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On Practical Asymptotically Optimal Cubature Formulas in Sobolev Space $\bar{L}_p^{(m)}(S_n)$

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Abstract. In investigating various problems related to the theory of approximate integration, partial differential equations, and other areas of analysis, the functional approach has proven to be highly effective. This approach involves treating the differential equation with boundary conditions as an operator acting in a specifically chosen functional space. The necessary information is derived from the properties of this operator. When addressing problems in approximate integration and differential equations, the appropriate selection of functional spaces is crucial for success. S.L. Sobolev exemplified this method clearly in his well-known study of polyharmonic equations. In our research, we examined cubature formulas within the Sobolev functional space $\bar{L}_p^{(m)}(S_n)$ for functions defined on n-dimensional unit sphere S. This issue requires careful attention when developing the most efficient formulas. In this paper, we discussed formulas that fulfill this criterion and refered to them as "practical" in accordance with N.S. Bakhvalov's terminology.

1. Introduction

Currently, many mathematical and physical models require the calculation of integrals in problems. In such cases, optimal quadrature formulas are needed to calculate the approximate value of the integral with high accuracy. Quadrature formulas are used to estimate definite integrals and are essential for solving differential and integral equations numerically. Many scientists have worked on the problems of approximate calculation of integrals and minimization of the norm of the error function, for example [10–21].

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These sources study optimality problems concerning specific spaces, with many focusing on Sobolev spaces [1]. L.V.Voitishek and N.I.Blinov developed algorithms and cubature formulas for certain polyhedral using their proposed methods [7].

This paper examines formulas known as practical asymptotic optimal cubature formulas. These formulas are considered the most efficient in the Sobolev space concerning the function's norm.

We extend functions $f(\theta) \in B$ to the entire space R^n, considering them constant on the rays going out of the center of sphere S and denote them by $\bar{f}(x)$.

Consider the error of the cubature formula:

$$\int\limits_S f(\theta)\, d\theta \approx \sum_{\lambda=1}^{N} C_\lambda f\left(\theta^{(\lambda)}\right), \tag{1.1}$$

defined on functions from B:

$$\ell_N[f] = <\ell_N, f> = \int\limits_S f(\theta)\, d\theta - \sum_{\lambda=1}^{N} C_\lambda f\left(\theta^{(\lambda)}\right) = \int\limits_{R^n} \ell_N(x) f(x)\, dx, \tag{1.2}$$

$$\ell_N(x) = \delta_S(1-r) - \sum_{\lambda=1}^{N} C_\lambda \delta\left(x - \theta^{(\lambda)}\right),$$

$\delta\left(x-\theta^{(\lambda)}\right)$, $\delta_S(1-r)$ are the Dirac delta-functions, $\sum\limits_{\lambda=1}^{N} C_\lambda = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, $r = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$.

The norm of the error functional in the space B^* is determined using the following formula [1,2]:

$$\left\|\ell_N | B^*\right\| = \sup_{f \in B, \|f\| \neq 0} \frac{\left|<\ell_N, f>\right|}{\left\|f|B\right\|}.$$

Definition 1.1. *By the space $L_2^m(S)$, we denote the Sobolev space of functions defined on the n-dimensional unit sphere S. The elements of this space are functions possessing square-integrable generalized derivatives up to order m, inclusive, with the norm defined by the equality:*

$$\left\|f | L_2^m(S)\right\|^2 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} a_{k,\ell}^2 k^m (k+n-2)^m, \tag{1.3}$$

here $2m > n$.

In the space $L_2^m(S)$ the norm of the error functional ℓ_N has the following form:

$$\left\|\ell_N | L_2^{m*}(S)\right\| = \left\{ \sum_{k=1}^{\infty} \sum_{\ell=1}^{\sigma(n,k)} \frac{\left[\sum\limits_{\lambda=1}^{N} C_\lambda Y_{k,\ell}(\theta)\right]^2}{k^m (k+n-2)^m} \right\}^{1/2} \tag{1.4}$$

where

$1 \le \ell \le \sigma(n,k)$, $a_{k,\ell} = \int\limits_S Y_{k,\ell}(\theta) f(\theta)\, d\theta$, $\sigma(n,k) = \frac{(k+n-3)!}{k!(n-2)!}(n+2k-2)$,

$Y_{k,\ell}(\theta)$ is the spherical harmonic of order k of form ℓ.

