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Adaptive Meir–Keeler Contractions in Double Controlled Metric Spaces: Theory and Applications to Caputo Fractional Equations

Ahmad Aloqaily[1], **Nabil Mlaiki**[1,*], **Thabet Abdeljawad**[1,2,3]

[1] *Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia*

[2] *Department of Fundamental Sciences, Faculty of Engineering and Architecture, Istanbul Gelisim University, Avcılar–Istanbul, 34310, Türkiye*

[3] *Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Garankuwa, Medusa 0204, South Africa*

[*] *Corresponding author:* nmlaiki@psu.edu.sa

Abstract. In this paper, we investigate fixed point results for a new class of Meir–Keeler type contractions in the setting of double controlled metric type spaces governed by two independent control functions. This framework strictly extends controlled metric type spaces and allows the treatment of nonlinear mappings that cannot be handled by a single control function. We establish existence and uniqueness fixed point theorems for generalized Meir–Keeler contractions in complete double controlled metric type spaces. As an application, we employ the obtained results to study the existence and uniqueness of solutions for nonlinear fractional differential equations involving the Caputo fractional derivative. Several illustrative examples, including non-Lipschitz nonlinearities, are provided to demonstrate the effectiveness of the proposed approach.

1. Introduction

Fixed point methods constitute a central analytical tool for establishing existence and uniqueness of solutions in nonlinear analysis. Over the past decades, numerous extensions of the classical Banach contraction principle have been developed in order to accommodate nonlinear behaviors that cannot be handled within standard metric frameworks [4],Such extensions are particularly important in applications involving nonuniform continuity, memory effects, or solution-dependent growth.

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One important research direction focuses on relaxing the axioms of metric spaces. The introduction of b-metric spaces allowed the classical triangle inequality to be weakened by a constant factor. This idea was subsequently generalized by permitting the control parameter to depend on points in the space, leading to the notion of controlled metric type spaces. These generalized structures have proven effective in addressing nonlinear mappings for which standard metric techniques fail.

More recently,Abdeljawad et al. [1] the concept of double controlled metric type spaces was proposed, where the triangle inequality is governed by two independent control functions. This framework provides additional flexibility by allowing asymmetric or nonuniform control of distances and strictly extends the class of controlled metric type spaces. As a consequence, several Banach-type fixed point results have been extended to this setting.

In parallel, Meir–Keeler contractions [3] have emerged as a powerful generalization of Banach contractions by replacing global Lipschitz conditions with distance-dependent contractive behavior. Although Meir–Keeler type results have been investigated in classical and controlled metric settings, their systematic study within double controlled metric type spaces remains largely unexplored.

Fractional differential equations represent another area where generalized fixed point techniques are particularly relevant. The presence of memory effects and nonlinear dynamics often prevents the direct application of classical contraction principles, especially in the absence of global Lipschitz conditions. This motivates the use of flexible metric structures together with weaker contractive assumptions.

Motivated by these observations, the aim of this paper is to establish new fixed point results for Meir–Keeler type contractions in the framework of double controlled metric type spaces and to demonstrate their applicability to nonlinear fractional differential equations involving the Caputo fractional derivative.

Main Contributions and Novelty. The main contributions of this paper are summarized as follows:

- We introduce Meir–Keeler type contractive conditions adapted to double controlled metric type spaces governed by two independent control functions. This framework cannot be reduced to classical metric or single-control settings.
- We prove new existence and uniqueness theorems for generalized Meir–Keeler contractions in complete double controlled metric type spaces, extending several known Banach and nonlinear contraction principles.
- We apply the abstract fixed point results to nonlinear fractional differential equations with Caputo derivatives, allowing nonlinearities that do not satisfy standard Lipschitz assumptions.
- We propose an adaptive, orbit-dependent choice of control functions, leading to a dynamic metric structure that responds to the behavior of the associated Picard iteration. This approach enhances the applicability of double controlled metric spaces to fractional models.

- We present a separation example showing that the generalized Meir–Keeler mechanism cannot, in general, be reduced to a single-control metric framework, thereby establishing the essential nature of double control.

2. Meir–Keeler Contractions in Double Controlled Metric Type Spaces

We now formalize the notion of Meir–Keeler type contractions within the context of double controlled metric type spaces. The goal is to capture distance reducing behavior that depends on the local geometry induced by two control functions rather than a single global constant.

A self-mapping $T : X \to X$ is said to exhibit Meir–Keeler type contractive behavior if, whenever the controlled distance between two points exceeds a given threshold, the image under T produces a strict reduction of that distance.

To encompass nonlinear effects, we further consider a generalized form in which the contractive condition depends not only on the distance between two points but also on their deviation from their images under the mapping.

Let (X, q) be a double controlled metric type space endowed with two control functions $\alpha, \mu : X \times X \to [1, \infty)$. Recall that q satisfies

$$q(x, y) \leq \alpha(x, z)\, q(x, z) + \mu(z, y)\, q(z, y), \quad \text{for all } x, y, z \in X,$$

as introduced in [1].

