

On Coherent Almost Distributive Lattices

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Abstract. The connection between coherent ADLs and other structures, including generalized Stone ADLs, moderately complemented ADLs, quasi-complemented ADLs, and normal ADLs, is examined. Coherent ADLs are presented as a separate class of ADLs. To determine when a subADL of a certain ADL demonstrates coherence, criteria are created. Conditions for intervals inside an ADL to be regarded as coherent subADLs are also described. The properties pertaining to filters and dense elements are also used in the study to characterize coherent ADLs.

1. INTRODUCTION

The notion of Almost Distributive Lattices (ADLs) was developed by Swamy and Rao in [10]. They deduced that the set of main ideals in an ADL, denoted as $\mathcal{PI}(\mathcal{L})$, constituted a distributive lattice. The work in [8] concentrated on a thorough examination of \mathcal{D} -filters in ADLs, looking at their significant characteristics. The characteristics of coherent lattices have been extensively investigated in [9].

The concept of coherent ADLs is introduced in this paper, and it is shown that generalized Stone ADLs fall within this category. It lists the prerequisites for identifying cohesive ADLs as generic Stone ADLs. The annulets and principal ideals within ADLs are used to examine relatively complemented ADLs, and standards are set for coherent ADLs to be categorized as relatively complemented ADLs. Additionally, sufficient conditions are given to demonstrate when a coherent ADL satisfies the characteristics of a normal ADL and when a quasi-complemented ADL

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can be deemed coherent. The study also investigates how the traits of coherent ADLs translate to their immediate product. Last but not least, coherent ADLs are further described using the characteristics of filters and \mathcal{D} -filters in ADLs.

2. PRELIMINARIES

This section gathers the basic terminology and essential results from [3, 10], which will be used throughout the manuscript.

Definition 2.1. [10] A structure $(\mathcal{L}, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is said to be an *Almost Distributive Lattice (ADL)* with zero when the following identities are satisfied for all $\iota, \vartheta, \rho \in \mathcal{L}$:

- (1) $\iota \vee 0 = \iota$;
- (2) $0 \wedge \iota = 0$;
- (3) $(\iota \vee \vartheta) \wedge \rho = (\iota \wedge \rho) \vee (\vartheta \wedge \rho)$;
- (4) $\iota \wedge (\vartheta \vee \rho) = (\iota \wedge \vartheta) \vee (\iota \wedge \rho)$;
- (5) $\iota \vee (\vartheta \wedge \rho) = (\iota \vee \vartheta) \wedge (\iota \vee \rho)$;
- (6) $(\iota \vee \vartheta) \wedge \vartheta = \vartheta$.

The relation \leq on \mathcal{L} is introduced by declaring that $\iota \leq \vartheta$ whenever $\iota = \iota \wedge \vartheta$, or equivalently when $\iota \vee \vartheta = \vartheta$. This yields a partial order on \mathcal{L} . An element ν that is maximal under this order is called a *maximal* element, and the collection of all such elements is denoted by $\mathfrak{M}(\mathcal{L})$.

Although ADLs share many structural features with distributive lattices, they generally lack commutativity of \vee and \wedge and do not require right distributivity of \vee over \wedge . As noted in [10], the presence of either of these additional properties would make \mathcal{L} a distributive lattice.

A nonempty subset $\mathcal{I} \subseteq \mathcal{L}$ is an *ideal* (respectively, a *filter*) if for all $\iota, \vartheta \in \mathcal{I}$ and $\varphi \in \mathcal{L}$, the elements $\iota \wedge \varphi$ and $\iota \vee \vartheta$ (respectively, $\varphi \vee \iota$ and $\iota \wedge \vartheta$) also lie in \mathcal{I} . An ideal (filter) is *maximal* when it contains every proper ideal (filter) of \mathcal{L} .

For a subset $\mathcal{G} \subseteq \mathcal{L}$, the ideal generated by \mathcal{G} is $[\mathcal{G}] := \{(\bigvee_{i=1}^n \iota_i) \wedge \varphi \mid \iota_i \in \mathcal{G}, \varphi \in \mathcal{L}, n \in \mathbb{N}\}$, and $[\iota]$ denotes the principal ideal generated by a single element ι . Similarly, the filter generated by \mathcal{G} is $[\mathcal{G}] := \{\varphi \vee (\bigwedge_{i=1}^n \iota_i) \mid \iota_i \in \mathcal{G}, \varphi \in \mathcal{L}, n \in \mathbb{N}\}$, with $[\iota]$ representing a principal filter.

The identities $[\iota] \vee [\vartheta] = [\iota \vee \vartheta]$ and $[\iota] \cap [\vartheta] = [\iota \wedge \vartheta]$ hold for all $\iota, \vartheta \in \mathcal{L}$. Hence the set of principal ideals $(\mathcal{PI}(\mathcal{L}), \vee, \cap)$ forms a sublattice of the distributive lattice $(\mathcal{I}(\mathcal{L}), \vee, \cap)$, the lattice of all ideals of \mathcal{L} . The lattice of all filters $(\mathcal{F}(\mathcal{L}), \vee, \cap)$ is also a bounded distributive lattice.

A prime ideal \mathcal{A} exists in an ADL precisely when its complement $\mathcal{L} \setminus \mathcal{A}$ is a prime filter [6]. Moreover, \mathcal{A} is a minimal prime ideal iff for every $\varphi \in \mathcal{A}$ there exists $\omega \notin \mathcal{A}$ satisfying $\varphi \wedge \omega = 0$; equivalently, $\varphi \notin \mathcal{A}$ iff $(\varphi)^* \subseteq \mathcal{A}$.

For any nonempty subset $\mathcal{G} \subseteq \mathcal{L}$, the annihilator set $\mathcal{G}^* = \{\varphi \in \mathcal{L} \mid \iota \wedge \varphi = 0 \text{ for all } \iota \in \mathcal{G}\}$ is an ideal of \mathcal{L} . In particular, $(\iota)^* = \{\iota\}^*$ for each $\iota \in \mathcal{L}$. An element σ is called *dense* when $(\sigma)^* = \{0\}$, and the set of all dense elements is denoted by \mathcal{D} . This set forms a filter in \mathcal{L} .

