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Analytic Study and Solutions on Some Classes for Difference Equations

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Abstract. This paper is devoted to finding the explicit form of the solutions of the rational difference equation

$$x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}\big(\pm 1 \pm x_{n-4}x_{n-6}\big)}, \qquad n = 0, 1, \dots,$$

where the initial conditions x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1} and x_0 are arbitrary positive constants. Specific closed-form expressions for the solutions of four distinct special cases of the equation are derived, corresponding to the choices of the signs in the denominator. For the cases with a positive constant term $(+1)$, the solutions are expressed as infinite products whose factors depend on the initial data in a structured periodic pattern. For the cases with a negative constant term (-1), the solutions are shown to be periodic with period 20. In all four cases, the unique equilibrium point $\bar{x} = 0$ is identified and its local asymptotic stability is analyzed; it is proven that the zero equilibrium is not locally asymptotically stable. Numerical simulations are provided to illustrate the theoretical results and to confirm the non-convergence behavior of the solutions.

1. Introduction

Difference equations serve as discrete counterparts to differential equations and are fundamental in modeling phenomena across various disciplines such as biology, economics, physics, and engineering. The study of nonlinear rational difference equations offers significant insights into the long-term behavior of discrete dynamical systems. Although low-order equations have been thoroughly explored, higher-order rational recursive sequences present a challenging and fertile area of research due to their intricate dynamics and the absence of a unified analytical approach. Determining explicit solutions, stability properties, and periodic structures of such equations is essential for both theoretical progress and practical applications.

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The objective of this paper is to analyze the behavior of solutions to the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}\left(\pm 1 \pm x_{n-4}x_{n-6}\right)}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the initial values x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, and x_0 are arbitrary positive constants.

The recurrence (1.1) involves terms from seven steps back, suggesting that the solution may exhibit periodic or multiplicative patterns involving past terms. Such high-order recurrences often lead to rich dynamical behaviors, including convergence to equilibrium, periodic oscillations, or more complex aperiodic dynamics. The aim of this study is to determine explicit solution forms for specific cases of Equation (1.1), to characterize their stability properties, and to provide numerical illustrations for each case. The analytical techniques developed here may also be applicable to other families of nonlinear difference equations with similar structural features.

Studying and solving high-order nonlinear rational recursive sequences is both challenging and rewarding [2, 4, 8, 12–14, 16–20, 32–35]. Recently, considerable attention has been directed toward understanding the qualitative properties of rational recursive sequences. Moreover, diverse nonlinear phenomena in science and engineering can be modeled by such equations, and their solutions contribute to the advancement of the theoretical framework. Nevertheless, a general method for addressing the global behavior of high-order rational difference equations remains elusive [21–25, 37, 40–43]. Therefore, further investigation into rational difference equations of order greater than one is warranted.

Many researchers have examined the behavior of solutions of various difference equations [1, 3, 5–7, 9–11, 15, 26–31, 36, 38, 39, 44, 46]. For instance, Elsayed et al. [37] established results concerning the dynamics and solution of the rational difference equation:

$$x_{n+1} = \frac{x_{n-1}x_{n-5}}{x_{n-3}\left(\pm 1 \pm x_{n-1}x_{n-5}\right)}.$$

Elsayed et al. [41] studied the behavior of trajectories of the dynamics:

$$x_{n+1} = \frac{x_{n}x_{n-4}}{x_{n-3}\left(\pm 1 \pm x_{n}x_{n-4}\right)}.$$

In [8], Aloqeili analyzed the equation:

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Simsek et al. [45] obtained the solution of the difference equation:

$$x_{n+1} = a + \frac{x_{n-3}}{1 - x_{n-1}}.$$

Çinar [16–18] derived solutions for the following discrete equations:

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

In [40], Ibrahim determined the solution form of the rational difference equation:

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}.$$

Karatas et al. [42] provided the solution form of the difference equation:

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

The paper is organized as follows. Section 2 provides essential definitions and preliminary results regarding equilibrium points, stability, and linearized equations. Section 3 examines the first special case $x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}(1+x_{n-4}x_{n-6})}$, deriving explicit solution formulas and analyzing the stability of the zero equilibrium. Section 4 considers the case $x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}(-1+x_{n-4}x_{n-6})}$, demonstrating the emergence of periodic solutions and establishing the non-convergence behavior. Section 5 studies an analogous case with a negative product term, $x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}(1-x_{n-4}x_{n-6})}$, where the solution is expressed as an infinite product with alternating signs. Section 6 addresses the remaining case $x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}(-1-x_{n-4}x_{n-6})}$, also showing periodic solutions with a distinct sign pattern. Finally, Section 7 summarizes the principal findings, discusses the implications of the results, and suggests directions for future research. Throughout the paper, theoretical results are illustrated with detailed numerical examples to confirm the analytical conclusions and visualize the solution dynamics.

2. Preliminaries

This section introduces the fundamental definitions, notations, and stability results that will serve as the theoretical foundation for the subsequent analysis. We begin by recalling standard concepts related to equilibrium points, local and global stability, and the linearization of difference equations?key tools for analyzing the long-term behavior of dynamical systems. Additionally, we present a classical sufficient condition for the asymptotic stability of linear difference equations, which will be applied to the linearized forms of our main equations. The established notation and theoretical framework are essential for the rigorous analytical discussions that follow, ensuring a consistent approach to examining solution existence, uniqueness, and stability characteristics.

Let I be an interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, x_{-k+2}, \dots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$. This existence and uniqueness guarantee forms the basis for all subsequent analysis of specific solution forms and their properties.

Definition 2.1. *(Equilibrium Point)*
A point $\bar{x} \in I$ is called an equilibrium point (or fixed point) of Equation (2.1) if $\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x})$. Equivalently, $x_n = \bar{x}$ for all $n \geq 0$ is a constant solution of Equation (2.1). The equilibrium represents a steady state of the dynamical system.

Definition 2.2. *(Stability)*

- *The equilibrium point $\bar{x}$ of Equation (2.1) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_0 \in I$, with*

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \delta,$$

  we have $|x_n - \bar{x}| < \varepsilon$ for all $n \geq -k$. This means that solutions starting sufficiently close to $\bar{x}$ remain close.
- *The equilibrium point $\bar{x}$ of Equation (2.1) is locally asymptotically stable if it is locally stable and, additionally, there exists $\gamma > 0$ such that for all initial conditions satisfying*

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \gamma,$$

  we have $\lim_{n\to\infty} x_n = \bar{x}$. This implies convergence to the equilibrium from nearby starting points.
- *The equilibrium point $\bar{x}$ of Equation (2.1) is a global attractor if for all initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, we have*

$$\lim_{n\to\infty} x_n = \bar{x}.$$

  The equilibrium attracts all trajectories, regardless of their starting values.
- *The equilibrium point $\bar{x}$ of Equation (2.1) is globally asymptotically stable if it is both locally stable and a global attractor. This represents the strongest form of stability.*
- *The equilibrium point $\bar{x}$ of Equation(2.1) is unstable if it is not locally stable. Small perturbations from the equilibrium lead to solutions that diverge away.*

The linearized equation of Equation (2.1) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-i}} y_{n-i}.$$

This linear approximation is crucial for determining local stability through the analysis of its characteristic roots.

Theorem 2.1. *Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, \ldots\}$. Then the condition $|p| + |q| < 1$ is sufficient for the asymptotic stability of the linear difference equation*

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, \ldots.$$

That is, all solutions of this equation converge to zero.

Remark 2.1. *Theorem 2.1 can be extended to a general linear equation of the form*

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n = 0, 1, \ldots, \tag{2.2}$$

where $p_1, p_2, \ldots, p_k \in \mathbb{R}$ *and* $k \in \{0, 1, 2, \ldots\}$*. Then Equation (2.2) is asymptotically stable provided that* $\sum_{i=0}^{k} |p_i| < 1$*. This condition ensures that all roots of the corresponding characteristic equation lie inside the unit disk.*

Definition 2.3. *A sequence* $\{x_n\}_{n=-k}^{\infty}$ *is said to be periodic with period p (where p is a positive integer) if*

$$x_{n+p} = x_n \quad \forall n \geq -k.$$

The smallest such positive integer p is called the minimal period. Periodic solutions represent oscillatory behavior and are important in understanding the long-term dynamics of nonlinear systems.

3. On the Difference Equation $x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}(1+x_{n-4}x_{n-6})}$

In this section, we analyze the first special case of equation (1.1), namely the recurrence with a positive constant and a positive product term in the denominator. We derive an explicit closed-form solution expressed as an infinite product, where each factor depends periodically on the index n and the initial values. The solution structure reveals a 20-step pattern, which we verify by induction. Furthermore, we examine the local stability of the unique equilibrium point $\bar{x} = 0$ and prove that it is not locally asymptotically stable. A numerical example is provided to illustrate the theoretical findings.

Let us consider the following special case of Equation (1.1):

$$x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}\left(1 + x_{n-4}x_{n-6}\right)}, \quad n = 0, 1, \ldots, \tag{3.1}$$

where the initial conditions $x_{-6} = g$, $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$ are arbitrary nonzero real numbers.

