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On Further Accurate Estimates for the Numerical Radii of Operators

Fadi Alrimawi[1], **Ahmad Al-Natoor**[2,*]

[1]*Department of Allied Sciences, Al-Ahliyya Amman University, Amman, Jordan*
[2]*Department of mathematics, Faculty of Sciences, Isra University, Amman 11622, Jordan*

[*]*Corresponding author:* ahmad.alnatoor@iu.edu.jo

Abstract. In this paper, we obtain some upper bounds for numerical radius of operators which generalize some well-known inequalities for classical numerical radius and refine some recent inequalities concerning the numerical radius inequalities of Hilbert space operators.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ be the C^*−algebra of all bounded linear operators on a complex Hilbert space $(\mathbb{H}, \langle\cdot,\cdot\rangle)$. The numerical range $W(A)$ of a bounded operator $A \in \mathbb{B}(\mathbb{H})$ is defined by $W(A) = \{|\langle Ax, x\rangle| : x \in \mathbb{H}, \|x\| = 1\}$. The numerical radius $w(A)$, is defined by

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathbb{H}, \|x\| = 1\}. \tag{1.1}$$

The usual operator norm of an operator A is defined to be

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{H}, \|x\| = 1\}.$$

It is known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$ which is equivalent to the usual operator norm $\|\cdot\|$. Moreover, for every $A \in \mathbb{B}(\mathbb{H})$, we have

$$\frac{1}{2}\|A\| \le w(A) \le \|A\|. \tag{1.2}$$

In 2003, Kittaneh [15] proved a refinement of the second inequality of the inequality (1.2) by obtaining that

$$w(A) \le \frac{1}{2}\||A| + |A^*|\|. \tag{1.3}$$

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Many refinements of the second inequality of the inequality (1.2) has been obtained. One of the most important improvements was given by Kittaneh [16] as follows:

$$w^2(A) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \tag{1.4}$$

In [11], Dragomir established the following inequality

$$w^p(B^*A) \leq \frac{1}{2}\left\||A|^{2p} + |B|^{2p}\right\| \text{ for } p \geq 1. \tag{1.5}$$

In [20], Sababheh and Moradi used the well-known Hermite-Hadamard inequality to perform the following numerical radius inequality

$$\begin{aligned} f(w(A)) &\leq \left\|\int_0^1 f\left((1-s)|A| + s|A^*|\right) ds\right\| && (1.6)\\ &\leq \frac{1}{2}\left\|f(|A|) + f(|A^*|)\right\|, && (1.7)\end{aligned}$$

for every $A \in \mathbb{B}(\mathbb{H})$, and for every increasing operator convex function $f : [0,\infty) \to [0,\infty)$.

Moradi and Sababheh (see [17]) proved the following refinement of the inequality (1.7):

$$f(w(A)) \leq \frac{1}{2}\left\|f\left(\frac{3|A| + |A^*|}{4}\right) + f\left(\frac{|A| + 3|A^*|}{4}\right)\right\|, \tag{1.8}$$

for all increasing convex function $f : [0,\infty) \to [0,\infty)$. In particular, they proved

$$w^2(A) \leq \frac{1}{32}\left\|(3|A| + |A^*|)^2 + (|A| + 3|A^*|)^2\right\|, \tag{1.9}$$

where the constant $\frac{1}{32}$ is the best possible.

For all increasing and operator convex function $f : [0,\infty) \to [0,\infty)$, the authors in [19] proved the following inequalities:

$$\begin{aligned} &f(w(B^*A))\\ &\leq \left\|\int_0^1 f\left((1-s)\left(\frac{2|A|^2 + |B|^2}{3}\right) + s\left(\frac{|A|^2 + 2|B|^2}{3}\right)\right) ds\right\| && (1.10)\\ &\leq \frac{1}{2}\left\|f\left(\frac{2|A|^2 + |B|^2}{3}\right) + f\left(\frac{|A|^2 + 2|B|^2}{3}\right)\right\|, && (1.11)\end{aligned}$$

and

$$\begin{aligned} &f(w(A))\\ &\leq \left\|\int_0^1 f\left((1-s)\left(\frac{2|A| + |A^*|}{3}\right) + s\left(\frac{|A| + 2|A^*|}{3}\right)\right) ds\right\| && (1.12)\\ &\leq \frac{1}{2}\left\|f\left(\frac{2|A| + |A^*|}{3}\right) + f\left(\frac{|A| + 2|A^*|}{3}\right)\right\|. && (1.13)\end{aligned}$$