2. Statement of the problem

In this paper, practical asymptotic optimal cubature formulas in the Sobolev space $\overline{L}_p^{(m)}(S_n)$ are considered. To solve the problem posed, first, we give the definition of the space $L_p^{(m)}(S_n)$.

Definition 2.1. *The space $L_p^{(m)}(S_n)$ consists of functions specified on the n-dimensional unit sphere S_n that have p-th power integrable generalized derivatives of order m. The norm of the element f in this space is respectively defined by the following formula[1,9]:*

$$\left\| f(\theta)\,|L_p^{(m)}(S_n) \right\| = \left\{ \int\limits_{S_n} \left\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left[D^{\alpha} f(\theta) \right]^2 \right\}^{\frac{p}{2}} d\theta \right\}^{\frac{1}{p}}, \tag{2.1}$$

in this formula
$|\alpha| = \alpha_1 + \alpha_2 + ... \alpha_n, \quad d\theta = d\theta_1 d\theta_2 ... d\theta_n, \quad D^{\alpha} f(\theta) = \frac{\partial^{|\alpha|} f(\theta_1,...,\theta_n)}{\partial\theta_1^{\alpha_1} \partial\theta_2^{\alpha_2} ... \partial\theta_n^{\alpha_n}}.$

On the n-dimensional unit sphere S_n, the cubature formula (1.1) becomes:

$$\int\limits_{S_n} f(\theta) d\theta \cong \sum_{\lambda=1}^{N} C_{\lambda} f(\theta^{(\lambda)}). \tag{2.2}$$

The error functional of the cubature formula (2.2) has the following form:

$$\ell_N(\theta) = \varepsilon_{S_n}(\theta) - \sum_{\lambda=1}^{N} C_{\lambda} \delta(\theta - \theta^{(\lambda)}), \tag{2.3}$$

where $\delta(\theta)$ is Dirac's delta-function.
The space $L_2^m(S_n)$ coincides with the similar Sobolev space $L_2^{(m)}(S_n)$ by the norm [3–5]:

$$\left\| f | L_2^{(m)}(S_n) \right\| = \left\{ \int\limits_{S_n} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(D^{\alpha} f(\theta) \right)^2 d\theta \right\}^{\frac{1}{2}}. \tag{2.4}$$

This means that norms (1.3) and (2.4) are equivalent.
Now instead of space $L_2^{(m)}(S_n)$, we will consider the space $\bar{L}_p^{(m)}(S_n)$, and define the norm as follows.

Definition 2.2. *Spaces $\bar{L}_p^{(m)}(S_n)$- are defined as, which norm of functions is determined by the following equality:*

$$\left\| f(\theta)\,|\bar{L}_p^{(m)}(S_n) \right\| = \left\{ \int\limits_{S_n} \left(\left(\frac{\partial^m f(\theta)}{\partial{\theta_1}^{m_1} \partial{\theta_2}^{m_2} ... \partial{\theta_n}^{m_n}} \right)^2 \right)^{\frac{p}{2}} d\theta \right\}^{\frac{1}{p}}, \tag{2.5}$$

where $m_1 + m_2 + ... + m_n = m, \quad m_i > 0, \quad i = 1, 2, ..., n, \quad d\theta = d\theta_1 d\theta_2 ... d\theta_n.$

Let in (2.1) $p = 2$, $n = 2$ and $m = 2$, then we obtain:

$$\int_{S_n}\sum_{|\alpha|=m}\frac{m!}{\alpha!}\left(\frac{\partial^m f(\theta)}{\partial\theta_1{}^{m_1}\partial\theta_2{}^{m_2}...\partial\theta_n{}^{m_n}}\right)^p d\theta=\int_{S_2}\sum_{\alpha_1+\alpha_2=2}\frac{2!}{\alpha_1!\alpha_2!}\left(\frac{\partial^2 f(\theta)}{\partial\theta_1^{\alpha_1}\partial\theta_2^{\alpha_2}}\right)^2 d\theta=$$

$$=\int_{S_2}\left[\left(\frac{\partial^2 f(\theta)}{\partial\theta_2^2}\right)^2+\frac{2!}{1!\cdot 1!}\left(\frac{\partial^2 f(\theta)}{\partial\theta_1\partial\theta_2}\right)^2+\left(\frac{\partial^2 f(\theta)}{\partial\theta_1^2}\right)^2 d\theta\right] \tag{2.6}$$

