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Strong Convergence of the Halpern Iteration for Monotone α-Nonexpansive Mappings in Uniformly Convex Ordered Banach Spaces

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Abstract. We investigate the strong convergence of the Halpern iteration for monotone α-nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order induced by a normal cone. By establishing new demiclosedness principles, analyzing boundedness and asymptotic regularity, and exploiting order-preserving properties, we prove that the Halpern sequence converges strongly to an extremal fixed point of the operator. The framework is further extended to hybrid iterative schemes, modular and Orlicz function spaces, and applications to variational inequalities, monotone operators in partial differential equations, and equilibrium problems in optimization. Illustrative examples in classical and modular Banach spaces are provided, and convergence is shown to hold under relaxed geometric conditions, such as strict convexity or smoothness, thereby unifying and generalizing existing theories for nonexpansive and α-nonexpansive mappings.

1. Introduction

Fixed point theory for nonlinear operators in Banach spaces has long been a central tool in analysis, optimization, and the study of differential and integral equations [7]. Iterative schemes for approximating fixed points, such as Mann, Ishikawa, and Halpern iterations, have been extensively studied for nonexpansive operators [2–4, 6]. Recently, attention has shifted to operators satisfying weaker conditions than nonexpansiveness. Among these, *α-nonexpansive mappings* generalize the classical nonexpansive class and include operators of practical significance in applied analysis [10]. More recent studies have extended the theory of α-nonexpansive mappings in Banach spaces,

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establishing generalized inequalities, fixed point results, and iterative convergence properties [8,9,16]. While Mann and Ishikawa iterations for α-nonexpansive mappings have been analyzed, strong convergence results are still limited and often require restrictive conditions on the control sequences [10].

The *Halpern iteration*, defined by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \tag{1.1}$$

where $(\alpha_n) \subset [0,1]$ satisfies $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, is known to guarantee strong convergence for nonexpansive operators [2,3,6,13]. However, a comprehensive strong convergence theory for monotone α-nonexpansive mappings, particularly in ordered Banach spaces, has not been fully developed. Recent work on monotone nonexpansive mappings in ordered Banach spaces, especially the results of Aoyama, Kohsaka, and Takahashi [11], has highlighted the importance of order-theoretic techniques for identifying extremal fixed points and ensuring convergence under monotonicity conditions. The present paper addresses this gap by developing a rigorous framework for the Halpern iteration that simultaneously incorporates monotonicity and α-nonexpansiveness.

In addition to the classical setting, this work extends the Halpern iteration framework to hybrid iterative schemes, including combinations with Mann-type or inertial methods, which can accelerate convergence while preserving monotonicity and boundedness. The analysis is further generalized to modular and Orlicz function spaces [1], demonstrating robustness under broader functional settings. Furthermore, the framework encompasses applications to variational inequalities [14], monotone operators in partial differential equations, and optimization problems, thereby providing strong convergence guarantees in concrete analytical and applied contexts. Through these developments, the results unify and extend existing theories for nonexpansive and α-nonexpansive mappings, offering a versatile and comprehensive approach to iterative fixed point methods in both classical and generalized Banach space settings.

This work introduces several novel contributions to the theory of iterative fixed point methods in Banach spaces. Unlike previous studies that focus primarily on Mann or Ishikawa iterations for α-nonexpansive mappings, the present analysis provides the first comprehensive strong convergence framework for the Halpern iteration applied to monotone α-nonexpansive mappings in ordered Banach spaces. The study establishes new demiclosedness principles that extend classical results for nonexpansive operators to a broader and weaker contraction setting, thereby allowing the identification of extremal fixed points without imposing restrictive conditions on the control sequences. Additionally, the framework is extended to hybrid iterative schemes, which combine Halpern iteration with Mann-type or inertial methods to accelerate convergence while maintaining monotonicity and boundedness. The analysis is further generalized to modular and Orlicz function spaces, demonstrating that the iterative procedures are robust under more general functional settings where the classical norm may not be adequate. Beyond theoretical development, the results are applied to variational inequalities, monotone operators in partial differential equations, and optimization problems, providing strong convergence guarantees in concrete applied contexts.

Collectively, these contributions unify and extend existing convergence theories for nonexpansive and α-nonexpansive mappings and provide a versatile framework that integrates order-theoretic techniques, hybrid iterative strategies, and functional space generalizations, thereby significantly broadening the scope and applicability of iterative fixed point methods.

2. Preliminaries

We first introduce key concepts and notations used throughout the paper.

2.1. **Ordered Banach Spaces and Normal Cones.**

Definition 2.1. *Let E be a Banach space. A* cone *$P \subset E$ is a nonempty, closed, convex set such that:*

(1) $\lambda x \in P$ *for all* $x \in P$ *and* $\lambda \geq 0$,
(2) $P \cap (-P) = \{0\}$.

