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Stability of Self-Adjoint Extensions of Symmetric Linear Relations under Relatively Bounded Perturbations on Hilbert Spaces

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Abstract. We investigate the stability of self-adjoint extensions of a closed symmetric linear relation S when subjected to a relatively bounded perturbation by another symmetric linear relation T. We demonstrate that the deficiency indices of S are stable under such perturbations, provided the relative bound is small enough. This result is fundamental, as it ensures that the perturbed relation $S+T$ possesses the same number of self-adjoint extensions as the original relation S. It is established that if T is relatively bounded with respect to S with a relative bound less than $\frac{1}{2}$, then S admits self-adjoint extensions if and only if $S+T$ does.

1. Introduction

The theory of self-adjoint extensions of symmetric operators is a cornerstone of modern mathematical physics, providing the rigorous foundation for quantum mechanics. The spectral theorem for self-adjoint operators establishes a correspondence between physical observables and self-adjoint operators, where the spectrum of the operator corresponds to the possible outcomes of a measurement. However, many differential operators that arise in quantum mechanics are only symmetric, not self-adjoint, on their natural domains. This necessitates the study of self-adjoint extensions, which represent the possible physical realizations of the system.

A fundamental question in this area is the stability of self-adjointness under perturbations. If we have a self-adjoint operator and we add a "small" perturbation, is the resulting operator still self-adjoint? The celebrated Kato-Rellich theorem provides a powerful answer: if the perturbation is relatively bounded with respect to the original operator with a relative bound less than 1, then self-adjointness is preserved.

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However, the classical Kato-Rellich theorem is formulated for single-valued operators. In many modern applications, including quantum field theory, boundary value problems, and the study of singular differential operators, the framework of single-valued operators is too restrictive. The more general framework of linear relations (or multi-valued operators) provides a more natural and powerful setting.

This manuscript extends the classical perturbation theory to the setting of symmetric linear relations. Our central goal is to investigate the stability of self-adjoint extensions under relatively bounded perturbations in this more general framework.

The existence and uniqueness of self-adjoint extensions are determined by the deficiency indices of the symmetric relation. We prove that these indices are stable under relatively bounded perturbations with a relative bound less than 1. This is a crucial step as it guarantees that the perturbed relation has the same number of self-adjoint extensions as the original one.

2. Preliminaries

This section provides the necessary background on the theory of linear relations in Hilbert spaces.

2.1. Basic Definitions.

This subsection introduces the fundamental concepts and terminology that underpin the study of linear relations and their associated structures. These definitions form the backbone of our theoretical development and provide the precise language necessary for the work that follows.

Definition 2.1. *A **linear relation** T in a Hilbert space H is a linear subspace of the product space $H \times H$. The elements of T are ordered pairs (u, v), where $u, v \in H$.*

Note that the concept of a linear relation generalizes the notion of the graph of a linear operator. We say that T is closed if it is closed as a subspace of $H \times H$.

Definition 2.2. *The domain, range, kernel (or null space), and multi-valued part of a linear relation T on a Hilbert space H are defined as follows:*

(i) $\mathrm{dom}(T) = \{u \in H \mid (u, v) \in T \text{ for some } v \in H\}$,
(ii) $\mathrm{ran}(T) = \{v \in H \mid (u, v) \in T \text{ for some } u \in H\}$,
(iii) $\ker(T) = \{u \in H \mid (u, 0) \in T\}$,
(iv) $\mathrm{mult}(T) = T(0) = \{v \in H \mid (0, v) \in T\}$.

We note here that a linear relation T is the graph of a single-valued linear operator if and only if its multi-valued part is trivial, that is, $T(0) = \{0\}$.

Definition 2.3. *Let S and T be two linear relations on a Hilbert space H. We define the sum $S + T$ of S and T and the adjoint T^* of T to be:*

(i) $S + T = \{(u, v + w) \mid (u, v) \in S \text{ and } (u, w) \in T\}$ with $\mathrm{dom}(S + T) = \mathrm{dom}(S) \cap \mathrm{dom}(T)$,
(ii) $T^* = \{(x, y) \in H \times H \mid \langle v, x\rangle = \langle u, y\rangle \text{ for all } (u, v) \in T\}$.

We say that T is symmetric if $T \subset T^*$ and that it is self-adjoint if $T = T^*$.

2.2. **Quotient Norm and Boundedness.** In order to handle the notion of boundedness for linear relations, we introduce the concept of the quotient norm. Let T be a linear relation on a Hilbert space H. By T_s we denote the single-valued operator defined on $\mathrm{dom}(T)$ with values in the quotient space $H/T(0)$ by

$$T_s(u) = [v] \quad \text{where } (u, v) \in T \text{ and } [v] \text{ is the equivalence class of } v \text{ in } H/T(0).$$

This is well-defined because if (u, v_1) and (u, v_2) are both in T, then $(0, v_1 - v_2) \in T$, which means $v_1 - v_2 \in T(0)$, so $[v_1] = [v_2]$ in the quotient space. The operator T_s will be called the Operator induced by T.

We say that a linear relation T on a Hilbert space H is bounded if

$$\|T_s\| < \infty. \tag{2.1}$$

See references [1], [3], [4], [6], and [8] for more details on the theory of linear relations.

3. Closed Linear Relations with Closed Range

In this section, we investigate fundamental properties of closed linear relations and examine how the topological structure of a linear relation's range influences the corresponding induced operator.

3.1. Some results on closed linear relations.

Theorem 3.1. *A linear relation T in a Hilbert space H is closed if and only if:*

(1) *$T(0)$ is a closed subspace of H, and*
(2) *The normed space $H_T = (\mathrm{dom}(T), \|\cdot\|_T)$, with the norm defined by $\|x\|_T = \|x\|_H + \|T_s(x)\|_{H/T(0)}$, is a Banach space.*

Proof. First, assume that T is a closed linear relation. We need to prove that

(1) $T(0)$ is a closed subspace of H, and
(2) H_T is a Banach space.