For $n=2$ and $m=2$, equality (2.5) takes the following form:

$$\left\|f(\theta)\,|\bar{L}_p^{(2)}(S_2)\right\|^p=\int_{S_2}\left(\left(\frac{\partial^2 f(\theta)}{\partial\theta_1^{m_1}\partial\theta_2^{m_2}}\right)^2\right)^{\frac{p}{2}} d\theta \tag{2.7}$$

It is evident that evaluating the right-hand side of (2.7) requires fewer computations than (2.6), since only mixed derivatives are involved in the norm (2.7).

3. Method of solving problems

We will prove the following theorem.

Theorem 3.1. *If for the cubature formula (2.2) over the space $\bar{L}_p^{(m)}(S_n)$, the error functional (2.3) the following holds [8]*

$$\ell_N(\theta)=\ell_{N_1}(\theta_1)\otimes\ell_{N_2}(\theta_2)\otimes...\otimes\ell_{N_n}(\theta_n)$$

and

$$\left\|\ell_{N_i}(\theta_i)\,|\bar{L}_p^{(m_i)*}(\omega_i)\right\|\le d_i\frac{1}{N_i^{m_i}}, \tag{3.1}$$

i.e.

$$\left\|\ell_{N_i}(\theta_i)\,|\bar{L}_p^{(m_i)*}(\omega_i)\right\|\le d_i o\left(h^{m_i}\right),\quad d_i \text{ are constants}, (i=1,2,...,n), \tag{3.2}$$

then

$$\left\|\ell_N(\theta)\,|\bar{L}_p^{(m)*}(S_n)\right\|\le d\cdot\frac{1}{\prod\limits_{i=1}^{n}N_i^{m_i}}, \tag{3.3}$$

or

$$\left\|\ell_N(\theta)\,|\bar{L}_p^{(m)*}(S_n)\right\|\le d\cdot o\left(h^m\right) \tag{3.4}$$

where

$$\ell_{N_i}(\theta_i)=\varepsilon_{\omega_i}(\theta_i)-\sum_{\lambda_i=1}^{N_i}C_{\lambda_i}\delta\left(\theta_i-\theta_i^{(\lambda_i)}\right)$$

$d=\prod\limits_{i=1}^{n}d_i$, $\quad m=m_1+m_2+...+m_n$, $\quad m_i$ *- are arbitrary* $(i=1,2,...,n)$, *i.e.,* $\quad 0\le m_i\le m \quad$ *and*

$$\omega_i=\begin{cases}[0,2\pi], & if\;\; i=n\\ [0,\pi], & if\;\; i=1,2,...,n-1.\end{cases}$$

The proof is obtained by the method of mathematical induction.
Let $n = 2$, then
$\theta = (\theta_1, \theta_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $m = m_1 + m_2$, $d\theta = d\theta_1 d\theta_2$, $f(\theta) = f(\theta_1, \theta_2)$ and $\ell_N(\theta) = \ell_{N_1}(\theta_1) \otimes \ell_{N_2}(\theta_2)$.
If we assume $n = 1$ in (2.5), then, we obtain

$$\left\| f(\theta_i) | \bar{L}_p^{(m_i)}(\omega_i) \right\| = \left\{ \int\limits_{\omega_i} \left(\left(\frac{\partial^{m_i} f(\theta_i)}{\partial \theta_i^{m_i}} \right)^2 \right)^{\frac{p}{2}} d\theta_i \right\}^{\frac{1}{p}}, \quad (i = 1, 2, ..., n).$$

Thus, we have

$$\begin{aligned} &\left| < \ell_N(\theta_1, \theta_2), f(\theta_1, \theta_2) > \right| = \left| < \ell_{N_2}(\theta_2), < \ell_{N_1}(\theta_1), f(\theta_1, \theta_2) >> \right| \leq \\ &\leq \left\| \ell_{N_2}(\theta_2) | \bar{L}_p^{(m_2)*}(\omega_i) \right\| \cdot \left\| < \ell_{N_1}(\theta_1), f(\theta_1, \theta_2) > | \bar{L}_p^{(m_2)}(\omega_i) \right\|. \end{aligned} \tag{3.5}$$

