

Analysis of Topological Degree Method for Solving Ambartsumian-Type Mathieu Fractional Differential Equations Under Φ -Hilfer Derivative

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Abstract. The Ambartsumian equation arises in astronomy and is used in the theory of surface brightness in the Milky Way. In this manuscript, we consider a new category of Ambartsumian-type Mathieu fractional differential systems involving the Φ -Hilfer fractional derivative. We study the existence and uniqueness of solutions of the above system using the topological degree method for condensing maps and Banach's contraction principle. Furthermore, Ulam-Hyers and generalized Ulam-Hyers stability are investigated, by using the generalized Gronwall's inequality and nonlinear analysis issue. As an application, we provide an example to illustrate the validity of our results.

1. INTRODUCTION

Fractional differential systems arise in mathematical models associated with various physical phenomena, such as synchronization of chaotic systems [1], BAM neural networks with time-varying delays [2], HIV-immune system dynamics with memory effects [3], anomalous diffusion [4], ecological processes [5], and co-infection of malaria and HIV/AIDS [6]. Several theoretical results on fractional differential equations involving Riemann-Liouville, Caputo, Hadamard, and Hilfer-type fractional derivative operators can be found in the literature [7–12] and the references cited therein. For a comprehensive overview of fractional calculus, we refer the reader to the monographs [13–16]. More recently, the concept of the ψ -Hilfer fractional derivative operator was introduced in [17] and has attracted considerable attention, as it generalizes several known fractional derivative operators under suitable choices of the parameters involved in its definition. Researchers used fixed point techniques and topological degree method to prove the existence of

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solutions for a variety of types of nonlinear fractional differential equations with various fractional operators, see [18–23]. In 2025, Lmou et al. [24], investigated the novel category of Ψ -Hilfer-type fractional differential equation involving topological degree method. They considered the following equation

$$\begin{aligned} {}^H\mathfrak{D}^{\omega,\sigma,\Psi}w(\tau) &= f(\tau, w(\tau)), \quad \tau \in [\zeta, \iota], \\ w(\zeta) &= 0, \quad w(\iota) = \sum_{i=1}^n \omega_i w(\xi_i). \end{aligned}$$

Where ${}^H\mathfrak{D}^{\omega,\sigma,\Psi}(\cdot)$ is the Ψ -Hilfer fractional derivative of order ω , $1 < \omega \leq 2$ and parameter σ , $0 \leq \sigma \leq 1$, $\omega_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, $\zeta < \xi_1 < \dots < \xi_n < \iota$, $f \in C([\zeta, \iota] \times \mathbb{R}, \mathbb{R})$.

The Mathieu equation is an important equation in mathematical physics, as it has numerous applications in several areas of the physical sciences [25]. Recently, motivated by [26], Amel Berhail et al. [27] investigated the existence results for the fractional Mathieu equation associated with the Hilfer–Katugampola fractional derivative, described by:

$$\begin{aligned} {}^\rho\mathcal{D}^{\alpha,\beta}y(t) + p(t)y(t) &= w(t, y(t), {}^\rho\mathcal{D}^{\alpha,\beta}y(t)), \quad t \in [0, T], \\ {}^\rho\mathcal{I}^{1-\gamma}y(0) &= \sum_{i=1}^m q_i y(e_i), \quad e_i \in (0, T], \end{aligned}$$

where $p(t) = a - 2b \cos(2t)$, a, b are real constants, ${}^\rho\mathcal{D}^{\alpha,\beta}(\cdot)$ is the Hilfer–Katugampola fractional derivative of order α , type $\beta \in [0, 1]$, and ${}^\rho\mathcal{I}^{1-\gamma}(\cdot)$ is the Riemann–Liouville fractional integral of order $1 - \gamma$, $\gamma = \alpha + \beta - \alpha\beta$, $\rho > 0$. Moreover, the function $w : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, e_i for $i = 0, 1, \dots, m$ are prefixed points satisfying $0 < e_1 \leq e_2 \leq \dots \leq e_m < T$, and q_i are real numbers.

On the other hand, in the deterministic setting, Ambartsumian derived the standard Ambartsumian equation [28], which describes the absorption of light by interstellar matter. In the deterministic setting of the theory of surface brightness in the Milky Way, the Ambartsumian delay equation is widely used. We consider in this paper the fractional Ambartsumian equation as in [29]:

$$\begin{aligned} \frac{d}{dt}\mathcal{A}(t) &= \frac{1}{\eta}\mathcal{A}\left(\frac{t}{\eta}\right) - \mathcal{A}(t), \quad \eta > 1, \\ \mathcal{A}(0) &= \mu, \end{aligned}$$

where η and μ are constants and the fluctuation of the surface brightness in the Milky Way is denoted by $\mathcal{A}(t)$. Because of its importance in astronomy, considerable attention has been devoted to obtaining accurate solutions of this equation. The Ambartsumian equation have received attention from many researchers. In particular, the existence and stability results for the Ambartsumian equation with Caputo fractional derivative was discussed in [30, 31].

Motivated by the above studies, we investigate the following equation, which represents an Ambartsumian-type Mathieu fractional differential equation formulated in the Φ -Hilfer sense

$$\begin{cases} {}^H\mathbb{D}_{c^+}^{\xi, \nu, \Phi} \mathcal{A}(t) + q(t)\mathcal{A}(t) = \mathbb{G}\left(t, \mathcal{A}\left(\frac{t}{\eta}\right), \mathcal{A}(t)\right), & t \in \mathbb{J} := [c, d], \quad \eta > 1, \\ \mathcal{A}(c) = 0, \quad \mathcal{A}(d) = \sum_{i=1}^n \sigma_i \mathcal{A}(\rho_i), \end{cases} \quad (1.1)$$

where $\mathbb{G}\left(t, \mathcal{A}\left(\frac{t}{\eta}\right), \mathcal{A}(t)\right) = \frac{1}{\eta} \mathcal{A}\left(\frac{t}{\eta}\right) - \mathcal{A}(t)$, $q(t) = a - 2b \cos(2t)$, a, b are constants, with order $\xi \in (1, 2]$ and type $\nu \in [0, 1]$, the Φ -Hilfer fractional derivative is represented by ${}^H\mathbb{D}_{c^+}^{\xi, \nu, \Phi}(\cdot)$. Here $\mathbb{G} : \mathbb{J} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, $\sigma_i \in \mathbb{R}$ such that for $i = 1, 2, \dots, n$, $c < \rho_1 < \rho_2 < \dots < \rho_n < d$.

In what follows, we present the contributions and novelty of this work:

- (i) The Ambartsumian-type Mathieu fractional differential system (1.1) is discussed under a Φ -Hilfer fractional derivative, which can be reduced to a well-known fractional derivatives (Riemann-Liouville, Caputo, Φ -Riemann-Liouville, Hilfer-Hadamard, Katugampola derivative), for different values of function Φ and parameters ν , see [17];
- (ii) The Ambartsumian-type Mathieu Φ -fractional differential system (1.1) is studied with nonlocal boundary condition;
- (iii) We establish the existence and uniqueness results for the new Ambartsumian-type Mathieu Φ -fractional differential system (1.1), along the Ulam-Hyers and generalized Ulam-Hyers stability, by applying topological degree method, Banach contraction principle, and the generalized Gronwall inequality.