To prove that $T(0)$ is closed, let's assume that $\{v_n\}$ is a sequence in $T(0)$ that converges to a vector $v \in H$. We must show that $v \in T(0)$. By the definition of $T(0)$, for each v_n, the pair $(0, v_n)$ is in the graph of T. This gives us a sequence of pairs $\{(0, v_n)\}$ in T. The first components of these pairs form the constant sequence $0, 0, \ldots$, which converges to 0. For the second components, the sequence $\{v_n\}$ converge to v by our initial assumption. So, we have $(0, v_n) \to (0, v)$ in the product space $H \times H$ with the graph norm. Since T is a closed linear relation, its graph is a closed set in $H \times H$. Therefore, the limit point $(0, v)$ must belong to the graph of T. By the definition of the multi-valued part, $(0, v) \in T$ implies that $v \in T(0)$. Thus, $T(0)$ contains all its limit points, which means $T(0)$ is a closed subspace of H.

To show that H_T is a Banach space, we must show that every Cauchy sequence in H_T converges to a limit within H_T. So, let $\{x_n\}$ be a Cauchy sequence in H_T. By the definition of the norm $\|\cdot\|_T$, for any $\epsilon > 0$, there exists an integer N such that for all $m, n > N$,

$$\|x_n - x_m\|_T = \|x_n - x_m\|_H + \|T_s\left(x_n - x_m\right)\|_{H/T(0)} < \epsilon$$

This inequality implies two separate conditions:

(i) $\|x_n - x_m\|_H < \epsilon$, which means $\{x_n\}$ is a Cauchy sequence in the Hilbert space H. Since H is complete, this sequence must converge to a limit. Let this limit be $x \in H$. So, $x_n \to x$.

(ii) $\|T_s\left(x_n - x_m\right)\|_{H/T(0)} < \epsilon$. Since T_s is a linear map, this is equivalent to $\|T_s(x_n) - T_s(x_m)\|_{H/T(0)} < \epsilon$. This shows that $\{T_s(x_n)\}$ is a Cauchy sequence in the quotient space $H/T(0)$.

Since $T(0)$ is a closed subspace of H, it follows that $H/T(0)$ is a Banach space. Because $\{T_s(x_n)\}$ is a Cauchy sequence in the complete space $H/T(0)$, it must converge to a limit in that space. Let this limit be $[y] \in H/T(0)$ for some $y \in H$. So, $T_s(x_n) \to [y]$.

Now we have established two convergences: $x_n \to x$ in H and $T_s(x_n) \to [y]$ in $H/T(0)$. We must use the fact that T is closed to show that $x \in \mathrm{dom}(T)$ and that $T_s(x) = [y]$.

By the definition of the operator part T_s, for each $x_n \in \mathrm{dom}(T)$, there exists a corresponding $y_n \in H$ such that $(x_n, y_n) \in T$ and $T_s(x_n) = [y_n]$. The convergence $T_s(x_n) \to [y]$ means that $\|[y_n] - [y]\|_{H/T(0)} \to 0$. By the definition of the quotient norm, this implies that $\inf_{z \in T(0)} \|y_n - y - z\|_H \to 0$. This allows us to find a sequence $\{z_n\}$ in $T(0)$ such that $\|y_n - y - z_n\|_H \to 0$.

Let's define a new sequence $y'_n = y_n - z_n$. We have $y'_n \to y$. Since $(x_n, y_n) \in T$ and $(0, z_n) \in T$ (because $z_n \in T(0)$), and T is a linear subspace, it follows that $(x_n, y_n - z_n) = (x_n, y'_n)$ is also in T. So, we have a sequence of pairs $\{(x_n, y'_n)\}$ in T such that $x_n \to x$ and $y'_n \to y$. Since T is a closed relation, its graph is closed, and therefore the limit point (x, y) must be in T. This implies two things:

(i) $x \in \mathrm{dom}(T)$, since there exists a y such that $(x, y) \in T$, and

(ii) $T_s(x) = [y]$.

We have shown that the Cauchy sequence $\{x_n\}$ from H_T converges to an element $x \in \mathrm{dom}(T)$ in the norm of H. To prove completeness, we must show that this convergence happens in the norm of H_T. We need to show that $\|x_n - x\|_T \to 0$. Consider

$$\|x_n - x\|_T = \|x_n - x\|_H + \|T_s(x_n) - T_s(x)\|_{H/T(0)}.$$

We already know that $x_n \to x$ in H, so the first term $\|x_n - x\|_H \to 0$. We also know that $T_s(x_n) \to [y]$ and we just proved that $T_s(x) = [y]$. Therefore, the second term $\|T_s(x_n) - T_s(x)\|_{H/T(0)} \to 0$. Since both terms go to zero, $\|x_n - x\|_T \to 0$. This shows that every Cauchy sequence in H_T converges to a limit in H_T, so H_T is a Banach space.

To prove the converse implication, let's assume that $T(0)$ is a closed subspace of H and that $H_T = (\mathrm{dom}(T), \|\cdot\|_T)$ is a Banach space. To show that T is closed, we must show that its graph is a closed set in $H \times H$. So, let $\{(x_n, y_n)\}$ be a sequence of pairs in T that converges to a limit $(x, y) \in H \times H$. We must show that this limit point (x, y) is also in T.

The convergence $(x_n, y_n) \to (x, y)$ means that $x_n \to x$ in H and $y_n \to y$ in H. Since (x_n, y_n) is in T, we know that $x_n \in \mathrm{dom}(T)$ for each n. To show that $\{x_n\}$ is a Cauchy sequence in H_T, we compute the norm

$$\|x_n - x_m\|_T = \|x_n - x_m\|_H + \|T_s(x_n) - T_s(x_m)\|_{H/T(0)}.$$

Since $x_n \to x$, the sequence $\{x_n\}$ is a Cauchy sequence in H, so the first term $\|x_n - x_m\|_H$ can be made arbitrarily small. For the second term, we use the definition of the induced operator T_s and the quotient norm:

$$\begin{aligned} T_s(x_n) = [y_n] \quad &\text{and} \quad T_s(x_m) = [y_m], \\ \|T_s(x_n) - T_s(x_m)\|_{H/T(0)} &= \|[y_n] - [y_m]\|_{H/T(0)} = \|[y_n - y_m]\|_{H/T(0)} \\ &= \inf_{z \in T(0)} \|y_n - y_m - z\|_H. \end{aligned}$$

Since the infimum is always less than or equal to any particular value, we can choose $z = 0 \in T(0)$ to get the inequality

$$\|T_s(x_n) - T_s(x_m)\|_{H/T(0)} \leq \|y_n - y_m\|_H.$$

Since $y_n \to y$, the sequence $\{y_n\}$ is a Cauchy sequence in H. Therefore, for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for $m, n > N$, $\|y_n - y_m\|_H < \epsilon$. This implies that $\|T_s(x_n) - T_s(x_m)\|_{H/T(0)}$ can also be made arbitrarily small.