An ADL \mathcal{L} is normal [7] when every prime ideal contains exactly one minimal prime ideal. It is quasi-complemented [5] if for every $\iota \in \mathcal{L}$ there exists $\iota' \in \mathcal{L}$ such that $\iota \wedge \iota' = 0$ and $\iota \vee \iota' \in \mathcal{D}$. Equivalently, the lattice is quasi-complemented exactly when for each $\varphi \in \mathcal{L}$ one can find $\varphi' \in \mathcal{L}$ such that $(\varphi)^{**} = (\varphi')^*$.

An ADL \mathcal{L} is a generalized Stone ADL [4] when the identity $(\varphi)^* \vee (\varphi)^{**} = \mathcal{L}$ holds for all $\varphi \in \mathcal{L}$. It is relatively complemented [3] if every interval $[\varphi, \omega]$ with $\varphi \leq \omega$ forms a complemented distributive lattice. According to [12], when \mathcal{L} contains maximal elements, relative complementedness is equivalent to the condition $B(\mathcal{L}) = \mathcal{L}$, where $B(\mathcal{L}) = \{\varphi \in \mathcal{L} \mid \varphi \wedge \omega = 0, \varphi \vee \omega \in \mathfrak{M}(\mathcal{L}), \omega \in \mathcal{L}\}$.

Theorem 2.1. [3] *An ADL \mathcal{L} is relatively complemented if and only if all prime ideals of \mathcal{L} are minimal.*

A unary map $\iota \mapsto \iota^*$ on \mathcal{L} is called a *pseudo-complementation* [11] when for all $\iota, \vartheta \in \mathcal{L}$: (1) $\iota \wedge \vartheta = 0$ implies $\iota^* \wedge \vartheta = \vartheta$, (2) $\iota \wedge \iota^* = 0$, (3) $(\iota \vee \vartheta)^* = \iota^* \wedge \vartheta^*$.

Every pseudo-complemented ADL is necessarily quasi-complemented.

Theorem 2.2. *An ADL \mathcal{L} is generalized Stone precisely when it is both normal and quasi-complemented.*

As introduced in [8], a filter \mathcal{H} is a \mathcal{D} -filter when $\mathcal{D} \subseteq \mathcal{H}$. The least \mathcal{D} -filter is \mathcal{D} itself. For any nonempty $\mathcal{G} \subseteq \mathcal{L}$, define $(\mathcal{G}, \mathcal{D}) = \{\varphi \in \mathcal{L} \mid \iota \vee \varphi \in \mathcal{D} \text{ for every } \iota \in \mathcal{G}\}$. One has $(\mathcal{L}, \mathcal{D}) = \mathcal{D}$, $(\mathcal{D}, \mathcal{D}) = \mathcal{L}$, and always $\mathcal{D} \subseteq (\mathcal{G}, \mathcal{D})$. For a single element ι , the notation (ι, \mathcal{D}) is used. For any maximal element ν , $(\nu, \mathcal{D}) = \mathcal{L}$. The set $(\mathcal{G}, \mathcal{D})$ forms a \mathcal{D} -filter.

Lemma 2.1. [8] *For any subsets $\mathcal{G}, \mathcal{K} \subseteq \mathcal{L}$, the following statements are valid:*

- (1) $\mathcal{G} \subseteq \mathcal{K} \Rightarrow (\mathcal{K}, \mathcal{D}) \subseteq (\mathcal{G}, \mathcal{D})$,
- (2) $\mathcal{G} \subseteq ((\mathcal{G}, \mathcal{D}), \mathcal{D})$,
- (3) $(\mathcal{G}, \mathcal{D}) = (((\mathcal{G}, \mathcal{D}), \mathcal{D}), \mathcal{D})$,
- (4) $\mathcal{G} \subseteq \mathcal{D} \iff (\mathcal{G}, \mathcal{D}) = \mathcal{L}$.

Since $([\varphi], \mathcal{D}) = (\varphi, \mathcal{D})$ for every $\varphi \in \mathcal{L}$, it follows immediately that $(0, \mathcal{D}) = \mathcal{D}$. This observation leads directly to the corresponding corollary.

3. COHERENT ADLS

This section presents the concept of coherent ADLs and investigates their connections to other types of ADLs, such as generalized Stone ADLs, relatively complemented ADLs, quasi-complemented ADLs, and normal ADLs. The properties of coherent ADLs are described using the notions of filters and \mathcal{D} -filters.

Definition 3.1. *An ADL \mathcal{L} is defined as coherent if, for each $\varphi, \omega \in \mathcal{L}$, $\varphi \vee \omega \in \mathcal{D}$ implies $(\varphi)^{**} \vee (\omega)^{**} = \mathcal{L}$.*

In a chain (a totally ordered set), every non-zero element is dense, which ensures that all chains are coherent ADLs. Furthermore, any dense ADL, characterized by $\{0\} = (\varphi)^*$ for each non-zero element φ in \mathcal{L} , is coherent. The next example highlights a non-trivial case of a coherent ADL.

Example 3.1. Let $\mathcal{L} = \{0, \iota, \vartheta, \rho, \sigma, \epsilon, \kappa, \alpha\}$ and define \vee, \wedge on \mathcal{L} as follows:

\wedge	0	ι	ϑ	ρ	σ	ϵ	κ	α
0	0	0	0	0	0	0	0	0
ι	0	ι	ϑ	ρ	σ	ϵ	κ	α
ϑ	0	ι	ϑ	ρ	σ	ϵ	κ	α
ρ	0	ρ	ρ	ρ	0	0	ρ	0
σ	0	σ	ϵ	0	σ	ϵ	α	α
ϵ	0	σ	ϵ	0	σ	ϵ	α	α
κ	0	κ	κ	ρ	α	α	κ	α
α	0	α	α	0	α	α	α	α

\vee	0	ι	ϑ	ρ	σ	ϵ	κ	α
0	0	ι	ϑ	ϑ	σ	ϵ	κ	α
ι	ι	ι	ι	ι	ι	ι	ι	ι
ϑ	ϑ	ϑ	ϑ	ϑ	ϑ	ϑ	ϑ	ϑ
ρ	ρ	ι	ϑ	ρ	ι	ϑ	κ	κ
σ	σ	ι	ι	ι	σ	σ	ι	σ
ϵ	ϵ	ϑ	ϑ	ϑ	ϵ	ϵ	ϑ	ϵ
κ	κ	ι	ϑ	κ	ι	ϑ	κ	κ
α	α	ι	ϑ	κ	σ	ϵ	κ	α

