Int. J. Anal. Appl. (2026), 24:120

International Journal of Analysis and Applications

Solutions of a Class of Operator Equation in Continuous Function Spaces and Several Applications

Chengbo Zhai[1,2,*], **Yong Zhou**[1]

[1]*School of Mathematics and Statistics, Shanxi University, Taiyuan 030006, Shanxi, P.R. China*
[2]*Key Laboratory of Complex Systems and Data Science of Ministry of Education, Shanxi University, Taiyuan 030006, Shanxi, China*

[*]*Corresponding author:* cbzhai@sxu.edu.cn

Abstract. This paper investigates an operator equation of the form $x = (Ax^s)^{1/s} + x_0$, where A is a linear operator defined in continuous function spaces, $s > 1$ is a real number and x_0 is a positive continuous function. We transform it into a fixed point problem and establish the existence and uniqueness of solutions by using some results for increasing concave operators in the interior of cones. Moreover, we give several applications in Caputo fractional differential problems, recursive preferences and wealth-consumption ratio.

1. Introduction

In this paper, we consider an operator equation of the following form

$$x = (Ax^s)^{1/s} + x_0, \tag{1.1}$$

where A is a linear operator, x_0 represents a positive continuous function and s is a nonzero real value. In recent years, the Equation 1.1 has emerged in a wide range of economic and financial applications, including but not limited to recursive utility models, state-dependent discounting in an optimal savings problem used in macroeconomic modeling, wealth-consumption ratio, see reference [1] and the references therein. Author et al. [1] investigated Equation 1.1 and gave some conditions to guarantee the existence and uniqueness of positive solutions under the important condition $r(A)^{\frac{1}{s}} < 1$, where $r(A)$ is the spectral radius of operator A. They transformed the problem into a fixed point problem of operator G by

$$Gy = \left((Ay)^{1/s} + x_0\right)^s.$$

Received: Feb. 5, 2026.

2020 *Mathematics Subject Classification.* 47H10; 26A33; 34A08; 34B15; 34B18; 47N10.

Key words and phrases. existence and uniqueness; concave operator; fixed point; positive solution; economics.

https://doi.org/10.28924/2291-8639-24-2026-120
ISSN: 2291-8639

By combining the partial order theory in Banach lattices with positive cones, they obtained the existence and uniqueness of positive solutions with the operator G is concave or convex, respectively. In literature, the existence and uniqueness of solutions to Equation 1.1 has been seldom seen. However, if we write Equation 1.1 as $x = Bx + b$ (where B is allowed to be nonlinear), then we get a generalization of operator equations discussed by [2], [3], [4], [5]. In these results, B is typically assumed to be α -concave (an operator B is α-concave if for $t \in (0,1)$, there is $\alpha \in (0,1)$ such that $B(tx) \geq t^{\alpha}Bx$), but it is evident that such a condition cannot hold under our hypotheses.

In this paper, we continue to investigate Equation 1.1 and obtain the existence and uniqueness of positive solutions under different conditions. Unlike prior works, we do not consider the spectral radius of mapping A. Instead, the solution problem of Equation 1.1 is transformed into a fixed-point problem for the operator F by

$$Fx = (Ax^s)^{1/s} + x_0.$$

We also discuss the operator G and obtain its properties. Based upon these, we give some new results for the operator F. In the rest of this paper, Section 2 gives some necessary definitions to support subsequent theoretical analysis, Section 3 presents the main theorems characterizing the existence and uniqueness of solutions to the Equation 1.1. In Section 4, we present several concrete applications to demonstrate the applicability and validity of our main results, which include ones in Caputo fractional differential problems, recursive preferences and wealth-consumption ratio.

2. Preliminaries

In this section, we list some concepts and known results which can be seen in [6], [7], [8], [9].

Let $(E, \|\cdot\|)$ be a real Banach space, $P \subset E$ is a cone. The space E is partly ordered by P, that is, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, we write $x < y$. If $\mathring{P} = \{x \in P \mid x$ is an interior point of $P\}$ is non-empty, then the cone P is called be solid. If $y - x \in \mathring{P}$, we write $x \ll y$ or $y \gg x$. Moreover, if there exists a constant $N > 0$ such that for $x, y \in E$ with $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$, then P is said to be a normal cone.