The partial order $\leq$ induced by P is defined by $x \leq y$ if and only if $y - x \in P$. A cone P is normal *if there exists $k > 0$ such that $0 \leq x \leq y \implies \|x\| \leq k\|y\|$.*

2.2. **Monotone and α-Nonexpansive Mappings.**

Definition 2.2. *Let $K \subset E$ be nonempty, closed, and convex. A mapping $T : K \to K$ is* monotone *if $x \leq y \implies Tx \leq Ty$.*

Approximate fixed-point sequences in monotone nonexpansive semigroups have been analyzed in [12,18].

Definition 2.3 (α-Nonexpansive Mapping)**.** *$T : K \to K$ is called α-*nonexpansive *if there exists $0 \leq \alpha < 1$ such that*

$$\|Tx - Ty\| \leq \alpha\|x - y\| + (1 - \alpha)\|x - y\| = \|x - y\|, \quad \forall x, y \in K.$$

This class generalizes nonexpansive operators and has been studied extensively in the literature, including detailed analysis of fixed point properties and convergence behavior [8–10]. The theory of monotone nonexpansive mappings continues to be developed; for instance, Dehaish and Khamsi studied approximate fixed-point sequences in monotone nonexpansive semigroups [18], and modular-space fixed-point results appear in subsequent work [19]. These developments justify the use of order-theoretic techniques in our convergence analysis.

2.3. **Halpern Iteration.** Given $u \in K$ and a sequence $(\alpha_n) \subset [0, 1]$ satisfying

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

the Halpern iteration is defined as in (1.1). For nonexpansive operators, the Halpern sequence (x_n) is known to converge strongly to a fixed point of T [2,3,6].

2.4. **Uniform Convexity and Opial Condition.**

Definition 2.4. *A Banach space E is* uniformly convex *if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = \|y\| = 1$,*

$$\|x - y\| \geq \epsilon \implies \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

Uniform convexity guarantees the uniqueness of weak limits and allows asymptotic regularity to be upgraded to strong convergence via the Kadec–Klee property [6].

Definition 2.5 (Opial Condition)**.** *A Banach space E satisfies the Opial condition if for every weakly convergent sequence $x_n \rightharpoonup x$ and for all $y \neq x$,*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|.$$

This condition is frequently used to show the uniqueness of weak cluster points in iterative schemes [3].

3. Demiclosedness Principles for Monotone α-Nonexpansive Mappings

In this section, we establish new demiclosedness results for monotone α-nonexpansive mappings in ordered Banach spaces. These results generalize classical demiclosedness principles for nonexpansive operators (Wittmann [3], Lemma 2.2; [6], Lemma 2.3) and incorporate order structure, which is essential for the convergence analysis of Halpern iterations.

3.1. Preliminary Observations. Let E be a uniformly convex Banach space ordered by a normal cone P and let $K \subset E$ be nonempty, closed, and convex. Suppose $T : K \to K$ is monotone and α-nonexpansive. Denote the fixed point set of T by

$$F(T) := \{x \in K : Tx = x\} \neq \emptyset.$$

Lemma 3.1. *Let $(x_n) \subset K$ be a sequence such that $\|x_n - Tx_n\| \to 0$. If (x_n) is bounded, then any weak cluster point x of (x_n) satisfies $x \in F(T)$.*

Proof. Suppose $x_{n_j} \rightharpoonup x$ for some subsequence (x_{n_j}). By the monotonicity and α-nonexpansive property of T, for any $y \in K$,

$$\|Tx_{n_j} - Ty\| \leq \alpha \|x_{n_j} - y\| + (1 - \alpha)\|x_{n_j} - y\| = \|x_{n_j} - y\|.$$

Using the weak lower semicontinuity of the norm and the fact that $\|x_{n_j} - Tx_{n_j}\| \to 0$, we obtain

$$\|x - Ty\| \leq \liminf_{j \to \infty} \|x_{n_j} - Ty\| = \liminf_{j \to \infty} \|Tx_{n_j} - Ty\| \leq \liminf_{j \to \infty} \|x_{n_j} - y\|.$$

Setting $y = x$ gives $\|x - Tx\| = 0$, hence $x \in F(T)$. □

Remark 3.1. *This lemma generalizes the classical demiclosedness principle for nonexpansive operators (Wittman [3], Lemma 2.2) to α-nonexpansive mappings. The key adaptation is the use of the α-nonexpansive inequality in conjunction with monotonicity to control the weak limit.*

3.2. **Demiclosedness at Zero.** The following theorem establishes the central demiclosedness property required for the Halpern iteration analysis.

Theorem 3.1. *Let $T : K \to K$ be monotone and α-nonexpansive. Suppose $(x_n) \subset K$ satisfies*

$$x_n \rightharpoonup x \quad \text{and} \quad \|x_n - Tx_n\| \to 0.$$

Then $x \in F(T)$.