Since both terms in the expression for $\|x_n - x_m\|_T$ can be made arbitrarily small, it follows that $\{x_n\}$ is a Cauchy sequence in H_T.

By assumption, H_T is a Banach space, so this Cauchy sequence must converge to a limit within H_T. Let this limit be $x' \in \mathrm{dom}(T)$. This means $\|x_n - x'\|_T \to 0$. This implies convergence in the underlying Hilbert space norm, so $\|x_n - x'\|_H \to 0$. We already know that $x_n \to x$ in H. By the uniqueness of limits, we must have $x = x'$. Therefore, the limit x of the original sequence is in $\mathrm{dom}(T)$.

Now we must show that the pair (x, y) is actually in T. Since $x_n \to x$ in the H_T norm, we have

$$\|x_n - x\|_T = \|x_n - x\|_H + \|T_s(x_n) - T_s(x)\|_{H/T(0)} \to 0$$

This implies that $T_s(x_n) \to T_s(x)$ in the quotient space $H/T(0)$.

We have two facts:

(i) $T_s(x_n) = [y_n]$ and $y_n \to y$, which implies $T_s(x_n) \to [y]$ (The map $v \mapsto [v]$ is continuous),

(ii) $T_s(x_n) \to T_s(x)$.

By the uniqueness of limits, we must have $T_s(x) = [y]$. By the definition of the induced operator T_s, this means that there exists a vector $y' \in H$ such that $(x, y') \in T$ and $[y'] = [y]$.

The condition $[y'] = [y]$ means that $y' - y \in T(0)$. Let $z = y' - y$. So, $z \in T(0)$, which implies $(0, z) \in T$.

We have $(x, y') \in T$ and $(0, z) \in T$. Since T is a linear subspace, it follows that

$$(x, y') - (0, z) = (x, y' - z) \in T$$

Substituting $z = y' - y$, we get:

$$(x, y' - (y' - y)) = (x, y) \in T$$

We have successfully shown that the limit point (x, y) is in the graph of T. This proves that T is a closed linear relation. □

Remark 3.1. *Let T be a closed linear relation on a Hilbert space H and let H_T be the Banach space defined in Lemma 3.1. Then the set $H_T \cap (\ker T)^\perp = \operatorname{dom}(T) \cap (\ker T)^\perp$ is a closed subspace of H_T.*

Proof. Let (x_n) be a sequence in $H_T \cap (\ker T)^\perp$ which converges to a point x in H_T with norm $\|\cdot\|_T$. We need to show that $x \in (\ker T)^\perp$. Since $x_n \to x$ in the norm $\|\cdot\|_T$, we see from

$$\|x_n - x\|_H \leq \|x_n - x\|_T \to 0,$$

that $x_n \to x$ in H with norm $\|\cdot\|_H$. Since $\ker(T)^\perp$ is a closed subspace of H and (x_n) is a sequence in $\ker(T)^\perp$ which converges to $x \in H$, we conclude that $x \in \ker(T)^\perp$. Hence $H_T \cap (\ker T)^\perp = \operatorname{dom}(T) \cap (\ker T)^\perp$ is a closed subspace of H_T. □

Remark 3.2. *Let T be a closed linear relation on a Hilbert space H and let $\widehat{T_s}$ be the restriction of the induced operator T_s onto the closed subspace $H_T \cap (\ker T)^\perp = \operatorname{dom}(T) \cap (\ker T)^\perp$ of H_T. Then $\operatorname{ran}\left(\widehat{T_s}\right) = \operatorname{ran}(T_s)$.*

Proof. Let $[y] \in \operatorname{ran}(T_s)$. Then there exists $x \in \operatorname{dom}(T_s)$ such that $T_s(x) = [y]$. Let $x = x_1 + x_2$ where $x_1 \in (\ker T)^\perp$ and $x_2 \in (\ker T)$. Then $x_1 = x - x_2 \in \operatorname{dom}(T)$ since $\operatorname{dom}(T)$ is a linear subspace of H. We conclude from

$$[y] = T_s(x) = T_s(x_1 + x_2) = T_s(x_1) + T_s(x_2) = T_s(x_1) + [0] = T_s(x_1)$$

that $\operatorname{ran}(T_s) \subset \operatorname{ran}\left(\widehat{T_s}\right)$. This completes the proof since $\operatorname{ran}\left(\widehat{T_s}\right) \subset \operatorname{ran}(T_s)$ by default. □

Lemma 3.1. *Let T be a closed linear relation on a Hilbert space H and let $\widehat{T_s}$ be the restriction of the induced operator T_s onto the closed subspace . If $x \in \operatorname{dom}(T) \cap (\ker(T))^\perp$ then $T_s(x) = 0$ in $H/T(0)$ implies that $x = 0$. That is, T_s is injective on $\operatorname{dom}(T) \cap (\ker(T))^\perp$.*

Proof. Let $x \in \operatorname{dom}(T) \cap (\ker(T))^\perp$ and assume $T_s(x) = 0$ in the quotient space $H/T(0)$. This means that the equivalence class of any element in the image of x is the zero class. Hence for any y such that $(x, y) \in T$, we must have that $y \in T(0)$. This means that $(0, y) \in T$. The linearity of T implies that $(x, y) - (0, y) = (x, 0) \in T$. Hence $x \in \ker(T)$. Since $x \in (\ker(T))^\perp$, it follows that $x = 0$. □

3.2. **Some results on linear relations with closed range.**

Theorem 3.2. *Let T be a linear relation on a Hilbert space H. If T is closed then T_s is closed. Moreover, if T is closed then T has a closed range if and only if the induced operator $T_s : \mathrm{dom}(T) \subseteq H \to H/T(0)$ has a closed range.*

Proof. That T_s is a closed operator if T is closed follows from [6, Proposition II.5.3].