The inequalities (1.10) and (1.11) refine the inequality (1.5). Also, the inequalities (1.12) and (1.13) refine the inequalities (1.6) and (1.7), respectively. In particular, they proved the following refinement of the inequality (1.9)

$$w^2(A) \leq \frac{1}{18} \left\| (2|A| + |A^*|)^2 + (|A| + 2|A^*|)^2 \right\|, \tag{1.14}$$

where the constant $\frac{1}{18}$ is the best possible.

For more generalizations, counterparts, and recent related results, the reader may refer to [2], [3], [4], [5], [6], [7], [8], [18], and [21].

It should be mentioned that operators and the numerical radius play a central role in functional analysis and operator theory, forming foundational tools for studying bounded linear transformations on Hilbert spaces in many areas (see, e.g., [22] and [23]). The numerical radius, in particular, offers a refined measure of operator size that often provides sharper bounds than the operator norm. These concepts are vital in perturbation theory [10], stability of dynamical systems [9], and quantum mechanics [14].

In this paper, we give refinements of the inequalities (1.3), (1.4), and (1.5). Also, generalizations of the inequalities (1.3), (1.8), (1.10), (1.11), (1.12), and (1.13) will be given.

2. Main results

To achieve our goal, we need the following three lemmas. The first lemma is known as the generalized mixed Schwarz inequality (see [12]). For the second and the third lemmas we refer to [1] and [13], respectively.

Lemma 2.1. *Let $A \in \mathbb{B}(\mathbb{H})$, and let $x, y \in \mathbb{H}$. Then*

$$\left| \langle Ax, y \rangle \right|^2 \leq \left\langle |A|^{2\alpha} x, x \right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle,$$

for $\alpha \in [0,1]$.

Lemma 2.2. *Let $A \in \mathbb{B}(\mathbb{H})$ be any positive operator, and let $x \in \mathbb{H}$ be a unit vector. Then*

$$\langle Ax, x \rangle^\alpha \leq \langle A^\alpha x, x \rangle,$$

for $\alpha \in [1, \infty)$, and

$$\langle A^\alpha x, x \rangle \leq \langle Ax, x \rangle^\alpha,$$

for $\alpha \in (0, 1]$.

Lemma 2.3. *Let $A \in \mathbb{B}(\mathbb{H})$ be a self adjoint operator whose spectrum $\sigma \subset [m, M]$ for some scalars $m \leq M$, and let $x \in \mathbb{H}$ be a unit vector. Then*

$$f(\langle Ax, x \rangle) \leq \langle f(A) x, x \rangle,$$

for every convex function f on $[m, M]$.

Our first main result can be seen in the following theorem. This result gives an interesting interpolating inequality involving numerical radii and spectral norms of operators in which the particular cases of this result give the inequalities (1.8) and (1.13).

Theorem 2.1. *Let $A \in \mathbb{B}(\mathbb{H})$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing convex function. Then*

$$f\left(w(A)\right) \leq \frac{1}{2}\left\| f\left(\alpha\left|A\right| + \beta\left|A^*\right|\right) + f\left(\beta\left|A\right| + \alpha\left|A^*\right|\right)\right\|, \tag{2.1}$$