Theorem 3.1. *Let* $\{x_n\}_{n=-6}^{\infty}$ *be a solution of Equation (3.1). Then for* $n = 0, 1, \ldots$*,*

$$x_{20n-6} = g\prod_{i=0}^{n-1} \frac{(1+4iac)(1+(4i+2)ce)(1+4ieg)(1+(4i+1)bd)(1+(4i+3)df)}{(1+(4i+2)ac)(1+4ice)(1+(4i+2)eg)(1+(4i+3)bd)(1+(4i+1)df)},$$

$$x_{20n-5} = f\prod_{i=0}^{n-1} \frac{(1+(4i+1)ac)(1+(4i+3)ce)(1+(4i+1)eg)(1+(4i+2)bd)(1+4idf)}{(1+(4i+3)ac)(1+(4i+1)ce)(1+(4i+3)eg)(1+4ibd)(1+(4i+2)df)},$$

$$x_{20n-4} = e\prod_{i=0}^{n-1} \frac{(1+(4i+2)ac)(1+4ice)(1+(4i+2)eg)(1+(4i+3)bd)(1+(4i+1)df)}{(1+4iac)(1+(4i+2)ce)(1+(4i+4)eg)(1+(4i+1)bd)(1+(4i+3)df)},$$

$$x_{20n-3} = d\prod_{i=0}^{n-1} \frac{(1+(4i+3)ac)(1+(4i+1)ce)(1+(4i+3)eg)(1+4ibd)(1+(4i+2)df)}{(1+(4i+1)ac)(1+(4i+3)ce)(1+(4i+1)eg)(1+(4i+2)bd)(1+(4i+4)df)},$$

$$x_{20n-2} = c\prod_{i=0}^{n-1} \frac{(1+4iac)(1+(4i+2)ce)(1+(4i+4)eg)(1+(4i+1)bd)(1+(4i+3)df)}{(1+(4i+2)ac)(1+(4i+4)ce)(1+(4i+2)eg)(1+(4i+3)bd)(1+(4i+1)df)},$$

$$x_{20n-1} = b\prod_{i=0}^{n-1} \frac{(1+(4i+1)ac)(1+(4i+3)ce)(1+(4i+1)eg)(1+(4i+2)bd)(1+(4i+4)df)}{(1+(4i+3)ac)(1+(4i+1)ce)(1+(4i+3)eg)(1+(4i+4)bd)(1+(4i+2)df)},$$

$$x_{20n} = a\prod_{i=0}^{n-1}\frac{(1+(4i+2)ac)(1+(4i+4)ce)(1+(4i+2)eg)(1+(4i+3)bd)(1+(4i+1)df)}{(1+(4i+4)ac)(1+(4i+2)ce)(1+(4i+4)eg)(1+(4i+1)bd)(1+(4i+3)df)},$$
$$x_{20n+1} = \frac{eg}{b\,(1+eg)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+3)ac)(1+(4i+1)ce)(1+(4i+3)eg)(1+(4i+4)bd)(1+(4i+2)df)}{(1+(4i+1)ac)(1+(4i+3)ce)(1+(4i+5)eg)(1+(4i+2)bd)(1+(4i+4)df)},$$
$$x_{20n+2} = \frac{df}{a\,(1+df)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+4)ac)(1+(4i+2)ce)(1+(4i+4)eg)(1+(4i+1)bd)(1+(4i+3)df)}{(1+(4i+2)ac)(1+(4i+4)ce)(1+(4i+2)eg)(1+(4i+3)bd)(1+(4i+5)df)},$$
$$x_{20n+3} = \frac{bc(1+eg)}{g\,(1+ce)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+1)ac)(1+(4i+3)ce)(1+(4i+5)eg)(1+(4i+2)bd)(1+(4i+4)df)}{(1+(4i+3)ac)(1+(4i+5)ce)(1+(4i+3)eg)(1+(4i+4)bd)(1+(4i+2)df)},$$
$$x_{20n+4} = \frac{ab\,(1+df)}{f\,(1+bd)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+2)ac)(1+(4i+4)ce)(1+(4i+2)eg)(1+(4i+3)bd)(1+(4i+3)df)}{(1+(4i+4)ac)(1+(4i+2)ce)(1+(4i+4)eg)(1+(4i+5)bd)(1+(4i+1)df)},$$
$$x_{20n+5} = \frac{ag\,(1+ce)}{b\,(1+eg)\,(1+ac)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+3)ac)(1+(4i+2)ce)(1+(4i+3)eg)(1+(4i+4)bd)(1+(4i+2)df)}{(1+(4i+5)ac)(1+4ice)(1+(4i+5)eg)(1+(4i+2)bd)(1+(4i+4)df)},$$
$$x_{20n+6} = \frac{efg\,(1+bd)}{ab\,(1+2eg)\,(1+df)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+4)ac)(1+(4i+2)ce)(1+(4i+4)eg)(1+(4i+5)bd)(1+(4i+3)df)}{(1+(4i+2)ac)(1+(4i+4)ce)(1+(4i+6)eg)(1+(4i+3)bd)(1+(4i+5)df)},$$
$$x_{20n+7} = \frac{bdf\,(1+ac)\,(1+eg)}{ag\,(1+2df)\,(1+ce)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+5)ac)(1+(4i+3)ce)(1+(4i+5)eg)(1+(4i+2)bd)(1+(4i+4)df)}{(1+(4i+3)ac)(1+(4i+5)ce)(1+(4i+3)eg)(1+(4i+4)bd)(1+(4i+6)df)},$$
$$x_{20n+8} = \frac{abc\,(1+df)\,(1+2eg)}{fg\,(1+2ce)\,(1+bd)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+2)ac)(1+(4i+4)ce)(1+(4i+6)eg)(1+(4i+3)bd)(1+(4i+5)df)}{(1+(4i+4)ac)(1+(4i+6)ce)(1+(4i+4)eg)(1+(4i+5)bd)(1+(4i+3)df)},$$
$$x_{20n+9} = \frac{ag\,(1+2df)\,(1+ce)}{f\,(1+2bd)\,(1+eg)\,(1+ac)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+3)ac)(1+(4i+5)ce)(1+(4i+3)eg)(1+(4i+4)bd)(1+(4i+6)df)}{(1+(4i+5)ac)(1+(4i+3)ce)(1+(4i+5)eg)(1+(4i+6)bd)(1+(4i+4)df)},$$
$$x_{20n+10} = \frac{fg\,(1+2ce)\,(1+bd)}{b\,(1+2ac)\,(1+2eg)\,(1+df)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+4)ac)(1+(4i+6)ce)(1+(4i+4)eg)(1+(4i+5)bd)(1+(4i+3)df)}{(1+(4i+6)ac)(1+(4i+4)ce)(1+(4i+6)eg)(1+(4i+3)bd)(1+(4i+5)df)},$$
$$x_{20n+11} = \frac{ef\,(1+ac)\,(1+eg)\,(1+2bd)}{a\,(1+3eg)\,(1+ce)\,(1+2df)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+5)ac)(1+(4i+3)ce)(1+(4i+5)eg)(1+(4i+6)bd)(1+(4i+4)df)}{(1+(4i+3)ac)(1+(4i+5)ce)(1+(4i+7)eg)(1+(4i+4)bd)(1+(4i+6)df)},$$
$$x_{20n+12} = \frac{bd\,(1+df)\,(1+2eg)\,(1+2ac)}{g\,(1+3df)\,(1+bd)\,(1+2ce)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+6)ac)(1+(4i+4)ce)(1+(4i+6)eg)(1+(4i+3)bd)(1+(4i+5)df)}{(1+(4i+4)ac)(1+(4i+6)ce)(1+(4i+4)eg)(1+(4i+5)bd)(1+(4i+7)df)},$$
$$x_{20n+13} = \frac{ac\,(1+2df)\,(1+ce)\,(1+3eg)}{f\,(1+3ce)\,(1+2bd)\,(1+eg)\,(1+ac)}\times\prod_{i=0}^{n-1}\frac{(1+(4i+3)ac)(1+(4i+5)ce)(1+(4i+7)eg)(1+(4i+4)bd)(1+(4i+6)df)}{(1+(4i+5)ac)(1+(4i+7)ce)(1+(4i+5)eg)(1+(4i+6)bd)(1+(4i+4)df)}.$$