Definition 2.1. *Let D be a convex set in E, and the operator $A : D \to E$.*
(i) If $A(tx + (1-t)y) \geq tAx + (1-t)Ay$ for $x, y \in D,\ x \leq y,\ 0 \leq t \leq 1$. Then A is said to be a concave operator *on D;*
(ii) If $A(tx + (1-t)y) \leq tAx + (1-t)Ay$ for $x, y \in D,\ x \leq y,\ 0 \leq t \leq 1$. Then A is said to be a convex operator *on D.*

Lemma 2.1. *([6], [7]) Let P be a normal solid cone and $T : P \to P$ be a concave operator satisfying $T\theta \gg \theta$. Then there exists $0 < \lambda^* \leq \infty$ such that when $0 \leq \lambda < \lambda^*$, the equation*

$$u = \lambda Tu \tag{2.1}$$

has a unique solution $u(\lambda)$ *in* P*; moreover,* $u(0) = \theta$ *and* $u(\lambda) \in \mathring{P}$ *for all* $0 < \lambda < \lambda^*$*, where* $\lambda^* = \sup\{\lambda \geq 0 \mid \exists u \in P \text{ such that } u = \lambda T u\}$.

Lemma 2.2. *([6], [7]) Let* P *be a normal cone,* $v > \theta$ *and the operator* $A : [\theta, v] \to E$ *is a concave increasing operator. If there exists a constant* ε *with* $0 < \varepsilon < 1$ *such that* $A\theta \geqslant \varepsilon v, Av \leqslant v$*, then* A *has a minimal fixed point* u^* *in* $[\theta, v]$ *with* $u^* > \theta$.

3. Main results

In this part, we discuss the correlation properties of the solutions for the operator Equation 1.1 in the space $E = C[a, b]$ $(a < b < +\infty)$. Apparently, E is a Banach space with the norm $\|f\| = \max_{a \leq t \leq b} |f(t)|$, where $f \in C[a, b]$. The standard cone P is

$$P = \{f \in E : f(t) \geq 0,\ t \in [a, b]\},$$

and then P is a normal cone, the interior of P is

$$\mathring{P} = \{f \in E : f(t) > 0,\ t \in [a, b]\}.$$

Firstly, we define two maps F, G on E by

$$Fx = (Ax^s)^{1/s} + x_0, \tag{3.1}$$

$$Gx = \left((Ax)^{1/s} + x_0\right)^s, \tag{3.2}$$

where $x, x_0 \in E, A : E \to E$ is a linear operator. Thus, we solve the Equation 1.1 by converting it into a fixed point problem associated with the map F.

It is noting that we suppose H to be the homeomorphism from P to itself defined by $Hx = x^s$, then F and G are topologically conjugate under H, in the sense that $H \circ F = G \circ H$. As a result, F has a unique fixed point in P if and only if the same is true for G, and fixed points of F and G have the same stability properties.

Lemma 3.1. *If* x *is the fixed point of* G*, then* $u = x^{\frac{1}{s}}$ *(*$s > 1$ *) is the fixed point of* F.

Proof. Let x be a fixed point of G ,that is $Gx = x$. From Equation 3.2, we have

$$\left[(Ax)^{\frac{1}{s}} + x_0\right]^s = x.$$

Taking the $\frac{1}{s}$-th power of both sides of the above equation, we can obtain

$$(Ax)^{\frac{1}{s}} + x_0 = x^{\frac{1}{s}}.$$

Let $u = x^{\frac{1}{s}}$, then $x = u^s$. Substituting $x = u^s$ into the above equation, we get

$$(Au^s)^{\frac{1}{s}} + x_0 = u.$$

This implies that u is a fixed point of F. The proof is completed.

Lemma 3.2. *Suppose that $A : P \to P$ is a continuous and positive linear operator, $x_0 \in \mathring{P}$. Then, $G : P \to P$ is a concave operator when $s > 1$.*

Proof. Let $t \in [a, b]$ and we define

$$\psi_t(r) = \left(r^{\frac{1}{s}} + x_0\right)^s, \quad r \geq 0.$$

Then, for any $y \in P$, $(Gy)(t)$ can be expressed as $\psi_t[(Ay)(t)]$. Evidently, ψ_t is an increasing function for all $t \in [a, b]$. This follows from the fact that

$$\psi_t'(r) = \left(r^{\frac{1}{s}} + x_0(t)\right)^{s-1} r^{\frac{1}{s}-1}.$$

Combining with $x_0(t) > 0, s > 1$ and $r \geq 0$, we know $\psi_t'(r) \geq 0$. Then, the second derivative of $\psi_t(r)$ is calculated as follows

