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Lower and Upper Solutions for Sequential Fractional Differential Systems by Monotone Iterative Techniques

Hasanen A. Hammad[1,2,*], **Manuel De la Sen**[3]

[1]*Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt*
[3]*Institute of Research and Development of Processes, Department of Electricity and Electronics, Faculty of Science and Technology, University of the Basque Country, 48940-Leioa (Bizkaia), Spain*

[*]*Corresponding author:* h.abdelwareth@qu.edu.sa

Abstract. This paper delves into the theoretical investigation of extremal solutions for a coupled sequential Caputo fractional differential system. We employ functional analysis to rigorously prove the existence of these solutions, combining the monotone iterative technique with the method of upper and lower solutions to establish sufficient conditions for finding the minimal and maximal solutions. To validate our theoretical results, we provide two illustrative examples.

1. Introduction

Fractional differential equations (DEs) represent a powerful generalization of classical differential equations, extending the concept of differentiation and integration to non-integer orders [1]. This extension provides a more accurate and nuanced description of various complex phenomena across diverse scientific and engineering disciplines, particularly those exhibiting memory effects and non-local interactions, which are often inadequately captured by integer-order models [2]. Applications of fractional DEs span fields such as viscoelasticity, anomalous diffusion, control theory, signal processing, and biological systems, highlighting their versatility in modeling real-world processes with intricate dynamics [3]. The theoretical study of fractional DEs involves sophisticated analytical techniques, and due to the frequent absence of analytical solutions, the development of robust and efficient numerical methods remains a crucial area of active research [4].

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In the literature, various techniques have been developed to define fractional integrals and derivatives, each with its own properties and applications. The Riemann-Liouville (RL) definition, one of the earliest and most widely used, extends the Cauchy formula for repeated integration to non-integer orders [5]. Another prominent definition is the Caputo fractional derivative, which differs from Riemann-Liouville (RL) by applying the integer-order derivative before the fractional integral, resulting in the derivative of a constant being zero and allowing for initial conditions similar to integer-order DEs [6]. The Grünwald-Letnikov definition, based on finite differences, provides a discrete approach and is particularly useful for numerical approximations [7]. Furthermore, the Hadamard fractional integral and derivative are defined with respect to a logarithmic function, making them suitable for problems on intervals starting from zero and exhibiting scale invariance [8]. Other notable definitions exist, such as the Miller-Ross and the Weyl fractional derivatives, each tailored for specific types of problems and offering unique advantages in modeling complex systems [9].

The $\delta-$fractional derivative extends the concept of fractional derivatives to the realm of distributions, particularly involving the Dirac delta function $\delta(\zeta)$. Unlike classical fractional derivatives of regular functions, defining the fractional derivative of a distribution requires careful consideration within the framework of generalized functions. Several approaches exist, often utilizing integral representations or Fourier/Laplace transforms to handle the singular nature of the delta function. These definitions find applications in various fields, including signal processing, physics (e.g., fractional quantum mechanics), and the study of anomalous diffusion processes where impulsive behavior is significant. The fractional derivative of the delta function can be expressed in terms of power-law functions and exhibits unique properties related to its integral and transform characteristics [10–13].

The $\delta-$Caputo and $\delta-$Hilfer fractional derivatives represent extensions of the standard Caputo and Hilfer fractional derivatives, respectively, specifically designed to handle distributions involving the Dirac delta function $\delta(\zeta)$. These definitions address the challenge of applying fractional differentiation to singular functions, which arise in various physical and engineering problems involving impulsive forces or instantaneous changes. The $\delta-$Caputo derivative typically modifies the standard Caputo definition by incorporating the distributional nature of $\delta(\zeta)$, often through convolution or distributional calculus, ensuring that the derivative of a constant remains zero even in the distributional sense. Similarly, the $\delta-$Hilfer derivative, which generalizes both RL and Caputo fractional derivatives, is adapted to accommodate the delta function, offering a flexible framework for modeling systems with both memory effects and impulsive behavior. These $\delta-$fractional operators find applications in areas like control theory, where impulsive control actions are modeled, and in the study of systems exhibiting both fractional dynamics and sudden perturbations [14–16].

On the other hand, a significant body of research has been dedicated to the theoretical analysis of sequential fractional DEs, particularly concerning the existence and uniqueness of solutions for various initial and boundary value problems (BVPs). These investigations often employ fixed-point (FP) theorems, such as Banach's contraction principle or Schauder's FP theorem, to establish the well-posedness of such problems. The sequential nature of these equations, involving compositions of fractional derivative operators, introduces complexities in their analysis and necessitates specialized techniques. For comprehensive treatments and further exploration of these topics, including diverse applications and methodologies, we direct the reader to the foundational works cited in [17–26] and the extensive references therein, which collectively represent a substantial contribution to the field of fractional calculus and its applications in modeling complex phenomena.

The monotone iterative technique, when coupled with the method of upper and lower solutions [27], stands as a powerful and fundamental approach for establishing the existence and constructing approximate solutions to a wide array of nonlinear differential and integral equations arising in applied mathematics. Beyond merely proving existence, this technique offers significant advantages by generating computable monotone sequences that converge to the extremal solutions within the bounds defined by the upper and lower solutions. This constructive aspect provides valuable insights into the solution behavior and facilitates numerical approximations. Recent advancements and applications of the monotone iterative method in various contexts can be found in the works presented in [28–31] and the extensive references contained within these studies, highlighting its continued relevance and utility in the field.

Inspired by the theoretical significance of sequential fractional DEs and recent advancements in the field [32–34], this study focuses on establishing the existence of extremal solutions for a specific BVP. Our investigation centers on $\delta-$Caputo coupled sequential fractional DEs, further incorporating nonlinear boundary conditions, which adds complexity and relevance to the analysis of such systems. The system takes the form

$$\begin{cases} \left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1 \, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\eta(\varrho) = F\left(\varrho, \eta(\varrho), \beta(\varrho)\right), \ \varrho \in J = [\tau_1, \tau_2], \\ \left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2 \, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\beta(\varrho) = G\left(\varrho, \beta(\varrho), \eta(\varrho)\right), \ \varrho \in J = [\tau_1, \tau_2], \\ \Omega_1\left(\eta(\tau_1), \eta(\tau_2)\right) = 0, \ \Omega_2\left(\beta(\tau_1), \beta(\tau_2)\right) = 0, \ \eta'(\tau_1) = \beta'(\tau_1) = 0, \end{cases} \tag{1.1}$$

where $F, G : [\tau_1, \tau_2] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\Omega_1, \Omega_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, μ_1, μ_2 are positive real numbers, ${}^{C}D^{\rho;\delta}_{\tau_1^+}$ and ${}^{C}D^{\sigma;\delta}_{\tau_1^+}$ are $\delta-$Caputo fractional derivative of orders $\rho \in (0, 1]$, and $\sigma \in (0, 1]$, respectively.