Next, assume that T is closed and has closed range. To prove that T_s has a closed range, let $\pi : H \to H/T(0)$ be the canonical quotient map, defined by $\pi(v) = [v]$. The range of the induced operator T_s is given by

$$\mathrm{ran}(T_s) = \{[v] \in H/T(0) : \exists u \in \mathrm{dom}(T) \text{ such that } (u, v) \in T\}.$$

This is precisely the image of the range of T under the quotient map π, that is,

$$\mathrm{ran}(T_s) = \pi(\mathrm{ran}(T)).$$

Let $\{[v_n]\}_{n=1}^{\infty}$ be a sequence in $\mathrm{ran}(T_s)$ that converges to some $[v] \in H/T(0)$. We need to show that $[v] \in \mathrm{ran}(T_s)$.

For each n, since $[v_n] \in \mathrm{ran}(T_s) = \pi(\mathrm{ran}(T))$, there exists an element $v'_n \in \mathrm{ran}(T)$ such that $[v_n] = [v'_n]$. The convergence $[v_n] \to [v]$ in the quotient space $H/T(0)$ means that

$$\lim_{n\to\infty} \left\| [v_n] - [v] \right\|_{H/T(0)} = 0$$

By the definition of the quotient norm, this is equivalent to

$$\lim_{n\to\infty} \inf_{w \in T(0)} \|v_n - v - w\|_H = 0. \tag{3.1}$$

Since $[v_n] = [v'_n]$, we have $v_n - v'_n \in T(0)$. If we set $u_n = v_n - v'_n$, then (3.1) can be written as

$$\lim_{n\to\infty} \inf_{w \in T(0)} \|v'_n - v - (w - u_n)\|_H = 0. \tag{3.2}$$

Since $T(0)$ is a linear subspace, the set of elements $\{w - u_n : w \in T(0)\}$ is the same as the set of elements $\{s : s \in T(0)\}$, which is $T(0)$ itself. It therefore follows that (3.2) is equivalent to

$$\lim_{n\to\infty} \inf_{s \in T(0)} \|v'_n - v - s\|_H = 0.$$

This means that for each n, we can choose an element $s_n \in T(0)$ such that the sequence $z_n = v'_n - s_n$ converges to v. Specifically, for any $\epsilon_n > 0$ such that $\epsilon_n \to 0$, we can find $s_n \in T(0)$ with $\|v'_n - v - s_n\|_H < \inf_{w' \in T(0)} \|v'_n - v - w'\|_H + \epsilon_n$. This ensures that $v'_n - s_n \to v$.

Let $z_n = v'_n - s_n$. Since $\mathrm{ran}(T)$ is a linear subspace and $T(0) \subseteq \mathrm{ran}(T)$, we have that $v'_n \in \mathrm{ran}(T)$ and $s_n \in T(0) \subseteq \mathrm{ran}(T)$, which implies $z_n \in \mathrm{ran}(T)$.

We have a sequence (z_n) in $\mathrm{ran}(T)$ and $z_n \to v$ in H. Since $\mathrm{ran}(T)$ is a closed subspace, the limit point v must also be in $\mathrm{ran}(T)$. Because $v \in \mathrm{ran}(T)$, its equivalence class $[v]$ is in the image $\pi(\mathrm{ran}(T))$. Therefore, $[v] \in \mathrm{ran}(T_s)$. This shows that $\mathrm{ran}(T_s)$ contains all its limit points, and is therefore a closed subspace of $H/T(0)$.

Now, assume that T is closed and T_s has a closed range. We know from above that the range of T is related to the range of T_s through the quotient map $\pi : H \to H/T(0)$ defined by $\pi(v) = [v]$. In particular,

$$\mathrm{ran}(T_s) = \pi(\mathrm{ran}(T)) = \{[v] : v \in \mathrm{ran}(T)\}.$$

Let (v_n) be a sequence in $\mathrm{ran}(T)$ converging to some $v \in H$. We must show that $v \in \mathrm{ran}(T)$.

Since π is continuous, $\pi(v_n) \to \pi(v)$. Since each $\pi(v_n)$ belongs to the closed set $\pi(\mathrm{ran}(T)$, their limit $\pi(v)$ must also be in $\pi(\mathrm{ran}(T))$.

This means there exists $u \in \mathrm{ran}(T)$ such that $\pi(v) = \pi(u)$, which implies $v - u \in T(0)$. Since $u \in \mathrm{ran}(T)$ and $T(0) \subset \mathrm{ran}(T)$ (as $\mathrm{ran}(T)$ is a linear subspace), we have

$$v = u + (v - u) \in \mathrm{ran}(T),$$

showing that $\mathrm{ran}(T)$ is closed in H. □

Lemma 3.2. *Let T be a closed linear relation on a Hilbert space H. Then* $\mathrm{ran}(T)$ *is closed if and only if there exists a constant $c > 0$ such that $\|T_s(x)\|_{H/T(0)} \geq c\,\|x\|_H$ for all $x \in \mathrm{dom}(T) \cap (\ker(T))^{\perp}$.*