Then $(\mathcal{L}, \vee, \wedge)$ is an ADL. Now, $(0)^* = \mathcal{L}, (\iota)^* = (\vartheta)^* = (\kappa)^* = \{0\}, (\rho)^* = \{0, \sigma, \epsilon, \alpha\}, (\sigma)^* = (\epsilon)^* = (\alpha)^* = \{0, \rho\}$. Hence $(0)^{**} = \{0\}, (\iota)^{**} = (\vartheta)^{**} = (\kappa)^{**} = \mathcal{L}, (\rho)^{**} = \{0, \rho\}, (\sigma)^{**} = (\epsilon)^{**} = (\alpha)^{**} = \{0, \sigma, \epsilon, \alpha\}$. Clearly ι, ϑ, κ are the dense elements of \mathcal{L} . It is easily observed that \mathcal{L} is a coherent ADL.

Proposition 3.1. All generalized Stone ADLs belong to the class of coherent ADLs.

Proof. Suppose \mathcal{L} is a generalized Stone ADL. Let $\varphi, \omega \in \mathcal{L}$ with $\varphi \vee \omega \in \mathcal{D}$. According to Theorem 2.2, \mathcal{L} is both normal and quasi-complemented. Therefore, there are elements $\varphi', \omega' \in \mathcal{L}$ such that $(\varphi)^{**} = (\varphi')^*$ and $(\omega)^{**} = (\omega')^*$. From this, we can conclude the following.

$$\begin{aligned}
 (\varphi)^{**} \vee (\omega)^{**} &= (\varphi')^* \vee (\omega')^* \\
 &= (\varphi' \wedge \omega')^* \\
 &= (\varphi' \wedge \omega')^{***} \\
 &= ((\varphi')^{**} \cap (\omega')^{**})^* \\
 &= ((\varphi)^* \cap (\omega)^*)^* \\
 &= ((\varphi \vee \omega)^*)^* \\
 &= \mathcal{L}.
 \end{aligned}$$

Therefore, for every $\varphi, \omega \in \mathcal{L}$ such that $\varphi \vee \omega \in \mathcal{D}$, we have $(\varphi)^{**} \vee (\omega)^{**} = \mathcal{L}$. This implies that \mathcal{L} is coherent. \square

The next result provides a series of equivalent circumstances that characterize when a coherent ADL is also a generalized Stone ADL.

Theorem 3.1. Given that \mathcal{L} is a coherent ADL, the statements listed below are mutually equivalent:

- (1) \mathcal{L} is generalized Stone;
- (2) any prime \mathcal{D} -filter is minimal;

- (3) any maximal filter is a minimal prime \mathcal{D} -filter;
- (4) \mathcal{L} is quasi-complemented.

Proof. (1) \Rightarrow (2) : Suppose condition (1) holds. Let \mathcal{A} be a prime filter of \mathcal{L} , $\sigma \in \mathcal{D}$, and $\varphi \in \mathcal{A}$. By (1), we have $(\varphi)^* \vee (\varphi)^{**} = \mathcal{L}$. Thus, σ belongs to $(\varphi)^* \vee (\varphi)^{**}$, meaning there exist $\iota \in (\varphi)^*$ and $\vartheta \in (\varphi)^{**}$ such that $\iota \vee \vartheta = \sigma \in \mathcal{D}$. Consequently, $\iota \wedge \varphi = 0$, and we find that $(\varphi)^* \subseteq (\vartheta)^{**}$. Now, suppose $\iota \in \mathcal{A}$. Then $\iota \wedge \varphi \in \mathcal{A}$, which leads to a contradiction. Therefore, ι cannot be an element of \mathcal{A} . Now,

$$\begin{aligned} \iota \vee \vartheta \in \mathcal{D} &\Rightarrow (\iota)^* \cap (\vartheta)^* = \{0\} \\ &\Rightarrow (\iota)^* \cap (\varphi)^* = \{0\} \\ &\Rightarrow (\iota \vee \varphi)^* = \{0\} \end{aligned}$$

This means that $\iota \vee \varphi \in \mathcal{D}$. Therefore, \mathcal{A} must be minimal.

(2) \Rightarrow (3) : It is evident because each maximal filter is also a prime \mathcal{D} -filter.

(3) \Rightarrow (4) : Assume (3) holds. Let $\varphi \in \mathcal{L}$. If $0 \notin [\varphi] \vee (\varphi, \mathcal{D})$, there is a maximal filter \mathcal{X} such that $[\varphi] \vee (\varphi, \mathcal{D}) \subseteq \mathcal{X}$. As a result, $\varphi \in \mathcal{X}$ and $(\varphi, \mathcal{D}) \subseteq \mathcal{X}$. According to condition (3), \mathcal{X} is a minimal prime \mathcal{D} -filter. However, since $(\varphi, \mathcal{D}) \subseteq \mathcal{X}$, it follows that $\varphi \notin \mathcal{X}$, which is a contradiction. Therefore, 0 must be in $[\varphi] \vee (\varphi, \mathcal{D})$. This implies that there is $\iota \in (\varphi, \mathcal{D})$ satisfying $\varphi \wedge \iota = 0$. Hence, $\varphi \vee \iota \in \mathcal{D}$, and therefore, \mathcal{L} is quasi-complemented.