The outline of this manuscript is organized as follows:

- In Section 2, we lay the groundwork for our subsequent analysis by providing essential definitions and lemmas. These foundational elements are crucial for establishing the rigor of our proofs and the clarity of our arguments.
- To provide a rigorous foundation, Section 3 defines the lower and upper solutions for the problem. Furthermore, we derive and present auxiliary lemmas that are essential for the proofs and contribute to the successful completion of our analysis.

- In Section 4, we leverage the power of the monotone iterative method to establish a concrete result concerning the existence of solutions for problem (1.1). Specifically, this method allows us to construct a sequence of approximations that converge to a solution, thereby demonstrating its existence.
- To solidify and visually demonstrate the validity of the theoretical results presented in Section 5, we have conducted a series of illustrative examples. These examples serve to provide concrete instances where the theoretical findings hold, thereby enhancing the reader's understanding and confidence in our conclusions.
- To finalize this study, Section 6 outlines the conclusions drawn from our analysis, while Section 7 provides a dedicated list of abbreviations, facilitating accurate interpretation of the presented information.

2. Basic facts

This section lays the groundwork for our subsequent analysis by introducing essential fractional calculus notations and concepts. Furthermore, we establish the definitions and lemmas that will serve as critical tools in the rigorous proofs presented later in this study.

Definition 2.1. *[32] Assume that $\eta : J \to \mathbb{R}$ is an integrable function with respect to (w.r.t.) another function $\delta : J \to \mathbb{R}$ where δ is an increasing differentiable function with $\delta'(\varrho) \neq 0$. The left-sided $\delta-$RL fractional integral of order $\rho > 0$ can be written as*

$$I^{\rho;\delta}_{\tau_1^+}\eta(\varrho) = \frac{1}{\Gamma(\rho)}\int_{\tau_1}^{\varrho} \delta'(\theta)\left[\delta(\varrho)-\delta(\theta)\right]^{\rho-1}\eta(\theta)d\theta,$$

for all $\varrho \in J$.

Definition 2.2. *[32] Assume that $\delta \in C^{\gamma}(J,\mathbb{R})$ $(\gamma > 0)$ is an increasing function with $\delta'(\varrho) \neq 0$, The left-sided $\delta-$RL fractional derivative of order $\rho > 0$ for the function $\eta \in C^{\gamma}(J,\mathbb{R})$ is described as*

$$\begin{aligned} {}^{C}D^{\rho;\delta}_{\tau_1^+}\eta(\varrho) &= \left(\frac{1}{\delta'(\varrho)}\frac{d}{d\varrho}\right)^{\gamma} I^{\gamma-\rho;\delta}_{\tau_1^+}\eta(\varrho) \\ &= \frac{1}{\Gamma(\gamma-\rho)}\left(\frac{1}{\delta'(\varrho)}\frac{d}{d\varrho}\right)^{\gamma}\int_{\tau_1}^{\varrho} \delta'(\theta)\left[\delta(\varrho)-\delta(\theta)\right]^{\gamma-\rho-1}\eta(\theta)d\theta, \end{aligned}$$

for all $\varrho \in J$, where $\gamma = [\rho]+1$.

Definition 2.3. *[32] Assume that $\delta \in C^{\gamma}(J,\mathbb{R})$ $(\gamma > 0)$ is an increasing function with $\delta'(\varrho) \neq 0$, The left-sided $\delta-$Caputo fractional derivative of order ρ for the function $\eta \in C^{\gamma}(J,\mathbb{R})$ is given by*

$${}^{C}D^{\rho;\delta}_{\tau_1^+}\eta(\varrho) = I^{\gamma-\rho;\delta}_{\tau_1^+}\left(\frac{1}{\delta'(\varrho)}\frac{d}{d\varrho}\right)^{\gamma}\eta(\varrho),$$

for all $\varrho \in J$, where

$$\gamma = \begin{cases} [\rho]+1, & \text{if } \rho \notin \mathbb{N}, \\ \rho, & \text{if } \rho \in \mathbb{N}. \end{cases}$$

For simplicity, we denote

$$\eta_{\delta}^{[\gamma]}(\varrho) = \left(\frac{1}{\delta'(\varrho)}\frac{d}{d\varrho}\right)^{\gamma}\eta(\varrho). \tag{2.1}$$

Remark 2.1. *It should be noted that:*

(i) *From the Definition 2.3 and the notion (2.1), we have*

$$ {}^{C}D_{\tau_1^+}^{\rho;\delta}\eta(\varrho) = \begin{cases} \int_{\tau_1}^{\varrho} \frac{\delta'(\theta)[\delta(\varrho)-\delta(\theta)]^{\gamma-\rho-1}}{\Gamma(\gamma-\rho)}\eta_{\delta}^{[\gamma]}(\theta)\,d\theta, & \text{if } \rho \notin \mathbb{N}, \\ \eta_{\delta}^{[\gamma]}(\varrho), & \text{if } \rho \in \mathbb{N}. \end{cases} \tag{2.2}$$

(ii) *The generalization presented in equation (2.2) specializes to the Caputo fractional derivative operator under the condition that $\delta(\varrho) = \varrho$.*

(iii) *The substitution of $\delta(\varrho) = \ln\varrho$ into equation (2.2) produces the Caputo-Hadamard fractional derivative.*

(iv) *If $\eta \in C^{\gamma}(J, \mathbb{R})$ $(\gamma > 0)$, The $\delta-$Caputo fractional derivative [32] of order ρ for η is calculated by*

$$ {}^{C}D_{\tau_1^+}^{\rho;\delta}\eta(\varrho) = D_{\tau_1^+}^{\rho;\delta}\left[\eta(\varrho) - \sum_{k=1}^{\gamma-1}\frac{\eta_{\delta}^{[k]}(\tau_1)\left(\delta(\varrho)-\delta(\tau_1)\right)^k}{k!}\right].$$

Lemma 2.1. *[17] Assume that $\rho, \nu > 0$. Then,*

(i) *for almost everywhere $\varrho \in J$,*

$$I_{\tau_1^+}^{\rho;\delta} I_{\tau_1^+}^{\nu;\delta}\eta(\varrho) = I_{\tau_1^+}^{\rho+\nu;\delta}\eta(\varrho), \text{ provided that } \eta \in L^1(J, \mathbb{R}),$$

(ii) *for $\varrho \in J$,*

$$I_{\tau_1^+}^{\rho;\delta} I_{\tau_1^+}^{\nu;\delta}\eta(\varrho) = I_{\tau_1^+}^{\rho+\nu;\delta}\eta(\varrho), \text{ provided that } \eta \in C(J, \mathbb{R}).$$

To facilitate the subsequent analysis and ensure a clear understanding of the operators involved, we now recall the crucial property that governs the composition rules for fractional $\delta-$integrals and $\delta-$derivatives. These rules are fundamental for manipulating and simplifying expressions involving these operators.