2. PRELIMINARIES

Definition 2.1. [17] Let $\Phi'(t) > 0$ and $\xi > 0$. Then the Φ -Riemann-Liouville fractional integral of order ξ of the function \mathbb{G} with respect to another function Φ on \mathbb{J} is given by

$$\mathbb{I}_{c^+}^{\xi, \Phi} \mathbb{G}(t) = \frac{1}{\Gamma(\xi)} \int_c^t \Phi'(s) (\Phi(t) - \Phi(s))^{\xi-1} \mathbb{G}(s) ds,$$

where $\Gamma(\cdot)$ indicates the Gamma function.

Definition 2.2. [17] Let $\Phi'(t) > 0$ and $\xi > 0$, $n \in \mathbb{N}$. Then the Φ -Riemann-Liouville fractional derivative of order ξ for an integrable function \mathbb{G} with respect to another function Φ on \mathbb{J} is given as follows:

$$\begin{aligned} {}^{RL}\mathbb{D}_{c^+}^{\xi, \Phi} \mathbb{G}(t) &= \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^n \mathbb{I}_{c^+}^{n-\xi, \Phi} \mathbb{G}(t) \\ &= \frac{1}{\Gamma(n-\xi)} \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^n \int_c^t \Phi'(s) (\Phi(t) - \Phi(s))^{n-\xi-1} \mathbb{G}(s) ds, \end{aligned}$$

where $n - 1 < \xi < n$, $n = [\xi] + 1$, and $[\xi]$ is an integer part of ξ .

Definition 2.3. [17] Let $n - 1 < \xi < n$ with $n \in \mathbb{N}$ and $\mathbb{G}, \Phi \in C^n(\mathbb{J}, \mathbb{R})$ two functions such that $\Phi'(t) > 0$ for every $t \in \mathbb{J}$. The Φ -Hilfer fractional derivative of a function \mathbb{G} of order ξ and type $\nu \in [0, 1]$,

is given as follows:

$$\begin{aligned} {}^H\mathbb{D}_{c^+}^{\xi, \nu, \Phi} \mathbf{G}(t) &= \mathbb{I}_{c^+}^{\nu(n-\xi), \Phi} \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^n \mathbb{I}_{c^+}^{(1-\nu)(n-\xi), \Phi} \mathbf{G}(t) \\ &= \mathbb{I}_{c^+}^{(\vartheta-\xi), \Phi} \mathbb{D}_{c^+}^{\vartheta, \Phi} \mathbf{G}(t), \end{aligned}$$

where $n-1 < \xi < n$, $n = [\xi] + 1$, and $\vartheta = \xi + \nu(n - \xi)$.

Lemma 2.1. [17] Let $\xi, \nu > 0$. Then we have

$$\mathbb{I}_{c^+}^{\xi, \Phi} \mathbb{I}_{c^+}^{\nu, \Phi} \mathbf{G}(t) = \mathbb{I}_{c^+}^{\xi+\nu, \Phi} \mathbf{G}(t), \quad t > c.$$

Proposition 2.1. [17] Let $c \geq 0$ and $\omega > c$. Then, we have

$$\begin{aligned} (i) \quad \mathbb{I}_{c^+}^{\xi, \Phi} (\Phi(s) - \Phi(c))^{\omega-1}(t) &= \frac{\Gamma(\omega)}{\Gamma(\omega + \xi)} (\Phi(s) - \Phi(c))^{\xi+\omega-1}(t). \\ (ii) \quad \mathbb{D}_{c^+}^{\xi, \Phi} (\Phi(s) - \Phi(c))^{\omega-1}(t) &= \frac{\Gamma(\omega)}{\Gamma(\omega - \xi)} (\Phi(s) - \Phi(c))^{\xi-\omega-1}(t). \end{aligned}$$

Lemma 2.2. [17] If $\mathbf{G} \in C^n(\mathbb{J}, \mathbb{R})$, $n-1 < \xi < n$, $0 \leq \nu \leq 1$ and $\vartheta = \xi + \nu(n - \xi)$, then

$$\mathbb{I}_{c^+}^{\xi, \Phi} ({}^H\mathbb{D}_{c^+}^{\xi, \nu, \Phi} \mathbf{G})(t) = \mathbf{G}(t) - \sum_{k=1}^n \frac{(\Phi(t) - \Phi(c))^{\vartheta-k}}{\Gamma(\vartheta - k + 1)} \mathbf{G}_{\Phi}^{[n-k]} \mathbb{I}_{c^+}^{(1-\nu)(n-\xi), \Phi} \mathbf{G}(t), \quad t \in \mathbb{J},$$

where $\mathbf{G}_{\Phi}^{[n-k]} \mathbf{G}(t) := \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^{n-k} \mathbf{G}(t)$.

Definition 2.4. [24] Let \mathbb{X} be a Banach space and \mathbb{O} a bounded subsets of \mathbb{X} . The Kuratowski measure of noncompactness is the map $\omega : \mathbb{O} \rightarrow \mathbb{R}_+$ given by

$$\omega(\mathbb{O}) = \inf\{\tau > 0 : \mathbb{O} \subseteq \bigcup_{i=1}^n \mathbb{O}_i \text{ and } \text{diam}(\mathbb{O}_i) \leq \tau\}.$$

Proposition 2.2. [24] Let $\mathbb{O}, \mathbb{O}_1, \mathbb{O}_2$ be bounded subsets of \mathbb{X} . Then the Kuratowski measure of noncompactness ω satisfies the following properties

- (i) \mathbb{O} is relatively compact $\iff \omega(\mathbb{O}) = 0$.
- (ii) $\omega(\alpha\mathbb{O}) = |\alpha|\omega(\mathbb{O})$, $\alpha \in \mathbb{R}$.
- (iii) $\omega(\mathbb{O}_1 + \mathbb{O}_2) \leq \omega(\mathbb{O}_1) + \omega(\mathbb{O}_2)$.
- (iv) $\mathbb{O}_1 \subset \mathbb{O}_2 \implies \omega(\mathbb{O}_1) \leq \omega(\mathbb{O}_2)$.
- (v) $\omega(\mathbb{O}_1 \cup \mathbb{O}_2) = \max\{\omega(\mathbb{O}_1), \omega(\mathbb{O}_2)\}$.
- (vi) $\omega(\mathbb{O}) = \omega(\overline{\mathbb{O}}) = \omega(\text{conv}\mathbb{O})$, where $\overline{\mathbb{O}}$ and $\text{conv}\mathbb{O}$ indicates the closure and the convex hull of the set \mathbb{O} , respectively.

Definition 2.5. [24] Let $\mathbb{P} : \mathbb{W} \subset \mathbb{X} \rightarrow \mathbb{X}$ be a continuous bounded map. The operator \mathbb{P} is said to be ω -Lipschitz if there exists $m \geq 0$ such that

$$\omega(\mathbb{P}(\mathbb{O})) < m\omega(\mathbb{O}), \quad \text{for all } \mathbb{O} \subset \mathbb{W}.$$

Moreover, if $m < 1$, then \mathbb{P} is a strict ω -contraction.