Let us calculate the following norm:

$$\begin{aligned} &\left\| < \ell_{N_1}(\theta_1), f(\theta_1, \theta_2) > | \bar{L}_p^{(m_2)}(\omega_i) \right\| = \\ &= \left\{ \int\limits_{\omega_i} \left| \frac{\partial^{m_2}}{\partial \theta_2^{m_2}} < \ell_{N_1}(\theta_1), f(\theta_1, \theta_2) > \right|^p d\theta_2 \right\}^{\frac{1}{p}} = \\ &= \left\{ \int\limits_{\omega_i} \left| < \ell_{N_1}(\theta_1), \frac{\partial^{m_2}}{\partial \theta_2^{m_2}} f(\theta_1, \theta_2) > \right|^p d\theta_2 \right\}^{\frac{1}{p}} \leq \\ &\leq \left\{ \int\limits_{\omega_i} \left[\left\| \ell_{N_1}(\theta_1) | \bar{L}_p^{(m_1)*}(\omega_i) \right\| \cdot \left\| \frac{\partial^{m_2}}{\partial \theta_2^{m_2}} f(\theta_1, \theta_2) | \bar{L}_p^{(m_1)}(\omega_i) \right\| \right]^p d\theta_2 \right\}^{\frac{1}{p}} = \\ &= \left\| \ell_{N_1}(\theta_1) | \bar{L}_p^{(m_1)*}(\omega_i) \right\| \cdot \left\{ \int\limits_{\omega_i} \left\{ \int\limits_{\omega_i} \left[\left[\frac{\partial^{m_1+m_2}}{\partial \theta_1^{m_1} \partial \theta_2^{m_2}} f(\theta_1, \theta_2) \right]^2 \right]^{\frac{p}{2}} d\theta_1 \right\} d\theta_2 \right\}^{\frac{1}{p}} = \\ &= \left\| \ell_{N_1}(\theta_1) | \bar{L}_p^{(m_1)*}(\omega_i) \right\| \cdot \left\| f(\theta) | \bar{L}_p^{(m)}(S_2) \right\|, \end{aligned} \tag{3.6}$$

where $\theta = (\theta_1, \theta_2)$ and $m = m_1 + m_2$.
Thus, from (3.5) and (3.6), we obtain:

$$\begin{aligned} &\left| < \ell_N(\theta_1, \theta_2), f(\theta_1, \theta_2) > \right| \leq \left\| \ell_{N_2}(\theta_2) | \bar{L}_p^{(m_2)*}(\omega_i) \right\| \cdot \\ &\cdot \left\| \ell_{N_1}(\theta_1) | \bar{L}_p^{(m_1)*}(\omega_i) \right\| \cdot \left\| f(\theta) | \bar{L}_p^{(m)}(S_2) \right\|. \end{aligned} \tag{3.7}$$

With (2.5), from (3.7) we obtain

$$\left\| \ell_N(\theta) | \bar{L}_2^{(m)*}(S_2) \right\| \leq \left\| \ell_{N_1}(\theta_1) | \bar{L}_p^{(m_1)*}(\omega_i) \right\| \cdot \left\| \ell_{N_2}(\theta_2) | \bar{L}_p^{(m_2)*}(\omega_i) \right\|. \tag{3.8}$$

Considering (3.1), from (3.8) we obtain

$$\left\|\ell_N(\theta)\,|\bar{L}_p^{(m)*}(S_2)\right\| \leq d_1 \cdot d_2 \cdot \frac{1}{N_1^{m_1} \cdot N_2^{m_2}}$$

i.e.,

$$\left\|\ell_N(\theta)\,|\bar{L}_p^{(m)*}(S_2)\right\| \leq d \cdot o\left(h^{m_1}\right) \cdot o\left(h^{m_2}\right), \tag{3.9}$$

where $d = d_1 \cdot d_2$.