(4) \Rightarrow (1) : Assume (4) holds. Let $\varphi \in \mathcal{L}$. Since \mathcal{L} is quasi-complemented, there is $\varphi' \in \mathcal{L}$ satisfying $\varphi \wedge \varphi' = 0$, $\varphi \vee \varphi' \in \mathcal{D}$. Given that \mathcal{L} is coherent, we have $(\varphi)^{**} \vee (\varphi')^{**} = \mathcal{L}$. Since $\varphi \wedge \varphi' = 0$, it follows that $(\varphi')^{**} \subseteq (\varphi)^*$. Therefore, we conclude:

$$\mathcal{L} = (\varphi)^{**} \vee (\varphi')^{**} \subseteq (\varphi)^{**} \vee (\varphi)^*.$$

Thus, \mathcal{L} is generalized Stone. □

Given that every pseudo-complemented ADL is quasi-complemented, the following result directly results from the previous theorem:

Corollary 3.1. *A pseudo-complemented ADL \mathcal{L} is coherent if and only if it is generalized Stone.*

Corollary 3.2. *Any quasi-complemented ADL that is coherent must be normal.*

Proof. This is a consequence of Theorem 3.1 and Theorem 2.2. □

Corollary 3.3. *A quasi-complemented ADL that is normal always must be coherent.*

Proof. This is a consequence of Theorem 3.1 and Theorem 2.2. □

Theorem 3.2. *An ADL \mathcal{L} is relatively complemented if and only if $(\varphi] \vee (\varphi)^* = \mathcal{L}$ for all $\varphi \in \mathcal{L}$.*

Proof. Suppose \mathcal{L} is relatively complemented. Then $B(\mathcal{L}) = \mathcal{L}$. Let $\varphi \in \mathcal{L}$. Since $B(\mathcal{L}) = \mathcal{L}$, there is $\varphi' \in \mathcal{L}$ satisfying $\varphi \wedge \varphi' = 0$ and $\varphi \vee \varphi' \in \mathfrak{M}(\mathcal{L})$. This implies $\varphi' \in (\varphi)^*$, and we also

have $(\varphi] \vee (\varphi'] = (\varphi \vee \varphi'] = \mathcal{L}$. Therefore, $(\varphi'] \subseteq (\varphi)^*$, which shows that $(\varphi] \vee (\varphi)^* = \mathcal{L}$. Conversely, assume this condition is true. Let $\varphi \in \mathcal{L}$. Then, $(\varphi)^* \vee (\varphi] = \mathcal{L}$. For any $m \in \mathfrak{M}(\mathcal{L})$, we find that $m \in (\varphi)^* \vee (\varphi]$, implying there exists $\iota \in (\varphi)^*$ satisfying $\iota \vee \varphi = m$. As $\iota \in (\varphi)^*$, it follows that $\iota \wedge \varphi = 0$, so ι is the complement of φ in \mathcal{L} . Hence $B(\mathcal{L}) = \mathcal{L}$. Thus, \mathcal{L} is relatively complemented. \square

Proposition 3.2. *Every relatively complemented ADL is generalized Stone.*

Proof. Let \mathcal{L} be relatively complemented and $\varphi \in \mathcal{L}$. Suppose $(\varphi)^* \vee (\varphi)^{**} \neq \mathcal{L}$. Then there is a prime ideal \mathcal{A} satisfying $(\varphi)^* \vee (\varphi)^{**} \subseteq \mathcal{A}$. This implies that $(\varphi)^* \subseteq \mathcal{A}$ and $\varphi \in (\varphi)^{**} \subseteq \mathcal{A}$. By our assumption, there is $\varphi' \in \mathcal{L}$ satisfying $\varphi \wedge \varphi' = 0$ and $\varphi \vee \varphi' \in \mathfrak{M}(\mathcal{L})$. Now, suppose $\varphi' \in \mathcal{A}$. Then $\varphi \vee \varphi' \subseteq \mathcal{A}$, which leads to a contradiction. Therefore, $\varphi' \notin \mathcal{A}$. This implies that \mathcal{A} is minimal. Since $(\varphi)^* \subseteq \mathcal{A}$, we conclude $\varphi \notin \mathcal{A}$, which is also a contradiction. Hence, $(\varphi)^* \vee (\varphi)^{**} = \mathcal{L}$, which shows that \mathcal{L} is generalized Stone. \square

Corollary 3.4. *Every relatively complemented ADL is coherent.*

Proof. This is a consequence of Proposition 3.1 and Proposition 3.2. \square

The reverse of Proposition 3.2 does not always hold. For instance, in the coherent ADL described in Example 3.1, \mathcal{L} fails to be complemented because the elements κ and α do not possess complements within \mathcal{L} . Nevertheless, certain equivalent conditions can guarantee that a coherent ADL is relatively complemented.

Theorem 3.3. *A coherent ADL \mathcal{L} is relatively complemented if and only if \mathcal{L} is quasi-complemented in which every principal ideal is annihilator.*

Proof. Assume a coherent ADL is quasi-complemented in which every principal ideal is annihilator. By Theorem 3.1, it gives \mathcal{L} is generalized Stone. Hence $(\varphi)^* \vee (\varphi)^{**} = \mathcal{L}$, for all $\varphi \in \mathcal{L}$. As each principal ideal is annihilator, we get $(\varphi] = (\varphi)^{**}$. Hence, $(\varphi] \vee (\varphi)^* = \mathcal{L}$. According to Theorem 3.2, this implies that \mathcal{L} is relatively complemented. On the other hand, suppose \mathcal{L} is relatively complemented. Its clear that \mathcal{L} is quasi-complemented. Let $\varphi \in \mathcal{L}$. By Proposition 3.2, \mathcal{L} is generalized Stone, meaning $(\varphi)^{**} \vee (\varphi)^* = \mathcal{L}$. By Theorem 3.2, we conclude $(\varphi] \vee (\varphi)^* = \mathcal{L}$. It is obvious that $(\varphi] \cap (\varphi)^* = \{0\}$ and $(\varphi)^{**} \cap (\varphi)^* = \{0\}$. Since $\mathfrak{I}(\mathcal{L})$ is distributive, we deduce $(\varphi] = (\varphi)^{**}$. Therefore, every principal ideal is annihilator. \square

Corollary 3.5. *A generalized Stone ADL is relatively complemented if and only if every principal ideal of an ADL is annihilator.*

Proof. This is a consequence of Theorem 3.1 and Theorem 3.3. \square

Proposition 3.3. *An ADL in which every maximal ideal is non-dense is a coherent ADL.*

Proof. Assume all maximal ideals of \mathcal{L} are non-dense. Let $\varphi, \omega \in \mathcal{L}$ such that $\varphi \vee \omega \in \mathcal{D}$. Assume, for contradiction, that $(\varphi)^{**} \vee (\omega)^{**} \neq \mathcal{L}$. Then, there exists a maximal ideal \mathcal{X} satisfying $(\varphi)^{**} \vee (\omega)^{**} \subseteq \mathcal{X}$