Definition 2.6. [24] $\mathbb{P} : \mathbb{W} \subset \mathbb{X} \rightarrow \mathbb{X}$ is called ω -condensing if

$$\omega(\mathbb{P}(\mathbb{O})) < \omega(\mathbb{O}),$$

for every bounded and non precompact subset \mathbb{O} of \mathbb{W} , with $\omega(\mathbb{O}) > 0$.

Definition 2.7. [24] Let $\mathbb{P} : \mathbb{W} \subset \mathbb{X} \rightarrow \mathbb{X}$. The map \mathbb{P} is called Lipschitz if there exists a constant $m \geq 0$ such that

$$\|\mathbb{P}\mathcal{A}_1 - \mathbb{P}\mathcal{A}_2\| \leq m\|\mathcal{A}_1 - \mathcal{A}_2\|, \quad \mathcal{A}_1, \mathcal{A}_2 \in \mathbb{W}.$$

Furthermore, \mathbb{P} is called a strict contraction, if $m < 1$.

Proposition 2.3. [24] If $\mathbb{P} : \mathbb{W} \subset \mathbb{X} \rightarrow \mathbb{X}$ is Lipschitz having constant m , then \mathbb{P} is ω -Lipschitz having the same constant m .

Proposition 2.4. [24] If $\mathbb{P}, \mathbb{Q} : \mathbb{W} \subset \mathbb{X} \rightarrow \mathbb{X}$ are ω -Lipschitz maps having constants m_1, m_2 respectively, then $\mathbb{P} + \mathbb{Q} : \mathbb{W} \rightarrow \mathbb{X}$ is ω -Lipschitz map having constant $m_1 + m_2$.

Proposition 2.5. [24] If $\mathbb{P} : \mathbb{W} \subset \mathbb{X} \rightarrow \mathbb{X}$ is compact, then \mathbb{P} is ω -Lipschitz having constant $m = 0$.

Theorem 2.1. [24] Let $\mathbb{T} : \mathbb{X} \rightarrow \mathbb{X}$ be ω -condensing and

$$\Pi_\varepsilon = \{\mathcal{A} \in \mathbb{X} : \mathcal{A} = \varepsilon\mathbb{T}\mathcal{A}, \text{ for all } 0 \leq \varepsilon \leq 1\}.$$

If Π_ε is a bounded set in \mathbb{X} , then there exists $r > 0$ such that $\Pi_\varepsilon \subset \mathbb{B}_r(0)$, then

$$\text{deg}(I - \varepsilon\mathbb{T}, \mathbb{B}_r(0), 0) = 1, \text{ for all } \varepsilon \in [0, 1].$$

Consequently, the operator \mathbb{T} has at least one fixed point and the set of fixed points of \mathbb{T} is contained in $\mathbb{B}_r(0)$.

3. MAIN RESULTS

Lemma 3.1. Let $c \geq 0, \xi \in (1, 2], \nu \in [0, 1], \vartheta = \xi + 2\nu - \xi\nu$ and $g \in C(\mathbb{J}, \mathbb{R})$. Then the function \mathcal{A} is a solution of the following boundary value problem:

$$\begin{cases} {}^H\mathbb{D}_{c^+}^{\xi, \nu, \Phi} \mathcal{A}(t) + q(t)\mathcal{A}(t) = g(t), & t \in \mathbb{J}, \\ \mathcal{A}(c) = 0, \quad \mathcal{A}(d) = \sum_{i=1}^n \sigma_i \mathcal{A}(\rho_i), & c < \rho_i < d, \end{cases} \quad (3.1)$$

if and only if

$$\begin{aligned} \mathcal{A}(t) = & \mathbb{I}_{c^+}^{\xi, \Phi} (g(t) - q(t)\mathcal{A}(t)) + \frac{(\Phi(t) - \Phi(c))^{\vartheta-1}}{\Theta\Gamma(\vartheta)} \left[\mathbb{I}_{c^+}^{\xi, \Phi} (g(t) - q(t)\mathcal{A}(t)) \right. \\ & \left. - \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} (g(t) - q(t)\mathcal{A}(t)) (\rho_i) \right], \end{aligned} \quad (3.2)$$

where

$$\Theta = \sum_{i=1}^n \sigma_i \frac{(\Phi(\rho_i) - \Phi(c))^{\vartheta-1}}{\Gamma(\vartheta)} - \frac{(\Phi(d) - \Phi(c))^{\vartheta-1}}{\Gamma(\vartheta)} \neq 0. \quad (3.3)$$

Proof. The equation (3.1) can be expressed as

$$\mathbb{I}_{c^+}^{\vartheta-\xi, \Phi} \mathbb{D}_{c^+}^{\vartheta, \Phi} \mathcal{A}(t) + q(t)\mathcal{A}(t) = g(t), \quad (3.4)$$

where $\vartheta = \xi + 2\nu - \xi\nu$, applying $\mathbb{I}_{c^+}^{\xi, \Phi}$ to both sides of equation (3.4) and by employing Lemma 2.2 we get

$$\mathcal{A}(t) = \mathbb{I}_{c^+}^{\xi, \Phi} g(t) + \frac{e_0}{\Gamma(\vartheta)} (\Phi(t) - \Phi(c))^{\vartheta-1} + \frac{e_1}{\Gamma(\vartheta-1)} (\Phi(t) - \Phi(c))^{\vartheta-2} - \mathbb{I}_{c^+}^{\xi, \Phi} [q\mathcal{A}](t). \quad (3.5)$$

Next, from using boundary condition $\mathcal{A}(0) = 0$ in equation (3.5), we come to $e_1 = 0$. Then, we have

$$\mathcal{A}(t) = \mathbb{I}_{c^+}^{\xi, \Phi} g(t) + \frac{e_0}{\Gamma(\vartheta)} (\Phi(t) - \Phi(c))^{\vartheta-1} - \mathbb{I}_{c^+}^{\xi, \Phi} [q\mathcal{A}](t). \quad (3.6)$$

Now, we exchange t by ρ_i and multiply by σ_i in equation (3.6), we get

$$\sigma_i \mathcal{A}(\rho_i) = \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} g(\rho_i) + \frac{e_0 \sigma_i}{\Gamma(\vartheta)} (\Phi(\rho_i) - \Phi(c))^{\vartheta-1} - \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} [q\mathcal{A}](\rho_i).$$

From $\mathcal{A}(d) = \sum_{i=1}^n \sigma_i \mathcal{A}(\rho_i)$, we write easily

$$\sum_{i=1}^n \sigma_i \mathcal{A}(\rho_i) = \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} g(\rho_i) + \frac{e_0}{\Gamma(\vartheta)} \sum_{i=1}^n \sigma_i (\Phi(\rho_i) - \Phi(c))^{\vartheta-1} - \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} [q\mathcal{A}](\rho_i). \quad (3.7)$$

