16	2.0011	0.00057

TABLE 4. Estimated values of λ_3 (EP1) in noise-free and noisy cases, where $\lambda_1 = \lambda_2 = 1$.

λ_3	$\hat{\lambda}_3$ (noise-free)	Relative Error	$\hat{\lambda}_3$ (noisy)	Relative Error
0.5	0.5000	5.5511e-16	0.50033	0.00066

TABLE 5. Estimated values of (λ_1, λ_2) (EP2) in noise-free and noisy cases, where $\lambda_3 = 1$.

$(\lambda_1 = 1, \lambda_2 = 1)$	$(\hat{\lambda}_1, \hat{\lambda}_2)$	Relative Error
Noise-Free	(1.0000, 1.0000)	(2.2204e-16, 1.199e-14)
Noisy	(1.0049, 1.0045)	(0.00486, 0.00453)

For the modulating functions, the normalized polynomial modulating functions have been used [1], which have the form:

$$\psi_r(x) = \frac{\bar{\psi}_r(x)}{||\bar{\psi}_r(x)||_{L^2}}, \qquad b \leq x \leq d \tag{4.2}$$

such that

$$\bar{\psi}_r(x) = (d - x)^{a+r}(b - x)^{a+R+1-r}, \qquad r = 1, \cdots, R \tag{4.3}$$

where R is the number of modulating functions and $a \in \mathbb{R}^+$ is a tuning parameter. These parameters, R and a, can be chosen according to the algorithm described in [4]. The integrals in the MFM solutions are approximated using the trapezoidal numerical integration method.

EP1: Table 2 shows the estimated value of λ_1, denoted by $\hat{\lambda}_1$, in noise-free and noisy cases, where different values for λ_1 have been considered.

Table 3 and Table 4 shows the estimated values of λ_2 and λ_3, respectively.

EP2: The estimated value of $\{\lambda_1, \lambda_2\}$, $\{\lambda_1, \lambda_3\}$, and $\{\lambda_2, \lambda_3\}$ are shown in Table 5, Table 6, and Table 7, respectively.

The results presented illustrate the efficiency and robustness of the MFM.

Table 6. Estimated values of (λ_1, λ_3) (EP2) in noise-free and noisy cases, where $\lambda_2 = 1$.

$(\lambda_1 = 1, \lambda_3 = 0.5)$	$(\hat{\lambda}_1, \hat{\lambda}_3)$	Relative Error
Noise-Free	(1.0000, 0.5000)	(1.7764e-15, 2.6645e-15)
Noisy	(1.0066, 0.50162)	(0.00665, 0.00324)

Table 7. Estimated values of (λ_2, λ_3) (EP2) in noise-free and noisy cases, where $\lambda_1 = 1$.

$(\lambda_2 = 0.5, \lambda_3 = 0.5)$	$(\hat{\lambda}_2, \hat{\lambda}_3)$ (noise-free)	Relative Error
Noise-Free	(0.5000, 0.5000)	(7.7716e-15, 7.3275e-15)
Noisy	(0.49929, 0.49867)	(0.00141, 0.00265)

5. Conclusion

In this paper, we initially generalized the logarithmic regularized Boussinesq equation (Log-RBE) (1.1) by considering model coefficients. Then, an exact Gaussian solitary wave solution for the generalized Log-RBE was derived. In addition, the implicit CNFD scheme (2.7) was applied to find a numerical solution to the exact one. Furthermore, we studied the identifiability of the model's coefficients, leading to two estimation problems. Then, the MFM has been applied to address these estimation problems.

Besides, all the theoretical findings were complemented by numerical experiments. Firstly, the implicit second-order centered CNFD scheme was used to approximate the solution of the generalized Log-RBE. Even though the scheme had to solve a non-linear system of equations, using the fixed point iteration scheme at each time step during the time integration, the scheme exhibits an efficient simulation of the approximated solution with a satisfying global error. The scheme's efficiency goes beyond its accuracy to the ease of its implementation and the short CPU time consumed to find the solution. This is due to the scheme's freedom of restrictions imposed, by conventional explicit methods, on the step-size limit, allowing to select large sizes of the time-step. Secondly, the estimation problems have been tested for different scenarios, and the results were satisfactory even in the presence of noisy measurements.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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