Int. J. Anal. Appl. (2026), 24:245

International Journal of Analysis and Applications

Existence and Uniqueness of Solution of Fractional Cauchy Problem

Muayyad Mahmood Khalil[1,*], Hayder Swadi Hamad[1], Estabraq Ismael Fadhil[2]

[1]*Department of Mathematics, College of Education for Pure Sciences, Tikrit University, Tikrit, Iraq*

[2]*Department of Mathematics, College of Education for Women, Tikrit University, Tikrit, Iraq*

**Corresponding author:* medomath80@tu.edu.iq

ABSTRACT. This paper deals with initial value problems (IVPs) for nonlinear first-order modified Caputo fractional differential equations (FDEs). The innovative role of this work is that the nonlinearity f is in L^p spaces, where $1 \leq p < \infty$, rather than in the conventional space of continuous functions. The equivalences between the FDEs and the corresponding integral equations are rigorously obtained. Related to the Banach contraction principle, we derive several new sets of necessary conditions for the existence and uniqueness of solutions to the considered system. These results precisely extend the existing literature by accommodating nonlinearities that exhibit jump discontinuities or integrable singularities, thereby contributing to the theoretical development of fractional calculus and providing broader analytical frameworks for further mathematical investigations.

1. Introduction

The mathematical modeling of a broad range of physical phenomena is characterized by Cauchy problems, which have attracted significant attention in both the science and engineering fields. There are numerous published articles solely focused on the study of the existence and uniqueness of solutions to fractional Cauchy problems on a finite interval. Cauchy problems involving modified Caputo fractional derivatives have been previously introduced and extensively studied both analytically and numerically [1-5]. In most of the previous papers, it is assumed that the nonlinear function f is continuous.

One of the pioneering and acclaimed works on obtaining existence and uniqueness results for modified Caputo FDEs with initial conditions was conducted by Diethelm and Ford in 2002

Received Apr. 15, 2026

2020 Mathematics Subject Classification. 34A08.

Key words and phrases. fractional differential equations; modified Caputo derivative; existence and uniqueness; L^p spaces; Banach contraction principle; fractional Cauchy problem.

https://doi.org/10.28924/2291-8639-24-2026-245

ISSN: 2291-8639

[1]. The existence theorem and the uniqueness theorem are based on Schauder's fixed point theorem and Weissinger's fixed point theorem, respectively. Li and Peng [6] discussed the reliability of the Laplace transform method for solving linear fractional Cauchy problems with constant coefficients and derived the analytic solution in terms of Mittag Leffler functions. Aghajani et al. [7] considered a nonlinear FDE of order $\alpha \in (0,1)$ with a homogeneous initial condition in Banach spaces. The proof of the existence theorem is grounded in the application of the Hausdorff measure of noncompactness alongside an extended variation of Darbo's fixed-point theorem. N'Guerekata [8] addressed Cauchy problems for nonlinear FDEs with nonlocal conditions in Banach spaces. The existence result was established utilizing the Krasnosel'skii fixed-point theorem, while existence and uniqueness result relies on Banach contraction principle. Webb [9] studied unique and exist solution of singular Cauchy problems containing a modified Caputo derivative. Recently, an IVP was studied in Lan and Webb [5]; the existence of nonnegative continuous solutions is proved in that paper when the nonlinearity f satisfies an L^p-Caratheodory condition for some $p > 1/\alpha$.

Sheng and Zhang [10] studied the solution theory in Sobolev space for fractional Cauchy problems, incorporating weakly singular and discontinuous f in the first variable. The existence and uniqueness of a global solution to a fractional neutral delay differential equation was discussed in [11]. Hernandez et al. [12] addressed the existence of mild solutions for a class of first-order abstract FDEs in Banach spaces. Moreover, three recent papers by Webb [13-15] are useful in this regard as well.

While the general strategy of converting FDEs into equivalent integral equations is well known, the precise theoretical improvement of our work lies in the explicit relaxation of the continuity and nonnegativity assumptions. Unlike Diethelm [1] who assumes f is continuous, or Lan and Webb [5] who restrict their focus to nonnegative solutions, our approach rigorously accommodates general nonlinearities in L^p spaces $(1 \leq p < \infty)$. This relaxation is structurally significant because it allows the mathematical model to capture physical processes by abrupt, non-continuous integrable source term that classical continuous models fail to address.

2. Preliminary and Basic Concepts and Definitions

Definition 2.1 ([13,16]). Let $\alpha \in \mathbb{R}_+$, and let $n = \lceil \alpha \rceil$. For $f \in L^1(a,b)$, the modified Caputo fractional derivative of order α is defined when $T_{n-1}[f;a]$ exists and $I_a^{n-\alpha}(f - T_{n-1}[f;a]) \in AC^n[a,b] \cap C^n[a,b]$ by

$$(D_{*a}^{\alpha} f)(t) = \left(D_a^{\alpha}(f - T_{n-1}[f;a])\right)(t) \text{ for a.e. (for all) } t \in [a,b],$$

where

$$T_{n-1}[f;a](t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k$$

is the Taylor polynomial of degree $n-1$ for f at a.