Proof. Assume that T is closed and has a closed range. Then $\mathrm{ran}(T_s)$, the range of T_s (which is also the range of the restriction of T_s onto $\mathrm{dom}(T) \cap (\ker(T))^{\perp}$ by Remark 3.2), is a closed subset of the Banach space $H/T(0)$ (by Theorem 3.2). Consider the Banach space H_T defined in Theorem 3.1. Since $\|T_s(x)\|_{H/T(0)} \leq \|x\|_H + \|T_s(x)\|_{H/T(0)} = \|x\|_T$, it follows that T_s is bounded as a linear operator on H_T to $\mathrm{ran}\,(T_s)$. Since $\mathrm{dom}(T) \cap (\ker(T))^{\perp}$ is a closed subspace of the Banach space H_T, $\mathrm{ran}(T_s)$ is closed (by Theorem 3.2), and T_s is injective on $\mathrm{dom}(T) \cap (\ker(T))^{\perp}$ (by Lemma 3.1), it follows by the Bounded Inverse Theorem that there exists a constant $c > 0$ such that $\|T_s(x)\|_{H/T(0)} \geq c\|x\|_T = c\left(\|x\|_H + \|T_s(x)\|_{H/T(0)}\right) \geq c\|x\|_H$. Hence

$$\|T_s(x)\|_{H/T(0)} \geq c\,\|x\|_H \text{ for all } x \in \mathrm{dom}(T) \cap (\ker(T))^{\perp}. \tag{3.3}$$

To prove the other implication, assume that T is a closed linear relation and there exists a constant $c > 0$ such that for all $x \in \mathrm{dom}(T) \cap (\ker(T))^{\perp}$, we have

$$\|T_s(x)\|_{H/T(0)} \geq c\,\|x\|_H.$$

Let us define an operator T_0 as the restriction of T_s to the subspace $\mathrm{dom}(T) \cap (\ker(T))^{\perp}$, that is,

$$T_0 = T_s|_{\mathrm{dom}(T)\cap(\ker(T))^{\perp}}.$$

The domain of T_0 is $D_0 = \mathrm{dom}(T) \cap (\ker(T))^{\perp}$, and its range is a subspace of $H/T(0)$. The given condition becomes

$$\|T_0(x)\|_{H/T(0)} \geq c\,\|x\|_H \quad \text{for all } x \in D_0.$$

This inequality implies that T_0 has a closed range. To prove this, let (y_n) be a sequence in $\mathrm{ran}(T_0)$, where $y_n = [v_n] \in H/T(0)$ for some $v_n \in H$, and suppose $y_n \to y = [v]$ for some $y = [v] \in H/T(0)$. We must show that $y = [v] \in \mathrm{ran}(T_0)$.

For each $y_n = [v_n]$, there exists an $x_n \in D_0$ such that $y_n = T_0(x_n) = [w_n]$ where $(x_n, w_n) \in T$. Since (y_n) is a Cauchy sequence, for any $\epsilon > 0$, there exists N such that for $n, m > N$,

$$\|[v_n] - [v_m]\|_{H/T(0)} < c\epsilon.$$

By the given inequality,

$$\|x_n - x_m\|_H \leq \frac{1}{c}\|T_0(x_n - x_m)\|_{H/T(0)} = \frac{1}{c}\|[v_n] - [v_m]\|_{H/T(0)} < \epsilon.$$

Thus (x_n) is a Cauchy sequence in H. Since H is complete, there exists $x \in H$ such that $x_n \to x$. Since each x_n belongs to the closed subspace $(\ker(T))^\perp$, their limit x must also satisfy $x \in (\ker(T))^\perp$.

Consider the sequence $(x_n, [w_n])$ in the graph of T_s, where $[w_n] = T_0(x_n)$. We have $x_n \to x$ and $[w_n] \to [v]$. Since T is a closed relation, its graph is closed in $H \times H$. The induced operator T_s inherits this closedness property (see Theorem 3.2). Hence, since $(x_n, [w_n])$ is a sequence in the graph of T_s with $x_n \to x$ and $[w_n] \to [v]$, then $(x, [v])$ is also in the graph of T_s.

Therefore, $x \in \operatorname{dom}(T_s) = \operatorname{dom}(T)$ and $[v] = T_s(x)$. Since $x \in \operatorname{dom}(T)$ and $x \in (\ker(T))^\perp$, we have $x \in D_0$, so $[v] = T_s(x) = T_0(x) \in \operatorname{ran}(T_0)$. This proves that $\operatorname{ran}(T_0)$ is closed. The conclusion then follows from Remark 3.2 and Theorem 3.2. □

4. Relative Boundedness

This sections establishes some key results on the relative boundedness of linear relations.

Lemma 4.1. . *Let S and T be linear relations on a Hilbert space H with $\operatorname{dom}(S) \subset \operatorname{dom}(T)$ and $T(0) \subset S(0)$. Then*

$$\|S_s(x) + T_s(x)\| \geq \|S_s(x)\| - \|T_s(x)\|.$$

See [7, Lemma 8] for the proof.

Definition 4.1. *Let S and T be two linear relations in a Hilbert space H. We say that T is relatively bounded with respect to S if $\operatorname{dom}(S) \subseteq \operatorname{dom}(T)$ and there exist non-negative constants a and b such that*

$$\|T_s x\|_{H/T(0)} \leq a\|S_s x\|_{H/S(0)} + b\|x\|_H \quad \text{for all } x \in \operatorname{dom}(S). \tag{4.1}$$

The infimum of a taken over all pairs a, b satisfying (4.1) *is called the relative bound of T with respect to S. We denote this infimum by $a_S(T)$. We say that T is strongly dominated by S if $a_S(T) < 1$.*

Theorem 4.1. *Let S be a closed symmetric linear relation on a Hilbert space H and let S_s be its induced operator, mapping $\operatorname{dom}(S)$ to the quotient space $H/S(0)$. Let $\pi : H \to H/S(0)$ be the canonical quotient map. For any non-real complex number $\lambda = \alpha + i\beta$, and for any $u \in \operatorname{dom}(S)$, the identity*

$$\|(S_s - \lambda\pi)u\|^2_{H/S(0)} = \|(S_s - \alpha\pi)u\|^2_{H/S(0)} + \beta^2\|u_0\|^2_H \tag{4.2}$$

where u_0 is the projection of u onto $S(0)^\perp$ holds.

Proof. Let $(u, v) \in S$. Since S is a closed relation, $S(0)$ is a closed subspace of H an therefore, u and v can be uniquely decomposed as

(i) $u = u_0 + u_1$, where $u_0 \in (S(0))^\perp$ and $u_1 \in S(0)$, and

(ii) $v = v_0 + v_1$, where $v_0 \in (S(0))^\perp$ and $v_1 \in S(0)$.