$$\begin{aligned}\psi_t''(r) &= \left(r^{\frac{1}{s}} + x_0(t)\right)^{s-2} r^{\frac{1}{s}-2}\left[\frac{s-1}{s} r^{\frac{1}{s}} + \frac{1-s}{s}\left(r^{\frac{1}{s}} + x_0(t)\right)\right] \\ &= \left(r^{\frac{1}{s}} + x_0(t)\right)^{s-2} r^{\frac{1}{s}-2}\left[\frac{1-s}{s} x_0(t)\right],\end{aligned}$$

so $\psi_t''(r) \leq 0$ when $s < 0$ or $s \geq 1$, that is to say, ψ_t is concave on $t \in [a, b]$.

Next we fix $y, z \in P$ and $\lambda \in [0, 1]$ and let $h := \lambda y + (1-\lambda)z$. Hence, fixing $t \in C[a, b]$, let $s \geq 1$ and combining the definition of the concave operator and the linearity of operator A, we will get

$$\begin{aligned}\varphi_t[(Ah)(t)] &= \varphi_t[\lambda(Ay)(t) + (1-\lambda)(Az)(t)] \\ &\geq \lambda\varphi_t[(Ay)(t)] + (1-\lambda)\varphi_t[(Az)(t)],\end{aligned}$$

accordingly, $(Gh)(t) \geq \lambda(Gy)(t) + (1-\lambda)(Gz)(t)$ is valid. So G is a concave operator on P when $s > 1$.

Theorem 3.1. *Suppose that $A : P \to P$ is a continuous and positive linear operator, $x_0 \in \mathring{P}$ and $s > 1$. Then, there exists $0 < \lambda^* \leq \infty$ such that when $0 \leq \lambda < \lambda^*$, the equation*

$$u = \lambda G u,$$

has a unique solution $u(\lambda)$ in P; moreover, $u(0) = \theta$ and $u(\lambda) \in \mathring{P}$ for all $0 < \lambda < \lambda^*$.

Poof. From Lemma 3.2, $G : P \to P$ is a concave operator when $s > 1$, and combining with Equation 3.2, we have

$$G\theta = ((A\theta)^{\frac{1}{s}} + x_0)^s = x_0^s \gg \theta.$$

Therefore, it follows immediately from Lemma 2.1 that the conclusion is valid.

Corollary 3.1. *If $u(\lambda)$ is a solution of the equation $u = \lambda G u$ in P, then $x(\lambda) = (u(\lambda))^{\frac{1}{s}}$ is a solution of the following Equation*

$$x = \lambda^{\frac{1}{s}} F x. \tag{3.3}$$

Proof. Let $u(\lambda)$ be a solution of $u = \lambda G u$. Then, according to Equation 3.2, we can obtain

$$u^{\frac{1}{s}} = \lambda^{\frac{1}{s}}[(Au)^{\frac{1}{s}} + x_0].$$

Let $x = u^{\frac{1}{s}}$, and combining with Equation 3.1, then

$$x = \lambda^{\frac{1}{s}}\left[(Ax^s)^{\frac{1}{s}} + x_0\right] = \lambda^{\frac{1}{s}} F x.$$

which means that Equation 3.3 has a unique solution $x(\lambda)$ in P.

Lemma 3.3. *Suppose that*
(i) A is a continuous and positive linear operator, $s > 1$ and $x_0 \in \mathring{P}$;
(ii) there exists $v_0 \in P$ such that $Av_0 \leq ((v_0)^{\frac{1}{s}} - x_0)^s$.
Then G has a unique fixed point $\bar{x}$ in $\mathring{P}$.

Proof. From Lemma 3.2, we know that $G : P \to P$ is a concave operator, so G is an increasing operator on P. According to (ii),

$$\left[(Av_0)^{\frac{1}{s}} + x_0\right]^s \leq v_0,$$

that is, $Gv_0 \leq v_0$. So $G : [\theta, v_0] \to P$ is a concave operator. Furthermore, we have $v_0 \geq Gv_0 \geq G\theta \gg \theta$, and we can take $0 < \varepsilon < 1$ sufficiently small such that $G\theta \geq \varepsilon v_0$. For any given $0 < \lambda \leq 1$, we can obtain

$$\lambda G\theta \geq \lambda \varepsilon v_0, \quad \lambda G v_0 \leq G v_0 \leq v_0.$$

By Lemma 2.2, the equation $u = \lambda G u$ has a minimal solution $u(\lambda)$ in $[\theta, v_0]$, and $u(\lambda) > \theta$. From Lemma 2.1, we get $\lambda^* \geq 1$. Therefore, G has a unique solution in P, that means, $G\bar{x} = \bar{x}$. It is worth noting that, given $x_0 \in \mathring{P}$ and $s > 1$, we have $Gx \gg \theta$ for all $x \in P$. As a direct consequence, the operator G admits a unique fixed point $\bar{x}$ in $\mathring{P}$.