Also

$$\mathcal{A}(d) = \frac{e_0}{\Gamma(\vartheta)} (\Phi(d) - \Phi(c))^{\vartheta-1} + \mathbb{I}_{c^+}^{\xi, \Phi} [g(d) - q(d)\mathcal{A}(d)]. \quad (3.8)$$

With the help of boundary condition $\mathcal{A}(d) = \sum_{i=1}^n \sigma_i \mathcal{A}(\rho_i)$ and considering equations (3.7) and (3.8), we derive:

$$\begin{aligned} & \frac{e_0}{\Gamma(\vartheta)} (\Phi(d) - \Phi(c))^{\vartheta-1} + \mathbb{I}_{c^+}^{\xi, \Phi} [g(t) - q(t)\mathcal{A}(t)](d) \\ &= \frac{e_0}{\Gamma(\vartheta)} \sum_{i=1}^n \sigma_i (\Phi(\rho_i) - \Phi(c))^{\vartheta-1} + \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} g(\rho_i) - \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} [q\mathcal{A}](\rho_i) \\ &= \frac{e_0}{\Gamma(\vartheta)} \sum_{i=1}^n \sigma_i (\Phi(\rho_i) - \Phi(c))^{\vartheta-1} + \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} [g(t) - q(t)\mathcal{A}(t)](\rho_i), \end{aligned}$$

which implies that

$$e_0 = \frac{1}{\Theta} \left[\mathbb{I}_{c^+}^{\xi, \Phi} [g(t) - q(t)\mathcal{A}(t)](d) - \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} [g(t) - q(t)\mathcal{A}(t)](\rho_i) \right]. \quad (3.9)$$

Substituting the value of e_0 in (3.6), we obtain (3.2). The converse follows by direct computation. This completes the proof. \square

Let us introduce the operators $\mathbb{P}, \mathbb{Q} : C(\mathbb{J}, \mathbb{R}) \rightarrow C(\mathbb{J}, \mathbb{R})$ by

$$(\mathbb{P}\mathcal{A})(t) = \mathbb{I}_{c^+}^{\xi, \Phi} \left[\mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) - q(t)\mathcal{A}(t) \right], \quad t \in \mathbb{J} \tag{3.10}$$

and

$$\begin{aligned} (\mathbb{Q}\mathcal{A})(t) = & \frac{(\Phi(t) - \Phi(c))^{\vartheta-1}}{|\Theta|\Gamma(\vartheta)} \left[\mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) - q(t)\mathcal{A}(t) \right) \right. \\ & \left. - \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) - q(t)\mathcal{A}(t) \right) (\rho_i) \right]. \end{aligned} \tag{3.11}$$

Then

$$\mathbb{T}\mathcal{A}(t) = \mathbb{P}\mathcal{A}(t) + \mathbb{Q}\mathcal{A}(t), \quad t \in \mathbb{J}. \tag{3.12}$$

We make the following growth and Lipschitz condition on \mathbb{G} as

(C₁) There exists a constant $\mathbb{L}_G > 0$ such that

$$\begin{aligned} & \left| \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) - \mathbb{G} \left(t, \overline{\mathcal{A}} \left(\frac{t}{\eta} \right), \overline{\mathcal{A}}(t) \right) \right| \\ & \leq \mathbb{L}_G \left\{ \left| \mathcal{A} \left(\frac{t}{\eta} \right) - \overline{\mathcal{A}} \left(\frac{t}{\eta} \right) \right| + |\mathcal{A}(t) - \overline{\mathcal{A}}(t)| \right\}. \end{aligned}$$

(C₂) There exists $\mathbb{H}_G > 0, \mathbb{M}_G > 0$ and \mathbb{K}_G such that

$$\left| \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) \right| \leq \mathbb{H}_G \left| \mathcal{A} \left(\frac{t}{\eta} \right) \right| + \mathbb{M}_G |\mathcal{A}(t)| + \mathbb{K}_G.$$

To abbreviate, let us establish.

$$\Xi = \frac{(\Phi(d) - \Phi(c))^{\vartheta-1}}{|\Theta|\Gamma(\vartheta)} \left(\frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} + \sum_{i=1}^n |\sigma_i| \frac{(\Phi(\rho_i) - \Phi(c))^\xi}{\Gamma(\xi + 1)} \right). \tag{3.13}$$

Lemma 3.2. *The operator \mathbb{Q} is ω -Lipschitz with constant $(\mathbb{L}_G + |a + 2b|)\Xi$. Moreover, fulfills the following growth condition*

$$\|\mathbb{Q}\mathcal{A}\| \leq \Xi((\mathbb{H}_G + \mathbb{M}_G + |a + 2b|)\|\mathcal{A}\| + \mathbb{K}_G), \tag{3.14}$$

where Ξ is mentioned by equation (3.13).

Proof. Let $\mathcal{A}, \overline{\mathcal{A}} \in C(\mathbb{J}, \mathbb{R})$, then we have

$$\begin{aligned} & |\mathbb{Q}\mathcal{A}(t) - \mathbb{Q}\overline{\mathcal{A}}(t)| \\ & \leq \frac{(\Phi(t) - \Phi(c))^{\vartheta-1}}{|\Theta|\Gamma(\vartheta)} \left[\mathbb{I}_{c^+}^{\xi, \Phi} \left(\left| \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) - \mathbb{G} \left(t, \overline{\mathcal{A}} \left(\frac{t}{\eta} \right), \overline{\mathcal{A}}(t) \right) \right| \right) \right. \\ & \quad \left. + |q(t)| |\mathcal{A}(t) - \overline{\mathcal{A}}(t)| + \sum_{i=1}^n |\sigma_i| \mathbb{I}_{c^+}^{\xi, \Phi} \left(\left| \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) - \mathbb{G} \left(t, \overline{\mathcal{A}} \left(\frac{t}{\eta} \right), \overline{\mathcal{A}}(t) \right) \right| \right) \right. \\ & \quad \left. + |q(t)| |\mathcal{A}(t) - \overline{\mathcal{A}}(t)| (\rho_i) \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\Phi(d) - \Phi(c))^{\vartheta-1}}{|\Theta|\Gamma(\vartheta)} \left[\frac{(\Phi(d) - \Phi(c))^{\xi}}{\Gamma(\xi + 1)} (\mathbb{L}_G + |a + 2b|) + \sum_{i=1}^n |\sigma_i| \frac{(\Phi(\rho_i) - \Phi(c))^{\xi}}{\Gamma(\xi + 1)} (\mathbb{L}_G + |a + 2b|) \right] \\
&\quad \times \|\mathcal{A} - \overline{\mathcal{A}}\| \\
&\leq \frac{(\mathbb{L}_G + |a + 2b|)(\Phi(d) - \Phi(c))^{\vartheta-1}}{|\Theta|\Gamma(\vartheta)} \left[\frac{(\Phi(d) - \Phi(c))^{\xi}}{\Gamma(\xi + 1)} + \sum_{i=1}^n |\sigma_i| \frac{(\Phi(\rho_i) - \Phi(c))^{\xi}}{\Gamma(\xi + 1)} \right] \|\mathcal{A} - \overline{\mathcal{A}}\| \\
&\leq (\mathbb{L}_G + |a + 2b|)\Xi \|\mathcal{A} - \overline{\mathcal{A}}\|.
\end{aligned}$$