Proof. We use an inductive proof for this rational recursive sequences. It is easy to see that for $n=0$, the result holds. Suppose that $n>0$ and that the assumption is satisfied for $n-1$. That is

$$x_{20n-26} = g\prod_{i=0}^{n-2}\frac{(1+(4i+1)bd)(1+(4i+2)ce)(1+(4i+3)df)(1+4ieg)(1+4iac)}{(1+(4i+1)df)(1+(4i+2)eg)(1+(4i+2)ac)(1+(4i+3)bd)(1+4ice)},$$
$$x_{20n-25} = f\prod_{i=0}^{n-2}\frac{(1+(4i+1)ac)(1+(4i+1)eg)(1+(4i+2)bd)(1+(4i+3)ce)(1+4idf)}{(1+(4i+1)ce)(1+(4i+2)df)(1+(4i+3)eg)(1+(4i+3)ac)(1+4ibd)},$$
$$x_{20n-24} = e\prod_{i=0}^{n-2}\frac{(1+(4i+1)df)(1+(4i+2)eg)(1+(4i+2)ac)(1+(4i+3)bd)(1+4ice)}{(1+(4i+1)bd)(1+(4i+2)ce)(1+(4i+3)df)(1+4ieg)(1+4iac)},$$
$$x_{20n-23} = d\prod_{i=0}^{n-2}\frac{(1+(4i+1)ce)(1+(4i+2)df)(1+(4i+3)eg)(1+(4i+3)ac)(1+4ibd)}{(1+(4i+1)ac)(1+(4i+1)eg)(1+(4i+2)bd)(1+(4i+3)ce)(1+(4i+4)df)},$$
$$x_{20n-22} = c\prod_{i=0}^{n-2}\frac{(1+(4i+1)bd)(1+(4i+2)ce)(1+(4i+3)df)(1+(4i+4)eg)(1+4iac)}{(1+(4i+1)df)(1+(4i+2)eg)(1+(4i+2)ac)(1+(4i+3)bd)(1+(4i+4)ce)},$$
$$x_{20n-21} = b\prod_{i=0}^{n-2}\frac{(1+(4i+1)ac)(1+(4i+1)eg)(1+(4i+2)bd)(1+(4i+3)ce)(1+(4i+4)df)}{(1+(4i+1)ce)(1+(4i+2)df)(1+(4i+3)eg)(1+(4i+3)ac)(1+(4i+4)bd)},$$

$$x_{20n-20} = a\prod_{i=0}^{n-2}\frac{(1+(4i+1)df)(1+(4i+2)eg)(1+(4i+2)ac)(1+(4i+3)bd)(1+(4i+4)ce)}{(1+(4i+1)bd)(1+(4i+2)ce)(1+(4i+3)df)(1+(4i+4)eg)(1+(4i+4)ac)},$$

$$x_{20n-19} = \frac{eg}{b(1+eg)}\prod_{i=0}^{n-2}\frac{(1+(4i+1)ce)(1+(4i+2)df)(1+(4i+3)eg)(1+(4i+3)ac)(1+(4i+4)bd)}{(1+(4i+1)ac)(1+(4i+2)bd)(1+(4i+3)ce)(1+(4i+4)df)(1+(3i+5)eg)},$$

$$x_{20n-18} = \frac{df}{a(1+df)}\prod_{i=0}^{n-2}\frac{(1+(4i+1)bd)(1+(4i+2)ce)(1+(4i+3)df)(1+(4i+4)eg)(1+(4i+4)ac)}{(1+(4i+2)eg)(1+(4i+2)ac)(1+(4i+3)bd)(1+(4i+4)ce)(1+(4i+5)df)},$$

$$x_{20n-17} = \frac{cb(1+eg)}{g(1+ce)}\prod_{i=0}^{n-2}\frac{(1+(4i+1)ac)(1+(4i+2)bd)(1+(4i+3)ce)(1+(4i+4)df)(1+)(4i+5)eg}{(1+(4i+2)df)(1+(4i+3)eg)(1+(4i+3)ac)(1+(4i+4)bd)(1+(4i+5)ce)},$$

$$x_{20n-16} = \frac{ab(1+df)}{f(1+bd)}\prod_{i=0}^{n-2}\frac{(1+(4i+2)eg)(1+(4i+2)ac)(1+(4i+3)bd)(1+(4i+4)ce)(1+(4i+5)df)}{(1+(4i+2)ce)(1+(4i+3)df)(1+(4i+4)eg)(1+(4i+4)ac)(1+(4i+5)bd)},$$

$$x_{20n-15} = \frac{ag(1+ce)}{b(1+eg)(1+ac)}\prod_{i=0}^{n-2}\frac{(1+(4i+2)df)(1+(4i+3)eg)(1+(4i+3)ac)(1+(4i+4)bd)(1+(4i+5)ce)}{(1+(4i+2)bd)(1+(4i+3)ce)(1+(4i+4)df)(1+(3i+5)eg)(1+(4i+5)ac)},$$

$$x_{20n-14} = \frac{egf(1+bd)}{ab(1+df)(1+2eg)}\prod_{i=0}^{n-2}\frac{(1+(4i+2)ce)(1+(4i+3)df)(1+(4i+4)eg)(1+(4i+4)ac)(1+(4i+5)bd)}{(1+(4i+2)ac)(1+(4i+3)bd)(1+(4i+4)ce)(1+(4i+5)df)(1+(4i+6)eg)},$$

$$x_{20n-13} = \frac{bdf(1+eg)(1+ac)}{ag(1+ce)(1+2df)}\prod_{i=0}^{n-2}\frac{(1+(4i+2)bd)(1+(4i+3)ce)(1+(4i+4)df)(1+(4i+5)eg)(1+(4i+5)ac)}{(1+(4i+3)eg)(1+(4i+3)ac)(1+(4i+4)bd)(1+(4i+5)ce)(1+(4i+6)df)},$$

$$x_{20n-12} = \frac{abc(1+df)(1+2eg)}{gf(1+bd)(1+2ce)}\prod_{i=0}^{n-2}\frac{(1+(4i+2)ac)(1+(4i+3)bd)(1+(4i+4)ce)(1+(4i+5)df)(1+(4i+6)eg)}{(1+(4i+3)df)(1+(4i+4)eg)(1+(4i+4)ac)(1+(4i+5)bd)(1+(4i+6)ce)},$$

$$x_{20n-11} = \frac{ag(1+ce)(1+2df)}{f(1+eg)(1+ac)(1+2bd)}\prod_{i=0}^{n-2}\frac{(1+(4i+3)eg)(1+(4i+3)ac)(1+(4i+4)bd)(1+(4i+5)ce)(1+(4i+6)df)}{(1+(4i+3)ce)(1+(4i+4)df)(1+(3i+5)eg)(1+(4i+5)ac)(1+(4i+6)bd)},$$

$$x_{20n-10} = \frac{gf(1+bd)(1+2ce)}{b(1+df)(1+2eg)(1+2ac)}\prod_{i=0}^{n-2}\frac{(1+(4i+3)df)(1+(4i+4)eg)(1+(4i+4)ac)(1+(4i+5)bd)(1+(4i+6)ce)}{(1+(4i+3)bd)(1+(4i+4)ce)(1+(4i+5)df)(1+(4i+6)eg)(1+(4i+6)ac)},$$

$$x_{20n-9} = \frac{ef(1+eg)(1+ac)(1+2bd)}{a(1+ce)(1+2df)(1+3eg)}\prod_{i=0}^{n-2}\frac{(1+(4i+3)ce)(1+(4i+4)df)(1+(4i+5)eg)(1+(4i+5)ac)(1+(4i+6)bd)}{(1+(4i+3)ac)(1+(4i+4)bd)(1+(4i+5)ce)(1+(4i+6)df)(1+(4i+7)eg)},$$

$$x_{20n-8} = \frac{bd(1+df)(1+2eg)(1+2ac)}{g(1+bd)(1+2ce)(1+3df)}\prod_{i=0}^{n-2}\frac{(1+(4i+3)bd)(1+(4i+4)ce)(1+(4i+5)df)(1+(4i+6)eg)(1+(4i+6)ac)}{(1+(4i+4)eg)(1+(4i+4)ac)(1+(4i+5)bd)(1+(4i+6)ce)(1+(4i+7)df)},$$

$$x_{20n-7} = \frac{ac(1+ce)(1+2df)(1+3eg)}{f(1+eg)(1+ac)(1+2bd)(1+3ce)}\prod_{i=0}^{n-2}\frac{(1+(4i+3)ac)(1+(4i+4)bd)(1+(4i+5)ce)(1+(4i+6)df)(1+(4i+7)eg}{(1+(4i+4)df)(1+(4i+5)eg)(1+(4i+5)ac)(1+(4i+6)bd)(1+(4i+7)ce)}.$$

Now, it follows from (3.1) that,

$$\begin{aligned}
x_{20n-6} &= \frac{x_{20n-11}x_{30n-13}}{x_{20n-8}(1+x_{20n-11}x_{30n-13})}\\
&= \frac{\frac{bd}{(1+2bd)}\prod_{i=0}^{n-2}\left(\frac{(1+(4i+2)bd)}{(1+(4i+6)bd}\right)}{\frac{bd(1+df)(1+2eg)(1+2ac)}{g(1+bd)(1+2ce)(1+3df)}\prod_{i=0}^{n-2}\frac{(1+(4i+3)bd)(1+(4i+4)ce)(1+(4i+5)df)(1+(4i+6)eg)(1+(4i+6)ac)}{(1+(4i+4)eg)(1+(4i+4)ac)(1+(4i+5)bd)(1+(4i+6)ce)(1+(4i+7)df)}\left(1+\frac{bd}{(1+2bd)}\prod_{i=0}^{n-2}\left(\frac{(1+(4i+2)bd)}{(1+(4i+6)bd}\right)\right)}\\
&= \frac{\frac{bd}{(1+(4n-2)bd)}}{\frac{bd(1+df)(1+2eg)(1+2ac)}{g(1+bd)(1+2ce)(1+3df)}\left(1+\frac{bd}{(1+(4n-2)bd)}\right)\prod_{i=0}^{n-2}\frac{(1+(4i+3)bd)(1+(4i+4)ce)(1+(4i+5)df)(1+(4i+6)eg)(1+(4i+6)ac)}{(1+(4i+4)eg)(1+(4i+4)ac)(1+(4i+5)bd)(1+(4i+6)ce)(1+(4i+7)df)}}.
\end{aligned}$$