This article is concerned with the following well-known fractional Cauchy problem

$$\begin{cases} D_{*0}^{\alpha}x(t) = f(t, x(t)) \text{ for a.e. } t \in [0, b] \\ x(0) = x_0 \end{cases} \tag{1.1}$$

where $b > 0$ is a finite real number, $\alpha \in (0,1), D_*^{\alpha}$ represents the modified Caputo fractional derivative of order α , f is a given nonlinear function that will be precisely defined later, and the initial data x_0 is given.

The so-called fundamental theorem of fractional calculus (FC) for the modified Caputo fractional derivative, commonly used in obtaining the equivalence result, is well established only for the space of continuous functions and for the L^{∞} space (see [13, 16]). We generalize it, without much difficulty, for fractional-order $\alpha \in (0,1)$, allowing for more general function spaces, particularly in L^p spaces, where $1 \le p < \infty$, as demonstrated in Lemmas 3.1 and 3.3 below.

The main goal of this research, achieved in section 3, A rigorous mathematical analysis is developed to prove the existence and uniqueness results of by assuming $f \in L^p$, which is a significant weaker constraint than continuity. Throughout this work, we consistently apply the Banach contraction principle in all our existence and uniqueness theorems. We first study the existence and uniqueness of solutions to (1.1) in the $L^1(0, b)$ space under the condition where f satisfies $t \mapsto f(t, x(t)) \in L^1(0, b)$ for every $x \in L^1(0, b)$ (see Theorem 3.9 below). This result provides a reassuring and affirmative answer to the concern raised in [18; Remark 3.1] regarding the existence of the initial condition in the L^1 space. Finally, by imposing the specific restriction $p > 1/\alpha$, we investigate the existence and uniqueness of solutions to (1.1) in $C[0, b]$, where f satisfies the conditions $t \mapsto f(t, x(t)) \in L^p(0, b)$ for every $x \in C[0, b]$ (see Theorems 3.14 and 3.15 below). Our results generalize those previous results in the literature obtained under the continuity condition or the Caratheodory-type condition on the nonlinear term f and fill the research gap (at least partially) concerning the study of FDEs in L^p spaces, where $1 \le p < \infty$. For the numerical simulations of the solutions, we employ the so-called predictor-corrector method; see [2-3] for a detailed study.

Throughout this article, we always assume $0 < \alpha < 1$ and define $(x - x_0)(t) := x(t) - x_0$, where x_0 is a constant.

Theorem 2.2 (Young's Inequality). Suppose $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ with $1 \le p \le \infty$.

Then for almost everywhere (a.e.) $t \in \mathbb{R}$, the function $s \mapsto f(t-s)g(s)$ is integrable on $\mathbb{R}$, and we define

$$(f \star g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds$$

Moreover, $f \star g \in L^p(\mathbb{R})$ and

$$\|f \star g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}$$

Lemma 2.3 ([13]). Let $0 < \alpha < 1$. The following mapping properties hold for I_a^α.

(i) For $1 \leq p \leq \infty$, I_a^α is a bounded linear operator from $L^p(a,b)$ into $L^p(a,b)$, and for every $f \in L^p(a,b)$,

$$\|I_a^\alpha f\|_{L^p} \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p}$$

(ii) I_a^α maps$C[a,b]$ into $C[a,b]$, and $\forall f \in C[a,b]$,

$$\|I_a^\alpha f\|_C \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\|f\|_C$$

(iii) I_a^α maps$AC[a,b]$ into $AC[a,b]$.

Lemma 2.4 ([5,13]). Let $0 < \alpha < 1$. For $p > 1/\alpha, I_a^\alpha$ be a compact operator from $L^p(a,b)$ into $C[a,b]$. Furthermore, $\lim_{t\to a^+} I_a^\alpha f(t) = 0$ for every f belongs to L^p.

Lemma 2.5 ([13]). For $0 \leq \gamma \leq \alpha < 1, I_a^\alpha$ is a bounded operator from $C_\gamma(a,b]$ into $C[a,b]$. Moreover, if $f(t) = (t-a)^{-\gamma}\tilde{f}(t)$ with $\tilde{f} \in C[a,b]$, then $\lim_{t\to a^+}(I_a^\alpha f)(t) = 0$ whenever $\gamma < \alpha$.

Lemma 2.6 ([13,16,17]). Let $\alpha \in \mathbb{R}_+$. The following assertions hold.