Definition 2.1. *Let $T : X \to X$ be a self-mapping. The mapping T is said to be a* Meir–Keeler contraction in a double controlled metric type space *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq q(x, y) < \varepsilon + \delta \quad \Longrightarrow \quad q(Tx, Ty) < \varepsilon,$$

for all $x, y \in X$.

The above definition extends the classical Meir–Keeler contraction [3] and generalizes recent results established in controlled metric type spaces.

To cover nonlinear behaviors, we introduce the following generalized version, inspired by nonlinear contractions in generalized metric frameworks [2].

Definition 2.2. *Let $T : X \to X$ be a self-mapping and define*

$$M(x, y) = \max\{q(x, y),\ q(x, Tx),\ q(y, Ty)\}.$$

Then T is called a generalized Meir–Keeler contraction in a double controlled metric type space *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \quad \Longrightarrow \quad q(Tx, Ty) < \varepsilon,$$

for all $x, y \in X$.

Remark 2.1. *Every Banach contraction in a double controlled metric type space is a generalized Meir–Keeler contraction. However, the converse does not hold in general. Hence, the introduced class strictly extends Banach-type contractions and includes a broader family of nonlinear mappings.*

Table 1. Overview of metric structures and corresponding contraction principles

Setting	Metric Structure	Control Functions	Contraction Type	Fixed Point Result
Classical metric space	Metric	None	Banach contraction	Banach principle [4]
Classical metric space	Metric	None	Meir–Keeler contraction	Meir–Keeler theorem [3]
Controlled metric type space	Generalized metric	Single control function	Meir–Keeler contraction	Fixed point results [2]
Double controlled metric type space	Generalized metric	Two control functions	Banach-type contraction	Known results [1]
Double controlled metric type space	**Generalized metric**	**Two independent control functions**	**Generalized Meir–Keeler contraction**	**This paper**

The following table summarizes the relationship between various metric frameworks and associated contraction principles. It highlights how the present setting extends earlier approaches by combining generalized contractive conditions with two independent control functions.

Remark 2.2. *Unlike Banach-type contractions in double controlled metric spaces, the Meir–Keeler framework adopted here allows contractive behavior to emerge locally along the Picard orbit rather than being imposed globally.*

2.1. **A separation example: why double control is essential. Purpose.** Table 1 suggests that the double–controlled framework is strictly more flexible than single–control (controlled metric type) settings. To make this strictness concrete (and not merely definitional), we give an example in which the contractive mechanism can be verified *only* when two independent controls are allowed: one control governs the "left leg" of the double triangle inequality and remains tame along the Picard orbit, while the other control absorbs a large right–leg distortion. Any attempt to collapse these two roles into a single control necessarily forces a uniformly large control along the orbit, which destroys the standard orbit–bounded hypotheses used in the single–control Meir–Keeler theory.

The space and the double controls. Let $X = [0, 1)$ and define

$$q(x, y) := \frac{|x - y|}{1 - |x - y|} \qquad (x, y \in X,\ x \neq y), \qquad q(x, x) = 0.$$

Then q is symmetric and $q(x, y) = 0 \Leftrightarrow x = y$. Moreover, q is a typical "metric–type" gauge that may exhibit large distortions near the boundary as $|x - y| \to 1^-$.

Define two independent control functions $\alpha, \mu : X \times X \to [1, \infty)$ by

$$\alpha(x, z) := 1 + \frac{1}{1 - z}, \qquad \mu(z, y) := 1 + \frac{1}{1 - y}.$$

The key feature is the *asymmetry*: α depends only on the intermediate point z (left leg), while μ depends only on the terminal point y (right leg). In particular, α stays moderate when z stays away from 1, even if y is very close to 1, whereas μ can become large as $y \to 1^-$.

A mapping with mild orbit and potentially large right–leg distortion. Define $T : X \to X$ by

$$T(x) = \frac{x}{2}.$$

For any initial point $x_0 \in X$, the Picard orbit $x_{n+1} = Tx_n$ satisfies $x_n = x_0/2^n \to 0$. Hence, along the orbit, intermediate points $z = x_{n+1}$ remain uniformly away from 1, so the left–control $\alpha(x_n, x_{n+1}) = 1 + \frac{1}{1-x_{n+1}}$ remains uniformly bounded. On the other hand, the space still contains points y arbitrarily close to 1, for which the right–control $\mu(\cdot, y) = 1 + \frac{1}{1-y}$ is arbitrarily large.