Taking supremum over t , we get

$$\|\mathbb{Q}\mathcal{A} - \mathbb{Q}\overline{\mathcal{A}}\| \leq (\mathbb{L}_G + |a + 2b|)\Xi \|\mathcal{A} - \overline{\mathcal{A}}\|.$$

Which implies that \mathbb{Q} is Lipschitz having constant $(\mathbb{L}_G + |a + 2b|)\Xi$ and by the Proposition 2.3, \mathbb{Q} is ω -Lipschitz with the same constant $(\mathbb{L}_G + |a + 2b|)\Xi$. Further for $\mathcal{A} \in C(\mathbb{J}, \mathbb{R})$ and by using (C_2) we get

$$\begin{aligned}
&|\mathbb{Q}\mathcal{A}(t)| \\
&\leq \frac{(\Phi(t) - \Phi(c))^{\vartheta-1}}{|\Theta|\Gamma(\vartheta)} \left[\mathbb{I}_{c^+}^{\xi, \Phi} \left(\left| \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) \right| + |q(t)| \|\mathcal{A}(t)\| \right) \right. \\
&\quad \left. + \sum_{i=1}^n |\sigma_i| \mathbb{I}_{c^+}^{\xi, \Phi} \left(\left| \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) \right| + |q(t)| \|\mathcal{A}(t)\| \right) (\rho_i) \right] \\
&\leq \frac{(\Phi(d) - \Phi(c))^{\vartheta-1}}{|\Theta|\Gamma(\vartheta)} (\mathbb{H}_G \|\mathcal{A}\| + \mathbb{M}_G \|\mathcal{A}\| + |a + 2b| \|\mathcal{A}\| + \mathbb{K}_G) \\
&\quad \times \left[\frac{(\Phi(d) - \Phi(c))^{\xi}}{\Gamma(\xi + 1)} + \sum_{i=1}^n |\sigma_i| \frac{(\Phi(\rho_i) - \Phi(c))^{\xi}}{\Gamma(\xi + 1)} \right] \\
&\leq \Xi ((\mathbb{H}_G + \mathbb{M}_G + |a + 2b|) \|\mathcal{A}\| + \mathbb{K}_G),
\end{aligned}$$

which, on taking supremum over t , yields

$$\|\mathbb{Q}\mathcal{A}\| \leq \Xi ((\mathbb{H}_G + \mathbb{M}_G + |a + 2b|) \|\mathcal{A}\| + \mathbb{K}_G).$$

□

Lemma 3.3. *The operator \mathbb{P} is continuous. In addition, \mathbb{P} fulfills the following growth condition*

$$\|\mathbb{P}\mathcal{A}\| \leq \frac{(\Phi(d) - \Phi(c))^{\xi}}{\Gamma(\xi + 1)} ((\mathbb{H}_G + \mathbb{M}_G + |a + 2b|) \|\mathcal{A}\| + \mathbb{K}_G), \quad \text{for all } \mathcal{A} \in C(\mathbb{J}, \mathbb{R}). \quad (3.15)$$

Proof. Let $\mathcal{A}_n, \mathcal{A} \in C(\mathbb{J}, \mathbb{R})$ be such that $\mathcal{A}_n \rightarrow \mathcal{A}$ ($n \rightarrow \infty$) in $C(\mathbb{J}, \mathbb{R})$. Then there exists $\lambda > 0$ such that $\|\mathcal{A}_n\| \leq \lambda$ for all $n \geq 1$, further by taking limits we get $\|\mathcal{A}\| \leq \lambda$. By employing the fact that \mathbb{G} is continuous and (C_2) , $\forall t \in \mathbb{J}$, we obtain that

$$\left| \mathbb{G} \left(t, \mathcal{A}_n \left(\frac{t}{\eta} \right), \mathcal{A}_n(t) \right) - \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) \right|$$

$$\begin{aligned} &\leq \left| \mathbf{G} \left(t, \mathcal{A}_n \left(\frac{t}{\eta} \right), \mathcal{A}_n(t) \right) \right| + \left| \mathbf{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) \right| \\ &\leq 2((\mathbf{H}_G + \mathbf{M}_G)\lambda + \mathbf{K}_G), \end{aligned}$$

and

$$\begin{aligned} |q(t)|\|\mathcal{A}_n(t) - \mathcal{A}(t)\| &\leq |q(t)|(|\mathcal{A}_n| + |\mathcal{A}(t)|) \\ &\leq 2\lambda|a + 2b|. \end{aligned}$$

The functions $s \rightarrow 2((\mathbf{H}_G + \mathbf{M}_G)\lambda + \mathbf{K}_G)$ and $s_1 \rightarrow 2\lambda|a + 2b|$ are integrable for $s, s_1 \in [0, t], t \in \mathbb{J}$, by helping Lebesgue Dominated Convergence theorem, we get

$$\begin{aligned} |\mathbb{P}\mathcal{A}_n(t) - \mathbb{P}\mathcal{A}(t)| &\leq \mathbb{I}_{c^+}^{\xi, \Phi} \left[\left| \mathbf{G} \left(t, \mathcal{A}_n \left(\frac{t}{\eta} \right), \mathcal{A}_n(t) \right) - \mathbf{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) \right| \right. \\ &\quad \left. + |q(t)|\|\mathcal{A}_n(t) - \mathcal{A}(t)\| \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence

$$\|\mathbb{P}\mathcal{A}_n(t) - \mathbb{P}\mathcal{A}(t)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Consequently, \mathbb{P} is continuous.

Also

$$\begin{aligned} |\mathbb{P}\mathcal{A}(t)| &\leq \mathbb{I}_{c^+}^{\xi, \Phi} \left[\left| \mathbf{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) \right| + |q(t)|\|\mathcal{A}(t)\| \right] \\ &\leq \mathbb{I}_{c^+}^{\xi, \Phi} [(\mathbf{H}_G + \mathbf{M}_G + |a + 2b|)\|\mathcal{A}\| + \mathbf{K}_G] \\ &\leq \frac{(\Phi(t) - \Phi(c))^\xi}{\Gamma(\xi + 1)} [(\mathbf{H}_G + \mathbf{M}_G + |a + 2b|)\|\mathcal{A}\| + \mathbf{K}_G]. \end{aligned}$$

Taking supremum over t , the above inequality becomes

$$\|\mathbb{P}\mathcal{A}\| \leq \frac{(\Phi(t) - \Phi(c))^\xi}{\Gamma(\xi + 1)} [(\mathbf{H}_G + \mathbf{M}_G + |a + 2b|)\|\mathcal{A}\| + \mathbf{K}_G].$$

□

Lemma 3.4. *The operator \mathbb{P} is compact. Consequently \mathbb{P} is ω -Lipschitz with zero constant.*

Proof. We claim that \mathbb{P} is compact, we define a bounded set Δ_1 such that $\Delta_1 \subset \mathbb{B}_p$, it is enough to prove that $\mathbb{P}(\Delta_1)$ is relatively compact in $C(\mathbb{J}, \mathbb{R})$. Let $\mathcal{A} \in \Delta_1 \subset \mathbb{B}_p$ and by using inequality (3.15), we get

$$\|\mathbb{P}\mathcal{A}\| \leq \frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} ((\mathbf{H}_G + \mathbf{M}_G + |a + 2b|)\|p\| + \mathbf{K}_G) = p. \tag{3.16}$$

It follows that $\mathbb{P}(\Delta_1) \subset \mathbb{B}_p$. So, we conclude that $\mathbb{P}(\Delta_1)$ is bounded.