(i) If $f \in C[a,b]$, then $(D_{*a}^\alpha I_a^\alpha f)(t) = f(t)$ for all $t \in [a,b]$.

(ii) If $f \in L^\infty(a,b)$, then $(D_{*a}^\alpha I_a^\alpha f)(t) = f(t)$ for a.e. $t \in [a,b]$.

Definition 2.7. ([13]). Let $\alpha \in \mathbb{R}_+$, and let $n = \lceil \alpha \rceil$. The Riemann-Liouville fractional derivative of order α is defined for $f \in L^1(a,b)$ with $I_a^{n-\alpha} f \in AC^n[a,b](C^n[a,b])$ by

$$(D_a^\alpha f)(t) = (D^n I_a^{n-\alpha} f)(t) \text{ for a.e. (for all) } t \in [a,b].$$

Note that $I_a^{n-\alpha}$ is always a case of a singular kernel. In particular, for $\alpha = 0$, we set $D_a^0 = \mathrm{I}$, the identity operator.

Definition 2.8. ([13). Let $\alpha \in \mathbb{R}_+$, and let $n = \lceil \alpha \rceil$. The Caputo fractional derivative of order α is defined for $f \in AC^n[a,b](C^n[a,b])$ by

$$({}^C D_a^\alpha f)(t) = (I_a^{n-\alpha} D^n f)(t) \text{ for a.e. (for all) } t \in [a,b].$$

3. Main Results

Before proceeding to main results, we will present some results similar to Lemma 2.6(i), in which functions are to be considered in L^p spaces for all $1 \leq p < \infty$.

Lemma 3.1. Let $f \in L^1(a,b)$ with $\lim_{t\to a^+}(I_a^\alpha f)(t) = 0$. Then there holds

$$(D_{*a}^\alpha I_a^\alpha f)(t) = f(t) \text{ for a.e. } t \in [a,b]$$

Proof. Since $f \in L^1(a,b)$, we infer from Lemma 2.3(i) that $I_a^\alpha f \in L^1(a,b)$. Now we have, for all $t \in [a,b]$,

$$(I_a^{1-\alpha} I_a^\alpha f)(t) = (I_a^1 f)(t)$$

It clearly shows that $I_a^{1-\alpha} I_a^\alpha f \in AC[a,b]$, since $f \in L^1(a,b)$.

We can rigorously apply the derivative definitions since absolute continuity guarantees differentiability almost everywhere (a.e).
Thus, the modified Caputo derivative exists a.e. Then we have

$$D_*^\alpha I_a^\alpha f(t) = D_a^\alpha I_a^\alpha f(t) - 0 = D I_a^1 f(t) = f(t)$$

Since $f \in L^1(a,b)$, by [17, Theorem 1.B], the above relation is valid for a.e. $t \in [a,b]$.

Remark 3.2. The starting point a is playing an important role. In Lemma 3.1, a condition $\lim_{t\to a^+} (I_a^\alpha f)(t) = 0$ is central. It is now evident that such a constraint on the nonlinearity is imperative for analyzing problem (1.1) in the L^1 space. In particular, if f is continuous, the condition then follows directly from Lemma 2.5 (with $\gamma = 0$).

Lemma 3.3. Let $f \in L^p(a,b)$ for some $p > 1/\alpha$. Then there holds

$$(D_{*a}^\alpha I_a^\alpha f)(t) = f(t) \text{ for a.e. } t \in [a,b]$$

Proof. Since $f \in L^p(a,b)$ with $p > 1/\alpha$, we infer from Lemma 2.4 that $I_a^\alpha f \in C[a,b]$ and also $(I_a^\alpha f)(0) = 0$. Clearly, the modified Caputo derivative exists a.e., and we have

$$(D_{*a}^\alpha I_a^\alpha f)(t) = D_a^\alpha (I_a^\alpha f - 0)(t) = D I_a^{1-\alpha} (I_a^\alpha f)(t) = D(I_a^1 f)(t) = f(t)$$

Since $f \in L^p(a,b)$, the above relation is valid for a.e. $t \in [a,b]$.

Remark 3.4. Note that the conclusions of both Lemmas 3.1 and 3.3 generalize that of Lemma 2.6.

Definition 3.5. A function $x \in L^1(0,b)$ is said to be a solution to problem (1.1) if $D_{*0}^\alpha x$ exists a.e. on $[0,b]$ and x satisfies (1.1).

In what follows, we now proceed to present our main results. Below we derive our first equivalence result. This will be essential in the sequel.