Claim 1 (Generalized Meir–Keeler contractivity holds). Let $M(x,y) = \max\{q(x,y), q(x,Tx), q(y,Ty)\}$ as in Definition 7. Then T is a generalized Meir–Keeler contraction on (X,q). Indeed, fix $\varepsilon > 0$ and set

$$\delta := \frac{\varepsilon}{2}.$$

If $\varepsilon \le M(x,y) < \varepsilon + \delta$, then in particular $q(x,y) < \varepsilon + \delta$. Since $|Tx - Ty| = \frac{1}{2}|x-y|$, we have the strict shrinking estimate

$$q(Tx,Ty) = \frac{|Tx-Ty|}{1-|Tx-Ty|} = \frac{\frac{1}{2}|x-y|}{1-\frac{1}{2}|x-y|} < \frac{|x-y|}{1-|x-y|} = q(x,y) < \varepsilon + \delta.$$

Moreover, the map $r \mapsto \frac{r/2}{1-r/2}$ is strictly smaller than $r \mapsto \frac{r}{1-r}$ on $(0,1)$, so once $q(x,y)$ lies in the Meir–Keeler window $[\varepsilon, \varepsilon+\delta)$, we obtain $q(Tx,Ty) < \varepsilon$. This verifies the generalized Meir–Keeler implication.

Claim 2 (Why one cannot "collapse" to a single effective control without losing applicability). In single–control (controlled metric type) approaches, one typically works with one control $\eta : X \times X \to [1,\infty)$ that must simultaneously govern *both legs* of the triangle inequality. If one attempts to dominate the above two–control geometry by a single control, a natural choice is

$$\eta(u,v) \ \ge \ \max\{\alpha(u,v), \mu(u,v)\} \ \ge \ 1 + \frac{1}{1-v}.$$

Consequently, even along the Picard orbit $x_n = x_0/2^n$, the quantity $\eta(x_{n+1}, y)$ must become arbitrarily large whenever y ranges near 1. This means that the *same* orbit estimates used to prove convergence in the single–control Meir–Keeler setting cannot be verified uniformly (the control unavoidably inherits the "large" right–leg behavior). In contrast, the double–control framework separates these two roles: the orbit estimates only need the left–control $\alpha(x_n, x_{n+1})$ (which remains bounded because $x_{n+1} \to 0$), while the right–control $\mu(\cdot, y)$ is allowed to be large for points y near 1 without destroying orbit boundedness.

Conclusion. This example demonstrates the conceptual necessity of two independent controls: the left control can track the Picard orbit geometry, while the right control absorbs distortions caused by points far from the orbit. Hence, the double–controlled framework is not a cosmetic reformulation of the single–control theory; it genuinely enlarges the class of settings where Meir–Keeler type arguments can be carried out.

3. Main Results: Meir–Keeler Fixed Points in Double Controlled Metric Type Spaces

We adapt the classical Meir–Keeler contraction to the setting of double controlled metric type spaces as follows, (X, q) denotes a complete double controlled metric type space with two control functions $\alpha, \mu : X \times X \to [1, \infty)$. That is, for all $x, y, z \in X$,

$$q(x, y) \leq \alpha(x, z)\, q(x, z) + \mu(z, y)\, q(z, y),$$

where $q(x, y) = 0$ if and only if $x = y$ and $q(x, y) = q(y, x)$ (cf. [1]).

We begin by recalling the iterative sequence associated with a self-mapping $T : X \to X$. For an arbitrary $x_0 \in X$, define the Picard sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots.$$

3.1. Meir–Keeler Type Definitions in Double Controlled Metric Settings.

Definition 3.1. *Let $T : X \to X$ be a self-mapping. We say that T is a* Meir–Keeler contraction in a double controlled metric type space *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq q(x, y) < \varepsilon + \delta \quad \Longrightarrow \quad q(Tx, Ty) < \varepsilon,$$

for all $x, y \in X$.

Remark 3.1. *We emphasize that the Meir–Keeler contractive condition itself is classical. The definition is recalled here in order to adapt it to the framework of double controlled metric type spaces and to study its fixed point consequences under the presence of two control functions.*

The above definition naturally extends the classical Meir–Keeler contraction [3] to the framework of double controlled metric type spaces.

To capture nonlinear behaviors, we introduce the following generalized form.

Definition 3.2. *Let $T : X \to X$ be a self-mapping and define*

$$M(x, y) = \max\{q(x, y),\ q(x, Tx),\ q(y, Ty)\}.$$

The mapping T is said to be a generalized Meir–Keeler contraction in a double controlled metric type space *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \quad \Longrightarrow \quad q(Tx, Ty) < \varepsilon,$$

for all $x, y \in X$.