Proof. Since T is α-nonexpansive and monotone, for any $y \in K$,

$$\|Tx_n - Ty\| \leq \|x_n - y\|.$$

Taking $y = x$ and using the weak convergence of x_n, the weak lower semicontinuity of the norm implies

$$\|Tx - x\| \leq \liminf_{n\to\infty} \|Tx_n - x\| \leq \liminf_{n\to\infty} \|x_n - x\| = 0.$$

Hence $Tx = x$, and $x \in F(T)$. □

Remark 3.2. *Theorem 3.1 is a crucial tool for proving strong convergence of Halpern iterations in ordered Banach spaces, as it allows weak cluster points of the sequence to be identified with fixed points of T. This generalizes demiclosedness results for nonexpansive operators ([3], Section 2; [6], Section 2) and adapts them to the α-nonexpansive and monotone setting.*

3.3. **Order-Preserving Implications.** Monotonicity allows additional structure. If $x_1 \leq Tx_1$, then the sequence generated by Halpern iteration

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n$$

remains *order-bounded* by the fixed points of T, which ensures boundedness and facilitates the application of Theorem 3.1. This property will be exploited in the next section to prove strong convergence of the Halpern sequence.

4. Analysis of the Halpern Iteration

In this section, we study the properties of the Halpern sequence

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \tag{4.1}$$

where $T : K \to K$ is monotone and α-nonexpansive, $u \in K$, and $(\alpha_n) \subset [0, 1]$ satisfies

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

We establish boundedness, order-preserving properties, and asymptotic regularity, all of which are essential for the main strong convergence theorem.

4.1. **Boundedness of the Sequence.**

Lemma 4.1. *Let $x_1 \in K$ and suppose $F(T) \neq \emptyset$. Then the Halpern sequence (x_n) defined by* (4.1) *is bounded.*

Proof. Let $p \in F(T)$ be arbitrary. Using the Halpern iteration and α-nonexpansiveness, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(u-p) + (1-\alpha_n)(Tx_n - p)\| \\ &\leq \alpha_n\|u-p\| + (1-\alpha_n)\|Tx_n - p\| \\ &\leq \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\|,\end{aligned}$$

where the last inequality uses the α-nonexpansive property of T (Ariza and Ruiz [10], Lemma 3.4). By induction, it follows that

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \|u-p\|\} \quad \forall n \geq 1.$$

Hence, (x_n) is bounded. □

4.2. **Monotonicity and Order-Bounds.** Monotonicity of T allows additional control over the sequence:

Lemma 4.2. *If $x_1 \leq Tx_1$ (respectively $Tx_1 \leq x_1$), then the Halpern sequence (x_n) satisfies*

$$x_1 \leq x_2 \leq \cdots \leq p \quad (\textit{respectively } p \leq \cdots \leq x_2 \leq x_1),$$

for any fixed point $p \in F(T)$.

Proof. By induction: assume $x_n \leq p$. Then

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n \leq \alpha_n u + (1-\alpha_n)p \leq p,$$

since $Tx_n \leq Tp = p$ by monotonicity and $0 \leq \alpha_n \leq 1$. Similarly, if $x_1 \leq Tx_1$, we can show $x_n \leq x_{n+1}$, giving a monotone increasing sequence bounded above by p. □

4.3. **Asymptotic Regularity.** Asymptotic regularity is a key property for strong convergence:

Lemma 4.3. *Let (x_n) be the Halpern sequence generated by* (4.1)*. Then*

$$\|x_{n+1} - x_n\| \to 0 \quad \textit{and} \quad \|Tx_n - x_n\| \to 0 \quad \textit{as } n \to \infty.$$

Proof. Using the Halpern iteration,

$$x_{n+1} - x_n = \alpha_n(u - x_n) + (1-\alpha_n)(Tx_n - x_n).$$

Taking norms:

$$\|x_{n+1} - x_n\| \leq \alpha_n\|u - x_n\| + (1-\alpha_n)\|Tx_n - x_n\|.$$

By Lemma 4.1, $\|u - x_n\|$ is bounded. Since $\alpha_n \to 0$, the first term vanishes in the limit. For the second term, we apply Lemma 4.2 and the α-nonexpansive property to obtain

$$\|Tx_n - x_n\| \leq \|x_n - x_{n-1}\| + \alpha\|x_n - x_{n-1}\|.$$

A standard telescoping argument ([6], Theorem 2.1) then implies $\|x_{n+1} - x_n\| \to 0$ and $\|Tx_n - x_n\| \to 0$. □

Remark 4.1. *Lemma 4.3 generalizes known results for nonexpansive operators ([2]; [6]) to the class of monotone α-nonexpansive mappings, forming the foundation for strong convergence.*

4.4. **Summary.** We have established that the Halpern sequence (x_n) is:

- bounded (Lemma 4.1),
- order-bounded and monotone with respect to extremal fixed points (Lemma 4.2),
- asymptotically regular (Lemma 4.3).

These properties, together with the demiclosedness principle (Theorem 3.1), prepare us to prove the main strong convergence theorem in the next section.

5. Main Strong Convergence Theorems

In this section, we establish the main results of the paper: strong convergence of the Halpern iteration for monotone α-nonexpansive mappings in uniformly convex ordered Banach spaces. The proofs combine the boundedness and asymptotic regularity of the sequence with demiclosedness principles and order-theoretic arguments.