It follows from

$$v - \lambda u = (v_0 + v_1) - \lambda(u_0 + u_1) = (v_0 - \lambda u_0) + (v_1 - \lambda u_1)$$

that

$$\|(S_s - \lambda\pi)u\|^2_{H/S(0)} = \|[v - \lambda u]\|^2_{H/S(0)} = \|v_0 - \lambda u_0\|^2.$$

and that

$$\|(S_s - \lambda\pi)u\|^2_{H/S(0)} = \|v_0 - \lambda u_0\|^2 = \|v_0\|^2 - \bar{\lambda}\langle v_0, u_0\rangle - \lambda\langle u_0, v_0\rangle + |\lambda|^2\|u_0\|^2. \tag{4.3}$$

Since $(u, v) = (u_0 + u_1, v_0 + v_1) \in S$ and $(0, v_1) \in S$, it follows that

$$(u_0 + u_1, v_0 + v_1) - (0, v_1) = (u_0 + u_1, v_0) \in S.$$

Since S is a symmetric relation and $(u_0 + u_1, v_0) \in S$, we see that $\langle v_0, u_0 + u_1\rangle \in \mathbb{R}$. From

$$\langle v_0, u_0 + u_1\rangle = \langle v_0, u_0\rangle + \langle v_0, u_1\rangle$$

and the fact that $v_0 \in (S(0))^\perp$ and $u_1 \in S(0)$ we conclude that $\langle v_0, u_0\rangle \in \mathbb{R}$.

Let $\langle v_0, u_0\rangle = R$, where $R \in \mathbb{R}$ and let $\lambda = \alpha + i\beta$. Identity (4.3) simplifies to

$$\begin{aligned}\|(S_s - \lambda\pi)u\|^2_{H/S(0)} &= \|v_0\|^2 - (\alpha - i\beta)R - (\alpha + i\beta)R + (\alpha^2 + \beta^2)\|u_0\|^2 \\ &= \|v_0\|^2 - 2\alpha R + (\alpha^2 + \beta^2)\|u_0\|^2.\end{aligned}$$

Using the same decomposition, the righthand side of identity (4.2) becomes

$$\|(S_s - \alpha\pi)u\|^2 = \|[v - \alpha u]\|^2 = \|v_0 - \alpha u_0\|^2 = \|v_0\|^2 - 2\alpha R + \alpha^2\|u_0\|^2.$$

Hence,

$$\|(S_s - \alpha\pi)u\|^2 + \beta^2\|u_0\|^2 = \|v_0\|^2 - 2\alpha R + \alpha^2\|u_0\|^2 + \beta^2\|u_0\|^2.$$

Hence identity (4.2) holds. □

Lemma 4.2. *Let T and S be closed linear relations in a Hilbert space H with and* $\mathrm{dom}(S) \subseteq \mathrm{dom}(T)$. *If T is relatively bounded with respect to S with relative bound $a > 0$, then it is also relatively bounded with respect to the relation $A = S - \lambda I$ for any non-real λ.*

Proof. We are given that T is relatively bounded with respect to S with relative bound $a > 0$. This means there exists a constant $b \geq 0$ such that for all $u \in \mathrm{dom}(T)$

$$\|T_s u\|_{H/T(0)} \leq a\|S_s u\|_{H/S(0)} + b\|u\|_H. \tag{4.4}$$

Let $A = S - \lambda I$, where $\lambda = \alpha + i\beta$ is a non-real complex number (so $\beta \neq 0$). The induced operator A_s of A is given by $A_s = S_s - \lambda\pi$, where $\pi : H \to H/S(0)$ is the canonical quotient map. Theorem 4.1 implies that

$$\|(S_s - \lambda\pi)u\|^2_{H/S(0)} = \|(S_s - \alpha\pi)u\|^2_{H/S(0)} + \beta^2\|u_0\|^2_H \tag{4.5}$$

where u_0 is the projection of u onto $S(0)^{\perp}$. Since $A_s = S_s - \lambda\pi$, identity (4.5) can be written as

$$\|A_s u\|^2_{H/S(0)} = \|(S_s - \alpha\pi)u\|^2_{H/S(0)} + \beta^2\|u_0\|^2_H. \tag{4.6}$$

Identity (4.6) directly implies that

$$|\beta|\|u_0\|_H \leq \|A_s u\|_{H/S(0)}. \tag{4.7}$$

Since

$$S_s u = (S_s - \lambda\pi)u + \lambda\pi(u) = A_s u + \lambda\pi(u),$$

the triangle inequality implies that

$$\|S_s u\|_{H/S(0)} \leq \|A_s u\|_{H/S(0)} + |\lambda|\|\pi(u)\|_{H/S(0)} = \|A_s u\|_{H/S(0)} + |\lambda|\|u_0\|_H$$

We use inequality (4.7) to get

$$\|S_s u\|_{H/S(0)} \leq \|A_s u\|_{H/S(0)} + \frac{|\lambda|}{|\beta|}\|A_s u\|_{H/S(0)} = \left(1 + \frac{|\lambda|}{|\beta|}\right)\|A_s u\|_{H/S(0)}. \tag{4.8}$$

Substitution of this bound for $\|S_s u\|$ into inequality (4.4) yields

$$\|T_s u\|_{H/T(0)} \leq a\left[\left(1 + \frac{|\lambda|}{|\beta|}\right)\|A_s u\|_{H/S(0)}\right] + b\|u\|_H. \tag{4.9}$$

This shows that T is relatively bounded with respect to $A = S - \lambda I$. The new bound is $a' = a\left(1 + \frac{|\lambda|}{|\beta|}\right)$ and the new constant is $b' = b$. □

Remark 4.1. *Inequality* (4.9) *shows that if* $a < \frac{1}{2}$ *and* $b = 0$*, then* $\|T_s u\|_{H/T(0)} \leq a'\|A_s u\|_{H/S(0)}$ *with* $a' < 1$.

5. Stability Analysis

This last section deals with the stability of certain quantities and properties of linear relations under relatively bounded perturbations.

5.1. **Stability of closedness.** The following theorem established the stability of closedness of a closed linear relation under relatively bounded perturbations.