Remark 3.2. *Every Banach contraction in a double controlled metric type space is a generalized Meir–Keeler contraction. However, the converse does not hold in general. Hence, this class strictly extends Banach-type contractions and allows a wider family of nonlinear mappings.*

3.2. **Main Fixed Point Theorem.**

Theorem 3.1. *Let (X, q) be a complete double controlled metric type space endowed with control functions $\alpha, \mu : X \times X \to [1, \infty)$. Let $T : X \to X$ be a generalized Meir–Keeler contraction.*

Assume that:

(1) *There exists $x_0 \in X$ such that the Picard sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ satisfies*

$$\sup_{m \geq 1} \limsup_{n \to \infty} \frac{\alpha(x_{n+1}, x_{n+2})}{\alpha(x_n, x_{n+1})} \mu(x_{n+1}, x_m) < 1.$$

(2) *For each $x \in X$, the limits $\lim_{n\to\infty} \alpha(x, x_n)$ and $\lim_{n\to\infty} \mu(x_n, x)$ exist and are finite.*

(3) *The mapping T is continuous on X.*

Then T admits a unique fixed point $\xi \in X$. Moreover, the Picard sequence $\{x_n\}$ converges to ξ with respect to q.

Proof. Let $\{x_n\}$ be the Picard sequence defined by $x_{n+1} = Tx_n$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point and we are done. Henceforth assume $x_n \neq x_{n+1}$ for all n.

Step 1: Monotonicity of successive distances. Set

$$d_n := q(x_n, x_{n+1}) = q(x_n, Tx_n) \quad (n \geq 0).$$

Apply the generalized Meir–Keeler condition (Definition 7) to the pair (x_n, x_{n+1}). Note that

$$M(x_n, x_{n+1}) = \max\{q(x_n, x_{n+1}),\ q(x_n, Tx_n),\ q(x_{n+1}, Tx_{n+1})\} = \max\{d_n,\ d_{n+1}\}.$$

If $d_{n+1} > d_n$, then $M(x_n, x_{n+1}) = d_{n+1}$ and hence for $\varepsilon = d_{n+1}$ we have

$$\varepsilon \leq M(x_n, x_{n+1}) < \varepsilon + \delta \quad \Rightarrow \quad q(Tx_n, Tx_{n+1}) < \varepsilon,$$

i.e. $d_{n+1} = q(x_{n+1}, x_{n+2}) < d_{n+1}$, a contradiction. Therefore $d_{n+1} \leq d_n$ for all n, so $\{d_n\}$ is decreasing and bounded below by 0. Hence $d_n \to r$ for some $r \geq 0$.

Step 2: Showing that the limit must be $r = 0$. Assume, for contradiction, that $r > 0$. Fix $\varepsilon := r$. By the generalized Meir–Keeler property, there exists $\delta > 0$ such that

$$r \leq M(u, v) < r + \delta \implies q(Tu, Tv) < r \qquad (\forall\, u, v \in X).$$

Since $d_n \to r$ and $d_{n+1} \leq d_n$, we also have $\max\{d_n, d_{n+1}\} \to r$. Hence there exists N such that for all $n \geq N$,

$$r \leq \max\{d_n, d_{n+1}\} < r + \delta.$$

Equivalently, for $n \geq N$,

$$r \leq M(x_n, x_{n+1}) < r + \delta.$$

Applying the Meir–Keeler implication with $(u, v) = (x_n, x_{n+1})$ yields

$$d_{n+1} = q(Tx_n, Tx_{n+1}) < r \qquad (n \geq N).$$

Taking $\limsup_{n\to\infty}$ on both sides gives

$$r = \lim_{n\to\infty} d_{n+1} \le \limsup_{n\to\infty} d_{n+1} \le r^- < r,$$

a contradiction. Therefore $r = 0$, i.e.

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = \lim_{n\to\infty} d_n = 0.$$

Step 3: Cauchy property using the double controlled triangle inequality. Fix integers $m > n$. Using the double controlled triangle inequality with $z = x_{n+1}$ gives

$$q(x_n, x_m) \le \alpha(x_n, x_{n+1})\, q(x_n, x_{n+1}) + \mu(x_{n+1}, x_m)\, q(x_{n+1}, x_m) = \alpha(x_n, x_{n+1})\, d_n + \mu(x_{n+1}, x_m)\, q(x_{n+1}, x_m). \tag{3.1}$$

Iterating (3.1) for $q(x_{n+1}, x_m), q(x_{n+2}, x_m), \ldots$ and repeatedly substituting yields an estimate of the form

$$q(x_n, x_m) \le \alpha(x_n, x_{n+1}) d_n + \sum_{k=n+1}^{m-1} \left(\alpha(x_k, x_{k+1}) d_k \prod_{j=n+1}^{k} \mu(x_j, x_m) \right), \tag{3.2}$$

(where the empty product is interpreted as 1).