Lemma 2.2. *[17] Assume that $\rho > 0$. The properties below hold:*

(1) *If $\eta \in C(J, \mathbb{R})$, then*

$$ {}^{C}D_{\tau_1^+}^{\rho;\delta} I_{\tau_1^+}^{\rho;\delta}\eta(\varrho) = \eta(\varrho), \ \varrho \in J.$$

(2) *If $\eta \in C^{\gamma}(J, \mathbb{R})$ and $\rho \in (\gamma - 1, \gamma)$, then*

$$I_{\tau_1^+}^{\rho;\delta}\, {}^{C}D_{\tau_1^+}^{\rho;\delta}\eta(\varrho) = \eta(\varrho) - \sum_{k=1}^{\gamma-1}\frac{\eta_{\delta}^{[k]}(\tau_1)\left(\delta(\varrho)-\delta(\tau_1)\right)^k}{k!}, \ \varrho \in J.$$

Lemma 2.3. *[3] Assume that $\rho \geq 0$, $\nu > 0$ and $\varrho > \tau_1$. The following relations are true:*

(i) $I_{\tau_1^+}^{\rho;\delta}\left(\delta(\varrho)-\delta(\tau_1)\right)^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\rho)}\left(\delta(\varrho)-\delta(\tau_1)\right)^{\nu+\rho-1}$;

(ii) $D_{\tau_1^+}^{\rho;\delta}\left(\delta(\varrho)-\delta(\tau_1)\right)^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\rho)}\left(\delta(\varrho)-\delta(\tau_1)\right)^{\nu-\rho-1}$;

(iii) $D_{\tau_1^+}^{\rho;\delta}\left(\delta(\varrho)-\delta(\tau_1)\right)^{k} = 0$, *for each $k \in \{0, 1, \cdots, \gamma-1\}$ and $\gamma \in \mathbb{N}$.*

3. Definitions and auxiliary lemmas

This section establishes the foundational definitions of lower and upper solutions for the problem under consideration. Additionally, we derive and present a set of auxiliary lemmas, which are instrumental in the subsequent proofs and contribute to the successful completion of our analysis.

Definition 3.1. *We say that the pair $(\eta_0, \beta_0) \in C(J, \mathbb{R}) \times C(J, \mathbb{R})$ is a lower solution to the problem (1.1) if it fulfills*

$$\begin{cases} \left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1\, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\eta_0(\varrho) \leq F\left(\varrho, \eta_0(\varrho), \beta_0(\varrho)\right),\ \varrho \in (\tau_1, \tau_2], \\ \left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2\, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\beta_0(\varrho) \leq G\left(\varrho, \beta_0(\varrho), \eta_0(\varrho)\right),\ \varrho \in (\tau_1, \tau_2], \\ \Omega_1\left(\eta_0(\tau_1), \eta_0(\tau_2)\right) \leq 0,\ \Omega_2\left(\beta_0(\tau_1), \beta_0(\tau_2)\right) \leq 0, \\ \eta_0'(\tau_1) = \beta_0'(\tau_1) = 0, \end{cases}$$

Definition 3.2. *We say that the pair $(f_0, g_0) \in C(J, \mathbb{R}) \times C(J, \mathbb{R})$ is an upper solution to the problem (1.1) if it fulfills*

$$\begin{cases} \left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1\, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)f_0(\varrho) \geq F\left(\varrho, f_0(\varrho), g_0(\varrho)\right),\ \varrho \in (\tau_1, \tau_2], \\ \left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2\, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)g_0(\varrho) \leq G\left(\varrho, g_0(\varrho), f_0(\varrho)\right),\ \varrho \in (\tau_1, \tau_2], \\ \Omega_1\left(f_0(\tau_1), f_0(\tau_2)\right) \geq 0,\ \Omega_2\left(g_0(\tau_1), g_0(\tau_2)\right) \geq 0, \\ f_0'(\tau_1) = g_0'(\tau_1) = 0. \end{cases}$$

Lemma 3.1. *Assume that $(\mathfrak{R}, \mathfrak{I}) \in C(J, \mathbb{R}) \times C(J, \mathbb{R})$. The integral representation of the unique solution to the following coupled sequential fractional DE*

$$\begin{cases} \left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1\, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\eta(\varrho) = \mathfrak{R}\left(\varrho\right),\ \varrho \in J, \\ \left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2\, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\beta(\varrho) = \mathfrak{I}\left(\varrho\right),\ \varrho \in J, \\ \eta'(\tau_1) = \beta'(\tau_1) = 0,\ \eta(\tau_1) = \eta_{\tau_1},\ \beta(\tau_1) = \beta_{\tau_1}, \end{cases} \tag{3.1}$$

is given by

$$\begin{cases} \eta(\varrho) = \eta_{\tau_1} + \int_{\tau_1}^{\varrho} \delta'(\theta) e^{-\mu_1(\delta(\varrho)-\delta(\theta))}\left(\int_{\tau_1}^{\theta} \frac{\delta'(s)[\delta(\varrho)-\delta(s)]^{\rho-1}}{\Gamma(\rho)}\mathfrak{R}\left(s\right)ds\right)d\theta, \\ \beta(\varrho) = \beta_{\tau_1} + \int_{\tau_1}^{\varrho} \delta'(\theta) e^{-\mu_2(\delta(\varrho)-\delta(\theta))}\left(\int_{\tau_1}^{\theta} \frac{\delta'(s)[\delta(\varrho)-\delta(s)]^{\sigma-1}}{\Gamma(\sigma)}\mathfrak{I}\left(s\right)ds\right)d\theta. \end{cases} \tag{3.2}$$

Proof. Applying the $\delta-$RL of order ρ and σ to both sides of the first and second equation of (3.1), respectively, we have

$$\begin{cases} I^{\rho;\delta}_{\tau_1^+}\, {}^{C}D^{\rho+1;\delta}_{\tau_1^+}\eta(\varrho) + \mu_1 I^{\rho;\delta}_{\tau_1^+}\, {}^{C}D^{\rho;\delta}_{\tau_1^+}\eta(\varrho) = I^{\rho;\delta}_{\tau_1^+}\mathfrak{R}\left(\varrho\right), \\ I^{\sigma;\delta}_{\tau_1^+}\, {}^{C}D^{\sigma+1;\delta}_{\tau_1^+}\beta(\varrho) + \mu_2 I^{\sigma;\delta}_{\tau_1^+}\, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\beta(\varrho) = I^{\sigma;\delta}_{\tau_1^+}\mathfrak{I}\left(\varrho\right). \end{cases}$$