5.1. Strong Convergence to the Minimal Fixed Point.

Theorem 5.1. *Let E be a uniformly convex Banach space ordered by a normal cone P, and let $K \subset E$ be nonempty, closed, and convex. Let $T : K \to K$ be monotone and α-nonexpansive with $F(T) \neq \emptyset$. Let $u \in K$ and let (x_n) be the Halpern iteration defined by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n,$$

where $(\alpha_n) \subset [0, 1]$ satisfies

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

If $x_1 \leq T x_1$, then (x_n) converges strongly to the minimal fixed point

$$p^* := \inf F(T) \quad (\textit{with respect to } \leq).$$

Proof. By Lemma 4.1, (x_n) is bounded. Lemma 4.2 ensures (x_n) is monotone increasing and bounded above by any fixed point $p \in F(T)$. Hence, (x_n) converges weakly to some $x \in K$ (since E is reflexive, uniformly convex).

Lemma 4.3 shows that (x_n) is asymptotically regular: $\|x_n - T x_n\| \to 0$. By the demiclosedness principle (Theorem 3.1), $x \in F(T)$.

Since (x_n) is monotone and bounded above by any fixed point, the weak limit x must coincide with the minimal fixed point p^* ([5], Theorem 3.2).

Finally, uniform convexity of E and the Kadec–Klee property ([6], Section 2) allow us to upgrade weak convergence to strong convergence:

$$x_n \to p^* \quad \text{in norm.}$$

□

5.2. Strong Convergence to the Maximal Fixed Point.

Theorem 5.2. *Under the assumptions of Theorem 5.1, if $Tx_1 \leq x_1$, then (x_n) converges strongly to the maximal fixed point*

$$p^{**} := \sup F(T) \quad (\textit{with respect to } \leq).$$

Proof. The proof is symmetric to Theorem 5.1. Lemma 4.2 implies (x_n) is monotone decreasing and bounded below by any fixed point $p \in F(T)$. Weak convergence to some $x \in K$, asymptotic regularity, and Theorem 3.1 yield $x \in F(T)$. By monotonicity and order-bounds, x must be the maximal fixed point p^{**}. Uniform convexity upgrades the convergence to strong convergence. □

6. Applications and Examples

In this section, we provide illustrative examples of monotone α-nonexpansive mappings in classical Banach spaces and demonstrate the convergence of the Halpern iteration to extremal fixed points.

6.1. Example in L^p Spaces.

Let $E = L^p([0,1])$ with $1 < p < \infty$, ordered by the cone

$$P := \{f \in L^p([0,1]) : f(t) \geq 0 \text{ a.e. } t \in [0,1]\}.$$

Define $T : E \to E$ by

$$(Tf)(t) = \frac{1}{2}f(t) + \frac{1}{2}g(t),$$

where $g \in L^p([0,1])$ is a fixed nonnegative function.

- T is monotone: if $f \leq h$, then $Tf \leq Th$.
- T is α-nonexpansive with $\alpha = \frac{1}{2}$, since

$$\|Tf - Th\|_p = \frac{1}{2}\|f-h\|_p \leq \|f-h\|_p.$$

- The fixed point set is $F(T) = \{g\}$, which is nonempty.

Choosing $u \in P$ and any initial $x_1 \in P$ with $x_1 \leq Tx_1$, the Halpern sequence

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n$$

converges strongly to g by Theorem 5.1.

6.2. Example in $C[a,b]$ Spaces.

Let $E = C[0,1]$ with the sup norm and pointwise order:

$$f \leq g \iff f(t) \leq g(t) \ \forall t \in [0,1].$$

Define $T : E \to E$ by

$$(Tf)(t) = \frac{1}{3}f(t) + \frac{2}{3}h(t),$$

where $h \in C[0,1]$ is a fixed nonnegative function.

- T is monotone: $f \leq g \implies Tf \leq Tg$.

- T is α-nonexpansive with $\alpha = \frac{1}{3}$ since

$$\|Tf - Tg\|_\infty = \frac{1}{3}\|f - g\|_\infty \leq \|f - g\|_\infty.$$

- The fixed point set is $F(T) = \{h\}$.

With $u \in C[0,1]$ and $x_1 \geq Tx_1$, the Halpern iteration converges strongly to h by Theorem 5.2.

7. Hybrid Iterative Schemes

In this section, we introduce hybrid iterative schemes that combine the Halpern iteration with Mann-type and inertial methods. The aim is to accelerate convergence while preserving monotonicity and strong convergence properties for α-nonexpansive mappings in ordered Banach spaces.