Now we use *Assumption (1)* of Theorem 9, namely

$$\sup_{m\ge 1} \limsup_{n\to\infty} \frac{\alpha(x_{n+1}, x_{n+2})}{\alpha(x_n, x_{n+1})}\, \mu(x_{n+1}, x_m) < 1.$$

Hence there exist a constant $\kappa \in (0, 1)$ and an index N_0 such that for all $m \ge 1$ and all $n \ge N_0$,

$$\frac{\alpha(x_{n+1}, x_{n+2})}{\alpha(x_n, x_{n+1})}\, \mu(x_{n+1}, x_m) \le \kappa. \tag{3.3}$$

From (3.3) one obtains inductively, for $k \ge n + 1$,

$$\alpha(x_k, x_{k+1}) \prod_{j=n+1}^{k} \mu(x_j, x_m) \le \alpha(x_n, x_{n+1})\, \kappa^{k-n}.$$

Substituting this bound into (3.2) gives, for all $m > n \ge N_0$,

$$q(x_n, x_m) \le \alpha(x_n, x_{n+1}) \left(d_n + \sum_{k=n+1}^{m-1} \kappa^{k-n} d_k \right) \le \alpha(x_n, x_{n+1}) \left(d_n + \sum_{k=n+1}^{\infty} \kappa^{k-n} d_k \right).$$

Since $d_k \to 0$ and $\sum_{k=n+1}^{\infty} \kappa^{k-n}$ converges, the right-hand side tends to 0 as $n \to \infty$, uniformly in $m > n$. Therefore $\{x_n\}$ is Cauchy in (X, q).

Step 4: Existence and identification of the fixed point. Because (X, q) is complete, there exists $\xi \in X$ with $x_n \to \xi$ in q. By continuity of T (Assumption 3), $Tx_n = x_{n+1} \to T\xi$. Limits are unique, hence $T\xi = \xi$.

Step 5: Uniqueness. If $\eta \ne \xi$ are fixed points, then $M(\xi, \eta) = q(\xi, \eta) > 0$ and the generalized Meir–Keeler condition applied to (ξ, η) implies

$$q(\xi, \eta) = q(T\xi, T\eta) < q(\xi, \eta),$$

a contradiction. Thus the fixed point is unique. □

Remark 3.3. *The conditions imposed on the control functions α and μ are automatically satisfied in many applications where the Picard sequence is bounded. In particular, the assumptions are compatible with adaptive control structures introduced later in Section 5.*

4. Application to Fractional Differential Equations

In this section, we apply the fixed point results obtained in Section 3 to study the existence and uniqueness of solutions for a nonlinear fractional differential equation involving the Caputo fractional derivative.

4.1. **Problem Formulation.** Fractional differential equations provide a natural framework for modeling memory-dependent processes. In many such problems, classical fixed point approaches based on Lipschitz continuity are insufficient. The present section illustrates how generalized Meir–Keeler contractions in double controlled metric spaces can be effectively used to address this limitation.

Let $0 < \gamma < 1$ and consider the following initial value problem for a fractional differential equation:

$$\begin{cases} {}^{C}D^{\gamma}x(t) = f(t, x(t)), \quad t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{4.1}$$

where ${}^{C}D^{\gamma}$ denotes the Caputo fractional derivative of order γ, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $x_0 \in \mathbb{R}$ is a given initial value.

It is well known (see, for instance, [5,6]) that problem

4.1 is equivalent to the following fractional integral equation:

$$x(t) = x_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} f(s, x(s))\, ds, \quad t \in [0, T]. \tag{4.2}$$

4.2. **Functional Setting.** Let $X = C([0, T], \mathbb{R})$ be the Banach space of continuous functions equipped with the metric

$$q(x, y) = \sup_{t \in [0,T]} |x(t) - y(t)|, \quad x, y \in X.$$

Define the control functions $\alpha, \mu : X \times X \to [1, \infty)$ by

$$\alpha(x, y) = 1 + \|x\|_\infty, \qquad \mu(x, y) = 1 + \|y\|_\infty,$$

where $\|x\|_\infty = \sup_{t \in [0,T]} |x(t)|$. Then (X, q) forms a complete double controlled metric type space (cf. [1]).

Comment on the choice of control functions. Although $q(x, y) = \|x - y\|_\infty$ is a standard metric, we deliberately equip (X, q) with two controls $\alpha(x, y) = 1 + \|x\|_\infty$ and $\mu(x, y) = 1 + \|y\|_\infty$ in order to fit the double-controlled framework and to allow orbit-dependent scaling. Since $\alpha, \mu \geq 1$, the classical triangle inequality implies the double controlled inequality automatically, while the dependence on $\|x\|_\infty$ and $\|y\|_\infty$ reflects the magnitude of trajectories and is compatible with the adaptive viewpoint of Section 5.

Define the operator $T : X \to X$ by

$$(Tx)(t) = x_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, x(s))\, ds. \tag{4.3}$$

Clearly, fixed points of T are solutions of the fractional integral equation
4.2, and hence of problem 4.1.

4.3. **Existence and Uniqueness Result.** We now impose the following assumption.

(H) There exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(r) < r$ for all $r > 0$ such that

$$|f(t,u) - f(t,v)| \le \phi(|u-v|), \quad \text{for all } t \in [0,T],\ u, v \in \mathbb{R}.$$

Interpretation of (H). Assumption (H) is a Meir–Keeler type continuity requirement: the increment $|f(t,u) - f(t,v)|$ is controlled by a comparison function φ with $\varphi(r) < r$ for $r > 0$. This is strictly weaker than global Lipschitz continuity and is designed to guarantee that the associated Volterra operator shrinks distances whenever they are above a given threshold.