where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

Proof. Let $x \in \mathbb{H}$. Then, we have

$$\begin{aligned}
& f\left(\left|\langle Ax, x\rangle\right|\right) \\
\leq\ & f\left(\sqrt{\langle\left|A\right| x, x\rangle\left\langle\left|A^*\right| x, x\right\rangle}\right) \quad \text{(by Lemma 2.1 with } \alpha = \tfrac{1}{2}\text{)} \\
\leq\ & f\left(\left\langle\left(\frac{\left|A\right| + \left|A^*\right|}{2}\right) x, x\right\rangle\right) \qquad \text{(by AM-GM inequality)} \\
=\ & f\left(\frac{1}{2}\left[\left\langle\left(\alpha\left|A\right| + \beta\left|A^*\right|\right) x, x\right\rangle + \left\langle\left(\beta\left|A\right| + \alpha\left|A^*\right|\right) x, x\right\rangle\right]\right) \qquad (2.2) \\
\leq\ & \frac{1}{2}\left[f\left(\left\langle\left(\alpha\left|A\right| + \beta\left|A^*\right|\right) x, x\right\rangle\right) + f\left(\left\langle\left(\beta\left|A\right| + \alpha\left|A^*\right|\right) x, x\right\rangle\right)\right] \\
\leq\ & \frac{1}{2}\left[\left\langle f\left(\alpha\left|A\right| + \beta\left|A^*\right|\right) x, x\right\rangle + \left\langle f\left(\beta\left|A\right| + \alpha\left|A^*\right|\right) x, x\right\rangle\right] \\
& \qquad\qquad\qquad \text{(by Lemma 2.3)} \\
=\ & \frac{1}{2}\left\langle\left[f\left(\alpha\left|A\right| + \beta\left|A^*\right|\right) + f\left(\beta\left|A\right| + \alpha\left|A^*\right|\right)\right] x, x\right\rangle.
\end{aligned}$$

Now, we get our result by taking supremum over $\|x\| = 1$ on both sides. □

Note that Theorem 2.1 gives a generalization of the inequalities (1.8) and (1.13). In fact, the inequality (1.8) can be retained from Theorem 2.1 by letting $\alpha = \frac{3}{4}$ and $\beta = \frac{1}{4}$, while the inequality (1.13) can be retained from Theorem 2.1 by letting $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$.

Applications of Theorem 2.1 are given in the following corollaries. In the first corollary we give a generalization of the first inequality of the inequality (1.3) which refines the second inequality of the inequality (1.2).

Corollary 2.1. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ and $p \in [1, 2]$, we have*

$$w^p(A) \leq \frac{1}{2}\left\|\left(\alpha\left|A\right| + \beta\left|A^*\right|\right)^p + \left(\beta\left|A\right| + \alpha\left|A^*\right|\right)^p\right\|. \tag{2.3}$$

In particular,

$$w(A) \leq \frac{1}{2}\left\|\left|A\right| + \left|A^*\right|\right\|.$$

Proof. The result follows from Theorem 2.1 by taking $f(t) = t^p, t \geq 0$ and $p \in [1, 2]$. The particular case follows directly either by taking $p = 1$, or by taking $\alpha = \beta$. □

Corollary 2.2. *Let $A \in \mathbb{B}(\mathbb{H})$ and $p \in [1,2]$. Then, for $k \geq 1$, we have*

$$w^p(A) \leq \frac{1}{2k^p} \left\| ((k-1)|A| + |A^*|)^p + (|A| + (k-1)|A^*|)^p \right\|, \tag{2.4}$$

where the constant $\frac{1}{2k^p}$ is the best possible. In particular, when $p = 2$, we have

$$w^2(A) \leq \frac{1}{2k^2} \left\| ((k-1)|A| + |A^*|)^2 + (|A| + (k-1)|A^*|)^2 \right\|. \tag{2.5}$$

Proof. The inequality (2.4) follows from the inequality (2.3) by taking $\alpha = \frac{k-1}{k}$. To prove the sharpness of the inequality (2.4), assume that the inequality (2.4) holds with another constant $c > 0$, that is

$$w^p(A) \leq c \left\| ((k-1)|A| + |A^*|)^p + (|A| + (k-1)|A^*|)^p \right\|. \tag{2.6}$$

Let A be any normal operator, then we have $w^p(A) = \|A\|^p$, then by the inequality (2.6), we deduce that $\frac{1}{2k^p} \leq c$, and this shows that the constant $\frac{1}{2k^p}$ is the best possible and thus the inequality (2.5) is sharp. □

The inequalities (1.9) and (1.14) can be obtained from the inequality (2.5) by taking $k = 4$ and $k = 3$, respectively.