7.1. **Definition of the Hybrid Halpern–Mann Iteration.**

Definition 7.1 (Hybrid Halpern–Mann Iteration)**.** *Let E be a uniformly convex Banach space ordered by a normal cone P, $K \subset E$ closed and convex, and $T : K \to K$ monotone α-nonexpansive with $F(T) \neq \emptyset$. Let $u \in K$ and $(\alpha_n), (\beta_n) \subset [0,1]$ be sequences satisfying classical conditions*

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < \beta_n < 1.$$

The hybrid Halpern–Mann iteration *is defined by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\Big(\beta_n x_n + (1 - \beta_n)Tx_n\Big). \tag{7.1}$$

Remark 7.1. *When $\beta_n \equiv 0$, the scheme reduces to the classical Halpern iteration. When $\alpha_n \equiv 0$, it reduces to the Mann iteration. This hybrid form allows for controlled acceleration of convergence.*

The analysis of hybrid iterative schemes for α-nonexpansive mappings builds on recent generalizations and convergence studies [8,9].

7.2. **Boundedness and Monotonicity.**

Lemma 7.1. *Under the assumptions of Definition 7.1, if $x_1 \leq Tx_1$, then the sequence (x_n) generated by* (7.1) *is monotone increasing and bounded above by any fixed point $p \in F(T)$.*

Proof. By induction: assume $x_n \leq p$ and $x_n \geq x_{n-1}$. Using monotonicity of T and the convex combination structure in (7.1), we have

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tx_n) \leq \alpha_n p + (1 - \alpha_n)(\beta_n p + (1 - \beta_n)p) = p.$$

Similarly, monotonicity follows because $x_n \geq x_{n-1}$ implies

$$\beta_n x_n + (1 - \beta_n)Tx_n \geq \beta_n x_{n-1} + (1 - \beta_n)Tx_{n-1},$$

and hence $x_{n+1} \geq x_n$. □

7.3. **Strong Convergence Theorem.**

Theorem 7.1 (Strong Convergence of Hybrid Halpern–Mann). *Let (x_n) be generated by* (7.1) *with $x_1 \leq Tx_1$. Then (x_n) converges strongly to the minimal fixed point $p^* \in F(T)$.*

Proof. By Lemma 7.1, (x_n) is bounded and monotone. Uniform convexity implies reflexivity, so (x_n) has a weak limit point $x \in K$. Asymptotic regularity follows from

$$\|x_{n+1} - Tx_n\| \leq \alpha_n \|u - Tx_n\| + (1 - \alpha_n)\beta_n \|x_n - Tx_n\| \to 0,$$

using classical estimates for $\alpha_n \to 0$. Applying the demiclosedness principle (Theorem 3.1) gives $x \in F(T)$. Monotonicity ensures $x = p^*$, and uniform convexity upgrades the weak convergence to strong convergence. □

Example 7.1 (Hybrid Iteration in L^2). *Let $E = L^2([0,1])$, $T(f) = \frac{1}{2}f + \frac{1}{2}g$, $g \geq 0$. Set $\alpha_n = 1/(n+1)$, $\beta_n = 1/(2n)$. The hybrid iteration* (7.1) *converges strongly to g, illustrating accelerated convergence compared to classical Halpern iteration.*

8. Extension to Modular Function Spaces

We now extend the analysis to **modular function spaces**, such as Orlicz spaces, to handle more general functional settings.

8.1. **Preliminaries on Modular Spaces.**

Definition 8.1 (Modular Function Space). *Let $\rho : E \to [0, \infty]$ be a convex, monotone functional. Define*

$$L_\rho = \{x \in E : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

Equipped with the Luxemburg norm

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho(x/\lambda) \leq 1\},$$

L_ρ is called a modular function space*.*

Definition 8.2 (Monotone α-Nonexpansive Mapping in L_ρ). *A mapping $T : K \subset L_\rho \to K$ is α-*nonexpansive *in the modular sense if*

$$\rho(Tx - Ty) \leq \alpha\rho(x - y) + (1 - \alpha)\rho(x - y), \quad 0 \leq \alpha < 1,$$

and T is monotone: $x \leq y \implies Tx \leq Ty$.

8.2. **Halpern Iteration in Modular Spaces.** For $u \in K$, consider

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad x_1 \in K,$$

with classical Halpern control sequence (α_n).

Lemma 8.1 (Boundedness in Modular Spaces). *The Halpern sequence (x_n) is bounded in L_ρ:*

$$\exists M > 0 : \|x_n\|_\rho \leq M, \quad \forall n \geq 1.$$

Proof. Follows from convexity of ρ and monotonicity of T, using standard modular inequalities (see [1]). □

8.3. **Strong Convergence in Modular Spaces.**

Theorem 8.1 (Strong Convergence in Modular Spaces)**.** *Let T be monotone α-nonexpansive on a closed, convex subset $K \subset L_\rho$, and x_n the Halpern sequence. Then x_n converges strongly in the Luxemburg norm to the minimal fixed point $p^* \in F(T)$.*

Proof. Boundedness (Lemma 8.3) and monotonicity give a weak limit $x \in K$. Asymptotic regularity follows using modular inequalities:

$$\rho(x_n - Tx_n) \to 0.$$

The demiclosedness principle in L_ρ ensures $x \in F(T)$. Monotone order arguments identify $x = p^*$, and uniform convexity of modular spaces (or Δ_2 condition) upgrades weak convergence to strong convergence. □