Theorem 4.1. *Assume that condition **(H)** holds. Then the operator T defined by* (4.3) *is a generalized Meir–Keeler contraction in the double controlled metric type space (X, q). Consequently, the fractional differential equation* (4.1) *admits a unique continuous solution on* $[0, T]$.

Proof. Let $x, y \in X$. Using condition **(H)** and definition (4.3), we obtain

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &\le \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} |f(s,x(s)) - f(s,y(s))|\, ds \\ &\le \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \phi(|x(s) - y(s)|)\, ds. \end{aligned}$$

Taking supremum over $t \in [0, T]$ and using the monotonicity of ϕ, we get

$$q(Tx, Ty) \le \frac{T^\gamma}{\Gamma(\gamma+1)}\, \phi\Big(q(x,y)\Big).$$

Let

$$c := \frac{T^\gamma}{\Gamma(\gamma+1)}.$$

Then the above estimate can be written as

$$q(Tx, Ty) \le c\, \phi\Big(q(x,y)\Big).$$

If $c \le 1$, define $\psi(r) := c\,\phi(r)$ for $r \ge 0$. Since $\phi(r) < r$ for all $r > 0$, we have $\psi(r) < r$ for all $r > 0$, and therefore T satisfies the generalized Meir–Keeler condition in (X, q). More generally, it is enough to assume that

$$c\,\phi(r) < r \quad \text{for all } r > 0,$$

which is typically ensured by choosing T sufficiently small (local-in-time). Moreover, T is continuous on X.

All assumptions of the main fixed point theorem in Section 3 are therefore satisfied.

Hence, T has a unique fixed point $\xi \in X$,

which is the unique continuous solution of problem 4.1. □

Remark 4.1. *If* $\frac{T^\gamma}{\Gamma(\gamma+1)} > 1$*, the conclusion still holds provided* $\frac{T^\gamma}{\Gamma(\gamma+1)}\phi(r) < r$ *for all* $r > 0$*. This is typically ensured by choosing* T *sufficiently small (local-in-time).*

4.4. **A Worked Example.** We illustrate the applicability of the main fixed point theorem by considering the following nonlinear fractional model.

Let $T > 0$, $0 < \gamma < 1$, and consider 4.1 with

$$f(t,x) = \frac{x}{1+|x|}, \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

It follows from the construction, f is continuous on $[0,T] \times \mathbb{R}$. Moreover, for all $u, v \in \mathbb{R}$,

$$\left| \frac{u}{1+|u|} - \frac{v}{1+|v|} \right| \leq |u-v|.$$

As a consequence, for every $t \in [0,T]$,

$$|f(t,u) - f(t,v)| \leq \phi(|u-v|), \quad \text{with} \quad \phi(r) = \frac{r}{1+r} \ \ (r \geq 0).$$

Note that ϕ is continuous, nondecreasing, and satisfies $\phi(r) < r$ for all $r > 0$. Therefore assumption **(H)** holds.

Define the operator $T : X \to X$ by

$$(Tx)(t) = x_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \frac{x(s)}{1+|x(s)|}\, ds, \qquad t \in [0,T],$$

where $X = C([0,T],\mathbb{R})$ and $q(x,y) = \|x-y\|_\infty$. Then, for $x, y \in X$ and $t \in [0,T]$, we obtain

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &\leq \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \left| \frac{x(s)}{1+|x(s)|} - \frac{y(s)}{1+|y(s)|} \right| ds \\
&\leq \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \phi(|x(s)-y(s)|)\, ds \\
&\leq \frac{\phi(\|x-y\|_\infty)}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1}\, ds \\
&= \frac{t^\gamma}{\Gamma(\gamma+1)} \phi(\|x-y\|_\infty).
\end{aligned}$$

Taking the supremum over $t \in [0,T]$ yields

$$q(Tx,Ty) \leq \frac{T^\gamma}{\Gamma(\gamma+1)} \phi\big(q(x,y)\big).$$

Since $\phi(r) < r$ for all $r > 0$, it follows that T satisfies a generalized Meir–Keeler contractive condition in the sense of Section 3. Consequently, by Theorem 4.1, problem (4.1) admits a unique solution $x \in C([0,T],\mathbb{R})$ on $[0,T]$.

4.5. **A Second Example (Non-Lipschitz Nonlinearity).** Let $T > 0$, $0 < \gamma < 1$, and consider (4.1) with the continuous nonlinearity

$$f(t,x) = \sqrt{|x|}, \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

Note that f is *not* globally Lipschitz on $\mathbb{R}$ (its derivative blows up at $x = 0$). We work on the closed subset

$$X_R = \Big\{x \in C([0,T],\mathbb{R}) : \|x\|_\infty \le R\Big\}, \quad R > 0,$$

endowed with the metric $q(x,y) = \|x - y\|_\infty$. Clearly, (X_R, q) is complete.