Hence, we have

$$x_{20n-6} = g\prod_{i=0}^{n-1}\frac{(1+(4i+1)bd)(1+(4i+2)ce)(1+(4i+3)df)(1+4ieg)(1+4iac)}{(1+(4i+1)df)(1+(4i+2)eg)(1+(4i+2)ac)(1+(4i+3)bd)(1+4ice)}.$$

Also, we see from Equation (3.1) that

$$\begin{aligned} x_{20n-5} &= \frac{x_{20n-10}x_{20n-12}}{x_{20n-7}(1+x_{20n-10}x_{20n-12})} \\ &= \frac{\frac{ac}{(1+2ac)}\prod_{i=0}^{n-2}\left(\frac{(1+(4i+2)ac)}{(1+(4i+6)ac}\right)}{\frac{ac(1+ce)(1+2df)(1+3eg)}{f(1+eg)(1+ac)(1+2bd)(1+3ce)}\prod_{i=0}^{n-2}\frac{(1+(4i+3)ac)(1+(4i+4)bd)(1+(4i+5)ce)(1+(4i+6)df)(1+(4i+7)eg}{(1+(4i+4)df)(1+(4i+5)eg)(1+(4i+5)ac)(1+(4i+6)bd)(1+(4i+7)ce)}\left(1+\frac{ac}{(1+2ac)}\prod_{i=0}^{n-2}\left(\frac{(1+(4i+2)ac)}{(1+(4i+6)ac}\right)\right)} \\ &= \frac{\frac{ac}{(1+(4n-2)ac)}}{\frac{ac(1+ce)(1+2df)(1+3eg)}{f(1+eg)(1+ac)(1+2bd)(1+3ce)}\left(1+\frac{ac}{(1+(4n-2)ac)}\right)\prod_{i=0}^{n-2}\frac{(1+(4i+3)ac)(1+(4i+4)bd)(1+(4i+5)ce)(1+(4i+6)df)(1+(4i+7)eg}{(1+(4i+4)df)(1+(4i+5)eg)(1+(4i+5)ac)(1+(4i+6)bd)(1+(4i+7)ce)}}. \end{aligned}$$

Hence, we have

$$x_{20n-5} = f\prod_{i=0}^{n-1}\frac{(1+(4i+1)ac)(1+(4i+1)eg)(1+(4i+2)bd)(1+(4i+3)ce)(1+4idf)}{(1+(4i+1)ce)(1+(4i+2)df)(1+(4i+3)eg)(1+(4i+3)ac)(1+4ibd)}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Theorem 3.2. *Equation (3.1) has one equilibrium point $\bar{x} = 0$ and this equilibrium point is not locally asymptotically stable.*

Proof. In this section we investigate the local stability character of the solutions of Equation (3.1). Equation (3.1) has a unique positive equilibrium point and is given by

$$\bar{x} = \frac{\bar{x}^2}{\bar{x}\,(1+\bar{x}^2)} = \frac{\bar{x}}{1+\bar{x}^2}, \quad \text{or also} \quad 1 = 1+\bar{x}^2,$$

then the unique equilibrium point is given by $\bar{x} = 0$.

Define the following function

$$\begin{gathered} f : (0,\infty)^3 \to (0,\infty) \\ f(u,v,w) = \frac{uw}{v\,(1+uw)}. \end{gathered}$$

Therefore it follows that

$$f_u(u,v,w) = \frac{w}{v\,(1+uw)^2}, \quad f_v(u,v,w) = -\frac{uw}{v^2\,(1+uw)}, \quad f_w(u,v,w) = \frac{u}{v\,(1+uw)^2}.$$

Then

$$f_u(\bar{x},\bar{x},\bar{x}) = \frac{1}{(1+\bar{x}^2)^2} = 1, \quad f_v(\bar{x},\bar{x},\bar{x}) = -\frac{1}{(1+\bar{x}^2)} = -1, \quad f_w(\bar{x},\bar{x},\bar{x}) = \frac{1}{(1+\bar{x}^2)^2} = 1.$$

The linearized equation of Equation (3.1) about $\bar{x}$ is

$$y_{n+1} - y_{n-6} + y_{n-5} - y_n = 0. \tag{3.2}$$

It follows from Theorem 2.1 that Equation (3.2) is not asymptotically stable. The proof is complete. □

For confirming the results of this section, we consider a numerical example (See Figure 1).

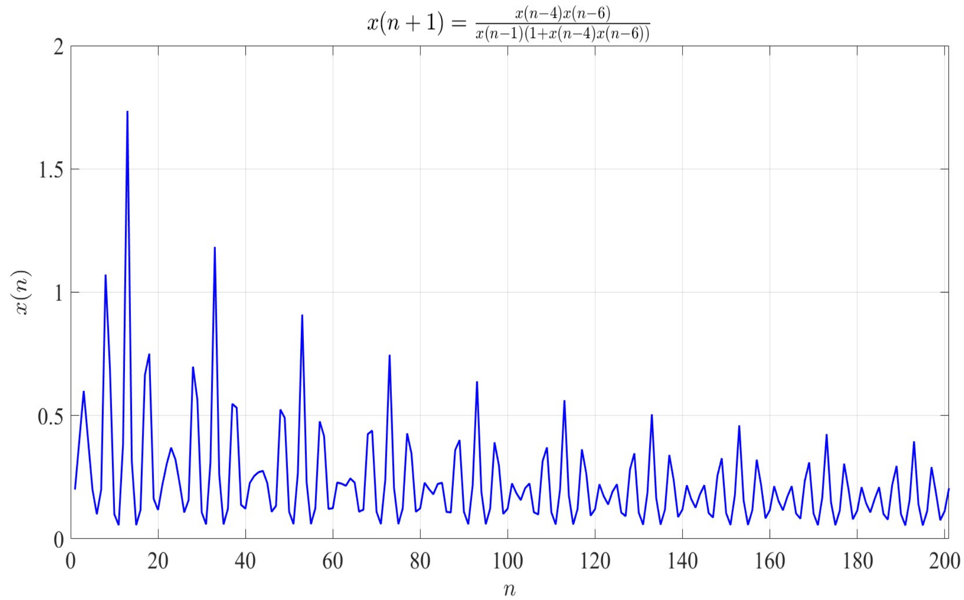

Figure 1. Behavior of the solution of system (3.1). It can be seen that the solution doesn't converge to zero which confirm the fact that the equilibrium point 0 is not locally asymptotically stable. The initial condition is given by $g = 0.2$, $f = 0.4$, $e = 0.6$, $d = 0.4$, $c = 0.2$, $b = 0.1$, and $a = 0.2$.

To summarize, we have obtained a closed-form solution for equation (3.1) expressed as an infinite product, and we have proven that the unique equilibrium $\bar{x} = 0$ is not locally asymptotically stable. The solution exhibits a clear periodic pattern over indices modulo 20, which is a consequence of the recurrence's structure. The numerical example further supports the theoretical conclusion that solutions do not converge to the equilibrium, highlighting the complex dynamics inherent in this rational difference equation.

4. On the Difference Equation $x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}(-1+x_{n-4}x_{n-6})}$

This section examines the second special case of Equation (1.1), characterized by a negative constant term and a positive product term in the denominator. Unlike the previous two cases, the dynamics here exhibit a qualitatively different behavior: the solution is shown to be bounded and purely periodic with period 20. We derive the exact periodic expressions for each subsequence indexed modulo 20 and verify them inductively. As before, the unique equilibrium $\bar{x} = 0$ is identified and its local asymptotic instability is established. A numerical example is provided to visualize the periodic trajectory.

Let us consider the following special case of Equation (1.1):

$$x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}\left(-1 + x_{n-4}x_{n-6}\right)}, \quad n = 0, 1, \ldots, \tag{4.1}$$

where the initial conditions $x_{-6} = g$, $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$ are arbitrary nonzero real numbers.