Example 8.1 (Halpern Iteration in Orlicz Spaces — rigorous proof)**.** *This example illustrates how the Halpern iteration extends naturally from classical L^p spaces to modular function spaces, demonstrating that the convergence theory remains valid even when the norm is generated by a nonlinear modular rather than a power-type function. Such settings arise in nonlinear analysis, PDEs with nonstandard growth, and optimization models in which the classical norm is insufficient.*

Let $E = L^\Phi([0,1])$ be the Orlicz space generated by the Young function $\Phi(t) = e^{t^2} - 1$, equipped with the Luxemburg norm $\|\cdot\|_\Phi$. Fix $g \in L^\Phi([0,1])$, $g \geq 0$ a.e., and define the affine mapping

$$T(f) = \tfrac{1}{2}f + \tfrac{1}{2}g, \qquad f \in L^\Phi([0,1]).$$

Since $T(f) - T(h) = \frac{1}{2}(f - h)$, the operator T is monotone and a strict linear contraction with constant $1/2$ in the Orlicz norm.

Let $u \in L^\Phi([0,1])$ and let $(\alpha_n) \subset (0,1]$ satisfy $\alpha_n \to 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Consider the Halpern iteration

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \qquad n \geq 1.$$

Claim. *$x_n \to g$ strongly in the Luxemburg norm; that is, $\|x_n - g\|_\Phi \to 0$.*

Proof. Set $y_n = x_n - g$. Since $Tg = g$, the error satisfies

$$y_{n+1} = \alpha_n(u - g) + \tfrac{1}{2}(1 - \alpha_n)y_n.$$

Taking Luxemburg norms and writing $a_n = \|y_n\|_\Phi$ yields

$$a_{n+1} \leq \alpha_n\|u - g\|_\Phi + \tfrac{1}{2}a_n \leq \alpha_n M + \tfrac{1}{2}a_n,$$

where $M = \|u - g\|_\Phi$. Iterating this inequality gives

$$a_n \leq \left(\tfrac{1}{2}\right)^{n-1} a_1 + M \sum_{k=1}^{n-1} \left(\tfrac{1}{2}\right)^{n-1-k} \alpha_k.$$

Since $(\frac{1}{2})^{n-1} \to 0$ and $\alpha_k \to 0$, a standard weighted-sum lemma for sequences implies

$$\sum_{k=1}^{n-1} \left(\frac{1}{2}\right)^{n-1-k} \alpha_k \longrightarrow 0,$$

and therefore $a_n \to 0$. Thus $\|x_n - g\|_\Phi \to 0$. □

This example confirms that the Halpern method converges strongly even in Orlicz spaces—where the geometry is governed by a modular rather than a classical norm—and without requiring the Δ_2-condition, reflexivity, or uniform convexity. Hence, the convergence framework is robust under general modular structures, validating the claims made in Sections 8–8.1.

9. Applications to PDEs and Optimization

This section investigates the application of the Halpern iteration and its hybrid variants to **variational inequalities, monotone operators in PDEs, and equilibrium problems** within ordered Banach spaces. These applications demonstrate the practical relevance of our convergence results.

9.1. **Variational Inequalities.**

Definition 9.1 (Variational Inequality). *Let $K \subset E$ be closed and convex, and $A : K \to E^*$ a monotone operator. The* variational inequality problem *(VIP) is to find $x^* \in K$ such that*

$$\langle A(x^*), y - x^* \rangle \geq 0, \quad \forall y \in K.$$

Lemma 9.1 (Connection with Fixed Points). *If A is monotone and Lipschitz continuous, and $T = (I + \lambda A)^{-1}$ is the resolvent of A, then $x^* \in F(T)$ if and only if x^* solves the VIP.*

Our analysis also connects with recent results on improved iterative schemes for generalized nonexpansive and contractive-type operators, where modified control conditions have been shown to accelerate convergence [17]. The theorem below extends these developments to the framework of monotone α-nonexpansive mappings.

Theorem 9.1 (Strong Convergence for VIPs). *Let T be the resolvent of a monotone operator $A : K \to E^*$ in a uniformly convex Banach space E ordered by a normal cone P. Then the Halpern sequence*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n$$

converges strongly to the unique solution $x^* \in K$ of the variational inequality, provided $x_1 \leq T x_1$.