Define the operator $T : X_R \to C([0,T],\mathbb{R})$ by

$$(Tx)(t) = x_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \sqrt{|x(s)|}\, ds, \qquad t \in [0,T].$$

Step 1 (Invariance). Assume $|x_0| \le R/2$ and choose $T > 0$ such that

$$\frac{T^\gamma}{\Gamma(\gamma+1)} \sqrt{R} \le \frac{R}{2}.$$

Then, for any $x \in X_R$ and $t \in [0,T]$,

$$|(Tx)(t)| \le |x_0| + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \sqrt{|x(s)|}\, ds \le \frac{R}{2} + \frac{t^\gamma}{\Gamma(\gamma+1)} \sqrt{R} \le R,$$

hence $T(X_R) \subseteq X_R$.

Step 2 (Generalized Meir–Keeler estimate). For $u, v \in [-R, R]$, one has the inequality

$$\Big| \sqrt{|u|} - \sqrt{|v|} \Big| \le \sqrt{|u-v|}.$$

Therefore, for $x, y \in X_R$ and $t \in [0,T]$,

$$\begin{aligned}
\|(Tx)(t) - (Ty)(t)\| &\le \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma 1} \Big| \sqrt{|x(s)|} - \sqrt{|y(s)|} \Big|\, ds \\
&\le \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \sqrt{|x(s) - y(s)|}\, ds \\
&\le \frac{t^\gamma}{\Gamma(\gamma+1)} \sqrt{\|x-y\|_\infty}.
\end{aligned}$$

Taking the supremum over $t \in [0,T]$ yields

$$q(Tx, Ty) \le \frac{T^\gamma}{\Gamma(\gamma+1)}\, \phi\big(q(x,y)\big), \quad \text{where} \quad \phi(r) = \sqrt{r}.$$

Step 3 (Meir–Keeler property). Fix $\varepsilon > 0$. Choose

$$\delta = \varepsilon^2 - \varepsilon > 0 \quad \text{whenever } \varepsilon > 1,$$

so that if $\varepsilon \le r < \varepsilon + \delta$, then $\sqrt{r} < \varepsilon$. More generally, since $\phi(r) = \sqrt{r}$ is continuous and satisfies $\phi(r) < r$ for all $r > 1$, the mapping T satisfies a generalized Meir–Keeler contractive behavior on

X_R in the sense of Section 3 (after restricting to the range where $q(x, y)$ is bounded away from 0 or choosing T small so that the prefactor $\frac{T^\gamma}{\Gamma(\gamma+1)} < 1$).

Consequently, by the fixed point theorem of Section 3 applied on the complete space (X_R, q), the fractional problem (4.1) with $f(t, x) = \sqrt{|x|}$ admits a (local-in-time) unique solution in X_R for T satisfying the above invariance condition.

4.6. **A Second Strong Example (Saturating–Oscillatory Nonlinearity).** Let $T > 0$, $0 < \gamma < 1$, and consider (4.1) with

$$f(t, x) = a(t)\,\frac{\sin x}{1 + |x|}, \qquad (t, x) \in [0, T] \times \mathbb{R},$$

where $a \in C([0, T], \mathbb{R})$ and $A := \|a\|_\infty = \sup_{t\in[0,T]} |a(t)| < \infty$.

Set $X = C([0, T], \mathbb{R})$ with $q(x, y) = \|x - y\|_\infty$ and define the operator

$$(Tx)(t) = x_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma-1} f(s, x(s))\, ds.$$

Then T is well-defined and continuous on X.

Claim: T satisfies the generalized Meir–Keeler condition of Section 3.

Let $u, v \in \mathbb{R}$. Using $|\sin u - \sin v| \le |u - v|$ and the elementary inequality

$$\left|\frac{1}{1 + |u|} - \frac{1}{1 + |v|}\right| \le \frac{|u - v|}{(1 + |u|)(1 + |v|)} \le |u - v|,$$

we obtain the estimate

$$\left|\frac{\sin u}{1 + |u|} - \frac{\sin v}{1 + |v|}\right| \le \left|\frac{\sin u - \sin v}{1 + |u|}\right| + |\sin v| \left|\frac{1}{1 + |u|} - \frac{1}{1 + |v|}\right| \le |u - v| + |u - v| = 2|u - v|.$$

Hence, for all $t \in [0, T]$ and $x, y \in X$,

$$|f(t, x(t)) - f(t, y(t))| \le A \left|\frac{\sin(x(t))}{1 + |x(t)|} - \frac{\sin(y(t))}{1 + |y(t)|}\right| \le 2A\, |x(t) - y(t)|.$$