Another related result can be seen in the following theorem.

Theorem 2.2. *Let $A \in \mathbb{B}(\mathbb{H})$ and let $f : [0,\infty) \to [0,\infty)$ be an increasing operator convex function. Then*

$$\begin{aligned} f(w(A)) &\leq \left\| \int_0^1 f\left((1-s)(\alpha|A| + \beta|A^*|) + s(\beta|A| + \alpha|A^*|)\right) ds \right\| && (2.7) \\ &\leq \frac{1}{2} \left\| f(\alpha|A| + \beta|A^*|) + f(\beta|A| + \alpha|A^*|) \right\| && (2.8) \\ &\leq \frac{1}{2} \left\| f(|A|) + f(|A^*|) \right\|, && (2.9) \end{aligned}$$

where $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

Proof. Let $x \in \mathbb{H}$. We have

$$\begin{aligned} & f(|\langle Ax, x\rangle|) \\ \leq\ & f\left(\frac{1}{2}\left[\langle(\alpha|A| + \beta|A^*|)x, x\rangle + \langle(\beta|A| + \alpha|A^*|)x, x\rangle\right]\right) \\ & \qquad\qquad \text{(by the inequality (2.2))} \\ \leq\ & \int_0^1 f\left((1-s)\langle(\alpha|A| + \beta|A^*|)x, x\rangle + s\langle(\beta|A| + \alpha|A^*|)x, x\rangle\right) ds. \\ =\ & \int_0^1 f\left(\langle(1-s)(\alpha|A| + \beta|A^*|)x, x\rangle + \langle s(\beta|A| + \alpha|A^*|)x, x\rangle\right) ds \\ =\ & \int_0^1 f\left(\langle[(1-s)(\alpha|A| + \beta|A^*|) + s(\beta|A| + \alpha|A^*|)]x, x\rangle\right) ds \\ \leq\ & \int_0^1 \langle f\left((1-s)(\alpha|A| + \beta|A^*|) + s(\beta|A| + \alpha|A^*|)\right)x, x\rangle ds \end{aligned}$$

$$\leq \left\langle \left(\int_0^1 \left((1-s) f\left((\alpha |A| + \beta |A^*|) \right) + s f\left((\beta |A| + \alpha |A^*|) \right) \right) ds \right) x, x \right\rangle \tag{2.10}$$

$$\leq \frac{1}{2} \left\langle \left[f\left(\alpha |A| + \beta |A^*| \right) + f\left(\beta |A| + \alpha |A^*| \right) \right] x, x \right\rangle \tag{2.11}$$

$$\leq \frac{1}{2} \left\langle \left[f\left(|A| \right) + f\left(|A^*| \right) \right] x, x \right\rangle . \tag{2.12}$$

Now, by taking supremum over $\|x\| = 1$ on both sides of the inequalities (2.10), (2.11), and (2.12), we get the inequalities (2.7), (2.8), and (2.9), respectively. □

Corollary 2.3. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for $p \in [1, 2]$, we have*

$$\begin{aligned} w^p(A) &\leq \left\| \int_0^1 \left((1-s) \left(\alpha |A| + \beta |A^*| \right) + s \left(\beta |A| + \alpha |A^*| \right) \right)^p ds \right\| \\ &\leq \frac{1}{2} \left\| \left(\alpha |A| + \beta |A^*| \right)^p + \left(\beta |A| + \alpha |A^*| \right)^p \right\| \\ &\leq \frac{1}{2} \left\| |A|^p + |A^*|^p \right\|, \end{aligned}$$

where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

Proof. The result follows from Theorem 2.2 by taking $f(t) = t^p$, $p \in [1, 2]$. □