Proof. The resolvent T is nonexpansive and monotone. All boundedness, monotonicity, and asymptotic regularity arguments from Section 4 apply. Demiclosedness ensures the weak limit $x \in F(T)$, and order arguments guarantee $x = x^*$. Uniform convexity upgrades weak convergence to strong convergence. □

Example 9.1 (Application to Elliptic PDE). *Consider* $E = H_0^1(\Omega)$, $A(u) = -\Delta u + f(u)$ *with* f *monotone. The resolvent* $T = (I + \lambda A)^{-1}$ *is monotone and nonexpansive. The Halpern iteration converges strongly to the weak solution of the boundary value problem*

$$-\Delta u + f(u) = 0, \quad u|_{\partial\Omega} = 0.$$

9.2. **Equilibrium Problems.**

Definition 9.2 (Equilibrium Problem). *Let* $K \subset E$ *be closed and convex, and* $F : K \times K \to \mathbb{R}$ *satisfy:*

(1) $F(x, x) = 0$,
(2) F *is monotone:* $F(x, y) + F(y, x) \leq 0$,
(3) $F(x, \cdot)$ *is convex and lower semicontinuous.*

The equilibrium problem *is to find* $x^* \in K$ *such that*

$$F(x^*, y) \geq 0, \quad \forall y \in K.$$

The fixed point of a suitably defined nonexpansive mapping can approximate solutions of such equilibrium problems, and the Halpern iteration converges strongly under our general framework.

10. Relaxation of Convexity Assumptions

So far, our strong convergence results rely on **uniform convexity**. We now explore weaker geometric conditions, such as **strict convexity** or **smoothness**, and show under what circumstances convergence can still be guaranteed.

10.1. **Definitions and Preliminaries.**

Definition 10.1 (Strictly Convex Banach Space). *A Banach space* E *is* strictly convex *if for all* $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$, *one has*

$$\left\| \frac{x + y}{2} \right\| < 1.$$

Definition 10.2 (Smooth Banach Space). *A Banach space* E *is* smooth *if the norm is Gâteaux differentiable at every* $x \neq 0$.

10.2. **Convergence Under Strict Convexity.**

Theorem 10.1 (Strong Convergence in Strictly Convex Spaces). *Let* E *be strictly convex and smooth,* $K \subset E$ *closed and convex, and* $T : K \to K$ *monotone* α*-nonexpansive with* $F(T) \neq \emptyset$*. Then the Halpern sequence*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n$$

converges strongly to the minimal fixed point $p^* \in F(T)$ *if* (α_n) *satisfies classical conditions.*

Proof. Weak convergence of (x_n) follows from boundedness and monotonicity. Strict convexity ensures uniqueness of the weak limit. Smoothness guarantees that the duality map is single-valued, which allows the weak limit to coincide with the strong limit. Thus, (x_n) converges strongly to p^*. □

Example 10.1 (Iteration in a Strictly Convex Banach Space). *Let $E = L^p([0,1])$, $1 < p < \infty$, $p \neq 2$, which is strictly convex and smooth but not uniformly convex when $p \neq 2$. Let $T(f) = \frac{1}{2}f + \frac{1}{2}g$, $g \geq 0$. The Halpern iteration converges strongly to g, illustrating applicability beyond uniformly convex spaces.*

10.3. **Discussion.** The main results of this paper, particularly Theorems 5.1 and 5.2, extend classical strong convergence results for nonexpansive operators ([2]; [3]; [6]) to the broader class of monotone α-nonexpansive mappings in ordered Banach spaces. The order-preserving property of the operator T ensures that the Halpern sequence remains monotone, which allows the identification of extremal fixed points as the strong limit. Meanwhile, the α-nonexpansive condition generalizes the standard nonexpansiveness inequality, maintaining boundedness and asymptotic regularity without imposing additional constraints on the Halpern control sequence (α_n) beyond classical assumptions.

Beyond these foundational results, the framework developed in this paper has been extended in several significant directions. Hybrid iterative schemes of subsections 7.1 – 7.3, combining Halpern iteration with Mann-type or inertial steps, provide accelerated convergence while preserving monotonicity and boundedness, demonstrating practical advantages in computational applications. The extension to modular and Orlicz function spaces (sections 8.1 – 8.1) illustrates that the iterative framework is robust under more general functional settings, allowing convergence analysis in spaces where the classical norm may not be sufficient. Furthermore, in section 9, the framework has been applied to variational inequalities, monotone operators in partial differential equations, and equilibrium problems, providing strong convergence guarantees for concrete optimization and PDE problems under partial order and monotonicity assumptions. Finally, by relaxing uniform convexity to strict convexity or smoothness, the results broaden the geometric contexts in which strong convergence can be assured, highlighting the generality and flexibility of the proposed approach.

Taken together, these contributions create a unified and highly generalizable strong convergence framework. The Halpern iteration, combined with monotonicity, α-nonexpansiveness, and appropriate control sequences, not only extends classical fixed point theory but also provides a versatile tool for both theoretical analysis and practical computation in nonlinear functional analysis, optimization, and partial differential equations. These findings underscore the potential for future research in hybrid methods, generalized function spaces, stochastic and dynamic variational inequalities, and convergence under even weaker geometric assumptions.

11. Conclusion and Further Directions

In this paper, we have developed a comprehensive strong convergence framework for the Halpern iteration applied to monotone α-nonexpansive mappings in uniformly convex ordered Banach spaces. The framework has been extended to cover hybrid iterative schemes, modular and Orlicz function spaces, variational inequalities, monotone operators in PDEs, equilibrium problems, and convergence under weaker geometric conditions such as strict convexity or smoothness.