Theorem 5.1. *Let A and B be linear relations on a Hilbert space H such that A is closed,* $\mathrm{dom}(B) \supset \mathrm{dom}(A)$ *and* $B(0) \subset A(0)$*. If B is strongly dominated by A, then $A + B$ defined on* $\mathrm{dom}(A)$ *is closed.*

Proof. Let a and b be such that $\left\|B_s(x)\right\|_{H/B(0)} \leq a\|x\|_H + b\|A_s(x)\|_{H/A(0)}$. Since $a_A(B) < 1$, we may assume that $0 < a < 1$. For simplicity, we write $\|B_s(x)\|$ for $\left\|B_s(x)\right\|_{H/B(0)}$, $\|A_s(x)\|$ for $\left\|A_s(x)\right\|_{H/A(0)}$, and $\|x\|$ for $\|x\|_H$. If $b \leq a$ then

$$\|B_s(x)\| \leq a\left(\|x\| + \|A_s(x)\|\right). \tag{5.1}$$

For the linear relation A, let $\|x\|_A$ denote the norm defined by $\|x\|_A = \|x\| + \|A_s(x)\|$ (see for example Theorem 3.1). Using Lemma 4.1 we see that

$$\begin{aligned}(1-a)\,\|x\|_A &= \|x\| + \|A_s(x)\| - (a\|x\| + \|A_s(x)\|) \\ &\le \|x\| + \|A_s(x)\| - \|B_s(x)\| \\ &\le \|x\| + \|A_s(x) + B_s(x)\| = \|x\| + \|(A_s + B_s)x\| = \|x\|_{A+B} \\ &\le \|x\| + \|A_s(x)\| + \|B_s(x)\| \\ &\le \|x\| + \|A_s(x)\| + a\left(\|x\| + \|A_(x)\|\right) \\ &= (1+a)\,(\|x\| + \|A_s(x)\|) \\ &= (1+a)\,\|x\|_A. \end{aligned} \tag{5.2}$$

We see from (5.2) that

$$(1-a)\,\|x\|_A \le \|x\|_{A+B} \le (1+a)\,\|x\|_A. \tag{5.3}$$

For $b > a$, consider qA and qB where $q = a(b)^{-1}$. Then

$$\|qB_s(x)\| = q\|B_s(x)\| \le bq\|x\| + a\|qA_s(x)\| = a\|x\| + a\|qA_s(x)\|,$$

that is,

$$\|qB_s(x)\| \le a\,(\|x\| + \|qA_s(x)\|)\,. \tag{5.4}$$

As before, inequality (5.4) implies that

$$(1-a)\,\|x\|_{qA} \le \|x\|_{qA+qB} \le (1+a)\,\|x\|_{qA},$$

that is,

$$(1-a)\,(\|x\| + q\|A_s(x)\|) \le \|x\| + q\|\,(A_s + B_s)\,(x)\| \le (1+a)\,(\|x\| + q\|A_s(x)\|)$$

or

$$(1-a)\,(\|x\| + \|A_s(x)\|) \le \|x\| + \|\,(A_s + B_s)\,(x)\| \le (1+a)\,(\|x\| + \|A_s(x)\|)\,.$$

Hence even in the case $b > a$, we have

$$(1-a)\,\|x\|_A \le \|x\|_{A+B} \le (1+a)\,\|x\|_A. \tag{5.5}$$

We see from (5.3) or (5.5) that the norms $\|x\|_A$ and $\|x\|_{B+A}$ are equivalent on $\mathrm{dom}(A)$. Since $\mathrm{dom}(A)$ is complete with respect to the norm $\|x\|_A$ by Theorem 3.1, it is also complete with respect to the norm $\|x\|_{B+A}$. The inclusion $B(0) \subset A(0)$ implies that $(A+B)(0) = A(0)$. Since $A(0)$ is closed by Theorem 3.1, we conclude that $A+B$ is closed by Theorem 3.1. □

5.2. **Stability of the Defect Number.** Let A be a linear relation on a Hilbert space H. By the defect number d_A of A, we mean the dimension of the kernel of A^*, that is, $d_A = \dim\ker(A^*)$. In this section we prove stability of the defect number under small relatively bounded perturbations.

The following lemma which is taken from [2], deals with orthogonality issues between certain subspaces of a Hilbert space H.

Lemma 5.1. *Let H_1 and H_2 be subspaces of a Hilbert space H such that $\dim H_1 > \dim H_2$. Then there exists $x \in H_1,\ x \neq 0$, such that $x \perp H_2$.*

Theorem 5.2. *Let $\mathcal{H}$ be a Hilbert space and let A be a closed linear relation on H with closed range. If B is a linear relation on H such that $\operatorname{dom}(B) \supset \operatorname{dom}(A)$, $B(0) \subset A(0)$, and for all $x \in \operatorname{dom}(A)$*

$$\|B_s(x)\| \leq a\|A_s(x)\|, \quad \textit{for some} \;\; a < 1, \tag{5.6}$$

then $A + B$ is closed and has closed range and $d_{A+B} = d_A$.

Proof. That $A + B$ is closed follows from Theorem 5.1. It follows from (5.6) and Lemma 4.1 that

$$\|(A_s + B_s)(x)\| \geq \|A_s(x)\| - \|B_s(x)\| \geq (1-a)\|A_s(x)\| \geq (1-a)c\|x\|. \tag{5.7}$$

That $A + B$ has closed range then follows from (5.7) and Lemma 3.2.