\mathcal{X} . This implies both $(\varphi)^{**} \subseteq \mathcal{X}$, $(\omega)^{**} \subseteq \mathcal{X}$. As a result, \mathcal{X}^* is contained in $(\varphi)^*$ and $(\omega)^*$, so $\mathcal{X}^* \subseteq (\varphi)^* \cap (\omega)^*$. Since $\varphi \vee \omega \in \mathcal{D}$, we know $(\varphi \vee \omega)^* = \{0\}$. Therefore, $\mathcal{X}^* \subseteq \{0\}$, which would make \mathcal{X} a dense ideal. This contradicts the assumption that all maximal ideals in \mathcal{L} are non-dense. Hence, the assumption $(\varphi)^{**} \vee (\omega)^{**} \neq \mathcal{L}$ must be false. It follows that $(\varphi)^{**} \vee (\omega)^{**} = \mathcal{L}$, and thus \mathcal{L} is coherent. \square

The reverse implication of Proposition 3.3 is invalid. Consider the coherent ADL presented in Example 3.1, where \mathcal{L} has two maximal ideals: $\mathcal{X}_1 = \{0, \rho, \kappa, \alpha\}$ and $\mathcal{X}_2 = \{0, \sigma, \epsilon, \alpha\}$. It is clear that $\mathcal{X}_1^* = \{0\}$, while $\mathcal{X}_2^* = \{0, \rho\} \neq \{0\}$.

We now present a significant result related to the direct products of coherent ADLs. To establish this, we first introduce a lemma with a simple proof.

Lemma 3.1. For $n \in \mathbb{N}$, consider $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ are n ADLs. For any $t_1 \in \mathcal{L}_1, t_2 \in \mathcal{L}_2, \dots, t_n \in \mathcal{L}_n$, The subsequent properties are valid:

- (1) $(t_1, t_2, \dots, t_n)^* = (t_1)^* \times (t_2)^* \times \dots \times (t_n)^*$;
- (2) $(t_1, t_2, \dots, t_n)^* \vee (\vartheta_1, \vartheta_2, \dots, \vartheta_n)^* = (t_1 \vee \vartheta_1, t_2 \vee \vartheta_2, \dots, t_n \vee \vartheta_n)^*$
- (3) $(t_1, t_2, \dots, t_n)^{**} = (t_1)^{**} \times (t_2)^{**} \times \dots \times (t_n)^{**}$.

Proof. (1) and (2) are straightforward.

(3) Let us represent $(\varphi)_i = (\varphi_1, \varphi_2, \dots, \varphi_n)$ here $\varphi_1 \in \mathcal{L}_1, \varphi_2 \in \mathcal{L}_2, \dots, \varphi_n \in \mathcal{L}_n$. For any $(t)_i \in \mathcal{L}_1 \times \mathcal{L}_2 \times \dots \times \mathcal{L}_n$, we get

$$\begin{aligned} (\varphi)_i \in (t)_i^{**} &\Leftrightarrow (t)_i^* \subseteq (\varphi)_i^* \\ &\Leftrightarrow (t_1)^* \times (t_2)^* \times \dots \times (t_n)^* \subseteq (\varphi_1)^* \times (\varphi_2)^* \times \dots \times (\varphi_n)^* \\ &\Leftrightarrow (t_i)^* \subseteq (\varphi_i)^* \quad \text{for } i = 1, 2, \dots, n \\ &\Leftrightarrow (t_i)^{**} \subseteq (\varphi_i)^{**} \quad \text{for } i = 1, 2, \dots, n \\ &\Leftrightarrow (\varphi)_i \in (\varphi)_i^{**} \subseteq (t)_i^{**}. \end{aligned}$$

Hence $(t_1, t_2, \dots, t_n)^{**} = (t_1)^{**} \times (t_2)^{**} \times \dots \times (t_n)^{**}$. \square

Theorem 3.4. Let $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ (where $n \in \mathbb{N}$) be a finite set of ADLs. Then the product ADL $\mathcal{L}_1 \times \mathcal{L}_2 \times \dots \times \mathcal{L}_n$ (with point-wise operations) is coherent iff $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ are coherent.

Proof. Assume that $\mathcal{L}_1 \times \mathcal{L}_2 \times \dots \times \mathcal{L}_n$ is coherent. Let $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n$ be the dense sets of $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ respectively. Let $\iota, \vartheta \in \mathcal{L}_1$ with $\iota \vee \vartheta \in \mathcal{D}_1$. Select $\sigma_2 \in \mathcal{D}_2, \sigma_3 \in \mathcal{D}_3, \dots, \sigma_n \in \mathcal{D}_n$. Then $(\iota, \sigma_2, \sigma_3, \dots, \sigma_n) \vee (\vartheta, \sigma_2, \sigma_3, \dots, \sigma_n) = (\iota \vee \vartheta, \sigma_2, \sigma_3, \dots, \sigma_n) \in \mathcal{D}_1 \times \mathcal{D}_2 \times \dots \times \mathcal{D}_n$. Since $\mathcal{L}_1 \times \mathcal{L}_2 \times \dots \times \mathcal{L}_n$ is coherent, it gives

$$(\iota, \sigma_2, \sigma_3, \dots, \sigma_n)^{**} \vee (\vartheta, \sigma_2, \sigma_3, \dots, \sigma_n)^{**} \in \mathcal{L}_1 \times \mathcal{L}_2 \times \dots \times \mathcal{L}_n.$$

Let $\zeta \in \mathcal{L}_1$. Then $(\zeta, \sigma_2, \sigma_3, \dots, \sigma_n) \in \mathcal{L}_1 \times \mathcal{L}_2 \times \dots \times \mathcal{L}_n$. Hence, there exists $(\mu_1, \mu_2, \mu_3, \dots, \mu_n) \in (\iota, \sigma_2, \sigma_3, \dots, \sigma_n)^{**}$ and $(\delta_1, \delta_2, \delta_3, \dots, \delta_n) \in (\vartheta, \sigma_2, \sigma_3, \dots, \sigma_n)^{**}$ with $(\zeta, \sigma_2, \sigma_3, \dots, \sigma_n) = (\mu_1, \mu_2, \mu_3, \dots, \mu_n) \vee (\delta_1, \delta_2, \delta_3, \dots, \delta_n)$. Hence $\zeta = \mu_1 \vee \delta_1$ where $\mu_1 \in (\iota)^{**}$ and $\delta_1 \in (\vartheta)^{**}$. Hence $(\iota)^{**} \vee (\vartheta)^{**} = \mathcal{L}_1$.