Corollary 2.4. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for $p \in [1, 2]$, we have*

$$\begin{aligned} w^p(A) &\leq \left\| \int_0^1 \left((1-s) \left(\frac{(k-1)|A| + |A^*|}{k} \right) + s \left(\frac{|A| + (k-1)|A^*|}{k} \right) \right)^p ds \right\| \\ &\leq \frac{1}{2k^p} \left\| \left((k-1)|A| + |A^*| \right)^p + \left(|A| + (k-1)|A^*| \right)^p \right\| \\ &\leq \frac{1}{2} \left\| |A|^p + |A^*|^p \right\|, \end{aligned}$$

where $k \geq 1$.

Proof. The result follows from Corollary 2.3 by taking $\alpha = \frac{k-1}{k}$. □

To illustrate our work, we give the following example.

Example 2.1. *Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$. Then $w(A) = 1.5$. Applying the inequalities in Corollary 2.4 when $k = 1.8$ and $p = 2$, we get*

$$w^2(A) \leq \left\| \int_0^1 \left((1-s) \left(\frac{0.8|A| + |A^*|}{1.8} \right) + s \left(\frac{|A| + 0.8|A^*|}{1.8} \right) \right)^2 ds \right\| \simeq 2.251 \tag{2.13}$$

$$\leq \frac{1}{2(1.8)^2} \left\| \left(0.8|A| + |A^*| \right)^2 + \left(|A| + 0.8|A^*| \right)^2 \right\| \simeq 2.253 \tag{2.14}$$

$$\leq \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\| = 2.5.$$

Using the inequalities (1.12), (1.9), and (1.14), the upper bound of $w^2(A)$ will be 2.259, 2.277, *and* 2.3125, *respectively. This shows that the inequality (2.13) gives a stronger upper bound of $w^2(A)$ than that given in the inequality (1.12), and the upper bound of $w^2(A)$ obtained from the inequality (2.14) is better than those given in the inequalities (1.9) and (1.14).*

It should be mentioned here that for every $A \in \mathbb{B}(\mathbb{H})$, many other stronger upper bounds of $w^2(A)$ can be obtained from Corollary 2.4 according to the choice of the value of k.

The following result gives a refinement and a generalization of the inequality (1.5) when $p \in [1, 2]$.

Theorem 2.3. *Let $A \in \mathbb{B}(\mathbb{H})$ and let $f : [0, \infty) \to [0, \infty)$ be an increasing operator convex function. Then*

$$\begin{aligned} f\left(w(B^*A)\right) &\leq \left\| \int_0^1 f\left((1-s)\left(\alpha |A|^2 + \beta |B|^2\right) + s\left(\beta |A|^2 + \alpha |B|^2\right)\right) ds \right\| \\ &\leq \frac{1}{2} \left\| f\left(\alpha |A|^2 + \beta |B|^2\right) + f\left(\beta |A|^2 + \alpha |B|^2\right) \right\| \qquad (2.15) \\ &\leq \frac{1}{2} \left\| f\left(|A|^2\right) + f\left(|B|^2\right) \right\|, \qquad (2.16) \end{aligned}$$

where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

Proof. Let $x \in \mathbb{H}$ with $\|x\| = 1$. Then by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} f\left(|\langle B^*Ax, x\rangle|\right) &= f\left(|\langle Ax, Bx\rangle|\right) \\ &\leq f\left(\|Ax\| \|Bx\|\right) \\ &= f\left(\left\langle |A|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle |B|^2 x, x\right\rangle^{\frac{1}{2}}\right) \\ &\leq f\left(\frac{\left\langle |A|^2 x, x\right\rangle + \left\langle |B|^2 x, x\right\rangle}{2}\right). \end{aligned}$$

The rest of the proof follows similarly to that given in Theorem 2.2 by replacing $|A|$ and $|A^*|$ by $|A|^2$ and $|B|^2$ in the inequality (2.2), respectively. □