Theorem 3.6. Let $f: (0,b) \times \Omega \to \mathbb{R}$ be a function such that $t \mapsto f(t, x(t)) \in L^1(0,b)$ for every $x \in L^1(0,b)$, $\Omega \subset \mathbb{R}$ is open set, then let $\lim_{t\to 0^+} I_0^\alpha f(t, x(t)) = 0$ for every $x \in L^1(0,b)$. Then $x \in L^1(0,b)$ with $I_0^{1-\alpha}(x - x(0)) \in AC[0,b]$ and $(I_0^{1-\alpha} x)(0) = 0$ solves (1.1) if and only if $x \in L^1(0,b)$ satisfies

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, x(s)\big) ds \tag{3.1}$$

for a.e. $t \in [0,b]$.

Proof. Necessity. Let $x \in L^1(0,b)$ with $I_0^{1-\alpha}(x - x(0)) \in AC[0,b]$ and $(I_0^{1-\alpha} x)(0) = 0$, and let x solves (1.1). Now

$$D_*^\alpha x(t) = f\big(t, x(t)\big)$$

$$\Longrightarrow D_0^\alpha \big(x - x(0)\big)(t) = f\big(t, x(t)\big) \Longrightarrow D I_0^{1-\alpha}(x - x_0)(t) = f\big(t, x(t)\big)$$

Integrating both sides, we get

$$I_0^{1-\alpha}(x - x_0)(t) = I_0^1 f\big(t, x(t)\big) \text{ for all } t \in [0,b] (\text{ since } (I_0^{1-\alpha} x)(0) = 0)$$

$$\Longrightarrow I_0^1 (x - x_0)(t) = I_0^{\alpha+1} f\big(t, x(t)\big)$$

Applying I_0^α yields

$$\Longrightarrow x(t) = x_0 + I_0^\alpha f\big(t, x(t)\big) \text{ for a.e. } t \in [0, b]$$

By Lemma 2.3 (i), because $f \in L^1(0, b)$, the integral operator rigorously maps it such that $I_0^\alpha f \in L^1(0, b)$, verifying the space constraint.

Sufficiency. Let $x \in L^1(0, b)$ satisfy a.e. the integral equation (3.1). Because $\lim_{t\to 0^+} I_0^\alpha f(t, x(t)) = 0$ is explicitly given by our hypothesis taking the limit as $t \longrightarrow 0^+$on both sides of (3.1) directly yields $x(0) = x_0$. Applying $I_0^{1-\alpha}$ on both sides of (3.1), and deriving confirms $D_{*0}^\alpha x(t) = f(t, x(t))$ for a.e. $t \in [0, b]$. This completes the proof.

Note that, according to [18, Theorem 2.3], it follows that the solution x belongs to the space $I_0^\alpha(L^1(0, b))$. We refer to [13, 19] for a detailed discussion on the space $I_a^\alpha(L^1(a, b))$.

Remark 3.7. It is worth noting that in theorem 3.6, the condition $\lim_{t\to 0^+} I_0^\alpha f(t, x(t)) = 0$ for every $x \in L^1(0, b)$ plays a significant role in obtaining the given IC $x(0) = x_0$. We can see that by the following very simple examples, the limit of $I_0^\alpha f(t, x(t))$ as $t \to 0^+$may equal a real number (zero or nonzero), or it may not exist at all. For instance, consider the power functions $t^{-\sigma}$ with $0 < \sigma < 1$ (so that $t^{-\sigma} \in L^1(0, b)$). In this case, two possibilities arise:

(i) If $0 < \sigma \le \alpha < 1$, then

$$\lim_{t\to 0^+} I_0^\alpha t^{-\sigma} = \begin{cases} \dfrac{\Gamma(1-\sigma)}{\Gamma(\alpha-\sigma+1)}, & \text{if } \sigma = \alpha \\ 0, & \text{if } \sigma < \alpha \end{cases}$$

(ii) For $0 < \alpha < \sigma < 1$, we have

$$I_0^\alpha t^{-\sigma} = \frac{\Gamma(1-\sigma)}{\Gamma(\alpha-\sigma+1)} t^{\alpha-\sigma}.$$

Clearly, the limit of $I_0^\alpha t^{-\sigma}$ as $t \to 0^+$does not exist.

Lemma 3.8. If the function $F \in L^1(0, b)$, then

$$\lim_{t\to 0^+} t^\alpha F(t) = l (l \in \mathbb{R}) \Longrightarrow \lim_{t\to 0^+} I_0^\alpha F(t) = l\Gamma(1-\alpha).$$

Proof. We note that a proof of this Lemma is closely similar to that of [13, Lemma 6.3] (also see [20, Lemma 3.2] and [17, Proposition 4.1]); therefore, we opt to omit the details here.

Note that in certain circumstances, Lemma 3.8 may provide an easy way to verify the key condition $\lim_{t\to 0^+} I_0^\alpha f(t, x(t)) = 0$ for every $x \in L^1(0, b)$.

Based on the Banach contraction principle, we below present our first existence and uniqueness result for system (1.1) in $L^1(0, b)$, under the (classical) Lipschitz condition on f.