From Lemma 2.2 (1) and the definition of $\eta_\delta^{[1]}$ in (2.1), we get

$$\begin{cases} \eta_\delta^{[1]}\left(\varrho\right) + \mu_1\eta(\varrho) = I^{\rho;\delta}_{\tau_1^+}\mathfrak{R}\left(\varrho\right) + a_0, \\ \beta_\delta^{[1]}\left(\varrho\right) + \mu_2\beta(\varrho) = I^{\sigma;\delta}_{\tau_1^+}\mathfrak{I}\left(\varrho\right) + b_0, \end{cases}$$

where $a_0, b_0 \in \mathbb{R}$. Based on the notation $\eta_{\delta}^{[1]}$ established in (2.1), we have

$$\begin{cases} \eta'(\varrho) + \mu_1\delta'(\varrho)\eta(\varrho) = \delta'(\varrho)\left(I_{\tau_1^+}^{\rho;\delta}\mathfrak{R}(\varrho) + a_0\right), \\ \beta'(\varrho) + \mu_2\delta'(\varrho)\beta(\varrho) = \delta'(\varrho)\left(I_{\tau_1^+}^{\sigma;\delta}\mathfrak{I}(\varrho) + b_0\right). \end{cases} \tag{3.3}$$

Multiplying both sides of the first equation of (2.1) by $e^{\mu_1(\delta(\varrho)-\delta(\tau_1))}$ and the second equation of (2.1) by $e^{\mu_2(\delta(\varrho)-\delta(\tau_1))}$, respectively, one can write

$$\begin{cases} \left(\eta(\varrho)e^{\mu_1(\delta(\varrho)-\delta(\tau_1))}\right)' = \delta'(\varrho)e^{\mu_1(\delta(\varrho)-\delta(\tau_1))}I_{\tau_1^+}^{\rho;\delta}\mathfrak{R}(\varrho) + a_0\delta'(\varrho)e^{\mu_1(\delta(\varrho)-\delta(\tau_1))}, \\ \left(\beta(\varrho)e^{\mu_2(\delta(\varrho)-\delta(\tau_1))}\right)' = \delta'(\varrho)e^{\mu_2(\delta(\varrho)-\delta(\tau_1))}I_{\tau_1^+}^{\sigma;\delta}\mathfrak{I}(\varrho) + b_0\delta'(\varrho)e^{\mu_2(\delta(\varrho)-\delta(\tau_1))}. \end{cases}$$

By performing the integration from τ_1 to ϱ, we find that

$$\begin{cases} \eta(\varrho) = a_1 e^{-\mu_1(\delta(\varrho)-\delta(\tau_1))} + \frac{a_0}{\mu_1} + \int_{\tau_1}^{\varrho}\delta'(\theta)e^{-\mu_1(\delta(\varrho)-\delta(\tau_1))}I_{\tau_1^+}^{\rho;\delta}\mathfrak{R}(\theta)\,d\theta, \\ \beta(\varrho) = b_1 e^{-\mu_2(\delta(\varrho)-\delta(\tau_1))} + \frac{b_0}{\mu_1} + \int_{\tau_1}^{\varrho}\delta'(\theta)e^{-\mu_2(\delta(\varrho)-\delta(\tau_1))}I_{\tau_1^+}^{\sigma;\delta}\mathfrak{I}(\theta)\,d\theta, \end{cases} \tag{3.4}$$

where a_1 and b_1 are arbitrary constants. Upon differentiation of equation (3.4), we arrive at

$$\begin{aligned} \eta'(\varrho) &= -a_1\mu_1\delta'(\varrho)e^{-\mu_1(\delta(\varrho)-\delta(\tau_1))} + \delta'(\varrho)I_{\tau_1^+}^{\rho;\delta}\mathfrak{R}(\varrho) \\ &\quad -\mu_1\delta'(\varrho)\int_{\tau_1}^{\varrho}\delta'(\theta)e^{-\mu_1(\delta(\varrho)-\delta(\theta))}I_{\tau_1^+}^{\rho;\delta}\mathfrak{R}(\theta)\,d\theta, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \beta'(\varrho) &= -b_1\mu_2\delta'(\varrho)e^{-\mu_2(\delta(\varrho)-\delta(\tau_1))} + \delta'(\varrho)I_{\tau_1^+}^{\sigma;\delta}\mathfrak{I}(\varrho) \\ &\quad -\mu_2\delta'(\varrho)\int_{\tau_1}^{\varrho}\delta'(\theta)e^{-\mu_2(\delta(\varrho)-\delta(\theta))}I_{\tau_1^+}^{\sigma;\delta}\mathfrak{I}(\theta)\,d\theta. \end{aligned} \tag{3.6}$$

Combining the initial conditions $\eta'(\tau_1) = \beta'(\tau_1) = 0$, $\eta(\tau_1) = \eta_{\tau_1}$, and $\beta(\tau_1) = \beta_{\tau_1}$ with equations (3.4)-(3.6) leads to

$$a_0 = \mu_1\eta_{\tau_1},\ a_1 = 0,\ b_0 = \mu_2\beta_{\tau_1},\ \text{and } b_1 = 0.$$

We arrive at (3.2) by substituting the determined values of a_0, a_1, b_0, and b_1 into (3.4). □

Lemma 3.2. *Suppose that $\rho, \sigma \in (0,1]$ and $u, v, \widetilde{u}, \widetilde{v} \in C(J, \mathbb{R})$. The linear fractional system*

$$\begin{cases} \left({}^{C}D_{\tau_1^+}^{\rho+1;\delta} + \mu_1\,{}^{C}D_{\tau_1^+}^{\rho;\delta}\right)\eta(\varrho) - u(\varrho)\eta(\varrho) = v(\varrho),\ \varrho \in J, \\ \left({}^{C}D_{\tau_1^+}^{\sigma+1;\delta} + \mu_2\,{}^{C}D_{\tau_1^+}^{\sigma;\delta}\right)\beta(\varrho) - \widetilde{u}(\varrho)\beta(\varrho) = \widetilde{v}(\varrho),\ \varrho \in J, \\ \eta'(\tau_1) = \beta'(\tau_1) = 0,\ \eta(\tau_1) = \eta_{\tau_1},\ \beta(\tau_1) = \beta_{\tau_1}, \end{cases} \tag{3.7}$$

has a unique solution $(\eta, \beta) \in C(J, \mathbb{R}) \times C(J, \mathbb{R})$, whenever

$$\|u\| \le \frac{\ell\mu_1\Gamma(\rho+1)}{(\delta(\tau_2)-\delta(\tau_1))^{\rho}} \text{ and } \|\widetilde{u}\| \le \frac{\ell\mu_2\Gamma(\sigma+1)}{(\delta(\tau_2)-\delta(\tau_1))^{\sigma}}. \tag{3.8}$$

where ℓ is positive constant with $\ell < 1$.