Theorem 3.2. *Suppose that*
(i) A is a continuous and positive linear operator, $s > 1$ and $x_0 \in \mathring{P}$;
(ii) there exists $v_0 \in P$ such that $Av_0 \leq ((v_0)^{\frac{1}{s}} - x_0)^s$.
Then F has a unique fixed point $\bar{x}$ in $\mathring{P}$.

Proof. It is obvious from Lemma 3.1 that the conclusion holds.

Remark 3.1. *Evidently, the conditions in our main results is not involved the spectral radius of the linear operator A.*

4. Several applications

In this section, we present several applications to illustrate our main results.

4.1. *Caputo* fractional differential equations

Fractional differential equations, particularly those defined using the Caputo derivative, have become key mathematical models for describing complex dynamic systems with memory and hereditary properties, owing to the fact that their derivative definition inherently incorporates classical initial value conditions from integer-order calculus. Based on the theory of fixed-point theorems in functional analysis, systematic research has been established on the existence and uniqueness of solutions to such equations. Current research has expanded from standard forms to complex cases involving time delay, impulses, and nonlocal boundary conditions, while theoretical analyses of solution positivity, stability, for example [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21] [22], [23], [24], [25], [26], [27], [28]. Here, we investigate a different problem of *Caputo* fractional differential Problem:

$$\begin{cases} -{}^{C}D_{a}^{\alpha}\left((u(t)-u_0(t))^s\right) = u^s(t), \;\; a < t < b, \\ u'(a) = u_0'(a), \;\; \beta\, {}^{C}D_{a}^{\alpha-1}\left((u(t)-u_0(t))^s\right)\Big|_{t=b} = u_0(\eta) - u(\eta), \end{cases} \tag{4.1}$$

where ${}^{C}D_{a}^{\alpha}$ denotes the α-order Caputo fractional derivative, $1 < \alpha \leq 2$, $\beta > 0$ and $a \leq \eta \leq b$, $s > 1$. Here, we use Theorem 3.2 to investigate the existence and uniqueness of positive solutions for the Problem 4.1.

In [29], the authors concerned with the problem of finding some Lyapunov-type inequalities for the following fractional Problem:

$$\begin{cases} -{}^{C}D_{a}^{\alpha}u(t) = y(t), \;\; a < t < b, \\ u'(a) = 0, \; \beta\, {}^{C}D_{a}^{\alpha-1}u(b) + u(\eta) = 0, \end{cases} \tag{4.2}$$

where ${}^{C}D_{a}^{\alpha}$ denotes the Caputo fractional derivative of order α, $1 < \alpha \leq 2$, $\beta > 0$ and $a \leq \eta \leq b$. By constructing the Green's function and systematically analyzing its monotonicity and extremal properties, they established a Lyapunov-type inequality for this problem and proved the existence of solutions for the Problem 4.2.

Definition 4.1. *([30]) For $f : [a,b] \to (-\infty,+\infty)$, the Riemann-liouville fractional integral $I_a^{\rho} f$ of order $\rho > 0$ is defined by*

$$I_a^{\rho} f\,(x) = \frac{1}{\Gamma(\rho)} \int_a^x \frac{f(t)}{(x-t)^{1-\rho}} dt$$

where $\Gamma(\rho)$ is the Gamma function.

Definition 4.2. *([30]) For $f : [a,b] \to (-\infty,+\infty)$ and $n-1 < \alpha < n (n \in \mathbb{N})$, the Caputo derivative of order α is defined as*

$${}^{C}D_{a}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $f^{(n)}(s)$ is the n - th derivative of $f(s)$.