Substituting into the integral operator yields, for $t \in [0, T]$,

$$\begin{aligned}
\|(Tx)(t) - (Ty)(t)\| &\le \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma-1}\, |f(s, x(s)) - f(s, y(s))|\, ds \\
&\le \frac{2A}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma-1}\, |x(s) - y(s)|\, ds \\
&\le \frac{2A}{\Gamma(\gamma)} \|x - y\|_\infty \int_0^t (t - s)^{\gamma-1} ds \\
&= \frac{2A\, t^\gamma}{\Gamma(\gamma + 1)}\, \|x - y\|_\infty.
\end{aligned}$$

Taking supremum over $t \in [0, T]$ gives

$$q(Tx, Ty) \le k\, q(x, y), \qquad k := \frac{2A\, T^\gamma}{\Gamma(\gamma + 1)}.$$

Choose $T > 0$ (or assume A is small) such that $k \in (0,1)$. Then T is a Banach contraction and therefore a generalized Meir–Keeler contraction in the sense of Section 3.

Consequently, by the main fixed point theorem of Section 3, the fractional problem (4.1) admits a unique solution $x \in C([0,T],\mathbb{R})$ on $[0,T]$.

5. Adaptive Control Functions in Double Controlled Metric Type Spaces

In many applications, fixed control functions may be too restrictive to capture the behavior of nonlinear operators. To address this issue, we introduce adaptive control functions that depend explicitly on the Picard orbit generated by the mapping. This construction allows the metric structure to adjust dynamically while preserving contractive behavior.

Motivated by this observation, we introduce the notion of *adaptive (orbit-dependent) control functions*, which adjust automatically to the behavior of the Picard iteration generated by the operator. This extension enhances the flexibility of the double controlled metric framework while preserving the validity of the fixed point results established earlier.

5.1. Definition of Adaptive Control Functions.

Let (X,q) be a complete metric-type space and let $T : X \to X$ be a self-mapping. Fix an initial point $x_0 \in X$ and define the Picard sequence

$$x_{n+1} = Tx_n, \qquad n = 0,1,2,\ldots.$$

Definition 5.1. *The functions* $\alpha_T, \mu_T : X \times X \to [1,\infty)$ *defined by*

$$\alpha_T(x,y) = 1 + \sup_{n\geq 0} q(x_n,x), \qquad \mu_T(x,y) = 1 + \sup_{n\geq 0} q(x_n,y),$$

are called adaptive control functions associated with T.

Clearly, α_T and μ_T depend on the operator T and its orbit $\{x_n\}$. If the Picard sequence is bounded, then α_T and μ_T are finite-valued.

5.2. Compatibility with Double Controlled Metric Structures.

Using the adaptive control functions, the inequality

$$q(x,y) \leq \alpha_T(x,z)\, q(x,z) + \mu_T(z,y)\, q(z,y), \qquad x,y,z \in X,$$

defines a double controlled metric-type structure that adapts naturally to the operator T.

Proposition 5.1. *Let* (X,q) *be complete and let* $T : X \to X$ *generate a bounded Picard sequence* $\{x_n\}$. *Then* (X,q) *equipped with the adaptive control functions* α_T *and* μ_T *forms a double controlled metric type space.*

Proof. Since $\alpha_T, \mu_T \geq 1$ by definition and the Picard sequence is bounded, both functions are finite-valued. The inequality follows directly from the definition of α_T and μ_T, together with the metric-type properties of q. □

5.3. **Preservation of Fixed Point Results.**

Remark 5.1. *All fixed point results proved in Section 3 remain valid when the fixed control functions α and μ are replaced by the adaptive control functions α_T and μ_T, provided the Picard sequence is bounded and the limits*

$$\lim_{n\to\infty} \alpha_T(x, x_n), \qquad \lim_{n\to\infty} \mu_T(x_n, x)$$

exist and are finite for each $x \in X$. These conditions are naturally satisfied in the fractional applications considered in Section 4, where the associated Volterra-type operators preserve boundedness of solutions on compact time intervals.

5.4. **Interpretation and Significance.** The introduction of adaptive control functions provides a mechanism by which the metric structure responds dynamically to the behavior of the operator. This feature is particularly useful in fractional models, where memory effects and nonlinear growth may lead to solution-dependent scaling. The adaptive framework allows one to retain Meir–Keeler contractive behavior without imposing unnecessarily strong global restrictions on the space or the operator.

This perspective suggests further applications to iterative algorithms, numerical schemes, and fractional problems with variable growth conditions.

6. Conclusion

In this paper, we developed fixed point results for Meir–Keeler type contractions in double controlled metric type spaces. The proposed framework extends existing theories by allowing two independent control functions and nonlinear contractive behavior.

Applications to fractional differential equations with Caputo derivatives demonstrated the usefulness of the theory in handling non-Lipschitz and memory-driven models. The introduction of adaptive control functions further enhances the flexibility of the framework.

Future work may address other generalized contractions in double controlled metric spaces, as well as applications to impulsive, delay, and stochastic fractional systems.

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