For the equicontinuity of \mathbb{P} , let $t_1, t_2 \in \mathbb{J}$ with $t_1 < t_2$ and for $\mathcal{A} \in \Delta_1$, we have

$$|\mathbb{P}\mathcal{A}(t_2) - \mathbb{P}\mathcal{A}(t_1)|$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\xi)} \int_c^{t_1} \Phi'(s) |(\Phi(t_2) - \Phi(s))^{\xi-1} - (\Phi(t_1) - \Phi(s))^{\xi-1}| \\
&\quad \times \left| \mathbb{G}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) - q(s)\mathcal{A}(s) \right| ds + \frac{1}{\Gamma(\xi)} \int_{t_1}^{t_2} \Phi'(s) (\Phi(t_2) - \Phi(s))^{\xi-1} \\
&\quad \times \left| \mathbb{G}\left(s, \mathcal{A}\left(\frac{s}{\eta}\right), \mathcal{A}(s)\right) - q(s)\mathcal{A}(s) \right| ds \\
&\leq \frac{((\mathbb{H}_G + \mathbb{M}_G + |a + 2b|)\|\mathcal{A}\| + \mathbb{K}_G)}{\Gamma(\xi + 1)} (\Phi(t_2) - \Phi(t_1))^\xi \\
&\rightarrow 0.
\end{aligned}$$

As $t_2 \rightarrow t_1$, $\|\mathbb{P}\mathcal{A}(t_2) - \mathbb{P}\mathcal{A}(t_1)\| \rightarrow 0$. This demonstrates the equicontinuity and compactness of the operator \mathbb{P} . According to the Arzela-Ascoli theorem, \mathbb{P} is compact as a consequence of Proposition 2.5, \mathbb{P} is ω -Lipschitz with zero constant. \square

Theorem 3.1. Assume that $(C_1) - (C_2)$ hold. If

$$(\mathbb{L}_G + |a + 2b|)\Xi < 1. \quad (3.17)$$

Then the equation (1.1) has at least one solution.

Proof. The operators \mathbb{P} , \mathbb{Q} and \mathbb{T} are bounded and continuous. Also, \mathbb{P} is ω -Lipschitz having constant $(\mathbb{L}_G + |a + 2b|)\Xi$ and \mathbb{Q} is ω -Lipschitz having zero constant. With the aid of inequality (3.17) and both proposition's 2.4 and 2.3, we deduce that \mathbb{T} is ω -condensing.

Now, we define $\Pi_\varepsilon = \{\mathcal{A} \in C(\mathbb{J}, \mathbb{R}) : \mathcal{A} = \varepsilon\mathbb{T}\mathcal{A}, \text{ for some } \varepsilon \in [0, 1]\}$.

Let $\mathcal{A} \in \Pi_\varepsilon$, then $\mathcal{A} = \varepsilon\mathbb{T}\mathcal{A} = \varepsilon(\mathbb{P}\mathcal{A} + \mathbb{Q}\mathcal{A})$, we come to

$$\begin{aligned}
\|\mathcal{A}\| &= \varepsilon\|\mathbb{P}\mathcal{A} + \mathbb{Q}\mathcal{A}\| \\
&\leq \|\mathbb{P}\mathcal{A}\| + \|\mathbb{Q}\mathcal{A}\| \\
&\leq \frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} [(\mathbb{H}_G + \mathbb{M}_G + |a + 2b|)\|\mathcal{A}\| + \mathbb{K}_G] \\
&\quad + \Xi [(\mathbb{H}_G + \mathbb{M}_G + |a + 2b|)\|\mathcal{A}\| + \mathbb{K}_G] \\
&\leq \left(\frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} + \Xi \right) [(\mathbb{H}_G + \mathbb{M}_G + |a + 2b|)\|\mathcal{A}\| + \mathbb{K}_G].
\end{aligned}$$

Hence Π_ε is bounded in $C(\mathbb{J}, \mathbb{R})$. Suppose Π_ε is not bounded, then we assume that $\kappa := \|\mathcal{A}\| \rightarrow \infty$ and from the preceding inequality, it follows that

$$1 \leq \lim_{\kappa \rightarrow +\infty} \frac{\left(\frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} + \Xi \right) [(\mathbb{H}_G + \mathbb{M}_G + |a + 2b|)\kappa + \mathbb{K}_G]}{\kappa} = 0,$$

which is a contradiction. Since all conditions of Theorem 2.1 hold. Then the operator \mathbb{T} has at least one fixed point which represents a solution to the equation (1.1). In addition, the set of the equation (1.1) is bounded in $C(\mathbb{J}, \mathbb{R})$. \square

In the next result, dealing with the uniqueness of the solution for the equation (1.1), is analogous to Banach’s contraction principle and stated below.

Theorem 3.2. *Let (C_1) be hold and if*

$$(\mathbb{L}_G + |a + 2b|) \left[\frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} + \Xi \right] < 1, \tag{3.18}$$

then the equation (1.1) has a unique solution $\mathcal{A} \in C(\mathbb{J}, \mathbb{R})$.

Proof. Let $\mathcal{A}, \overline{\mathcal{A}} \in C(\mathbb{J}, \mathbb{R})$, then

$$\begin{aligned} & |\mathbb{T}\mathcal{A}(t) - \mathbb{T}\overline{\mathcal{A}}(t)| \\ & \leq \mathbb{I}_{c^+}^{\xi, \Phi} \left[\left| \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) - \mathbb{G} \left(t, \overline{\mathcal{A}} \left(\frac{t}{\eta} \right), \overline{\mathcal{A}}(t) \right) \right| + |q(t)| |\mathcal{A}(t) - \overline{\mathcal{A}}(t)| \right] \\ & \quad + \frac{(\Phi(d) - \Phi(c))^{\vartheta-1}}{|\Theta|\Gamma(\vartheta)} \left[\mathbb{I}_{c^+}^{\xi, \Phi} \left(\left| \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) - \mathbb{G} \left(t, \overline{\mathcal{A}} \left(\frac{t}{\eta} \right), \overline{\mathcal{A}}(t) \right) \right| \right. \right. \\ & \quad \left. \left. + |q(t)| |\mathcal{A}(t) - \overline{\mathcal{A}}(t)| \right) + \sum_{i=1}^n |\sigma_i| \mathbb{I}_{c^+}^{\xi, \Phi} \left(\left| \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) - \mathbb{G} \left(t, \overline{\mathcal{A}} \left(\frac{t}{\eta} \right), \overline{\mathcal{A}}(t) \right) \right| \right. \right. \\ & \quad \left. \left. + |q(t)| |\mathcal{A}(t) - \overline{\mathcal{A}}(t)| \right) (\rho_i) \right]. \end{aligned}$$