Theorem 4.1. *Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Equation (4.1). Assume that $ac \neq 1$, $bd \neq 1$, $ce \neq 1$, $df \neq 1$, and $eg \neq 1$. Then, the solution of Equation (4.1) is bounded and periodic of period 20 given by:*

$$\begin{aligned}
x_{20n-6} &= g, & x_{20n-5} &= f,\\
x_{20n-4} &= e, & x_{20n-3} &= d,\\
x_{20n-2} &= c, & x_{20n-1} &= b,\\
x_{20n} &= a, & x_{20n+1} &= \frac{eg}{b(-1+eg)},\\
x_{20n+2} &= \frac{df}{a(-1+df)}, & x_{20n+3} &= \frac{bc(-1+eg)}{g(-1+ce)},\\
x_{20n+4} &= \frac{ab(-1+df)}{f(-1+bd)}, & x_{20n+5} &= \frac{ag(-1+ce)}{b(-1+eg)(-1+ac)},\\
x_{20n+6} &= \frac{efg(-1+bd)}{ab(-1+df)}, & x_{20n+7} &= \frac{bdf(-1+eg)(-1+ac)}{ag(-1+ce)},\\
x_{20n+8} &= \frac{abc(-1+df)}{fg(-1+bd)}, & x_{20n+9} &= \frac{ag(-1+ce)}{f(-1+eg)(-1+ac)},\\
x_{20n+10} &= \frac{fg(-1+bd)}{b(-1+df)}, & x_{20n+11} &= \frac{ef(-1+ac)}{a(-1+ce)},\\
x_{20n+12} &= \frac{bd}{g(-1+bd)}, & x_{20n+13} &= \frac{ac}{f(-1+ac)}.
\end{aligned}$$

Proof. For $n = 0$, the result holds. Now suppose that our assumption holds for $n-1$. That is;

$$\begin{aligned}
x_{20n-26} &= g, & x_{20n-25} &= f,\\
x_{20n-24} &= e, & x_{20n-23} &= d,\\
x_{20n-22} &= c, & x_{20n-21} &= b,\\
x_{20n-20} &= a, & x_{20n-19} &= \frac{eg}{b(-1+eg)},\\
x_{20n-18} &= \frac{df}{a(-1+df)}, & x_{20n-17} &= \frac{bc(-1+eg)}{g(-1+ce)},\\
x_{20n-16} &= \frac{ab(-1+df)}{f(-1+bd)}, & x_{20n-15} &= \frac{ag(-1+ce)}{b(-1+eg)(-1+ac)},\\
x_{20n-14} &= \frac{efg(-1+bd)}{ab(-1+df)}, & x_{20n-13} &= \frac{bdf(-1+eg)(-1+ac)}{ag(-1+ce)},\\
x_{20n-12} &= \frac{abc(-1+df)}{fg(-1+bd)}, & x_{20n-11} &= \frac{ag(-1+ce)}{f(-1+eg)(-1+ac)},\\
x_{20n-10} &= \frac{fg(-1+bd)}{b(-1+df)}, & x_{20n-9} &= \frac{ef(-1+ac)}{a(-1+ce)},\\
x_{20n-8} &= \frac{bd}{g(-1+bd)}, & x_{20n-7} &= \frac{ac}{f(-1+ac)}.
\end{aligned}$$

Now it follows from Equation (4.1) that

$$\begin{aligned}
x_{20n-6} &= \frac{x_{20n-11}x_{20n-13}}{x_{20n-8}\left(-1+x_{20n-11}x_{20n-13}\right)}\\
&= \frac{\frac{ag(-1+ce)}{f(-1+eg)(-1+ac)}\frac{bdf(-1+eg)(-1+ac)}{ag(-1+ce)}}{\frac{bd}{g(-1+bd)}\left(-1+\frac{ag(-1+ce)}{f(-1+eg)(-1+ac)}\frac{bdf(-1+eg)(-1+ac)}{ag(-1+ce)}\right)}
\end{aligned}$$

$$= \frac{bd}{\frac{bd}{g(-1+bd)}(-1+bd)} = g.$$

Similarly

$$\begin{aligned} x_{20n-5} &= \frac{x_{20n-10}x_{20n-12}}{x_{20n-7}\left(-1+x_{20n-10}x_{20n-12}\right)} \\ &= \frac{\frac{fg(-1+bd)}{b(-1+df)}\frac{abc(-1+df)}{fg(-1+bd)}}{\frac{ac}{f(-1+ac)}\left(-1+\frac{fg(-1+bd)}{b(-1+df)}\frac{abc(-1+df)}{fg(-1+bd)}\right)} \\ &= \frac{ac}{\frac{ac}{f(-1+ac)}(-1+ac)} \\ &= f. \end{aligned}$$

Similarly

$$\begin{aligned} x_{20n-4} &= \frac{x_{20n-9}x_{20n-11}}{x_{20n-6}\left(-1+x_{20n-9}x_{20n-11}\right)} \\ &= \frac{\frac{ef(-1+ac)}{a(-1+ce)}\frac{ag(-1+ce)}{f(-1+eg)(-1+ac)}}{g\left(-1+\frac{ef(-1+ac)}{a(-1+ce)}\frac{ag(-1+ce)}{f(-1+eg)(-1+ac)}\right)} \\ &= \frac{\frac{eg}{(-1+eg)}}{g\left(-1+\frac{eg}{(-1+eg)}\right)} \\ &= e. \end{aligned}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Theorem 4.2. *Equation (4.1) has a unique equilibrium point which is* 0 *and this equilibrium point is not locally asymptotically stable.*

Proof. The proof is similar to the one of Theorem 3.2 and will be omitted. □

For confirming the results of this section, we consider the following numerical example (See Figure 2).

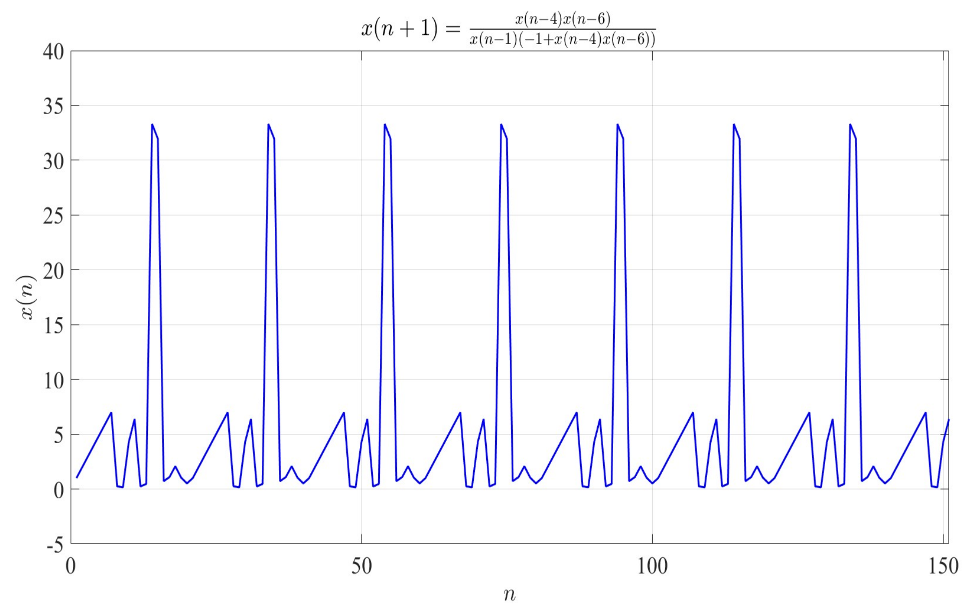

FIGURE 2. Behavior of the solution of system (4.1). It can be seen that the solution is periodic and doesn't converge to zero which confirm the fact that the equilibrium point 0 is not locally asymptotically stable. The initial condition is given by $g = 1$, $f = 2$, $e = 3$, $d = 4$, $c = 5$, $b = 6$, and $a = 7$.

Thus, we have established that the solution of Equation (4.1) is periodic with period 20 and that the unique equilibrium $\bar{x} = 0$ is unstable. The periodic behavior emerges directly from the structure of the recurrence when the constant term in the denominator is negative, in contrast to the infinite-product solutions obtained for the positive-constant cases. This result further illustrates the sensitivity of the system's long-term dynamics to the signs of the parameters, reinforcing the need for case-specific analysis in higher-order rational difference equations.

5. On the Difference Equation $x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}(1-x_{n-4}x_{n-6})}$

In this section, we turn to the third variant of Equation (1.1), where the denominator contains a positive constant and a negative product term. The analysis follows a structure similar to that of Section 3, but the change in sign leads to distinct solution formulas. We establish explicit expressions for the solution, again in the form of an infinite product, where the signs within the factors alternate according to a systematic pattern. The zero equilibrium remains the unique fixed point, and its local asymptotic instability is demonstrated. A numerical simulation is included to confirm the non-convergent behavior of the solution.

Let us consider a third special case of Equation (1.1) given by:

$$x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}\left(1 - x_{n-4}x_{n-6}\right)}, \quad n = 0, 1, \ldots, \tag{5.1}$$

where the initial conditions $x_{-6} = g$, $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$ are arbitrary nonzero real numbers.