Theorem 3.9. Let f satisfies the conditions in Theorem 3.6. If, in addition, there exists a number $L > 0$ provided

$$|f(t, x_1) - f(t, x_2)| \le L|x_1 - x_2|$$

for a.e. $t \in (0,b)$ and all $x_1, x_2 \in \Omega$, then system (1.1) has a unique solution in $L^1(0,b)$, provided the contraction constant

$$\frac{Lb^\alpha}{\Gamma(\alpha+1)} < 1. \tag{3.2}$$

Proof. We define a nonlinear operator T by $Tx(t) = x_0 + I_0^\alpha f\big(t, x(t)\big)$. Let $x, y \in L^1(0,b)$. Then we have

$$\|Tx - Ty\|_{L^1} \leq I_0^\alpha \|f(\cdot,x) - f(\cdot,y)\|_{L^1} \leq L I_0^\alpha \|x-y\|_{L^1} \leq \frac{Lb^\alpha}{\Gamma(\alpha+1)}\|x-y\|_{L^1}$$

Since $\frac{Lb^\alpha}{\Gamma(\alpha+1)} < 1, T$ is strictly a contraction operator on $L^1(0,b)$**.** The Banach contraction principle rigorously guarantees the existence of a unique fixed point, thereby ensuring a unique solution.

We now direct our focus towards a specific instance of the nonlinear function f, where the resulting solution exhibits continuity. More precisely, $f(t,x(t)) = t^{-\gamma} g(t,x(t))$ for $0 \leq \gamma < 1$.

Theorem 3.10. Let $0 \leq \gamma < \alpha < 1$. Let $g: [0,b] \times \mathbf{\Omega} \to \mathbb{R}$ be a continuous function, the set $\mathbf{\Omega} \subset \mathbb{R}$ is a bounded. Then $x \in C[0,b]$ with $I_0^{1-\alpha}(x - x(0)) \in AC[0,b]$ solves

$$\begin{cases} D_{*0}^\alpha x(t) = t^{-\gamma} g(t,x(t)) & \text{for a.e. } t \in [0,b] \\ x(0) = x_0 \end{cases} \tag{3.3}$$

if and only if $x \in C[0,b]$ satisfies

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} g(s,x(s)) ds \tag{3.4}$$

for all $t \in [0,b]$.

Proof. Let $G(t) = t^{-\gamma} g(t,x(t))$ for all $t \in (0,b]$. Note that $G \in L^1(0,b)$. Since $0 \leq \gamma < 1$ and g is continuous, the function $G \in C_\gamma(0,b]$. Since $\gamma < \alpha$, we infer from Lemma 2.5 that $I_0^\alpha G \in C[0,b]$ and also $\lim_{t\to 0^+} I_0^\alpha G(t) = 0$. The remain of the proof is analogous to Theorem 3.6; therefore, the details are omitted here, except for mentioning that $x \in C[0,b]$, since $I_0^\alpha G \in C[0,b]$.

Theorem 3.10 can also be found in [9, Theorem 4.6] (also see [15, Proposition 5.3]). However, the existence and uniqueness theorem for problem (3.3) is not studied therein. We now present the theorem.

Theorem 3.11. Let $0 \leq \gamma < \alpha < 1$. Let $g: [0,b] \times \mathbf{\Omega} \to \mathbb{R}$ be a continuous function, the set $\mathbf{\Omega} \subset \mathbb{R}$ is a bounded. If, in addition, there exists a bounded function $\varphi: [0,b] \to [0,\infty)$ provided that

$$|g(t,x_1) - g(t,x_2)| \leq \varphi(t)|x_1 - x_2|$$

$\forall t \in [0,b]$ and $x_1, x_2 \in \mathbf{\Omega}$, then problem (3.3) has a unique solution in $C[0,b]$, provided the contraction constant

$$\frac{1}{\Gamma(\alpha)} B(\alpha, 1-\gamma) b^{\alpha-\gamma} \varphi^* < 1 \tag{3.5}$$

where $\varphi^* = \sup_{t\in[0,b]} \{|\varphi(t)|\}$.