Proof. By Lemma 3.1, the problem (3.7) can be reformulated as the integral equations:

$$\begin{cases} \eta\left(\varrho\right) = \eta_{\tau_1} + \int_{\tau_1}^{\varrho} \delta'(\theta) e^{-\mu_1(\delta(\varrho)-\delta(\theta))} \left(\int_{\tau_1}^{\theta} \frac{\delta'(s)[\delta(\theta)-\delta(s)]^{\rho-1}}{\Gamma(\rho)} \left(u(s)\eta(s) + v(s)\right) ds \right) d\theta \\ \beta\left(\varrho\right) = \beta_{\tau_1} + \int_{\tau_1}^{\varrho} \delta'(\theta) e^{-\mu_2(\delta(\varrho)-\delta(\theta))} \left(\int_{\tau_1}^{\theta} \frac{\delta'(s)[\delta(\theta)-\delta(s)]^{\sigma-1}}{\Gamma(\sigma)} \left(\widetilde{u}(s)\beta(s) + \widetilde{v}(s)\right) ds \right) d\theta. \end{cases}$$

Define the mapping $\Theta : C(J,\mathbb{R}) \times C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as $\Theta(\eta,\beta) = \Theta_1\left(\eta\right)$ and $\Theta(\beta,\eta) = \Theta_2\left(\beta\right)$, where

$$\Theta_1\eta\left(\varrho\right) = \eta_{\tau_1} + \int_{\tau_1}^{\varrho} \delta'(\theta) e^{-\mu_1(\delta(\varrho)-\delta(\theta))} \left(\int_{\tau_1}^{\theta} \frac{\delta'(s)\left[\delta(\theta)-\delta(s)\right]^{\rho-1}}{\Gamma\left(\rho\right)} \left(u(s)\eta(s) + v(s)\right) ds \right) d\theta,$$

and

$$\Theta_2\beta\left(\varrho\right) = \beta_{\tau_1} + \int_{\tau_1}^{\varrho} \delta'(\theta) e^{-\mu_2(\delta(\varrho)-\delta(\theta))} \left(\int_{\tau_1}^{\theta} \frac{\delta'(s)\left[\delta(\theta)-\delta(s)\right]^{\sigma-1}}{\Gamma\left(\sigma\right)} \left(\widetilde{u}(s)\beta(s) + \widetilde{v}(s)\right) ds \right) d\theta,$$

for all $\varrho \in J$. We will now demonstrate that the operator Θ has a unique FP by proving it is a contraction mapping.

Assume that $\eta, \widetilde{\eta} \in C(J,\mathbb{R})$ and $\varrho \in J = \left[\tau_1, \tau_2\right]$. Then, by (3.8), we get

$$\begin{aligned} \left|\Theta_1\left(\eta\right)\left(\varrho\right) - \Theta_1\left(\widetilde{\eta}\right)\left(\varrho\right)\right| &\leq \int_{\tau_1}^{\varrho} \delta'(\theta) e^{-\mu_1(\delta(\varrho)-\delta(\theta))} \\ &\quad \times \left(\int_{\tau_1}^{\theta} \frac{\delta'(s)\left[\delta(\theta)-\delta(s)\right]^{\rho-1}}{\Gamma\left(\rho\right)} \left|u(s)\right| \left|\eta(s) - \eta(s)\right| dr \right) d\theta \\ &\leq \frac{\left(\delta(\tau_2) - \delta(\tau_1)\right)^{\rho}}{\Gamma\left(\rho+1\right)} \left\|u\right\| \left\|\eta - \widetilde{\eta}\right\| \int_{\tau_1}^{\varrho} \delta'(\theta) e^{-\mu_1(\delta(\varrho)-\delta(\theta))} \\ &\leq \frac{\left(\delta(\tau_2) - \delta(\tau_1)\right)^{\rho}}{\mu_2\Gamma\left(\rho+1\right)} \left\|u\right\| \left\|\eta - \widetilde{\eta}\right\| \\ &\leq \ell \left\|\eta - \widetilde{\eta}\right\| \end{aligned}$$

Hence,

$$\left\|\Theta_1\left(\eta\right) - \Theta_1\left(\widetilde{\eta}\right)\right\| \leq \ell \left\|\eta - \widetilde{\eta}\right\|. \tag{3.9}$$

Similarly, for $\beta, \widetilde{\beta} \in C(J,\mathbb{R})$, we have

$$\left\|\Theta_2\left(\beta\right) - \Theta_2\left(\widetilde{\beta}\right)\right\| \leq \ell \left\|\beta - \widetilde{\beta}\right\|. \tag{3.10}$$

It follows that from (3.9) and (3.10) that Θ_1 and Θ_2 is contraction mappings (since $\ell < 1$). Hence, the mapping Θ is a contraction. Applying Banach's FP theorem, we conclude that Θ possesses a unique FP, which corresponds to the unique solution of problem (3.7). Hence, the proof is complete. □

Lemma 3.3. *Let $u, \widetilde{u} \in C(J, \mathbb{R}_+)$ such that (3.8) holds. If $\alpha, \lambda \in C(J, \mathbb{R})$ such that the inequalities below are true*

$$\begin{cases} \left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1 \, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\alpha(\varrho) \geq u(\varrho)\alpha(\varrho), \ \varrho \in J, \\ \left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2 \, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\lambda(\varrho) \geq \widetilde{u}(\varrho)\lambda(\varrho), \ \varrho \in J, \\ \alpha(\tau_1) \geq 0, \ \alpha'(\tau_1) = 0, \ \lambda(\tau_1) \geq 0, \ \lambda'(\tau_1) = 0, \end{cases}$$

then $\alpha(\varrho) \geq 0$ and $\lambda(\varrho) \geq 0$ on J.

Proof. Assume that

$$\begin{cases} \left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1 \, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\alpha(\varrho) - u(\varrho)\alpha(\varrho) = v(\varrho), \\ \left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2 \, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\lambda(\varrho) - \widetilde{u}(\varrho)\lambda(\varrho) = \widetilde{v}(\varrho), \\ \eta'(\tau_1) = \beta'(\tau_1) = 0, \ \alpha(\tau_1) = \eta_{\tau_1}, \ \lambda(\tau_1) = \beta_{\tau_1}, \end{cases}$$

where $v, \widetilde{v} \in C(J, \mathbb{R})$. Clearly, $v(\varrho) \geq 0$, $\widetilde{v}(\varrho) \geq 0$, $\eta_{\tau_1} \geq 0$, and $\eta_{\tau_1} \geq 0$. Here, we prove our result by the contradiction. So, we assume that $\alpha(\varrho) \geq 0$ and $\lambda(\varrho) \geq 0$ on the interval J are not true. Then, there is at least $\varrho_0 \in J$ such that $\alpha(\varrho_0) < 0$ and $\lambda(\varrho_0) < 0$.