Lemma 4.1. *([29]) Suppose $y \in C[a,b]$. A function $u \in C[a,b]$ is a solution of the Problem 4.2 if and only if it satisfies the integral equation*

$$u(t) = \int_a^b G(t,s) y(s)\, ds,$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \beta + H_{\eta}(s) - H_t(s),$$

where for $r \in [a,b]$, $H_r : [a,b] \to \mathbb{R}$ is the function defined as

$$H_r(s) = \begin{cases} \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{for } a \le s \le r \le b, \\ 0, & \text{for } a \le r < s \le b. \end{cases}$$

Lemma 4.2. *([29]) Let $\beta\Gamma(\alpha) \ge (b-\eta)^{\alpha-1}$. Then the Green's function in Lemma 4.1 satisfies the following properties:*
(i) $G(t,s) \ge 0, t,s \in [a,b]$;
(ii) $\max\{G(t,s) : t,s \in [a,b]\} = \beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}$;
(iii) $\min\{G(t,s) : t,s \in [a,b]\} = \beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}$.

Theorem 4.1. *Let $\beta\Gamma(\alpha) \ge (b-\eta)^{\alpha-1}$. Suppose $u_0 \in \mathring{P}$ and there exists $v_0 \in P$ such that $\int_a^b G(t,\tau)v_0(\tau)\,d\tau \le [(v_0(t))^{\frac{1}{s}} - u_0(t)]^s, t \in [a,b]$. Then the Problem 4.1 has a unique solution $u \in \mathring{P}$, and it satisfies the integral equation*

$$u(t) = \left(\int_a^b G(t,\tau)\, u^s(\tau)\, d\tau\right)^{\frac{1}{s}} + u_0(t),$$

where $G(t,s)$ is the Green's function given by Lemma 4.1.

Proof. Let $w(t) = u(t) - u_0(t)$ (where u is a solution of the Problem 4.1), then Problem 4.1 can be transformed into the following form

$$\begin{cases} -{}^C D_a^{\alpha}(w(t))^s = u^s(t), \ \ a < t < b, \\ w'(a) = 0, \ \ \beta\, {}^C D_a^{\alpha-1}(w(t))^s\big|_{t=b} + w(\eta) = 0. \end{cases}$$

According to Lemma 4.1, we can get

$$(w(t))^s = (u(t) - u_0(t))^s = \int_a^b G(t,\tau)\, u^s(\tau) d\tau.$$

Taking the $\frac{1}{s}$-th power of both sides for above equation and rearranging terms, we can obtain

$$u(t) = \left(\int_a^b G(t,\tau)\, u^s(\tau) d\tau\right)^{\frac{1}{s}} + u_0(t),$$

where $G(t,s)$ is the Green's function given by Lemma 4.1. Next, we define an operator A in the following form

$$Au(t) = \int_a^b G(t,\tau)\, u(\tau)\, d\tau, \ u \in P.$$

According to Lemma 4.2, $G(t,s) \ge 0$. It is evident that $A : P \to P$ is a continuous and positive linear operator. Meanwhile, we can obtain

$$Av_0(t) = \int_a^b G(t,\tau)\, v_0(\tau)\, d\tau \le [(v_0(t))^{\frac{1}{s}} - u_0(t)]^s.$$

Therefore, the Problem 4.1 have a unique solution in $\mathring{P}$ by Theorem 3.2.

Example 4.1. *We consider a different fractional Problem in* $C[1, \frac{3}{2}]$:

$$\begin{cases} -{}^{C}D^{\frac{3}{2}}\left((u(t)-2)^2\right) = u^2(t), \quad 1 < t < \frac{3}{2}, \\ u'(1) = 0, \ {}^{C}D_1^{\frac{1}{2}}\,(u(t)-2)^2\,|_{t=\frac{3}{2}} = 2 - u(\frac{5}{4}), \end{cases} \tag{4.3}$$

where $u_0(t) = 2$, $\alpha = \frac{3}{2}$, $s = 2$, $\eta = \frac{5}{4}$, $\beta = 1$. *Thus, we have*

$$u(1) = u_0'(1) = 0,$$

$$\beta\Gamma(\alpha) = \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2},$$

$$\beta(b-\eta)^{\alpha-1} = \left(\frac{3}{2} - \frac{5}{4}\right)^{\frac{3}{2}-1} = \frac{1}{2}.$$

Hence, $\beta\Gamma(\alpha) > (b-\eta)^{\alpha-1}$. *Further, we can obtain*

$$\max\left\{G(t,s) : t, s \in [1, \frac{3}{2}]\right\} = \beta + \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)} = 1 + \frac{\left(\frac{1}{4}\right)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} = 1 + \frac{1}{\sqrt{\pi}}.$$