Proof. Considering of Theorem 3.10, it is enough to prove the assertion for Eqn. (3.4) in the space $C[0,b]$. We define a nonlinear operator $T: C[0,b] \to C[0,b]$ by

$$(Tx)(t)x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\gamma}g(s,x(s))ds$$

for all $t \in [0,b]$. Since g is continuous and $\gamma < \alpha$, together with Lemma 2.5, $Tx \in C[0,b]$. We now prove that T is a contraction operator on $C[0,b]$. Let $x, \bar{x} \in C[0,b]$ be arbitrary functions. Then we get, $\forall t \in [0,b]$,

$$\begin{aligned}|(Tx)(t)-(T\bar{x})(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\gamma}|g(s,x(s))-g(s,\bar{x}(s))|ds\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\gamma}\varphi(s)|x(s)-\bar{x}(s)|ds\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\gamma}ds\varphi^*\|x-\bar{x}\|_C\\ &= \frac{1}{\Gamma(\alpha)}B(\alpha,1-\gamma)t^{\alpha-\gamma}\varphi^*\|x-\bar{x}\|_C\end{aligned}$$

Taking the supremum over $t \in [0,b]$, this gives

$$\|Tx - T\bar{x}\|_C \le \frac{1}{\Gamma(\alpha)}B(\alpha,1-\gamma)b^{\alpha-\gamma}\varphi^*\|x-\bar{x}\|_C$$

for every $x, \bar{x} \in C[0,b]$, which guarantees, considering of Eqn. (3.5), that T is a contraction operator on $C[0,b]$. Hence the Banach contraction principle asserts that T possesses a unique fixed point in $C[0,b]$, which serves as the solution to problem (3.3). This concludes the proof.

Theorem 3.12. Assume $f: (0,b) \times \mathbf{\Omega} \to \mathbb{R}$ be a function provided that $t \mapsto f(t,x(t)) \in L^p(0,b)$ with $p > 1/\alpha$ for every $x \in C[0,b]$, the set $\mathbf{\Omega} \subset \mathbb{R}$ is a bounded. Then $x \in C[0,b]$ with $I_0^{1-\alpha}(x - x(0)) \in AC[0,b]$ solves Eqn. (1.1) if and only if $x \in C[0,b]$ satisfies

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\big(s,x(s)\big)ds \tag{3.6}$$

for all $t \in [0,b]$.

Proof. Since $t \mapsto f(t,x(t)) \in L^p(0,b)$ with $p > 1/\alpha$ for every $x \in C[0,b]$, the integral $I_0^\alpha f(t,x(t)) \in C[0,b]$ and $\lim_{t\to 0^+} I_0^\alpha f(t,x(t)) = 0$ too according to Lemma 2.4. The remain of the proof is analogous to Theorem 3.6; therefore, the details are omitted here, except for mentioning that $x \in C[0,b]$, since $I_0^\alpha f(t,x(t)) \in C[0,b]$.

The linear version of Theorem 3.12 can be found in the very recent article [5, Lemma 4]. However, the existence and uniqueness theorem for problem (1.1) involving f of the specified nature is not studied therein. We now present the theorem.

Remark 3.13. The integral equation (3.6) differs from (3.1). The difference is that (3.6) holds for all $t \in [0, b]$, whereas (3.1) holds for a.e. $t \in [0, b]$.

Theorem 3.14. Let $f: (0, b) \times \mathbf{\Omega} \to \mathbb{R}$ be a function such that $t \mapsto f(t, x(t)) \in L^p(0, b)$ with $p > 1/\alpha$ for every $x \in C[0, b]$, the set $\mathbf{\Omega} \subset \mathbb{R}$ is a bounded. If, in addition, we can imply existence of a function $\psi \in L^{1/\sigma}((0, b), [0, \infty))$ with $\sigma \in (0, \alpha)$ provided that

$$|f(t, x_1) - f(t, x_2)| \leq \psi(t)|x_1 - x_2|$$

for a.e. $t \in (0, b)$ and all $x_1, x_2 \in \mathbf{\Omega}$, hence system (1.1) has a unique solution in $C[0, b]$, such that the contraction constant

$$\frac{1}{\Gamma(\alpha)} \left(\frac{1 - \sigma}{\alpha - \sigma}\right)^{1-\sigma} b^{\alpha-\sigma} \|\psi\|_{L^{\frac{1}{\sigma}}} < 1 \tag{3.7}$$

Proof. In view of Theorem 3.12, it suffices to show the assertion for (3.6) in the space $C[0, b]$. We define a nonlinear operator $\mathcal{G}: C[0, b] \to C[0, b]$ by

$$(\mathcal{G}x)(t) := x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x(s)) ds \tag{3.8}$$

for all $t \in [0, b]$. Clearly, if $x \in C[0, b]$, since $t \mapsto f(t, x(t)) \in L^p(0, b)$ with $p > 1/\alpha$ for every $x \in C[0, b]$, therefore the integral $I_0^\alpha f(t, x(t)) \in C[0, b]$ and also $\mathcal{G}x \in C[0, b]$ by using Lemma 2.4,. We now show that $\mathcal{G}$ is a contraction operator on $C[0, b]$. Suppose $x, \tilde{x} \in C[0, b]$ be arbitrary functions. Then $\forall t \in [0, b]$,

$$\begin{aligned}
|(\mathcal{G}x)(t) - (\mathcal{G}\tilde{x})(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |f(s, x(s)) - f(s, \tilde{x}(s))| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \psi(s)|x(s) - \tilde{x}(s)| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \psi(s) ds \|x - \tilde{x}\|_C \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\frac{1 - \sigma}{\alpha - \sigma}\right)^{1-\sigma} t^{\alpha-\sigma} \|\psi\|_{L^{1/\sigma}} \|x - \tilde{x}\|_C
\end{aligned}$$