We can assume, without loss of generality, that $\alpha(\varrho_0) = \min_{\varrho \in J} \alpha(\varrho)$ and $\lambda(\varrho_0) = \min_{\varrho \in J} \lambda(\varrho)$. Applying Lemma 3.2, we obtain that

$$\begin{aligned} \alpha(\varrho) &= \eta_{\tau_1} + \int_{\tau_1}^{\varrho} \delta'(\theta) e^{-\mu_1(\delta(\varrho)-\delta(\theta))} \\ &\quad \times \left(\int_{\tau_1}^{\theta} \frac{\delta'(s)\left[\delta(\theta)-\delta(s)\right]^{\rho-1}}{\Gamma(\rho)} \left(u(s)\alpha(s) + v(s)\right) ds \right) d\theta \\ &\geq \alpha(\varrho_0) \int_{\tau_1}^{\varrho} \delta'(\theta) e^{-\mu_1(\delta(\varrho)-\delta(\theta))} \\ &\quad \times \left(\int_{\tau_1}^{\theta} \frac{\delta'(s)\left[\delta(\theta)-\delta(s)\right]^{\rho-1}}{\Gamma(\rho)} u(s)\, ds \right) d\theta, \end{aligned}$$

For $\varrho = \varrho_0$, we have

$$\alpha(\varrho_0) \geq \alpha(\varrho_0) \int_{\tau_1}^{\varrho_0} \delta'(\theta) e^{-\mu_1(\delta(\varrho)-\delta(\theta))} \left(\int_{\tau_1}^{\theta} \frac{\delta'(s)\left[\delta(\theta)-\delta(s)\right]^{\rho-1}}{\Gamma(\rho)} u(s)\, ds \right) d\theta$$

Since $\alpha(\varrho_0) < 0$, we have

$$1 \leq \int_{\tau_1}^{\varrho_0} \delta'(\theta) e^{-\mu_1(\delta(\varrho)-\delta(\theta))} \left(\int_{\tau_1}^{\theta} \frac{\delta'(s)\left[\delta(\theta)-\delta(s)\right]^{\rho-1}}{\Gamma(\rho)} u(s)\, ds \right) d\theta. \tag{3.11}$$

Similarly, since $\lambda(\varrho_0) < 0$, we can write

$$1 \leq \int_{\tau_1}^{\varrho_0} \delta'(\theta) e^{-\mu_2(\delta(\varrho)-\delta(\theta))} \left(\int_{\tau_1}^{\theta} \frac{\delta'(s)\left[\delta(\theta)-\delta(s)\right]^{\sigma-1}}{\Gamma(\sigma)} \widetilde{u}(s)\, ds \right) d\theta. \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\|u\| \geq \frac{\ell\mu_1 \Gamma(\rho+1)}{\left(\delta(\tau_2)-\delta(\tau_1)\right)^{\rho}} \text{ and } \left\|\widetilde{u}\right\| \geq \frac{\ell\mu_2 \Gamma(\sigma+1)}{\left(\delta(\tau_2)-\delta(\tau_1)\right)^{\sigma}}.$$

This is contrary to the established result in (3.8). □

4. Extremal solutions

The primary objective of this section is to prove the existence of a solution for problem (1.1). To achieve this, we will apply the monotone iterative method, a well-established technique that yields existence results through the construction of convergent sequences.

We commence by stating the assertions required to analyze the lower and upper solutions of the proposed problem.

(A_1) For all $\varrho \in J$, there exist $\eta_0, \beta_0, f_0, g_0 \in C(J, \mathbb{R})$ $(J = [\tau_1, \tau_2])$ such that (η_0, β_0) and (f_0, g_0) are lower and upper solutions to the system (1.1), respectively, with $\eta_0(\varrho) \leq f_0(\varrho)$ and $\beta_0(\varrho) \leq g_0(\varrho)$;

(A_2) The functions $F, G : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exist $u, \widetilde{u} \in C(J, \mathbb{R}_+)$ fulfill (3.8) such that

$$F(\varrho, \varphi_2, \psi_2) - F(\varrho, \varphi_1, \psi_1) \geq u(\varrho)\left[(\varphi_2 - \varphi_1) + (\psi_2 - \psi_1)\right],$$

for $\eta_0 \leq \varphi_1 \leq \varphi_2 \leq f_0$, $\beta_0 \leq \psi_1 \leq \psi_2 \leq g_0$, and

$$G(\varrho, \psi_2, \varphi_2) - G(\varrho, \psi_1, \varphi_1) \geq \widetilde{u}(\varrho)\left[(\psi_2 - \psi_1) + (\varphi_2 - \varphi_1)\right],$$

for $\eta_0 \leq \psi_1 \leq \psi_2 \leq f_0$, $\beta_0 \leq \varphi_1 \leq \varphi_2 \leq g_0$.

(A_3) For $\eta_0(\tau_1) \leq \varkappa_1 \leq \varkappa_2 \leq f_0(\tau_1)$, $\beta_0(\tau_1) \leq \varkappa_1 \leq \varkappa_2 \leq g_0(\tau_1)$, $\eta_0(\tau_2) \leq z_1 \leq z_2 \leq f_0(\tau_2)$, and $\beta_0(\tau_2) \leq z_1 \leq z_2 \leq g_0(\tau_2)$, there exist $r_1, r_2 > 0$ and $r_3, r_4 \geq 0$ such that

$$\Omega_1(\varkappa_2, z_2) - \Omega_1(\varkappa_1, z_1) \leq r_1(\varkappa_2 - \varkappa_1) - r_3(z_2 - z_1),$$

and

$$\Omega_2(\varkappa_2, z_2) - \Omega_2(\varkappa_1, z_1) \leq r_2(\varkappa_2 - \varkappa_1) - r_4(z_2 - z_1),$$

where $\Omega_1, \Omega_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Now, we present our main theorem in this part.

Theorem 4.1. *Under the hypotheses (A_1)-(A_3), there exist monotone iterative sequences $\{\eta_\gamma\}$, $\{\beta_\gamma\}$, $\{f_\gamma\}$, and $\{g_\gamma\}$ that exhibit uniform convergence on J to the extremal solutions of (3.8) in $[\eta_0, f_0]$, and $[\beta_0, g_0]$, where*

$$\begin{cases} [\eta_0, f_0] = \{\varphi \in C(J, \mathbb{R}) : \eta_0(\varrho) \leq \varphi(\varrho) \leq f_0(\varrho)\}, \\ [\beta_0, g_0] = \{\psi \in C(J, \mathbb{R}) : \beta_0(\varrho) \leq \psi(\varrho) \leq g_0(\varrho)\}, \end{cases}$$

for all $\varrho \in J$.