To prove the converse, we use the fact that $\ker(A^*) = \operatorname{ran}(A)^\perp$. Now, let's suppose that $d_{A+B} < d_A$. Then Lemma 5.1 implies that there exists $0 \neq f \in H \ominus \operatorname{ran}(A)$ such that $f \perp H \ominus \operatorname{ran}(A+B)$. The last condition implies that $f \in \operatorname{ran}(A+B)$ so that there exists an element $y \in \operatorname{dom}(A)$ such that $f \in (A+B)(y)$. Since $f \perp \operatorname{ran}(A)$,

$$\langle f, v\rangle = 0 \quad \text{for every } v \in A(y). \tag{5.8}$$

In particular, let $f = u + s$ with $u \in A(y)$ and $s \in B(y)$ and decompose u and s as $u = u_1 + u_2$ and $s = s_1 + s_2$ where $u_1 \in A(0)^\perp$, $u_2 \in A(0)$ and $s_1 \in \overline{B(0)}^\perp$, $s_2 \in \overline{B(0)}$. Since

$$A(y) = u + A(0) = u_1 + A(0),$$

it follows that $u_1 \in A(y)$ and (5.8) implies that

$$\langle u_1, u_1\rangle = -\langle s_1, u_1\rangle. \tag{5.9}$$

Similarly, if $d_{A+B} > d_A$, we can find $0 \neq g \in H \ominus \operatorname{ran}(A+B)$ such that $g \perp H \ominus \operatorname{ran}(A)$. This implies that $g \in \operatorname{ran}(A)$ and so there exists $z \in \operatorname{dom}(A)$ such that $g \in A(z)$. Since $g \perp \operatorname{ran}(A+B)$,

$$\langle g, w\rangle = 0 \quad \text{for every } w \in (A+B)(z). \tag{5.10}$$

Decompose g as $g = g_1 + g_2$ with $g_1 \in A(0)$ and $g_2 \in A(0)^\perp$. Since

$$A(z) = g_1 + A(0),$$

it follows that

$$g_1 \in A(z). \tag{5.11}$$

For $h \in B(z)$ with decomposition $h = h_1 + h_2 \in A(0)^\perp \oplus A(0)$, (5.10) implies that

$$\langle g, g+h\rangle = \langle g_1, g_1\rangle + \langle g_2, g_2\rangle + \langle g_1, h_1\rangle + \langle g_2, h_1\rangle = 0, \tag{5.12}$$

and

$$\langle g, g_1+h\rangle = \langle g_1, g_1\rangle + \langle g_1, h_1\rangle + \langle g_2, h_1\rangle = 0, \tag{5.13}$$

Equalities (5.12) and (5.13) imply that $\langle g_2, g_2\rangle = 0$ and therefore $g_2 = 0$ so that $g = g_1 \in A(0)^\perp$. Equality (5.12) can now be rewritten as

$$\langle g, g\rangle = -\langle g, h_1\rangle. \tag{5.14}$$

We see from (5.9) and (5.14) that

$$\|u_1\|^2 \leq \|s_1\|\,\|u_1\| \tag{5.15}$$

and

$$\|g\|^2 \leq \|h_1\|\,\big\|g\big\|. \tag{5.16}$$

Using the fact that for a closed subspace N of H, the quotient space H/N is isometrically isomorphic to $N^\perp$ via the isometry $\psi : H/N \to N^\perp$ defined by $[x] \mapsto P^\perp x$ where $P^\perp$ denotes the orthogonal projection of H onto $N^\perp$, and the fact that $A(0)^\perp \subset B(0)^\perp$, we obtain the inequalities

$$\|A_s(y)\|^2 \leq \big\|B_s(y)\big\|\,\big\|A_s(y)\big\| \leq a\|A_s(y)\|^2 \tag{5.17}$$

and

$$\|A_s(z)\|^2 \leq \big\|B_s(z)\big\|\,\big\|A_s(z)\big\| \leq a\|A_s(z)\|^2 \tag{5.18}$$

using (5.15) and (5.16) respectively. Since $0 < a < 1$, the contradictions in (5.17) and (5.18) imply that $d_{A+B} = d_A$. □

5.3. **Stability of Selfadjoint extensions.** This section demonstrates a primary application of the stability of the defect numbers under relatively bounded perturbations. It establishes the stability of self-adjoint extensions under relatively bounded perturbations. First we define the deficiency indices.

Definition 5.1. *Let T be a linear relation on a Hilbert space H. The spaces $N_+ = \ker(T^* - iI)$ and $N_- = \ker(S^* + iI)$ are called the deficiency subspaces of T while their dimensions, $\dim(N_\pm)$, which we denote by $n_\pm(T)$, are called the deficiency indices of T.*

Recall that a symmetric linear relation S on a Hilbert space H is said to have a self-adjoint extension if there exist a self-adjoint relation $\widetilde{S}$ on H such that $S \subset \widehat{S}$

Remark 5.1. *It is well known (see for example [5]) that a symmetric linear relation on a Hilbert space has self-adjoint extensions if and only if $n_+(T) = n_-(T)$ and that the extension is unique if and only if $n_+(T) = n_-(T) = 0$.*

The following theorem established the stability of self-adjoint extensions under relatively bounded perturbations.

Theorem 5.3. *Let S and T be symmetric linear relation on a Hilbert space H such that S is closed,* $\mathrm{dom}(S) \subset \mathrm{dom}(T)$, $T(0) \subset S(0)$. *If there exists a constant $a < \frac{1}{2}$ such that*

$$\|T_s(x)\|_{H/T(0)} \leq \|S_s(x)\|_{H/T(0)} \text{ for all } x \in \mathrm{dom}(S),$$

that is, T is relatively bounded with respect to S with relative bound $a < \frac{1}{2}$ and $b = 0$, then S has self-adjoint extensions if and only if $S + T$ has self-adjoint extensions.

Proof. Let $A = S \mp iI$ and $B = T$. Then A is closed by Theorem 5.1. Since both A and B are symmetric, it follows that $A + B$ is also symmetric. By Lemma 4.2, T is relatively bounded with respect to A with relative bound $a_1 < 1$. Theorem 5.2 implies that for such relations, $\dim\ker(A^*) = \dim\ker((A + T)^*)$. We identify the terms:

(i) $\ker(A^*) = \ker((S \mp iI)^*) = \ker(S^* \pm iI) = \mathcal{N}_\pm(S)$,

(ii) $\ker((A + T)^*) = \ker((S + T \mp iI)^*) = \ker((S + T)^* \pm iI) = \mathcal{N}_\pm(S + T)$.

The equality of dimensions is therefore $\dim(N_\pm(S)) = \dim(N_\pm(S + T))$, which is precisely the statement $n_\pm(S) = n_\pm(S + T)$. The conclusion then follows from Remark 5.1. □

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