Thus \mathcal{L}_1 is coherent. In a similar manner, it can be shown that $\mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n$ are coherent ADLs. The converse can be derived from the fact that $(l_1, l_2, \dots, l_n)^{**} = (l_1)^{**} \times (l_2)^{**} \times \dots \times (l_n)^{**}$ for any $(l_1, l_2, \dots, l_n) \in \mathcal{L}_1 \times \mathcal{L}_2 \times \dots \times \mathcal{L}_n$. \square

Lemma 3.2. *If an ADL \mathcal{L} is relatively complemented, then every chain in \mathcal{L} contains no more than three elements.*

Proof. Assume \mathcal{L} is relatively complemented. Let $\varphi, \omega, \zeta \in \mathcal{L} \setminus \{0\}$ with $0 < \varphi < \omega < \zeta$. Clearly, φ is an element of the interval $[0, \varphi \vee \omega]$. By our assumption, there is an element $\delta \in \mathcal{L}$ satisfying $\varphi \wedge \delta = 0$, $\varphi \vee \delta = \varphi \vee \omega = \omega$. Given $\varphi \wedge \delta = 0$, it follows that $\omega = \varphi \vee \delta$ must belong to $\mathfrak{M}(\mathcal{L})$, which leads to a contradiction. Thus, every chain in \mathcal{L} has no more than three elements. \square

A subADL \mathcal{I} of an ADL \mathcal{L} is referred to as a \mathcal{D} -subADL if $0 \in \mathcal{I}$, $\mathcal{I} \cap \mathcal{D} \neq \emptyset$. An ideal \mathcal{T} of an ADL is said to be a \mathcal{D} -ideal if $\mathcal{T} \cap \mathcal{D} \neq \emptyset$. It is evident that every \mathcal{D} -ideal is also a \mathcal{D} -subADL.

Theorem 3.5. *An ADL \mathcal{L} satisfies the following equivalent conditions:*

- (1) *each \mathcal{D} -subADL is coherent;*
- (2) *for every $\varphi, \omega \in \mathcal{L} \setminus \{0\}$, $\varphi \wedge \omega = 0$ implies $\varphi \vee \omega \in \mathfrak{M}(\mathcal{L})$;*
- (3) *\mathcal{L} is a dense ADL or \mathcal{L} is relatively complemented.*

Proof. (1) \Rightarrow (2) : Assume (1). Let φ, ω non-zero elements of \mathcal{L} such that $\varphi \wedge \omega = 0$. Suppose $\varphi \vee \omega \notin \mathfrak{M}(\mathcal{L})$. Choose a non-maximal element $\zeta \in \mathcal{L}$ such that $\varphi \vee \omega < \zeta$. Consider the subADL $\mathcal{L}_1 = \{0, \varphi, \omega, \varphi \vee \omega, \zeta\}$. It is clear that $\varphi \vee \omega \in \mathcal{D}_1$, meaning \mathcal{L}_1 is a \mathcal{D} -subADL. Now, we compute $(\varphi)^{**} \vee (\omega)^{**} = \{0, \varphi\} \vee \{0, \omega\} = \mathcal{L}_1 \setminus \{\zeta\} \neq \mathcal{L}_1$. Hence \mathcal{L}_1 is not coherent, It leads a contradicts. Therefore, $\varphi \vee \omega \in \mathfrak{M}(\mathcal{L})$.

(2) \Rightarrow (3) : Assume (2). Let \mathcal{L} be a non-dense ADL. In this scenario, $\{0\}$ does not qualify as a prime ideal of \mathcal{L} . Let \mathcal{A} be a prime ideal of \mathcal{L} . If \mathcal{A} is not minimal, then there exists a minimal prime ideal \mathcal{X} such that $\mathcal{X} \subseteq \mathcal{A}$. Choose an element $0 \neq \varphi \in \mathcal{X}$. If $(\varphi)^* \cap \mathcal{A} \neq \{0\}$, then there is some $\omega \in (\varphi)^* \cap \mathcal{A}$, meaning $\varphi \wedge \omega = 0$ and $\omega \in \mathcal{A}$. According to the hypothesis, we have $\varphi \vee \omega \in \mathfrak{M}(\mathcal{L})$. Since $\varphi \in \mathcal{A}$, it follows that $\varphi \vee \omega \in \mathcal{A}$, which leads to a contradiction. Therefore, $(\varphi)^* \cap \mathcal{A} = \{0\} \subseteq \mathcal{X}$. Since \mathcal{X} is a prime ideal and $\mathcal{X} \subseteq \mathcal{A}$, it must be that $(\varphi)^* \subseteq \mathcal{X}$, which contradicts the minimality of \mathcal{X} . Thus, \mathcal{A} must be minimal. By Theorem 2.1, \mathcal{L} is relatively complemented.

(3) \Rightarrow (1) : Assume (3). Let \mathcal{L}_1 be a \mathcal{D} -subADL of \mathcal{L} , and let \mathcal{D}_1 denote the set of dense elements in \mathcal{L}_1 . If \mathcal{L} is dense, the statement holds trivially. Suppose \mathcal{L} is relatively complemented. By Lemma 3.2, any chain in \mathcal{L} contains maximum three elements. Let $\varphi, \omega \in \mathcal{L}_1$ such that $\varphi \vee \omega \in \mathcal{D}_1$.

If either $\varphi \in \mathcal{D}_1$ or $\omega \in \mathcal{D}_1$, then clearly $(\varphi)_{\mathcal{L}_1}^{**} \vee (\omega)_{\mathcal{L}_1}^{**} = \mathcal{L}_1$.