Corollary 2.5. *Let $A \in \mathbb{B}(\mathbb{H})$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. Then, for $p \in [1, 2]$, we have*

$$\begin{aligned} w^p(B^*A) &\leq \left\| \int_0^1 \left((1-s)\left(\alpha |A|^2 + \beta |B|^2\right) + s\left(\beta |A|^2 + \alpha |B|^2\right)\right)^p ds \right\| \\ &\leq \frac{1}{2} \left\| \left(\alpha |A|^2 + \beta |B|^2\right)^p + \left(\beta |A|^2 + \alpha |B|^2\right)^p \right\| \qquad (2.17) \\ &\leq \frac{1}{2} \left\| |A|^{2p} + |B|^{2p} \right\|. \qquad (2.18) \end{aligned}$$

In particular,

$$\begin{aligned} w^2(B^*A) &\leq \left\| \int_0^1 \left((1-s)\left(\alpha |A|^2 + \beta |B|^2\right) + s\left(\beta |A|^2 + \alpha |B|^2\right)\right)^2 ds \right\| \\ &\leq \frac{1}{2} \left\| \left(\alpha |A|^2 + \beta |B|^2\right)^2 + \left(\beta |A|^2 + \alpha |B|^2\right)^2 \right\| \end{aligned}$$

$$\leq \frac{1}{2}\left\| |A|^4 + |B|^4 \right\|.$$

Proof. The result follows from Theorem 2.3 by taking $f(t) = t^p, p \in [1,2]$. □

Corollary 2.6. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for $p \in [1,2]$, we have*

$$\begin{aligned} w^p(B^*A) &\leq \left\| \int_0^1 \left((1-s)\left(\frac{(k-1)|A|^2 + |B|^2}{k} \right) + s\left(\frac{|A|^2 + (k-1)|B|^2}{k} \right) \right)^p ds \right\| \\ &\leq \frac{1}{2k^p} \left\| \left((k-1)|A|^2 + |B|^2 \right)^p + \left(|A|^2 + (k-1)|B|^2 \right)^p \right\| \\ &\leq \frac{1}{2}\left\| |A|^{2p} + |B|^{2p} \right\|, \end{aligned}$$

where $k \geq 1$.

Proof. The result follows from the inequalities (2.17) and (2.18) by taking $\alpha = \frac{k-1}{k}$. □

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References

[1] M.W. Alomari, On Cauchy-Schwarz Type Inequalities and Applications to Numerical Radius Inequalities, arXiv:2009.01839, 2020. https://doi.org/10.48550/arXiv.2009.01839.

[2] M.W. Alomari, Numerical Radius Inequalities for Hilbert Space Operators, Complex Anal. Oper. Theory 15 (2021), 111. https://doi.org/10.1007/s11785-021-01161-z.

[3] F. Alrimawi, Some Singular Value Inequalities for Convex Functions of Matrices, Int. J. Anal. Appl. 24 (2026), 55. https://doi.org/10.28924/2291-8639-24-2026-55.

[4] F. Alrimawi, H. Kawariq, On Some Generalized Numerical Radius Inequalities for Hilbert Space Operators, J. Math. Comput. Sci. 32 (2023), 257–262. https://doi.org/10.22436/jmcs.032.03.06.

[5] F. Alrimawi, F.A. Abushaheen, R. Alkhateeb, Improved Lower Bounds for Numerical Radius via Cartesian Decomposition, J. Math. Comput. Sci. 33 (2023), 169–175. https://doi.org/10.22436/jmcs.033.02.05.

[6] F. Alrimawi, H. Kawariq, F.A. Abushaheen, Generalized-Weighted Numerical Radius Inequalities for Schatten p-Norms, Int. J. Math. Comput. Sci. 17 (2022), 1463–1473.

[7] F. Alrimawi, O. Hirzallah, F. Kittaneh, Norm Inequalities Involving the Weighted Numerical Radii of Operators, Linear Algebr. Appl. 657 (2023), 127–146. https://doi.org/10.1016/j.laa.2022.10.018.