Taking the supremum over t , we get

$$\begin{aligned} & \|\mathbb{T}\mathcal{A} - \mathbb{T}\overline{\mathcal{A}}\| \\ & \leq \frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} (\mathbb{L}_G + |a + 2b|) \|\mathcal{A} - \overline{\mathcal{A}}\| + \frac{(\Phi(d) - \Phi(c))^{\vartheta-1}}{|\Theta|\Gamma(\vartheta)} \left[\frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} (\mathbb{L}_G + |a + 2b|) \right. \\ & \quad \left. + \sum_{i=1}^n |\sigma_i| \frac{(\Phi(\rho_i) - \Phi(c))^\xi}{\Gamma(\xi + 1)} (\mathbb{L}_G + |a + 2b|) \right] \|\mathcal{A} - \overline{\mathcal{A}}\| \\ & \leq (\mathbb{L}_G + |a + 2b|) \left[\frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} + \Xi \right] \|\mathcal{A} - \overline{\mathcal{A}}\|. \end{aligned}$$

Since $(\mathbb{L}_G + |a + 2b|) \left[\frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} + \Xi \right] < 1$, the operator \mathbb{T} is a contraction. Hence we conclude the proof by applying Banach’s contraction principle. \square

4. STABILITY THEORY

We are ready to discuss the Ulam-Hyers (U-H) stability and generalized Ulam-Hyers (g-U-H) stability for the equation (1.1) by this section.

Definition 4.1. *Equation (1.1) is said to be U-H stable if there is a constant $C_G > 0$ such that for each $\epsilon > 0$ and each solution $\mathcal{A} \in C(\mathbb{J}, \mathbb{R})$ of the inequality*

$$\left| {}^H\mathbb{D}_{c^+}^{\xi, \nu, \Phi} \mathcal{A}(t) + q(t)\mathcal{A}(t) - \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) \right| \leq \epsilon, \quad t \in \mathbb{J}, \tag{4.1}$$

there exists a solution $\mathcal{A}_1 \in C(\mathbb{J}, \mathbb{R})$ of equation (1.1) with

$$|\mathcal{A}_1(t) - \mathcal{A}(t)| \leq C_G \epsilon, \quad t \in \mathbb{J}.$$

Definition 4.2. Equation (1.1) is said to be g -U-H stable if there exists a function $Y \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $Y(0) = 0$ such that for each solution $\mathcal{A} \in C(\mathbb{J}, \mathbb{R})$ of the inequality (4.1), there exists a solution $\mathcal{A}_1 \in C(\mathbb{J}, \mathbb{R})$ of (1.1) with

$$|\mathcal{A}(t) - \mathcal{A}_1(t)| \leq Y(\epsilon), \quad t \in \mathbb{J}.$$

Remark 4.1. Definition 4.1 \implies Definition 4.2.

Remark 4.2. A function $\mathcal{A} \in C(\mathbb{J}, \mathbb{R})$ is a solution of (4.1) if and only if there exists a function $w \in C(\mathbb{J}, \mathbb{R})$ such that

- (i) $|w(t)| \leq \epsilon, \quad t \in \mathbb{J},$
- (ii) ${}^H\mathbb{D}_{c^+}^{\xi, \nu, \Phi} \mathcal{A}(t) + q(t)\mathcal{A}(t) = \mathbb{G}\left(t, \mathcal{A}\left(\frac{t}{\eta}\right), \mathcal{A}(t)\right) + w(t), \quad t \in \mathbb{J}.$

Theorem 4.1. Assume that (C_1) is fulfilled. Then, the equation (1.1) will be U-H stable as well as g -U-H stable.

Proof. Consider \mathcal{A}_1 to be a solution of the inequality (4.1) and \mathcal{A} be a solution of (1.1). Then, there exists $w \in C(\mathbb{J}, \mathbb{R})$ such that

$${}^H\mathbb{D}_{c^+}^{\xi, \nu, \Phi} \mathcal{A}_1(t) + q(t)\mathcal{A}_1(t) = \mathbb{G}\left(t, \mathcal{A}_1\left(\frac{t}{\eta}\right), \mathcal{A}_1(t)\right) + w(t), \quad t \in \mathbb{J}, \quad (4.2)$$

and $|w(t)| \leq \epsilon, t \in \mathbb{J}$. In view of Lemma 3.1, that the solution of (4.2) can be written as follows

$$\begin{aligned} \mathcal{A}_1(t) &= \frac{(\Phi(t) - \Phi(c))^{\vartheta-1}}{\Theta\Gamma(\vartheta)} \left[\mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G}\left(t, \mathcal{A}_1\left(\frac{t}{\eta}\right), \mathcal{A}_1(t)\right) - q(t)\mathcal{A}_1(t) \right) \right. \\ &\quad \left. - \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G}\left(t, \mathcal{A}_1\left(\frac{t}{\eta}\right), \mathcal{A}_1(t)\right) - q(t)\mathcal{A}_1(t) \right) (\rho_i) \right] \\ &\quad + \mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G}\left(t, \mathcal{A}_1\left(\frac{t}{\eta}\right), \mathcal{A}_1(t)\right) - q(t)\mathcal{A}_1(t) \right) + \mathbb{I}_{c^+}^{\xi, \Phi} w(t). \end{aligned} \quad (4.3)$$

If we take

$$\begin{aligned} \mathfrak{B}_{\mathcal{A}_1} &= \mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G}\left(t, \mathcal{A}_1\left(\frac{t}{\eta}\right), \mathcal{A}_1(t)\right) - q(t)\mathcal{A}_1(t) \right) \\ &\quad - \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G}\left(t, \mathcal{A}_1\left(\frac{t}{\eta}\right), \mathcal{A}_1(t)\right) - q(t)\mathcal{A}_1(t) \right) (\rho_i), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{B}_{\mathcal{A}} &= \mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G}\left(t, \mathcal{A}\left(\frac{t}{\eta}\right), \mathcal{A}(t)\right) - q(t)\mathcal{A}(t) \right) \\ &\quad - \sum_{i=1}^n \sigma_i \mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G}\left(t, \mathcal{A}\left(\frac{t}{\eta}\right), \mathcal{A}(t)\right) - q(t)\mathcal{A}(t) \right) (\rho_i). \end{aligned}$$