Proof. For any $\eta_0, \beta_0, f_0, g_0 \in C(J, \mathbb{R})$, assume that

$$\begin{cases} \left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1 \, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\eta_{\gamma+1}(\varrho) = F\left(\varrho, \eta_\gamma(\varrho), \beta_\gamma(\varrho)\right) + u(\varrho)\left[\left(\eta_{\gamma+1}(\varrho) - \eta_\gamma(\varrho)\right)\right], \\ \left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2 \, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\beta_{\gamma+1}(\varrho) = G\left(\varrho, \beta_\gamma(\varrho), \eta_\gamma(\varrho)\right) + \widetilde{u}(\varrho)\left[\left(\beta_{\gamma+1}(\varrho) - \beta_\gamma(\varrho)\right)\right], \\ \eta'_{\gamma+1}(\tau_1) = \beta'_{\gamma+1}(\tau_1) = 0, \\ \eta_{\gamma+1}(\tau_1) = \eta_\gamma(\tau_1) - \frac{1}{r_1}\Omega_1\left(\eta_\gamma(\tau_1), \eta_\gamma(\tau_2)\right), \\ \beta_{\gamma+1}(\tau_1) = \beta_\gamma(\tau_1) - \frac{1}{r_2}\Omega_2\left(\beta_\gamma(\tau_1), \beta_\gamma(\tau_2)\right). \end{cases} \tag{4.1}$$

and

$$\begin{cases} \left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1 \, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)f_{\gamma+1}(\varrho) = F\left(\varrho, f_\gamma(\varrho), g_\gamma(\varrho)\right) + u(\varrho)\left[\left(f_{\gamma+1}(\varrho) - f_\gamma(\varrho)\right)\right], \\ \left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2 \, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)g_{\gamma+1}(\varrho) = G\left(\varrho, g_\gamma(\varrho), f_\gamma(\varrho)\right) + \widetilde{u}(\varrho)\left[\left(g_{\gamma+1}(\varrho) - g_\gamma(\varrho)\right)\right], \\ f'_{\gamma+1}(\tau_1) = g'_{\gamma+1}(\tau_1) = 0, \\ f_{\gamma+1}(\tau_1) = f_\gamma(\tau_1) - \frac{1}{r_3}\Omega_1\left(f_\gamma(\tau_1), g_\gamma(\tau_2)\right), \\ g_{\gamma+1}(\tau_1) = g_\gamma(\tau_1) - \frac{1}{r_4}\Omega_2\left(g_\gamma(\tau_1), f_\gamma(\tau_2)\right). \end{cases} \tag{4.2}$$

According to Lemma 3.2, the system of equations (4.1) and (4.2) admits unique solutions in $C(J, \mathbb{R})$. The rest of the proof will be divided onto the following steps:

St. 1: We will demonstrate that $\left(\eta_\gamma, \beta_\gamma\right)$ and $\left(f_\gamma, g_\gamma\right)$ (for $\gamma \geq 1$) constitute lower and upper solutions, respectively, of problem (1.1). Moreover, for $\varrho \in J$,

$$\eta_0(\varrho) \leq \eta_1(\varrho) \leq \cdots \leq \eta_\gamma(\varrho) \leq \cdots \leq f_\gamma(\varrho) \leq \cdots \leq f_1(\varrho) \leq f_0(\varrho)$$

and

$$\beta_0(\varrho) \leq \beta_1(\varrho) \leq \cdots \leq \beta_\gamma(\varrho) \leq \cdots \leq g_\gamma(\varrho) \leq \cdots \leq g_1(\varrho) \leq g_0(\varrho).$$

Put $\alpha(\varrho) = \eta_1(\varrho) - \eta_0(\varrho)$ and $\lambda(\varrho) = \beta_1(\varrho) - \beta_0(\varrho)$. Using (4.1) and Definition 3.1, we have

$$\begin{aligned} \left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1 \, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\alpha(\varrho) &= F(\varrho, \eta_0(\varrho), \beta_0(\varrho)) \\ &\quad - \left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1 \, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\eta_0(\varrho) + u(\varrho)\,\alpha(\varrho) \\ &\geq u(\varrho)\,\alpha(\varrho), \end{aligned}$$

and

$$\begin{aligned} \left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2 \, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\lambda(\varrho) &= G(\varrho, \beta_0(\varrho), \eta_0(\varrho)) \\ &\quad - \left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2 \, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\beta_0(\varrho) + \widetilde{u}(\varrho)\,\lambda(\varrho) \\ &\geq \widetilde{u}(\varrho)\,\lambda(\varrho). \end{aligned}$$

From the conditions (4.1), we have $\alpha'(\tau_1)=0$, $\lambda'(\tau_1)=0$,

$$\begin{aligned}\alpha(\tau_1) &= \eta_1(\tau_1)-\eta_0(\tau_1)\\ &= \eta_0(\tau_1)-\frac{1}{r_1}\Omega_1(\eta_0(\tau_1),\eta_0(\tau_2))-\eta_0(\tau_1)\\ &= -\frac{1}{r_1}\Omega_1(\eta_0(\tau_1),\eta_0(\tau_2))\geq 0 \text{ (since } \frac{1}{r_1}<0),\end{aligned}$$

and similarly, $\lambda(\tau_1)\geq 0$. By Lemma 3.3, $\alpha(\varrho)\geq 0$ and $\lambda(\varrho)\geq 0$ on J. Hence,

$$\eta_0(\varrho)\leq\eta_1(\varrho).$$

Similarly, we can prove that

$$\beta_0(\varrho)\leq\beta_1(\varrho).$$

Again, set $\alpha(\varrho)=f_1(\varrho)-\eta_1(\varrho)$ and $\lambda(\varrho)=g_1(\varrho)-\beta_1(\varrho)$. From (4.1), (4.2) and (A_2), we can write

$$\begin{aligned}\left({}^{C}D^{\rho+1;\delta}_{\tau_1^+}+\mu_1\,{}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\alpha(\varrho) &= F(\varrho,f_0(\varrho),g_0(\varrho))-F(\varrho,\eta_0(\varrho),\beta_0(\varrho))\\ &\quad +u(\varrho)\left[(f_1(\varrho)-f_0(\varrho))\right]-u(\varrho)\left[(\eta_1(\varrho)-\eta_0(\varrho))\right]\\ &\geq u(\varrho)\left[(f_0(\varrho)-\eta_0(\varrho))+(g_0(\varrho)-\beta_0(\varrho))\right]\\ &\quad +u(\varrho)\left[(f_1(\varrho)-f_0(\varrho))\right]-u(\varrho)\left[(\eta_1(\varrho)-\eta_0(\varrho))\right]\\ &= u(\varrho)\left[f_1(\varrho)-\eta_1(\varrho)\right]+u(\varrho)\left[g_0(\varrho)-\beta_0(\varrho)\right]\\ &\geq u(\varrho)\left[f_1(\varrho)-\eta_1(\varrho)\right]=u(\varrho)\alpha(\varrho).\end{aligned}$$