Theorem 5.1. *Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Equation (5.1). Then for $n = 0, 1, \ldots,$*

$$
\begin{aligned}
x_{20n-6} &= g\prod_{i=0}^{n-1}\frac{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-4ieg)(1-4iac)}{(1-(4i+1)df)(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-4ice)},\\
x_{20n-5} &= f\prod_{i=0}^{n-1}\frac{(1-(4i+1)ac)(1-(4i+1)eg)(1-(4i+2)bd)(1-(4i+3)ce)(1-4idf)}{(1-(4i+1)ce)(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-4ibd)},\\
x_{20n-4} &= e\prod_{i=0}^{n-1}\frac{(1-(4i+1)df)(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-4ice)}{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-4ieg)(1-4iac)},\\
x_{20n-3} &= d\prod_{i=0}^{n-1}\frac{(1-(4i+1)ce)(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-4ibd)}{(1-(4i+1)ac)(1-(4i+1)eg)(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)},\\
x_{20n-2} &= c\prod_{i=0}^{n-1}\frac{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-(4i+4)eg)(1-4iac)}{(1-(4i+1)df)(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)},\\
x_{20n-1} &= b\prod_{i=0}^{n-1}\frac{(1-(4i+1)ac)(1-(4i+1)eg)(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)}{(1-(4i+1)ce)(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)},\\
x_{20n} &= a\prod_{i=0}^{n-1}\frac{(1-(4i+1)df)(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)}{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)},\\
x_{20n+1} &= \frac{eg}{b(1-eg)}\prod_{i=0}^{n-1}\frac{(1-(4i+1)ce)(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)}{(1-(4i+1)ac)(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)(1-(3i+5)eg)},\\
x_{20n+2} &= \frac{df}{a(1-df)}\prod_{i=0}^{n-1}\frac{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)}{(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)},\\
x_{20n+3} &= \frac{cb(1-eg)}{g(1-ce)}\prod_{i=0}^{n-1}\frac{(1-(4i+1)ac)(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)(1-(4i+5)eg}{(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)},\\
x_{20n+4} &= \frac{ab(1-df)}{f(1-bd)}\prod_{i=0}^{n-1}\frac{(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)}{(1-(4i+2)ce)(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)},\\
x_{20n+5} &= \frac{ag(1-ce)}{b(1-eg)(1-ac)}\prod_{i=0}^{n-1}\frac{(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)}{(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)(1-(3i+5)eg)(1-(4i+5)ac)},\\
x_{20n+6} &= \frac{egf(1-bd)}{ab(1-df)(1-2eg)}\prod_{i=0}^{n-1}\frac{(1-(4i+2)ce)(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)}{(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)(1-(4i+6)eg)},\\
x_{20n+7} &= \frac{bdf(1-eg)(1-ac)}{ag(1-ce)(1-2df)}\prod_{i=0}^{n-1}\frac{(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)(1-(4i+5)eg)(1-(4i+5)ac)}{(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)(1-(4i+6)df)},\\
x_{20n+8} &= \frac{abc(1-df)(1-2eg)}{gf(1-bd)(1-2ce)}\prod_{i=0}^{n-1}\frac{(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)(1-(4i+6)eg)}{(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)(1-(4i+6)ce)},\\
x_{20n+9} &= \frac{ag(1-ce)(1-2df)}{f(1-eg)(1-ac)(1-2bd)}\prod_{i=0}^{n-1}\frac{(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)(1-(4i+6)df)}{(1-(4i+3)ce)(1-(4i+4)df)(1-(3i+5)eg)(1-(4i+5)ac)(1-(4i+6)ce)},\\
x_{20n+10} &= \frac{gf(1-bd)(1-2ce)}{b(1-df)(1-2eg)(1-2ac)}\prod_{i=0}^{n-1}\frac{(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)(1-(4i+6)ce)}{(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)(1-(4i+6)eg)(1-(4i+6)ac)},\\
x_{20n+11} &= \frac{ef(1-eg)(1-ac)(1-2bd)}{a(1-ce)(1-2df)(1-3eg)}\prod_{i=0}^{n-1}\frac{(1-(4i+3)ce)(1-(4i+4)df)(1-(4i+5)eg)(1-(4i+5)ac)(1-(4i+6)bd)}{(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)(1-(4i+6)df)(1-(4i+7)eg)},\\
x_{20n+12} &= \frac{bd(1-df)(1-2eg)(1-2ac)}{g(1-bd)(1-2ce)(1-3df)}\prod_{i=0}^{n-1}\frac{(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)(1-(4i+6)eg)(1-(4i+6)ac)}{(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)(1-(4i+6)ce)(1-(4i+7)df)},\\
x_{20n+13} &= \frac{ac(1-ce)(1-2df)(1-3eg)}{f(1-eg)(1-ac)(1-2bd)(1-3ce)}\prod_{i=0}^{n-1}\frac{(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)(1-(4i+6)df)(1-(4i+7)eg}{(1-(4i+4)df)(1-(4i+5)eg)(1-(4i+5)ac)(1-(4i+6)bd)(1-(4i+7)ce)}.
\end{aligned}
$$

Proof. We use an inductive proof for this rational recursive sequences. It is easy to see that for $n = 0$, the result holds. Suppose that $n > 0$ and that the assumption is satisfied for $n-1$. That is

$$x_{20n-26}=g\prod_{i=0}^{n-2}\frac{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-4ieg)(1-4iac)}{(1-(4i+1)df)(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-4ice)},$$

$$x_{20n-25}=f\prod_{i=0}^{n-2}\frac{(1-(4i+1)ac)(1-(4i+1)eg)(1-(4i+2)bd)(1-(4i+3)ce)(1-4idf)}{(1-(4i+1)ce)(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-4ibd)},$$

$$x_{20n-24}=e\prod_{i=0}^{n-2}\frac{(1-(4i+1)df)(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-4ice)}{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-4ieg)(1-4iac)},$$

$$x_{20n-23}=d\prod_{i=0}^{n-2}\frac{(1-(4i+1)ce)(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-4ibd)}{(1-(4i+1)ac)(1-(4i+1)eg)(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)},$$

$$x_{20n-22}=c\prod_{i=0}^{n-2}\frac{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-(4i+4)eg)(1-4iac)}{(1-(4i+1)df)(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)},$$

$$x_{20n-21}=b\prod_{i=0}^{n-2}\frac{(1-(4i+1)ac)(1-(4i+1)eg)(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)}{(1-(4i+1)ce)(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)},$$

$$x_{20n-20}=a\prod_{i=0}^{n-2}\frac{(1-(4i+1)df)(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)}{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)},$$

$$x_{20n-19}=\frac{eg}{b(1-eg)}\prod_{i=0}^{n-2}\frac{(1-(4i+1)ce)(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)}{(1-(4i+1)ac)(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)(1-(3i+5)eg)},$$

$$x_{20n-18}=\frac{df}{a(1-df)}\prod_{i=0}^{n-2}\frac{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)}{(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)},$$

$$x_{20n-17}=\frac{cb(1-eg)}{g(1-ce)}\prod_{i=0}^{n-2}\frac{(1-(4i+1)ac)(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)(1-)(4i+5)eg}{(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)},$$

$$x_{20n-16}=\frac{ab(1-df)}{f(1-bd)}\prod_{i=0}^{n-2}\frac{(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)}{(1-(4i+2)ce)(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)},$$

$$x_{20n-15}=\frac{ag(1-ce)}{b(1-eg)(1-ac)}\prod_{i=0}^{n-2}\frac{(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)}{(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)(1-(3i+5)eg)(1-(4i+5)ac)},$$

$$x_{20n-14}=\frac{egf(1-bd)}{ab(1-df)(1-2eg)}\prod_{i=0}^{n-2}\frac{(1-(4i+2)ce)(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)}{(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)(1-(4i+6)eg)},$$

$$x_{20n-13}=\frac{bdf(1-eg)(1-ac)}{ag(1-ce)(1-2df)}\prod_{i=0}^{n-2}\frac{(1-(4i+2)bd)(1-(4i+3)ce)(1-(4i+4)df)(1-(4i+5)eg)(1-(4i+5)ac)}{(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)(1-(4i+6)df)},$$

$$x_{20n-12}=\frac{abc(1-df)(1-2eg)}{gf(1-bd)(1-2ce)}\prod_{i=0}^{n-2}\frac{(1-(4i+2)ac)(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)(1-(4i+6)eg)}{(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)(1-(4i+6)ce)},$$

$$x_{20n-11}=\frac{ag(1-ce)(1-2df)}{f(1-eg)(1-ac)(1-2bd)}\prod_{i=0}^{n-2}\frac{(1-(4i+3)eg)(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)(1-(4i+6)df)}{(1-(4i+3)ce)(1-(4i+4)df)(1-(3i+5)eg)(1-(4i+5)ac)(1-(4i+6)bd)},$$

$$x_{20n-10}=\frac{gf(1-bd)(1-2ce)}{b(1-df)(1-2eg)(1-2ac)}\prod_{i=0}^{n-2}\frac{(1-(4i+3)df)(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)(1-(4i+6)ce)}{(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)(1-(4i+6)eg)(1-(4i+6)ac)},$$

$$x_{20n-9}=\frac{ef(1-eg)(1-ac)(1-2bd)}{a(1-ce)(1-2df)(1-3eg)}\prod_{i=0}^{n-2}\frac{(1-(4i+3)ce)(1-(4i+4)df)(1-(4i+5)eg)(1-(4i+5)ac)(1-(4i+6)bd)}{(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)(1-(4i+6)df)(1-(4i+7)eg)},$$

$$x_{20n-8}=\frac{bd(1-df)(1-2eg)(1-2ac)}{g(1-bd)(1-2ce)(1-3df)}\prod_{i=0}^{n-2}\frac{(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)(1-(4i+6)eg)(1-(4i+6)ac)}{(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)(1-(4i+6)ce)(1-(4i+7)df)},$$

$$x_{20n-7}=\frac{ac(1-ce)(1-2df)(1-3eg)}{f(1-eg)(1-ac)(1-2bd)(1-3ce)}\prod_{i=0}^{n-2}\frac{(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)(1-(4i+6)df)(1-(4i+7)eg}{(1-(4i+4)df)(1-(4i+5)eg)(1-(4i+5)ac)(1-(4i+6)bd)(1-(4i+7)ce)}.$$