Taking $v_0(t) = ct$, *where* $t \in [1, \frac{3}{2}]$ *and* $c > 2$ *is a positive constant satisfying*

$$c^{\frac{1}{2}}\left[1 - \sqrt{\frac{5}{8}\left(1 + \frac{1}{\sqrt{\pi}}\right)}\right] \geq 2. \tag{4.4}$$

Then, we can get

$$\int_1^{\frac{3}{2}} G(t,\tau)v_0(\tau)\,d\tau = c\int_1^{\frac{3}{2}} G(t,\tau)\tau\,d\tau \leq c\left(1 + \frac{1}{\sqrt{\pi}}\right)\int_1^{\frac{3}{2}} \tau\,d\tau = \frac{5}{8}c\left(1 + \frac{1}{\sqrt{\pi}}\right).$$

Based on Equation 4.4, we can conclude that

$$c^{\frac{1}{2}} - 2 \geq c^{\frac{1}{2}}\sqrt{\frac{5}{8}\left(1 + \frac{1}{\sqrt{\pi}}\right)}.$$

So

$$\left[(v_0(t))^{\frac{1}{s}} - u_0(t)\right]^s = (c^{\frac{1}{2}}t^{\frac{1}{2}} - 2)^2 > (c^{\frac{1}{2}} - 2)^2 \geq \frac{5}{8}c\left(1 + \frac{1}{\sqrt{\pi}}\right).$$

The above derivation demonstrates that

$$\int_1^{\frac{3}{2}} G(t,\tau)v_0(\tau)\,d\tau \leq \left[(v_0(t))^{\frac{1}{s}} - u_0(t)\right]^s.$$

Therefore, it follows from Theorem 4.1 that the Problem 4.3 has a unique positive solution in P and satisfies the following integral equation

$$u(t) = \left(\int_1^{\frac{3}{2}} G(t,\tau)u^2(\tau)d\tau\right)^{\frac{1}{2}} + 2,$$

where $G(t,s)$ *is the Green's function given by Lemma 4.1.*

4.2. Recursive preferences

Epstein - Zin preferences play an important role in studying mainstream asset pricing models in finance (for example [31], [32], [33]). In [34], a universal specification is adopted in macroeconomics

$$v = ((1-\beta)c^{\rho} + \beta(\mathcal{R}v)^{\rho})^{1/\rho}\,, \tag{4.5}$$

where $c \in \mathring{P} \subset C(T)$ stands for consumption in every state, $\rho, \beta > 0$ are parameters. The operator $\mathcal{R}$ acts on $v \in \mathring{P}$ as $\mathcal{R}v = (Qv^{\alpha})^{1/\alpha}$, where α is a nonzero parameter and Q is an irreducible Markov operator. The unknown function v represents the lifetime utility in each state of the world. The authors took $w = v^{\alpha}$, $s = \alpha/\rho$, then the above formula is transformed into

$$w = \left\{(1-\beta)c^{\rho} + \beta(Qw)^{\rho/\alpha}\right\}^{\alpha/\rho} = \left\{(1-\beta)c^{\rho} + (\beta^{s}Qw)^{1/s}\right\}^{s},$$

They have already established that Equation 4.5 has a solution when $\beta < 1$. Meanwhile, by defining the linear operator $A := \beta^{s}Q$ in [1], the authors proved the existence and uniqueness of a positive solution to Equation 4.5 under the condition that the spectral radius of A satisfies $r(A)^{1/s} < 1$.

In this paper, if there exists $v_0 \in P$ such that

$$(\beta^{s}Qv_0)^{\frac{1}{s}} \leq v_0^{\frac{1}{s}} - (1-\beta)c^{\rho},$$

which means $Av_0 \leq \left(v_0^{\frac{1}{s}} - (1-\beta)c^{\rho}\right)^{s}$. By Theorem 3.2, the Equation 4.5 has a unique positive solution in P. Here, we do not involve the spectral radius of the operator A.

4.3. Wealth-consumption ratio

In asset pricing, the wealth-consumption ratio of the representative agent is an important object. In the above 4.2, let w, the equilibrium wealth-consumption ratio under Epstein-Zin preferences, be satisfied with

$$\beta^{s}Qw^{s} = (w-1)^{s}, \tag{4.6}$$

where Q is an irreducible and power compact linear operator. By defining $A := \beta^{s}Q$, we can rephrase Equation 4.6 as

$$w = (Aw^{s})^{1/s} + 1,$$

which is the same form of the Equation 1.1. In [1], the authors demonstrate that a unique strictly positive wealth-consumption ratio exists if and only if the linear operator A satisfies $r(A)^{1/s} < 1$. Here, if there exists $v_0 \in P$ satisfying

$$(\beta^{s}Qv_0)^{\frac{1}{s}} \leq v_0^{\frac{1}{s}} - 1,$$

that is $Av_0 \leq \left(v_0^{\frac{1}{s}} - 1\right)^{s}$, then the Equation 4.6 has a unique positive solution in P. Also, we do not consider the spectral radius of the operator A.