For $n = k$ we have

$$\left|<\ell_N(\theta), f(\theta)>\right| = \left|<\ell_N(\theta_1,\theta_2,...,\theta_k), f(\theta_1,\theta_2,...,\theta_k)>\right| =$$

$$\left|<\ell_{N_k}(\theta_k), <\ell_{N_{k-1}}(\theta_{k-1}), ..., <\ell_{N_1}(\theta_1), f(\theta_1,\theta_2,...,\theta_k)>, ..., >\right| \leq$$

$$\leq \left\|\ell_{N_k}(\theta_k)\,|\bar{L}_p^{(m_k)*}(\omega_i)\right\| \cdot \left\|\ell_{N_{k-1}}(\theta_{k-1})\,|\bar{L}_p^{(m_{k-1})*}(\omega_i)\right\| ...$$

$$\cdot \left\|<\ell_{N_1}(\theta_1), f(\theta_1,\theta_2,...,\theta_k)>|\bar{L}_p^{(m_1)*}(\omega_i)\right\| \leq$$

$$\leq \left\|\ell_{N_k}(\theta_k)\,|\bar{L}_p^{(m_k)*}(\omega_i)\right\| ... \left\|\ell_{N_1}(\theta_1)\,|\bar{L}_p^{(m_1)*}(\omega_i)\right\| \cdot \left\|f(\theta)\,|\bar{L}_p^{(m)}(S_k)\right\|, \tag{3.10}$$

where $\omega_i = \begin{cases} [0,2\pi], & if \;\; i = k \\ [0,\pi], & if \;\; i = \overline{1,k-1} \end{cases}$.

From (3.10), taking into account (2.5), we obtain:

$$\left\|\ell_N(\theta)\,|\bar{L}_p^{(m)*}(S_k)\right\| \leq \left\|\ell_{N_1}(\theta_1)\,|\bar{L}_p^{(m_1)*}(\omega_i)\right\| ... \left\|\ell_{N_k}(\theta_k)\,|\bar{L}_p^{(m_k)*}(\omega_i)\right\| \tag{3.11}$$

Then, keeping in mind (3.1), from (3.11) we get:

$$\left\|\ell_N(\theta)\,|\bar{L}_p^{(m)*}(S_k)\right\| \leq d\frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_k^{m_k}} \tag{3.12}$$

or taking into account (3.2), from (3.10), we have

$$\left\|\ell_N(\theta)\,|\bar{L}_p^{(m)*}(S_k)\right\| \leq d \cdot o\left(h^{m_1}\right) ... o\left(h^{m_k}\right),$$

where $d = \prod_{i=1}^{k} d_i$.

We use the assumption that the theorem is true for $n = k$ to establish the result for $n = k+1$. Thus, for $n = k+1$ with account for (2.5), from (3.10), we have

$$\left|<\ell_{k+1}(\theta_1,\theta_2,...,\theta_{k+1}), f(\theta_1,\theta_2,...,\theta_{k+1})>\right| =$$

$$= \left|<\ell_{N_1}(\theta_1), <\ell_{N_2}(\theta_2), <, ..., <\ell_{N_{k+1}}(\theta_{k+1}), f(\theta_1,\theta_2,...,\theta_{k+1})> ... >\right| \leq$$

$$\leq \left\|\ell_{N_1}(\theta_1)\,|\bar{L}_p^{(m_1)*}(\omega_i)\right\| ... \left\|\ell_{N_k}(\theta_k)\,|\bar{L}_p^{(m_k)*}(\omega_i)\right\| \cdot$$

$$\cdot \left\|<\ell_{N_{k+1}}(\theta_{k+1}), f(\theta_1,\theta_2,...,\theta_{k+1})>|\bar{L}_p^{(m_{k+1})*}(\omega_i)\right\| \tag{3.13}$$

With (2.5) and (3.10), from (3.13) we obtain:

$$\left\| \ell_N(\theta) |\bar{L}_p^{(m)*}(S_{k+1}) \right\| \leq \left\| \ell_{N_1}(\theta_1) |\bar{L}_p^{(m_1)*}(\omega_i) \right\| ... \cdot \left\| \ell_{N_k}(\theta_k) |\bar{L}_p^{(m_k)*}(\omega_i) \right\| \cdot \left\| \ell_{N_{k+1}}(\theta_{k+1}) |\bar{L}_p^{(m_{k+1})*}(\omega_i) \right\|, \tag{3.14}$$

where $\omega_i = \begin{cases} [0, 2\pi], & if \;\; i = k+1, \\ [0, \pi], & if \;\; i = 1, 2, ..., k. \end{cases}$

Using (3.1), from (3.14) we obtain:

$$\left\| \ell_N(\theta) |\bar{L}_p^{(m)*}(S_{k+1}) \right\| \leq d \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_{k+1}^{m_{k+1}}}, \tag{3.15}$$

and with (3.2) and (3.15), we have

$$\left\| \ell_N(\theta) |\bar{L}_p^{(m)*}(S_{k+1}) \right\| \leq d \cdot o(h^{m_1}) ... o(h^{m_{k+1}}),$$

where $d = \prod_{i=1}^{k+1} d_i$.