Which implies that

$$\begin{aligned}
 & |\mathcal{A}_1(t) - \mathcal{A}(t)| \\
 &= \left| \frac{(\Phi(t) - \Phi(c))^{\vartheta-1}}{\Theta\Gamma(\vartheta)} \mathfrak{B}_{\mathcal{A}_1} + \mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G} \left(t, \mathcal{A}_1 \left(\frac{t}{\eta} \right), \mathcal{A}_1(t) \right) - q(t)\mathcal{A}_1(t) \right) + \mathbb{I}_{c^+}^{\xi, \Phi} w(t) \right. \\
 &\quad \left. - \frac{(\Phi(t) - \Phi(c))^{\vartheta-1}}{\Theta\Gamma(\vartheta)} \mathfrak{B}_{\mathcal{A}} - \mathbb{I}_{c^+}^{\xi, \Phi} \left(\mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) - q(t)\mathcal{A}(t) \right) \right| \\
 &\leq \frac{(\Phi(t) - \Phi(c))^{\vartheta-1}}{\Theta\Gamma(\vartheta)} |\mathfrak{B}_{\mathcal{A}_1} - \mathfrak{B}_{\mathcal{A}}| + \mathbb{I}_{c^+}^{\xi, \Phi} \left[\left| \mathbb{G} \left(t, \mathcal{A}_1 \left(\frac{t}{\eta} \right), \mathcal{A}_1(t) \right) - \mathbb{G} \left(t, \mathcal{A} \left(\frac{t}{\eta} \right), \mathcal{A}(t) \right) \right| \right. \\
 &\quad \left. + |q(t)| |\mathcal{A}_1(t) - \mathcal{A}(t)| \right] + \mathbb{I}_{c^+}^{\xi, \Phi} |w(t)|,
 \end{aligned}$$

since $\mathcal{A}_1(b) = \mathcal{A}(b) = 0$, we can see that $|\mathfrak{B}_{\mathcal{A}_1} - \mathfrak{B}_{\mathcal{A}}| = 0$. Then, by (C₁) and remark 4.2, we obtain

$$\begin{aligned}
 |\mathcal{A}_1(t) - \mathcal{A}(t)| &\leq \frac{(\mathbb{L}_G + |a + 2b|)}{\Gamma(\xi)} \int_c^t \Phi'(s) (\Phi(t) - \Phi(s))^{\xi-1} |\mathcal{A}_1(s) - \mathcal{A}(s)| ds \\
 &\quad + \frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)},
 \end{aligned}$$

since $\frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)}$ is a constant on \mathbb{J} , the above inequality is converted into

$$|\mathcal{A}_1(t) - \mathcal{A}(t)| \leq \frac{\epsilon(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} \mathbb{E}_\xi \left((\mathbb{L}_G + |a + 2b|) (\Phi(d) - \Phi(c))^\xi \right),$$

by considering

$$\mathbb{C}_G := \frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} \mathbb{E}_\xi \left((\mathbb{L}_G + |a + 2b|) (\Phi(d) - \Phi(c))^\xi \right),$$

so, we get

$$|\mathcal{A}_1(t) - \mathcal{A}(t)| \leq \mathbb{C}_G \epsilon.$$

It follows from Definition 4.1, clearly the equation (1.1) is U-H stable. □

Remark 4.3. choosing $Y(\epsilon) = \mathbb{C}_G(\epsilon)$; $Y(0) = 0$, we can state that the equation (1.1) is g-U-H stable.

5. AN ILLUSTRATIVE EXAMPLE

Consider the following Ambartsumian-type Φ -Hilfer Mathieu fractional differential equation with boundary condition:

$$\begin{cases}
 {}^H\mathbb{D}_{0^+}^{\frac{7}{4}, \frac{2}{3}, \frac{t}{6}} \mathcal{A}(t) + q(t)\mathcal{A}(t) = \frac{1}{7}\mathcal{A}\left(\frac{t}{7}\right) - \mathcal{A}(t), & t \in [0, \pi] \\
 \mathcal{A}(0) = 0, \quad \mathcal{A}(\pi) = \frac{21}{4}\mathcal{A}\left(\frac{2\pi}{13}\right) + \frac{19}{5}\mathcal{A}\left(\frac{3\pi}{7}\right),
 \end{cases} \tag{5.1}$$

where

$$q(t) = \frac{115}{83} - \frac{2\sqrt{329}}{15} \cos(2t), \quad t \in (0, \pi].$$

Clearly, $\xi = \frac{7}{4} \in (1, 2]$, $\nu = \frac{2}{5} \in [0, 1]$, $\Phi(t) = \frac{e^t}{6}$, $\sigma_1 = \frac{21}{4}$, $\sigma_2 = \frac{19}{5}$, $\rho_1 = \frac{2\pi}{13}$, $\rho_2 = \frac{3\pi}{7}$ and $a = \frac{115}{83}$, $b = \frac{\sqrt{329}}{15}$.

We define the map $\mathbb{G} : [0, \pi] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\mathbb{G}\left(t, \mathcal{A}\left(\frac{t}{\eta}\right), \mathcal{A}(t)\right) = \frac{1}{7}\mathcal{A}\left(\frac{t}{\eta}\right) - \mathcal{A}(t),$$

where $\eta = 7$. For $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{R}$, we have

$$\begin{aligned} & \left| \mathbb{G}\left(t, \mathcal{A}_1\left(\frac{t}{\eta}\right), \mathcal{A}_1(t)\right) - \mathbb{G}\left(t, \mathcal{A}_2\left(\frac{t}{\eta}\right), \mathcal{A}_2(t)\right) \right| \\ & \leq \frac{1}{7} \left\{ \left| \mathcal{A}_1\left(\frac{t}{\eta}\right) - \mathcal{A}_2\left(\frac{t}{\eta}\right) \right| + |\mathcal{A}_1(t) - \mathcal{A}_2(t)| \right\}. \end{aligned}$$

Thus, condition (C_1) holds with $\mathbb{L}_{\mathbb{G}} = \frac{1}{7} > 0$. In addition, for every $t \in [0, \pi]$ and $\mathcal{A} \in \mathbb{R}$, we have

$$\left| \mathbb{G}\left(t, \mathcal{A}\left(\frac{t}{\eta}\right), \mathcal{A}(t)\right) \right| \leq \frac{1}{7} \left[\left| \mathcal{A}\left(\frac{t}{\eta}\right) \right| + |\mathcal{A}(t)| + 1 \right].$$

Hence, condition (C_2) with $\mathbb{H}_{\mathbb{G}} = \mathbb{M}_{\mathbb{G}} = \mathbb{K}_{\mathbb{G}} = \frac{1}{7}$.

To deal with the uniqueness we use the data given above, we get

$$(\mathbb{L}_{\mathbb{G}} + |a + 2b|) \left[\frac{(\Phi(d) - \Phi(c))^\xi}{\Gamma(\xi + 1)} + \Xi \right] \approx 0.79 < 1.$$

Accordingly, by Theorem 3.2, the equation (5.1) has a unique solution in $C([0, \pi], \mathbb{R})$. Further, it follows from Theorem 4.1 that the equation (5.1) is U-H stable and consequently it is also g-U-H stable.

6. CONCLUSION

Using topological degree theory for condensing maps and Banach's contraction principle, we established the existence and uniqueness of solutions for an Ambartsumian-type Φ -Hilfer Mathieu fractional differential equation with boundary conditions. Moreover, the Ulam–Hyers and generalized Ulam–Hyers stability of the considered problem were investigated. To the best of our knowledge, this is the first work addressing an Ambartsumian-type Mathieu fractional differential equation with boundary conditions. Consequently, the obtained results extend and generalize existing results in the literature and contribute to the qualitative analysis of Φ -Hilfer fractional differential equations.

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