Now, it follows from (5.1) that,

$$
\begin{aligned}
x_{20n-6} &= \frac{x_{20n-11}x_{30n-13}}{x_{20n-8}(1-x_{20n-11}x_{30n-13})} \\
&= \frac{\frac{bd}{(1-2bd)}\prod_{i=0}^{n-2}\left(\frac{(1-(4i+2)bd)}{(1-(4i+6)bd}\right)}{\frac{bd(1-df)(1-2eg)(1-2ac)}{g(1-bd)(1-2ce)(1-3df)}\prod_{i=0}^{n-2}\frac{(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)(1-(4i+6)eg)(1-(4i+6)ac)}{(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)(1-(4i+6)ce)(1-(4i+7)df)}\left(1-\frac{bd}{(1-2bd)}\prod_{i=0}^{n-2}\left(\frac{(1-(4i+2)bd)}{(1-(4i+6)bd}\right)\right)} \\
&= \frac{\frac{bd}{(1-(4n-2)bd)}}{\frac{bd(1-df)(1-2eg)(1-2ac)}{g(1-bd)(1-2ce)(1-3df)}\left(1-\frac{bd}{(1-(4n-2)bd)}\right)\prod_{i=0}^{n-2}\frac{(1-(4i+3)bd)(1-(4i+4)ce)(1-(4i+5)df)(1-(4i+6)eg)(1-(4i+6)ac)}{(1-(4i+4)eg)(1-(4i+4)ac)(1-(4i+5)bd)(1-(4i+6)ce)(1-(4i+7)df)}}.
\end{aligned}
$$

Hence, we have

$$x_{20n-6} = g\prod_{i=0}^{n-1}\frac{(1-(4i+1)bd)(1-(4i+2)ce)(1-(4i+3)df)(1-4ieg)(1-4iac)}{(1-(4i+1)df)(1-(4i+2)eg)(1-(4i+2)ac)(1-(4i+3)bd)(1-4ice)}.$$

Also, we see from Equation (5.1) that

$$
\begin{aligned}
x_{20n-5} &= \frac{x_{20n-10}x_{20n-12}}{x_{20n-7}(1-x_{20n-10}x_{20n-12})} \\
&= \frac{\frac{ac}{(1-2ac)}\prod_{i=0}^{n-2}\left(\frac{(1-(4i+2)ac)}{(1-(4i+6)ac}\right)}{\frac{ac(1-ce)(1-2df)(1-3eg)}{f(1-eg)(1-ac)(1-2bd)(1-3ce)}\prod_{i=0}^{n-2}\frac{(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)(1-(4i+6)df)(1-(4i+7)eg}{(1-(4i+4)df)(1-(4i+5)eg)(1-(4i+5)ac)(1-(4i+6)bd)(1-(4i+7)ce)}\left(1-\frac{ac}{(1-2ac)}\prod_{i=0}^{n-2}\left(\frac{(1-(4i+2)ac)}{(1-(4i+6)ac}\right)\right)} \\
&= \frac{\frac{ac}{(1-(4n-2)ac)}}{\frac{ac(1-ce)(1-2df)(1-3eg)}{f(1-eg)(1-ac)(1-2bd)(1-3ce)}\left(1-\frac{ac}{(1-(4n-2)ac)}\right)\prod_{i=0}^{n-2}\frac{(1-(4i+3)ac)(1-(4i+4)bd)(1-(4i+5)ce)(1-(4i+6)df)(1-(4i+7)eg}{(1-(4i+4)df)(1-(4i+5)eg)(1-(4i+5)ac)(1-(4i+6)bd)(1-(4i+7)ce)}}.
\end{aligned}
$$

Hence, we have

$$x_{20n-5} = f\prod_{i=0}^{n-1}\frac{(1-(4i+1)ac)(1-(4i+1)eg)(1-(4i+2)bd)(1-(4i+3)ce)(1-4idf)}{(1-(4i+1)ce)(1-(4i+2)df)(1-(4i+3)eg)(1-(4i+3)ac)(1-4ibd)}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Theorem 5.2. *Equation (5.1) has one equilibrium point* $\bar{x} = 0$ *and this equilibrium point is not locally asymptotically stable.*

Proof. In this section we investigate the local stability character of the solutions of Equation (5.1). Equation (5.1) has a unique positive equilibrium point and is given by

$$\bar{x} = \frac{\bar{x}^2}{\bar{x}\,(1-\bar{x}^2)} = \frac{\bar{x}}{1-\bar{x}^2}, \quad \text{or also} \quad 1 = 1-\bar{x}^2,$$

then the unique equilibrium point is given by $\bar{x} = 0$.

Define the following function

$$
\begin{aligned}
&f : (0,\infty)^3 \to (0,\infty) \\
&f(u,v,w) = \frac{uw}{v\,(1-uw)}.
\end{aligned}
$$

Therefore it follows that

$$f_u(u,v,w) = \frac{w}{v\,(1-uw)^2}, \quad f_v(u,v,w) = -\frac{uw}{v^2\,(1-uw)}, \quad f_w(u,v,w) = \frac{u}{v\,(1-uw)^2}.$$

Then

$$f_u(\bar{x},\bar{x},\bar{x}) = \frac{1}{(1-\bar{x}^2)^2} = 1, \quad f_v(\bar{x},\bar{x},\bar{x}) = -\frac{1}{(1-\bar{x}^2)} = -1, \quad f_w(\bar{x},\bar{x},\bar{x}) = \frac{1}{(1-\bar{x}^2)^2} = 1.$$

The linearized equation of Equation (5.1) about $\bar{x}$ is

$$y_{n+1} - y_{n-6} + y_{n-5} - y_n = 0. \tag{5.2}$$

It follows from Theorem 2.1 that Equation (5.2) is not asymptotically stable. The proof is complete. □

For confirming the results of this section, we consider a numerical example (See Figure 3).

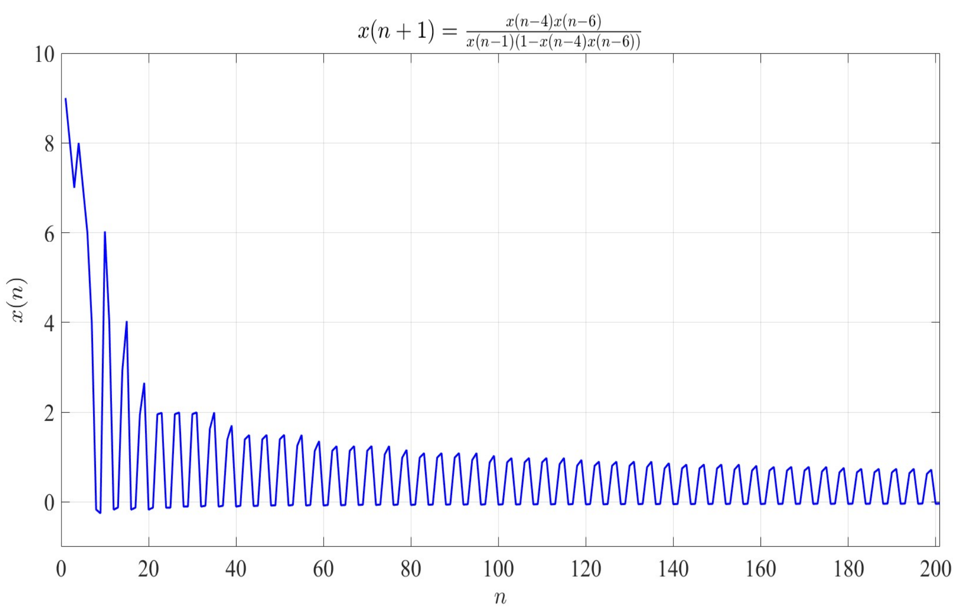

Figure 3. Behavior of the solution of system (5.1). It can be seen that the solution doesn't converge to zero which confirm the fact that the equilibrium point 0 is not locally asymptotically stable. The initial condition is given by $g = 9$, $f = 8$, $e = 7$, $d = 8$, $c = 7$, $b = 6$, and $a = 4$.

In summary, we have derived an explicit infinite-product solution for Equation (5.1) and demonstrated that the unique equilibrium $\bar{x} = 0$ is not locally asymptotically stable. The structure of the solution, with alternating signs in the factors, reflects the influence of the negative product term in the denominator. The analysis confirms that, despite the change in sign compared to the case studied in Section 3, the qualitative instability of the zero equilibrium persists, and the solution exhibits bounded, non-convergent behavior.

6. On the Difference Equation $x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}(-1-x_{n-4}x_{n-6})}$

In this final analytical section, we study the fourth and last variant of Equation (1.1), where both the constant and the product term in the denominator are negative. The resulting recurrence again yields a periodic solution with period 20, though the explicit formulas differ in sign from those obtained in Section 5. We present the complete periodic form of the solution and prove its validity using induction. The zero equilibrium remains unstable, in line with the behavior observed in the previous cases. A numerical simulation is included to illustrate the periodic nature of the solution and to corroborate the theoretical results.

Let us consider the following special case of Equation (1.1):

$$x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}\left(-1 - x_{n-4}x_{n-6}\right)}, \quad n = 0, 1, \ldots, \tag{6.1}$$

where the initial conditions $x_{-6} = g, x_{-5} = f, x_{-4} = e, x_{-3} = d, x_{-2} = c, x_{-1} = b, x_0 = a$ are arbitrary nonzero real numbers.