Funding: This paper is supported by Fundamental Research Program of Shanxi Province (202303021221068).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J. Stachurski, O. Wilms, J. Zhang, Unique Solutions to Power-Transformed Affine Systems, J. Math. Anal. Appl. 550 (2025), 129515. https://doi.org/10.1016/j.jmaa.2025.129515.

[2] M. Berzig, B. Samet, Positive Fixed Points for a Class of Nonlinear Operators and Applications, Positivity 17 (2012), 235–255. https://doi.org/10.1007/s11117-012-0162-z.

[3] C.B. Zhai, C. Yang, C.M. Guo, Positive Solutions of Operator Equations on Ordered Banach Spaces and Applications, Comput. Math. Appl. 56 (2008), 3150–3156. https://doi.org/10.1016/j.camwa.2008.09.005.

[4] C.B. Zhai, C. Yang, X.Q. Zhang, Positive Solutions for Nonlinear Operator Equations and Several Classes of Applications, Math. Z. 266 (2009), 43–63. https://doi.org/10.1007/s00209-009-0553-4.

[5] C. Zhai, L. Wang, φ-(h, e)-Concave Operators and Applications, J. Math. Anal. Appl. 454 (2017), 571–584. https://doi.org/10.1016/j.jmaa.2017.05.010.

[6] D. Guo, Semi-ordered Methods in Nonlinear Analysis, Shandong Science and Technology Press, 2000 (in Chinese).

[7] Y. Du, Fixed Points of a Class of Non-Compact Operators and Applications, Acta Math. Sin. Chin. Ser. 32 (1989), 618–627.

[8] Y. Du, Fixed Points of Increasing Operators in Ordered Banach Spaces and Applications, Appl. Anal. 38 (1990), 1–20.

[9] D. Guo, Nonlinear Functional Analysis, Higher Education Press, Beijing, 2015.

[10] C. Zhai, L. Xu, Properties of Positive Solutions to a Class of Four-Point Boundary Value Problem of Caputo Fractional Differential Equations with a Parameter, Commun. Nonlinear Sci. Numer. Simul. 19 (2014), 2820–2827. https://doi.org/10.1016/j.cnsns.2014.01.003.

[11] C. Zhai, P. Li, Nonnegative Solutions of Initial Value Problems for Langevin Equations Involving Two Fractional Orders, Mediterr. J. Math. 15 (2018), 164. https://doi.org/10.1007/s00009-018-1213-x.

[12] J. Ren, C. Zhai, Stability Analysis for Generalized Fractional Differential Systems and Applications, Chaos Solitons Fractals 139 (2020), 110009. https://doi.org/10.1016/j.chaos.2020.110009.

[13] J. Ren, C. Zhai, Solvability for p-Laplacian Generalized Fractional Coupled Systems with Two-sided Memory Effects, Math. Methods Appl. Sci. 43 (2020), 8797–8822. https://doi.org/10.1002/mma.6545.

[14] L. Sajedi, N. Eghbali, H. Aydi, Impulsive Coupled System of Fractional Differential Equations with Caputo–Katugampola Fuzzy Fractional Derivative, J. Math. 2021 (2021), 7275934. https://doi.org/10.1155/2021/7275934.

[15] R. Agarwal, S. Hristova, D. O'Regan, Stability of Generalized Proportional Caputo Fractional Differential Equations by Lyapunov Functions, Fractal Fract. 6 (2022), 34. https://doi.org/10.3390/fractalfract6010034.

[16] Z. Baitiche, C. Derbazi, G. Wang, Monotone Iterative Method for Nonlinear Fractional p-Laplacian Differential Equation in Terms of ψ-Caputo Fractional Derivative Equipped with a New Class of Nonlinear Boundary Conditions, Math. Methods Appl. Sci. 45 (2021), 967–976. https://doi.org/10.1002/mma.7826.