[8] P. Bhunia, K. Paul, Some Improvements of Numerical Radius Inequalities of Operators and Operator Matrices, Linear Multilinear Algebr. 70 (2020), 1995–2013. https://doi.org/10.1080/03081087.2020.1781037.

[9] Y. Chen, Y. Wei, Numerical Radius for the Asymptotic Stability of Delay Differential Equations, Linear Multilinear Algebr. 65 (2016), 2306–2315. https://doi.org/10.1080/03081087.2016.1273313.

[10] M.T. Chien, H. Nakazato, Perturbation of the Q-Numerical Radius of a Weighted Shift Operator, J. Math. Anal. Appl. 345 (2008), 954–963. https://doi.org/10.1016/j.jmaa.2008.05.015.

[11] S.S. Dragomir, Power Inequalities for the Numerical Radius of a Product of Two Operators in Hilbert Spaces, Sarajev. J. Math. 5 (2024), 269–278. https://doi.org/10.5644/sjm.05.2.10.

[12] T. Furuta, A Simplified Proof of Heinz Inequality and Scrutiny of Its Equality, Proc. Am. Math. Soc. 97 (1986), 751–753. https://doi.org/10.1090/s0002-9939-1986-0846001-3.

[13] J. Pecaric, T. Furuta, J.M. Hot, Y. Seo, Mond-Pecaric Method in Operator Inequalitie, Element Publishing House, Zagreb, Croatia, 2005.

[14] K. He, J.C. Hou, Applying the Theory of Numerical Radius of Operators to Obtain Multi-Observable Quantum Uncertainty Relations, Acta Math. Sin. Engl. Ser. 38 (2022), 1241–1254. https://doi.org/10.1007/s10114-022-1474-y.

[15] F. Kittaneh, A Numerical Radius Inequality and an Estimate for the Numerical Radius of the Frobenius Companion Matrix, Stud. Math. 158 (2003), 11–17. https://doi.org/10.4064/sm158-1-2.

[16] F. Kittaneh, Numerical Radius Inequalities for Hilbert Space Operators, Studia Math. 168 (2005), 73–80.

[17] H.R. Moradi, M. Sababheh, More Accurate Numerical Radius Inequalities (II), Linear Multilinear Algebr. 69 (2019), 921–933. https://doi.org/10.1080/03081087.2019.1703886.

[18] H.R. Moradi, F. Alrimawi, M. Sababheh, New Norm and Numerical Radius Bounds via Advanced Cauchy-Schwarz Inequalities, Ann. Univ. Ferrara 72 (2026), 20. https://doi.org/10.1007/s11565-026-00644-1.

[19] T. Qawasmeh, A. Qazza, R. Hatamleh, M.W. Alomari, R. Saadeh, Further Accurate Numerical Radius Inequalities, Preprint, (2023). https://doi.org/10.20944/preprints202304.1255.v1.

[20] M. Sababheh, H.R. Moradi, More Accurate Numerical Radius Inequalities (I), Linear Multilinear Algebr. 69 (2019), 1964–1973. https://doi.org/10.1080/03081087.2019.1651815.

[21] M. Sattari, M.S. Moslehian, T. Yamazaki, Some Generalized Numerical Radius Inequalities for Hilbert Space Operators, Linear Algebr. Appl. 470 (2015), 216–227. https://doi.org/10.1016/j.laa.2014.08.003.

[22] G. Vinti, L. Zampogni, Approximation by Means of Nonlinear Kantorovich Sampling Type Operators in Orlicz Spaces, J. Approx. Theory 161 (2009), 511–528. https://doi.org/10.1016/j.jat.2008.11.011.

[23] M.S. Yaseen, F. Alrimawi, H. Ibrahim, Modeling Sentence Meaning Using Linear Operators in a Vector Space Semantics Framework: An Interdisciplinary Approach, Forum Linguist. Stud. 7 (2025), 891–898. https://doi.org/10.30564/fls.v7i5.9553.