Taking the supremum over $t \in [0, b]$, this gives

$$\|\mathcal{G}x - \mathcal{G}\tilde{x}\|_C \leq \frac{1}{\Gamma(\alpha)} \left(\frac{1 - \sigma}{\alpha - \sigma}\right)^{1-\sigma} b^{\alpha-\sigma} \|\psi\|_{L^{1/\sigma}} \|x - \tilde{x}\|_C$$

for every $x, \tilde{x} \in C[0, b]$, which guarantees, considering of (3.7), that means $\mathcal{G}$ is a contraction operator on $C[0, b]$. Therefore, the Banach contraction principle asserts that $\mathcal{G}$ possesses a unique fixed point in $C[0, b]$, which serves as the solution to problem (1.1). We conclude the proof.

In articles [21,22], the author elegantly modified the (classical) Lipschitz condition, which also serves as a fractional analog of the classical Nagumo's condition [23]. With this assumption regarding f.

Theorem 3.15. Let $f:(0,b)\times\mathbf{\Omega}\to\mathbb{R}$ be a function provided that $t\mapsto f(t,x(t))\in L^p(0,b)$ with $p>1/\alpha$ for every $x\in C[0,b]$, the set $\mathbf{\Omega}\subset\mathbb{R}$ is a bounded. If, in addition, there exists an $M>0$ such that

$$|f(t,x_1)-f(t,x_2)|\le Mt^{-\alpha}|x_1-x_2|$$

for a.e. $t\in(0,b)$ and all $x_1,x_2\in\mathbf{\Omega}$, then problem (1.1) has a unique solution in $C[0,b]$, such that the contraction constant

$$M\Gamma(1-\alpha)<1. \tag{3.9}$$

Proof. Here, we only need to show that $\mathcal{G}$ defined in (3.8) is a contraction operator on $C[0,b]$. Let $x,\tilde{x}\in C[0,b]$ be arbitrary functions. Then we obtain, $\forall t\in[0,b]$,

$$\begin{aligned}|(\mathcal{G}x)(t)-(\mathcal{G}\tilde{x})(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s))-f(s,\tilde{x}(s))|ds\\ &\le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\alpha}|x(s)-\tilde{x}(s)|ds\\ &\le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\alpha}ds\|x-\tilde{x}\|_C\\ &= \frac{M}{\Gamma(\alpha)}B(\alpha,1-\alpha)\|x-\tilde{x}\|_C\end{aligned}$$

Taking the supremum over $t\in[0,b]$, this gives

$$\|Gx-G\tilde{x}\|_C\le M\Gamma(1-\alpha)\|x-\tilde{x}\|_C$$

for every $x,\tilde{x}\in C[0,b]$, which ensures, considering of Eqn. (3.9), that means G is a contraction operator on $C[0,b]$. Thus, Banach contraction principle asserts that G possesses a unique fixed point in $C[0,b]$, which serves as the solution to problem (1.1). We conclude the proof.

Remark 3.16. Since $Mt^{-\alpha}\in L^1(0,b)$ for $0<\alpha<1$, the generalized Lipschitz condition imposed on f in Theorem 3.15 is entirely different from the one considered in Theorem 3.14.

Remark 3.17. The results presented in this paper cannot be studied with the usual Caputo fractional derivative because the requirements to define it (see Definition 2.8) are not fulfilled under the aforementioned assumptions in each theorem. Even when dealing with a continuous nonlinearity, the same issues persist unchanged. For more detailed discussions about its severe disadvantages, we refer to [13,17].

4. Examples

Example 4.1. Consider the nonlinear problem

$$D_*^\alpha x(t)=\left|\frac{1}{4}-t\right|^{-0.95}+(1+t)^{-2}+\frac{1}{6}\cos(x(t)),x(0)=3 \tag{3.10}$$

for a.e. $t\in[0,1]$, where $b=1,x_0=3$, and

$$f(t,x(t))=\left|\frac{1}{4}-t\right|^{-0.95}+(1+t)^{-2}+\frac{1}{6}\cos(x(t))$$

Clearly, the function $t \mapsto f(t,x(t)) \mapsto \left|\frac{1}{4} - t\right|^{-0.95} + (1+t)^{-2} + \frac{1}{6}\cos(x(t))$ belongs to $L^1(0,1)$ for every $x \in L^1(0,1)$. Furthermore, for all $\alpha \in (0,1)$, the condition $\lim_{t\to 0^+} I_0^\alpha f(t,x(t)) = 0$ holds for every $x \in L^1(0,1)$ by Lemma 3.8.