Now suppose that neither $\varphi \in \mathcal{D}_1$ nor $\omega \in \mathcal{D}_1$. Assume $0 < \varphi \leq \varphi \vee \omega$. If $\varphi = \varphi \vee \omega$, then $\varphi \in \mathcal{D}_1$, it leads a contradiction. Therefore, $0 < \varphi < \varphi \vee \omega$. Hence $\varphi \vee \omega$ must be the greatest element in \mathcal{L}_1 because any chain has at most three elements,. Thus, $\varphi \vee \omega \in (\varphi)_{\mathcal{L}_1}^{**} \vee (\omega)_{\mathcal{L}_1}^{**} = \mathcal{L}_1$. \square

Theorem 3.6. *In an ADL \mathcal{L} , the following conditions are mutually equivalent:*

- (1) *\mathcal{L} is coherent;*

- (2) any proper \mathcal{D} -ideal is a coherent subADL;
- (3) for any $\sigma \in \mathcal{D}$, $[0, \sigma]$ is a coherent sublattice.

Proof. (1) \Rightarrow (2) : Assume condition (1). Let \mathcal{T} be a \mathcal{D} -ideal of \mathcal{L} with $\mathcal{T} \neq \mathcal{L}$. Suppose $\varphi, \omega \in \mathcal{T}$ such that $\varphi \vee \omega \in \mathcal{D}_{\mathcal{T}} \subseteq \mathcal{D}$. Since \mathcal{L} is coherent, we have $(\varphi)^{**} \vee (\omega)^{**} = \mathcal{L}$. Define $(\iota)_{\mathcal{T}}^{**} = \mathcal{T} \cap (\iota)^{**}$ for any $\iota \in \mathcal{T}$. Clearly, $(\iota)_{\mathcal{T}}^{**}$ is an ideal in \mathcal{T} , and $(\iota)_{\mathcal{T}}^{**}$ represents the annulet of ι within \mathcal{T} . Now, we can write:

$$\mathcal{T} = \mathcal{T} \cap \mathcal{L} = \mathcal{T} \cap \{(\varphi)^{**} \vee (\omega)^{**}\} = \{\mathcal{T} \cap (\varphi)^{**}\} \vee \{\mathcal{T} \cap (\omega)^{**}\} = (\varphi)_{\mathcal{T}}^{**} \vee (\omega)_{\mathcal{T}}^{**}$$

Thus, \mathcal{T} is a coherent subADL of \mathcal{L} .

(2) \Rightarrow (3) : This is clear because $[0, \sigma]$ is a proper \mathcal{D} -ideal for each $\sigma \in \mathcal{D}$.

(3) \Rightarrow (1) : The proof follows by choosing $\sigma \in \mathfrak{M}(\mathcal{L})$. □

Definition 3.2. Given any non-void subset \mathcal{G} of an ADL \mathcal{L} , we define:

$$\mathcal{G}^{\tau} = \{\varphi \in \mathcal{L} \mid (\iota)^{**} \vee (\varphi)^{**} = \mathcal{L} \text{ for all } \iota \in \mathcal{G}\}$$

It is clear that $\mathcal{L}^{\tau} = \mathcal{D}$ and $\{0\}^{\tau} = \mathcal{D}$. For each $\iota \in \mathcal{L}$, we represent $\{\iota\}^{\tau}$ as $(\iota)^{\tau}$. Thus, it is evident that $\mathcal{D} = (0)^{\tau}$ and $\mathcal{L} = (m)^{\tau}$ for any $m \in \mathfrak{M}(\mathcal{L})$.

Proposition 3.4. For each non-void subset \mathcal{G} in \mathcal{L} , \mathcal{G}^{τ} constitutes a \mathcal{D} -filter.

Proof. It is evident that $\mathcal{D} \subseteq \mathcal{G}^{\tau}$. Let $\varphi, \omega \in \mathcal{G}^{\tau}$. For any $\iota \in \mathcal{G}$, we observe that

$$(\varphi \wedge \omega)^{**} \vee (\iota)^{**} = \{(\varphi)^{**} \cap (\omega)^{**}\} \vee (\iota)^{**} = \{(\varphi)^{**} \vee (\iota)^{**}\} \cap \{(\omega)^{**} \vee (\iota)^{**}\} = \mathcal{L} \cap \mathcal{L} = \mathcal{L}.$$

Thus, $\varphi \wedge \omega \in \mathcal{G}^{\tau}$. Next, assume $\varphi \in \mathcal{G}^{\tau}$ and $\varphi \leq \omega$. Then, for every $\iota \in \mathcal{G}$, we have $(\varphi)^{**} \vee (\iota)^{**} = \mathcal{L}$, and it follows that $(\varphi)^{**} \subseteq (\omega)^{**}$. For any $\rho \in \mathcal{G}$, we conclude

$$\mathcal{L} = (\varphi)^{**} \vee (\rho)^{**} \subseteq (\omega)^{**} \vee (\rho)^{**}.$$

Therefore, $\omega \in \mathcal{G}^{\tau}$. This shows that \mathcal{G}^{τ} is a \mathcal{D} -filter. □

The next lemma is a simple consequence of the definition provided earlier.

Lemma 3.3. For each two non-void subsets \mathcal{G} and \mathcal{K} of an ADL \mathcal{L} , the subsequent properties are true:

- (1) $\mathcal{G}^{\tau} = \bigcap_{\iota \in \mathcal{G}} (\iota)^{\tau}$;
- (2) $\mathcal{G} \cap \mathcal{G}^{\tau} \subseteq \mathcal{D}$;
- (3) $\mathcal{G} \subseteq \mathcal{K}$ implies $\mathcal{K}^{\tau} \subseteq \mathcal{G}^{\tau}$;
- (4) $\mathcal{G} \subseteq \mathcal{G}^{\tau\tau}$;
- (5) $\mathcal{G}^{\tau\tau\tau} = \mathcal{G}^{\tau}$;
- (6) $\mathcal{G}^{\tau} = \mathcal{L}$ iff $\mathcal{G} \subseteq \mathcal{D}$.

The following result holds in the case of filters.

Proposition 3.5. For any filters \mathcal{H}, \mathcal{J} of \mathcal{L} , $(\mathcal{H} \vee \mathcal{J})^{\tau} = \mathcal{H}^{\tau} \cap \mathcal{J}^{\tau}$.