Thus, we obtain inequalities (3.3) and (3.4), i.e.:

$$\left\| \ell_N(\theta) |\bar{L}_p^{(m)*}(S_n) \right\| \leq d \frac{1}{N_1^{m_1} \cdot N_2^{m_2} ... N_n^{m_n}}, \quad d \text{ - is a constant.} \tag{3.16}$$

Or taking into account (3.2), from (3.16) we have

$$\left\| \ell_N(\theta) |\bar{L}_p^{(m)*}(S_n) \right\| \leq d \cdot o(h^{m_1}) ... o(h^{m_n}), \tag{3.17}$$

where $d = \prod_{i=1}^{n} d_i$.

Since $o(h^{m_1}) ... o(h^{m_n}) = o(h^{m_1 + m_2 + ... + m_n}) = o(h^m)$, from (3.17), we obtain

$$\left\| \ell_N(\theta) |\bar{L}_p^{(m)*}(S_n) \right\| \leq d \cdot o(h^m), \tag{3.18}$$

which is what was to be proven.

It follows that the norm of the error functional (2.3) of the cubature formula (2.2) in the space $\bar{L}_p^{(m)*}(S_n)$ admits an upper bound.

Thus, the cubature formula (2.2) over Sobolev factor space $\bar{L}_p^{(m)*}(S_n)$ with the error functional (2.3) is asymptotic optimal. According to N.S.Bakhvalov's expression, such formulas are called practical [6].

For illustration, we take one specific example for $n = 2$.

Let

$$f(\theta_1, \theta_2) = e^{a\theta_1} \left(\frac{1}{2} - b\theta_2^2 \right)^{\frac{3}{2}}, \quad \text{where} \quad a \neq 0 \quad \text{and} \quad b \neq 0.$$

It is obvious that derivatives $\frac{\partial^{m-1} f(\theta_1, \theta_2)}{\partial \theta_1^{m-1}}$ and $\frac{\partial f(\theta_1, \theta_2)}{\partial \theta_2}$ are continuous on S_2, but $\frac{\partial^2 f(\theta_1, \theta_2)}{\partial \theta_2^2}$ has a singularity on S_2. Therefore, it is clear from condition $m = m_1 + m_2$ that $m_1 = m - 1$, $m_2 = 1$, since $m - 1 + 1 = m$. Hence, $f(\theta_1, \theta_2) \in \bar{L}_p^{(m)}(S_n)$, for $m_1 = m - 1$, $m_2 = 1$. Thus, it is clear that $\frac{\partial^2 f(\theta_1, \theta_2)}{\partial \theta_2^2}$ is not continuous on S_2, since for $m_1 = m - 2$ and $m_2 = 2$, $f(\theta_1, \theta_2) \notin L_p^{(m)}(S_n)$, but conditions $m = m_1 + m_2$ are satisfied, i.e., $m - 2 + 2 = m$.

4. Conclusion

The challenge of accurately approximating definite integrals emerges because numerous scientific and technological problems are expressed through integral and differential equations. The solutions to these equations are frequently given in the form of definite integrals, which are not always amenable to exact evaluation. Numerous mathematicians have investigated these problems, leading to the development of various methods for deriving optimal quadrature formulas. One notable approach is the method introduced by S.L.Sobolev, which utilizes the concept of an extremal function for the error functional.

In our research, we examined cubature formulas within the Sobolev functional space $\bar{L}_p^{(m)}(S_n)$ for functions defined on n-dimensional unit sphere S.

Thus, based on the N.S.Bakhvalov theorem, we proved that the cubature formula in question is asymptotically optimal within this space. However, a major drawback of multidimensional cubature is the rapid increase in the number of integration nodes as the dimension rises, leading to high computational costs. This issue requires careful attention when developing the most efficient formulas. In this paper, we discussed formulas that fulfill this criterion and refered to them as "practical" in accordance with N.S. Bakhvalov's terminology [6].

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