Using the mean-value theorem on the cosine term, we obtain the Lipschitz constant $L = \frac{1}{6}$. Hence,

$$\frac{Lb^\alpha}{\Gamma(\alpha+1)} = \frac{1/6}{\Gamma(\alpha+1)} < 1 \text{ for all } \alpha \in (0,1)$$

Thus, by Theorem 3.9, this problem has a unique solution in $L^1(0,1)$ for all $\alpha \in (0,1)$.

Example 4.2. Consider the nonlinear problem

$$D_*^\alpha x(t) = t^{-0.5} + e^{-t} \cdot \frac{2}{9}\ln(1+|x(t)|)\,, x_0 = 1 \tag{3.11}$$

for a.e. $t \in [0,1]$, where $b = 1, \gamma = 0.5, x_0 = 1$. Write the right-hand side in the form $t^{-\gamma} g(t,x(t))$:

$$f(t,x(t)) = t^{-0.5}\left(1 + t^{0.5}e^{-t}\frac{2}{9}\ln(1+|x(t)|)\right)$$

Here $g(t,x) = 1 + t^{0.5}e^{-t}\frac{2}{9}\ln(1+|x|)$, which is continuous.

Since $\left|\frac{d}{dx}\ln(1+|x|)\right| \le 1$, we may take $\varphi(t) = \frac{2}{9}t^{0.5}e^{-t}$. On $[0,1]$ we have $t^{0.5}e^{-t} < 1$, hence $\varphi^* \le \frac{2}{9}$; set $\varphi^* = \frac{2}{9}$.

The contraction constant becomes

$$\frac{1}{\Gamma(\alpha)}B(\alpha, 1-\gamma)b^{\alpha-\gamma}\varphi^* = \frac{2/9}{\Gamma(\alpha)}B(\alpha, 0.5).$$

For $\alpha = 0.8$ this value is less than 1. Consequently, by Theorem 3.11, the problem has a unique solution in $C[0,1]$ whenever the contraction constant is less than 1 (which holds for $\alpha \in (0.5,1)$ satisfying the inequality).

Example 4.3. Consider the nonlinear problem

$$D_*^\alpha x(t) = \frac{5}{1+t^2} + \frac{1}{5}t^{-1/4}\frac{|x(t)|}{1+|x(t)|}, x_0 = 2 \tag{3.12}$$

for a.e. $t \in [0,1]$, where $b = 1, x_0 = 2$, and

$$f(t,x(t)) = \frac{5}{1+t^2} + \frac{1}{5}t^{-1/4}\frac{|x(t)|}{1+|x(t)|}$$

Clearly $t \mapsto f(t,x(t)) \in L^p(0,1)$ for a suitable p (depending on α). Using $\left|\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right| \le |x - y|$, we verify the generalized Lipschitz condition with $\psi(t) = \frac{1}{5}t^{-1/4}$.

Now compute $\|\psi\|_{L^{1/\sigma}} = \left(\int_0^1 \left(\frac{1}{5}s^{-1/4}\right)^{1/\sigma} ds\right)^\sigma$. The integral converges if $\frac{1}{4\sigma} < 1$, i.e. $\sigma > \frac{1}{4}$. Choose $\sigma = 0.3$; then

$$\|\psi\|_{L^{1/\sigma}} = \frac{1}{5}\left(\frac{1}{1-0.25/0.3}\right)^{0.3}$$

We finally check the condition

$$\frac{1}{\Gamma(\alpha)}\left(\frac{1-\sigma}{\alpha-\sigma}\right)^{1-\sigma} b^{\alpha-\sigma}\|\psi\|_{L^{1/\sigma}} < 1$$

For $\alpha \in [0.8,1)$ the left-hand side is indeed less than 1. Hence, by Theorem 3.14, the problem possesses a unique solution in $C[0,1]$.

5. Conclusions

In this article, we have established existence and uniqueness results for the fractional Cauchy problem in L^p spaces under the weak integrability assumption on f. The rigorous equivalence and the use of the Banach contraction principle generalize previous works that required continuity or Caratheodory conditions. Crucially, these generalizations have significant mathematical implications. By explicitly relaxing the restriction to continuous functions, our theoretical framework directly supports applications in modern control theory, viscoelastic materials, biological population models with singular growth, anomalous diffusion modeling and many engineering problems, where driving forces frequently exhibit discontinuous, impulsive, or abruptly changing behaviors. Such physical phenomena fall naturally into L^p spaces, making our results highly applicable where classical theorems fail. Future work will extend this framework to investigate stability theory for impulsive fractional equations and systems subject to nonlocal boundary conditions, further bridging the gap between abstract operator theory and applied physics.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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