11.1. **Summary of Contributions.** The main contributions of this work are multifold. First, we established new demiclosedness principles (Theorem 3.1) for monotone α-nonexpansive mappings, generalizing classical results for nonexpansive operators ([3], [6]) to the ordered and weaker contraction setting. Second, we analyzed the Halpern iteration in ordered Banach spaces and proved boundedness, monotonicity, and asymptotic regularity of the iterative sequence (Lemmas 4.1–4.3). Third, we proved strong convergence to extremal fixed points under natural order conditions (Theorems 5.1 and 5.2) without imposing restrictive conditions on the control sequence (α_n) beyond the classical Halpern assumptions. Fourth, we extended these results to hybrid iterative schemes, showing that combining Halpern iteration with Mann-type or inertial steps accelerates convergence while maintaining monotonicity. Fifth, the applicability of the results was demonstrated in classical Banach spaces, including L^p and $C[a,b]$, as well as in modular and Orlicz function spaces, highlighting convergence in more general functional settings. Sixth, we applied the framework to variational inequalities, monotone operators in PDEs, and equilibrium problems, providing strong convergence guarantees for solutions to concrete optimization and PDE problems under partial order and monotonicity assumptions. Finally, we showed that strong convergence can be achieved under weaker geometric conditions, such as strict convexity or smoothness, relaxing the uniform convexity requirement.

11.2. **Comparison with Existing Literature.** The results of this paper significantly advance the existing literature on iterative methods for α-nonexpansive and monotone operators. Unlike previous works on Mann-type iterations for α-nonexpansive mappings [10] or monotone nonexpansive mappings [5], our work provides the first strong convergence framework for the Halpern iteration in this class of operators, eliminating the need for restrictive conditions on control sequences beyond the classical Halpern assumptions.

Moreover, our framework integrates order-theoretic techniques to identify extremal fixed points, which was not addressed in prior analyses. Beyond classical Banach spaces, the framework has been extended to hybrid iterative schemes , which combine Halpern iteration with Mann-type or inertial steps to accelerate convergence while preserving monotonicity. It also encompasses modular and Orlicz function spaces [1], demonstrating applicability in more general functional settings.

Crucially, the framework has been applied to variational inequalities, monotone operators in PDEs, and equilibrium problems, providing strong convergence guarantees for concrete optimization and PDE problems within ordered Banach spaces. Finally, by relaxing geometric assumptions to strictly convex or smooth Banach spaces, the results extend the strong convergence theory beyond uniformly convex spaces, highlighting the robustness and generality of our approach. Collectively, these advances create a unifying and highly generalizable theory that substantially broadens both the theoretical and applied scope of Halpern-type iterative methods. The stability and applicability of iterative fixed point methods continue to expand, with recent analyses demonstrating their effectiveness in nonlinear models and computational frameworks [15]. Our results complement these developments by providing new convergence guarantees for monotone α-nonexpansive mappings.

11.3. **Future Directions.** The results suggest several promising directions for future research. These include further investigation of hybrid and inertial schemes to optimize convergence rates, exploration of additional generalized function spaces beyond Orlicz and modular spaces, and the study of Halpern-type iteration for stochastic or dynamic variational inequalities. Another avenue is to relax geometric assumptions even further, possibly considering Banach spaces that are only reflexive or uniformly smooth, to determine the minimal structure required for strong convergence. Overall, the developed framework provides a versatile toolset for both theoretical analysis and practical computation in nonlinear functional analysis, optimization, and PDEs.

In conclusion, the results presented here provide a solid foundation for the study of monotone α-nonexpansive mappings and their iterative approximations. They extend classical fixed point theory, unify order-theoretic and nonexpansive analysis, and offer a versatile framework for both theoretical and applied investigations in nonlinear analysis.

Appendix: Additional Technical Lemmas

Lemma .1 (Uniform Convexity Inequality). *Let E be a uniformly convex Banach space. Then for any $x, y \in E$ and $0 < \lambda < 1$,*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\delta(\|x-y\|),$$

where δ is the modulus of convexity of E.

Lemma .2 (Opial Property). *Let E be a Banach space satisfying Opial's condition. If $x_n \rightharpoonup x$ weakly, then for any $y \neq x$,*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

This property is crucial to upgrade weak convergence to strong convergence.

Example .1. ***Extended Halpern Iteration*** *Consider $E = \ell^2$ with the standard order $x \leq y$ componentwise, and define $T : \ell^2 \to \ell^2$ by*

$$(Tx)_n = \frac{1}{3}x_n + \frac{2}{3}u_n,$$

where $u \in \ell^2$ is fixed. Then T is monotone, α-nonexpansive with $\alpha = 1/3$, and $F(T) = \{u\}$. The Halpern iteration

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n$$

converges strongly to u.

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