Proof. It is evident that $(\mathcal{H} \vee \mathcal{J})^\tau \subseteq \mathcal{H}^\tau \cap \mathcal{J}^\tau$. On the other hand, let $x \in \mathcal{H}^\tau \cap \mathcal{J}^\tau$. Let $\rho \in \mathcal{H} \vee \mathcal{J}$ be any element. Then, $\rho = j \wedge \mathfrak{a}$ for some $j \in \mathcal{H}$ and $\mathfrak{a} \in \mathcal{J}$. Now,

$$(\varphi)^{**} \vee (\rho)^{**} = (\varphi)^{**} \vee (j \wedge \mathfrak{a})^{**} = (\varphi)^{**} \vee \{(j)^{**} \cap (\mathfrak{a})^{**}\} = \{(\varphi)^{**} \vee (j)^{**}\} \cap \{(\varphi)^{**} \vee (\mathfrak{a})^{**}\} = \mathcal{L} \cap \mathcal{L} = \mathcal{L}.$$

Thus, $\varphi \in (\mathcal{H} \vee \mathcal{J})^\tau$, which implies $(\mathcal{H} \vee \mathcal{J})^\tau = \mathcal{H}^\tau \cap \mathcal{J}^\tau$. \square

The following result is a straightforward outcome of the previous results.

Corollary 3.6. *Let $\iota, \vartheta \in \mathcal{L}$. The subsequent properties hold:*

- (1) $\iota \leq \vartheta$ implies $(\iota)^\tau \subseteq (\vartheta)^\tau$;
- (2) $(\iota \wedge \vartheta)^\tau = (\iota)^\tau \cap (\vartheta)^\tau$;
- (3) $(\iota)^\tau = \mathcal{L}$ iff $\iota \in \mathcal{D}$;
- (4) $\iota \in \mathcal{D}^\tau$ implies $\iota \vee \vartheta \in \mathcal{D}$;
- (5) $(\iota)^* = (\vartheta)^*$ implies $(\iota)^\tau = (\vartheta)^\tau$.

For each filter \mathcal{H} in \mathcal{L} , it is clear that $\mathcal{H}^\tau \subseteq (\mathcal{H}, \mathcal{D})$. However, we present a collection of equivalent conditions under which the reverse inclusion holds, providing a characterization of coherent ADLs.

Theorem 3.7. *The subsequent propositions are equivalent in an ADL \mathcal{L} :*

- (1) \mathcal{L} is a coherent ADL;
- (2) For every two filters \mathcal{H} and \mathcal{J} of \mathcal{L} , $\mathcal{H} \cap \mathcal{J} \subseteq \mathcal{D}$ iff $\mathcal{H} \subseteq \mathcal{J}^\tau$;
- (3) for every filter \mathcal{H} of \mathcal{L} , $\mathcal{H}^\tau = (\mathcal{H}, \mathcal{D})$;
- (4) for every $\iota \in \mathcal{L}$, $(\iota)^\tau = (\iota, \mathcal{D})$.

Proof. (1) \Rightarrow (2) : Assume (1). Let \mathcal{H} and \mathcal{J} be two filters in \mathcal{L} . Suppose that $\mathcal{H} \cap \mathcal{J} \subseteq \mathcal{D}$. Let $x \in \mathcal{H}$. For every $\iota \in \mathcal{J}$, we have $\varphi \vee \iota \in \mathcal{H} \cap \mathcal{J} \subseteq \mathcal{D}$. This implies that $\varphi \vee \iota \in \mathcal{D}$. Since \mathcal{L} is coherent, we obtain $(\varphi)^{**} \vee (\iota)^{**} = \mathcal{L}$ for all $\iota \in \mathcal{J}$. Therefore, $\varphi \in \mathcal{J}^\tau$, and thus $\mathcal{H} \subseteq \mathcal{J}^\tau$.

On the other hand, suppose $\mathcal{H} \subseteq \mathcal{J}^\tau$. Let $\varphi \in \mathcal{H} \cap \mathcal{J}$. Since $\varphi \in \mathcal{H}$, we know $\varphi \in \mathcal{J}^\tau$, implying $\varphi \in \mathcal{J} \cap \mathcal{J}^\tau \subseteq \mathcal{D}$. Therefore, $\mathcal{H} \cap \mathcal{J} \subseteq \mathcal{D}$.

(2) \Rightarrow (3) : Assume condition (2). Let \mathcal{H} be a filter in \mathcal{L} . Clearly, $\mathcal{H}^\tau \subseteq (\mathcal{H}, \mathcal{D})$.

Now, suppose $\varphi \in (\mathcal{H}, \mathcal{D})$. For each $\iota \in \mathcal{H}$, we have

$$\begin{aligned} \varphi \vee \iota \in \mathcal{D} &\Rightarrow [\varphi] \cap [\iota] \subseteq \mathcal{D} \\ &\Rightarrow [\varphi] \subseteq [\iota]^\tau \subseteq (\iota)^\tau && \text{by 2} \\ &\Rightarrow [\varphi] \subseteq \bigcap_{\iota \in \mathcal{H}} (\iota)^\tau = \mathcal{H}^\tau \\ &\Rightarrow \varphi \in \mathcal{H}^\tau \end{aligned}$$

which gives that $(\mathcal{H}, \mathcal{D}) \subseteq \mathcal{H}^\tau$. Hence $(\mathcal{H}, \mathcal{D}) = \mathcal{H}^\tau$.

(3) \Rightarrow (4) : Assume condition (3). Let $\iota \in \mathcal{L}$. It is evident that $(\iota)^\tau \subseteq (\iota, \mathcal{D})$. Now, if $\varphi \in (\iota, \mathcal{D})$,

since $([\iota], \mathcal{D}) = (\iota, \mathcal{D})$, by condition (3), we conclude that $\varphi \in ([\iota], \mathcal{D}) = ([\iota])^\tau$. Since $\{\iota\} \subseteq [\iota]$, it follows that $\varphi \in ([\iota])^\tau \subseteq (\{\iota\})^\tau = (\iota)^\tau$.

(4) \Rightarrow (1) : Let condition (4) hold. Suppose $\iota, \vartheta \in \mathcal{L}$ and $\iota \vee \vartheta \in \mathcal{D}$. Then, we have $\iota \in (\vartheta, \mathcal{D}) = (\vartheta)^\tau$.

This leads to $(\iota)^{**} \vee (\vartheta)^{**} = \mathcal{L}$, implying that \mathcal{L} is coherent. \square

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