[17] C. Wu, Comparison Principles for Systems of Caputo Fractional Order Ordinary Differential Equations, Chaos Solitons Fractals 171 (2023), 113437. https://doi.org/10.1016/j.chaos.2023.113437.

[18] M. Li, J. Wang, The Existence and Averaging Principle for Caputo Fractional Stochastic Delay Differential Systems, Fract. Calc. Appl. Anal. 26 (2023), 893–912. https://doi.org/10.1007/s13540-023-00146-3.

[19] L. Zhang, N. Zhang, B. Zhou, Solutions and Stability for p-Laplacian Differential Problems with Mixed Type Fractional Derivatives, Int. J. Nonlinear Sci. Numer. Simul. 24 (2022), 2677–2692. https://doi.org/10.1515/ijnsns-2021-0204.

[20] R. Fan, N. Yan, C. Yang, C. Zhai, Qualitative Behaviour of a Caputo Fractional Differential System, Qual. Theory Dyn. Syst. 22 (2023), 143. https://doi.org/10.1007/s12346-023-00836-6.

[21] L. Xue, Q. Sun, D. O'Regan, J. Xu, Existence of Nontrivial Solutions for an Integral Boundary Value Problem Involving the Caputo-Fabrizio-Type Fractional Derivative, J. Appl. Anal. Comput. 15 (2025), 1786–1802. https://doi.org/10.11948/20240411.

[22] J. Ren, Z. Du, C. Zhai, A New Caputo Fractional Differential Equation with Infinite-Point Boundary Conditions: Positive Solutions, Fractal Fract. 9 (2025), 466. https://doi.org/10.3390/fractalfract9070466.

[23] L. Ma, Y. Xu, A Modified Nishihara Model with Nonlinear Time-Varying Viscosity via ψ-Caputo Fractional Derivative for Salt Rock, Appl. Math. Model. 149 (2026), 116296. https://doi.org/10.1016/j.apm.2025.116296.

[24] F. Evirgen, S. Uçar, N. Özdemir, Mathematical Analysis and Optimal Control of a Caputo Fractional Diabetes System with Parameter Identification, J. Comput. Appl. Math. 477 (2026), 117151. https://doi.org/10.1016/j.cam.2025.117151.

[25] X. Zhang, Z. Hao, M. Bohner, Positive Solutions of Semipositone Singular Three-Points Boundary Value Problems for Nonlinear Fractional Differential Equations, Nonlinear Anal.: Real World Appl. 87 (2026), 104425. https://doi.org/10.1016/j.nonrwa.2025.104425.

[26] H. Li, Y. Chen, Existence and Uniqueness of Positive Solutions for a New System of Fractional Differential Equations, Discret. Dyn. Nat. Soc. 2020 (2020), 2617272. https://doi.org/10.1155/2020/2617272.

[27] B. Ahmad, S.K. Ntouyas, A. Alsaedi, K.A. Alalwi, Sequential Fractional Differential Equations With Parametric Type Anti-Periodic Boundary Conditions, J. Appl. Anal. Comput. 15 (2025), 2921–2934. https://doi.org/10.11948/20240511.

[28] P. Agarwal, H.A. Saeed, A. Alsaedi, B. Ahmad, On a Fully Coupled System of Nonlinear Multi-Term Fractional Differential Equations With Integral-Multipoint Boundary Conditions, Filomat 39 (2025), 9827–9849. https://doi.org/10.2298/FIL2528827A.

[29] I.J. Cabrera, J. Rocha, K.B. Sadarangani, Lyapunov Type Inequalities for a Fractional Thermostat Model, RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 112 (2016), 17–24. https://doi.org/10.1007/s13398-016-0362-7.

[30] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[31] R. Gomez-Cram, A. Yaron, How Important Are Inflation Expectations for the Nominal Yield Curve?, Rev. Financ. Stud. 34 (2020), 985–1045. https://doi.org/10.1093/rfs/hhaa039.

[32] M. Marinacci, L. Montrucchio, Unique Tarski Fixed Points, Math. Oper. Res. 44 (2019), 1174–1191. https://doi.org/10.1287/moor.2018.0959.

[33] F. Schorfheide, D. Song, A. Yaron, Identifying Long-Run Risks: A Bayesian Mixed-Frequency Approach, Econometrica 86 (2018), 617–654. https://doi.org/10.3982/ECTA14308.

[34] S. Basu, B. Bundick, Uncertainty Shocks in a Model of Effective Demand, Econometrica 85 (2017), 937–958. https://doi.org/10.3982/ECTA13960.