Theorem 6.1. *Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Equation (6.1). Then the solution of Equation (6.1) is bounded and periodic of period 20 given by:*

$$\begin{aligned}
x_{20n-6} &= g, & x_{20n-5} &= f,\\
x_{20n-4} &= e, & x_{20n-3} &= d,\\
x_{20n-2} &= c, & x_{20n-1} &= b,\\
x_{20n} &= a, & x_{20n+1} &= -\frac{eg}{b(1+eg)},\\
x_{20n+2} &= -\frac{df}{a(1+df)}, & x_{20n+3} &= \frac{bc(1+eg)}{g(1+ce)},\\
x_{20n+4} &= \frac{ab(1+df)}{f(1+bd)}, & x_{20n+5} &= -\frac{ag(1+ce)}{b(1+eg)(1+ac)},\\
x_{20n+6} &= \frac{efg(1+bd)}{ab(1+df)}, & x_{20n+7} &= -\frac{bdf(1+eg)(1+ac)}{ag(1+ce)},\\
x_{20n+8} &= \frac{abc(1+df)}{fg(1+bd)}, & x_{20n+9} &= -\frac{ag(1+ce)}{f(1+eg)(1+ac)},\\
x_{20n+10} &= \frac{fg(1+bd)}{b(1+df)}, & x_{20n+11} &= \frac{ef(1+ac)}{a(1+ce)},\\
x_{20n+12} &= -\frac{bd}{g(1+bd)}, & x_{20n+13} &= -\frac{ac}{f(1+ac)}.
\end{aligned}$$

Proof. For $n = 0$, the result holds. Now suppose that our assumption holds for $n-1$. That is

$$\begin{aligned}
x_{20n-26} &= g, & x_{20n-25} &= f,\\
x_{20n-24} &= e, & x_{20n-23} &= d,\\
x_{20n-22} &= c, & x_{20n-21} &= b,\\
x_{20n-20} &= a, & x_{20n-19} &= -\frac{eg}{b(1+eg)},\\
x_{20n-18} &= -\frac{df}{a(1+df)}, & x_{20n-17} &= \frac{bc(1+eg)}{g(1+ce)},
\end{aligned}$$

$$x_{20n-16} = \frac{ab(1+df)}{f(1+bd)}, \quad x_{20n-15} = -\frac{ag(1+ce)}{b(1+eg)(1+ac)},$$
$$x_{20n-14} = \frac{efg(1+bd)}{ab(1+df)}, \quad x_{20n-13} = -\frac{bdf(1+eg)(1+ac)}{ag(1+ce)},$$
$$x_{20n-12} = \frac{abc(1+df)}{fg(1+bd)}, \quad x_{20n-11} = -\frac{ag(1+ce)}{f(1+eg)(1+ac)},$$
$$x_{20n-10} = \frac{fg(1+bd)}{b(1+df)}, \quad x_{20n-9} = \frac{ef(1+ac)}{a(1+ce)},$$
$$x_{20n-8} = -\frac{bd}{g(1+bd)}, \quad x_{20n-7} = -\frac{ac}{f(1+ac)}.$$

Now it follows from Equation (6.1) that

$$\begin{aligned} x_{20n-6} &= \frac{x_{20n-11}x_{20n-13}}{x_{20n-8}\left(-1 - x_{20n-11}x_{20n-13}\right)} \\ &= \frac{\frac{ag(1+ce)}{f(1+eg)(1+ac)}\frac{bdf(1+eg)(1+ac)}{ag(1+ce)}}{-\frac{bd}{g(1+bd)}\left(-1 - \frac{ag(1+ce)}{f(1+eg)(1+ac)}\frac{bdf(1+eg)(1+ac)}{ag(1+ce)}\right)} \\ &= \frac{bd}{\frac{bd}{g(1+bd)}\left(1+bd\right)} = g. \end{aligned}$$

Similarly

$$\begin{aligned} x_{20n-5} &= \frac{x_{20n-10}x_{20n-12}}{x_{20n-7}\left(-1 - x_{20n-10}x_{20n-12}\right)} \\ &= \frac{\frac{fg(1+bd)}{b(1+df)}\frac{abc(1+df)}{fg(1+bd)}}{-\frac{ac}{f(1+ac)}\left(-1 - \frac{fg(1+bd)}{b(1+df)}\frac{abc(1+df)}{fg(1+bd)}\right)} \\ &= \frac{ac}{\frac{ac}{f(1+ac)}\left(1+ac\right)} = f. \end{aligned}$$

Similarly

$$\begin{aligned} x_{20n-4} &= \frac{x_{20n-9}x_{20n-11}}{x_{20n-6}\left(-1 - x_{20n-9}x_{20n-11}\right)} \\ &= \frac{-\frac{ef(1+ac)}{a(1+ce)}\frac{ag(1+ce)}{f(1+eg)(1+ac)}}{g\left(-1 + \frac{ef(1+ac)}{a(1+ce)}\frac{ag(1+ce)}{f(1+eg)(1+ac)}\right)} \\ &= \frac{-\frac{eg}{(1+eg)}}{g\left(-1 + \frac{eg}{(1+eg)}\right)} = e. \end{aligned}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

□

Theorem 6.2. *Equation (6.1) has a unique equilibrium point which is* 0 *and this equilibrium point is not locally asymptotically stable.*

Proof. The proof is similar to the one of Theorem 3.2 and will be omitted. □

For confirming the results of this section, we consider the following numerical example (See Figure 4).

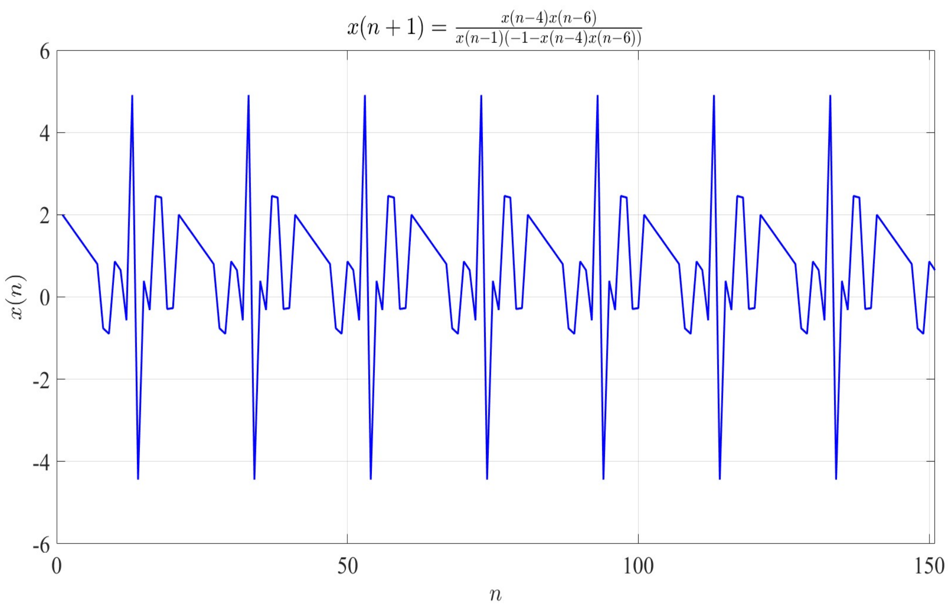

FIGURE 4. Behavior of the solution of system (6.1). It can be seen that the solution periodic and doesn't converge to zero which confirm the fact that the equilibrium point 0 is not locally asymptotically stable. The initial condition is given by $g = 2$, $f = 1.8$, $e = 1.6$, $d = 1.4$, $c = 1.2$, $b = 1$, and $a = 0.8$.

This completes our analysis of the fourth special case of Equation (1.1). We have shown that the solution is periodic with period 20 and that the unique equilibrium $\bar{x} = 0$ is not locally asymptotically stable. The periodic structure, together with the instability of the zero equilibrium, highlights how the sign combination in the denominator can lead to bounded, non-convergent dynamics even when the initial conditions are strictly positive. The results of this section, along with those of Sections 3–4, provide a comprehensive picture of the solution behaviors across all four sign variations of the original equation.

7. Conclusion

We have investigated, in this paper, the behavior of the solution of the higher-order nonlinear difference equation

$$x_{n+1} = \frac{x_{n-4}x_{n-6}}{x_{n-1}\left(\pm 1 \pm x_{n-4}x_{n-6}\right)}, \qquad n = 0, 1, \ldots,$$

under arbitrary positive real initial conditions. Closed-form expressions for the solutions of all four sign combinations have been obtained. In the cases with $+1$ in the denominator, the solutions take the form of infinite products whose factors exhibit a clear periodic dependence on the initial values, revealing a structured combinatorial pattern over indices modulo 20. In the cases with -1 in the denominator, the solutions are purely periodic with period 20, a property that highlights the sensitivity of the dynamics to the sign of the constant term. In each scenario, the unique equilibrium $\bar{x} = 0$ was shown to be unstable, and numerical simulations confirmed the non-convergent, often oscillatory or bounded periodic, behavior of the trajectories.

These results extend the known theory of rational difference equations and provide a complete analytical description for a family of seventh-order recurrences. The methods employed-particularly the inductive derivation of product forms and the linearized stability analysis-can be adapted to study similar equations with different delays or sign structures. Future work may focus on exploring boundedness, persistence, and global attractivity for more general nonlinearities, or on applications where such higher-order recursions naturally arise.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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