Analogously,

$$\left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+}+\mu_1\,{}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\lambda(\varrho)\geq\widetilde{u}(\varrho)\lambda(\varrho).$$

By the conditions of (4.1), (4.2) and (A_3), we have $\alpha'(\tau_1)=0$, $\lambda'(\tau_1)=0$, and

$$\begin{aligned}\alpha(\tau_1) &= f_1(\varrho)-\eta_1(\varrho)\\ &= \left[f_0(\tau_1)-\frac{1}{r_1}\Omega_1(f_0(\tau_1),g_0(\tau_2))\right]-\left[\eta_0(\tau_1)-\frac{1}{r_1}\Omega_1(\eta_0(\tau_1),\eta_0(\tau_2))\right]\\ &= (f_0(\tau_1)-\eta_0(\tau_1))-\frac{1}{r_1}\left(\Omega_1(f_0(\tau_1),g_0(\tau_2))-\Omega_1(\eta_0(\tau_1),\eta_0(\tau_2))\right)\\ &\geq (f_0(\tau_1)-\eta_0(\tau_1))-\frac{1}{r_1}\left[r_1(f_0(\tau_1)-\eta_0(\tau_1))-r_3(g_0(\tau_2)-\eta_0(\tau_2))\right]\\ &= (f_0(\tau_1)-\eta_0(\tau_1))-(f_0(\tau_1)-\eta_0(\tau_1))+\frac{r_3}{r_1}(g_0(\tau_2)-\eta_0(\tau_2))\\ &= \frac{r_3}{r_1}(g_0(\tau_2)-\eta_0(\tau_2))\geq 0.\end{aligned}\tag{4.3}$$

In the same manner, one can write

$$\lambda(\varrho)\geq\frac{r_4}{r_2}(f_0(\tau_2)-\beta_0(\tau_2))\geq 0.\tag{4.4}$$

It follows from (4.3), (4.4) and Lemma 3.3 that $\eta_1(\varrho)\leq f_1(\varrho)$ and $\beta_1(\varrho)\leq g_1(\varrho)$.

We proceed to demonstrate that $(\eta_1(\varrho), \beta_1(\varrho))$ and $(f_1(\varrho), g_1(\varrho))$ constitute lower and upper solutions of (1.1), respectively. Given that (η_0, β_0) and (f_0, g_0) are lower and upper solutions of (1.1), Hypotheses (A_3) and (A_4) imply that

$$\begin{aligned}\left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1\, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\eta_1(\varrho) &= F(\varrho, \eta_0(\varrho), \beta_0(\varrho)) + u(\varrho)\left[(\eta_1(\varrho) - \eta_0(\varrho))\right] \\ &\leq F(\varrho, \eta_1(\varrho), \beta_1(\varrho)),\end{aligned}$$

and

$$\begin{aligned}\left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_2\, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\beta_1(\varrho) &= G(\varrho, \beta_0(\varrho), \eta_0(\varrho)) + \widetilde{u}(\varrho)\left[(\beta_1(\varrho) - \beta_0(\varrho))\right] \\ &\leq G(\varrho, \beta_1(\varrho), \eta_1(\varrho)).\end{aligned}$$

Furthermore, $\eta_1'(\tau_1) = 0$, $\eta_1'(\tau_1) = 0$,

$$\begin{aligned}0 &= r_1(\eta_1(\tau_1) - \eta_0(\tau_1)) + \Omega_1(\eta_0(\tau_1), \eta_0(\tau_2)) \\ &\geq \Omega_1(\eta_1(\tau_1), \eta_1(\tau_2)) + r_3(\eta_1(\tau_2) - \eta_0(\tau_2)),\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}0 &= r_2(\beta_1(\tau_1) - \beta_0(\tau_1)) + \Omega_2(\beta_0(\tau_1), \beta_0(\tau_2)) \\ &\geq \Omega_2(\beta_1(\tau_1), \beta_1(\tau_2)) + r_4(\beta_1(\tau_2) - \beta_0(\tau_2)).\end{aligned} \tag{4.6}$$

Hence, we deduce from (4.5), and (4.6) that

$$\Omega_1(\eta_1(\tau_1), \eta_1(\tau_2)) \leq 0 \text{ and } \Omega_2(\beta_1(\tau_1), \beta_1(\tau_2)) \leq 0.$$

Consequently, the pair (η_1, β_1) is lower solution of (1.1). Similarly, one can prove (f_1, g_1) is upper solution of (1.1). By inductive argument, we prove that for all $\gamma \geq 1$, $\left(\eta_\gamma, \beta_\gamma\right)$ and $\left(f_\gamma, g_\gamma\right)$ are lower and upper solutions to (1.1), respectively, and that the following relations are satisfied:

$$\eta_0(\varrho) \leq \eta_1(\varrho) \leq \cdots \leq \eta_\gamma(\varrho) \leq \cdots \leq f_\gamma(\varrho) \leq \cdots \leq f_1(\varrho) \leq f_0(\varrho)$$

and

$$\beta_0(\varrho) \leq \beta_1(\varrho) \leq \cdots \leq \beta_\gamma(\varrho) \leq \cdots \leq g_\gamma(\varrho) \leq \cdots \leq g_1(\varrho) \leq g_0(\varrho).$$

St. 2: We show that the sequences $\left(\{\eta_\gamma\}, \{\beta_\gamma\}\right)$ and $\left(\{f_\gamma\}, \{g_\gamma\}\right)$ uniformly converge to the limit functions (η, β) and (f, g), respectively. Since $\left(\{\eta_\gamma\}, \{\beta_\gamma\}\right)$ are monotone nondecreasing and bounded above by (f_0, g_0), respectively, and since $\left(f_\gamma, g_\gamma\right)$ are monotone nonincreasing and bounded below by (η_0, β_0), respectively, then, the pointwise limits (η^*, β^*) and (f^*, g^*) exist. Applying Dini's theorem [35] to the continuous function sequences $\left(\{\eta_\gamma\}, \{\beta_\gamma\}\right)$ and $\left(\{f_\gamma\}, \{g_\gamma\}\right)$ on $[\tau_1, \tau_2]$, yields uniform convergence. That is,

$$\lim_{\gamma\to\infty} \eta_\gamma(\varrho) = \eta^*(\varrho), \ \lim_{\gamma\to\infty} \beta_\gamma(\varrho) = \beta^*(\varrho), \ \lim_{\gamma\to\infty} f_\gamma(\varrho) = f^*(\varrho), \text{ and } \lim_{\gamma\to\infty} g_\gamma(\varrho) = g^*(\varrho),$$

uniformly on $\varrho \in J$ and the limit functions (η^*, β^*), and (f^*, g^*) satisfy the system (1.1). Additionally, (η^*, β^*), and (f^*, g^*) fulfill the inequalities:

$$\eta_0 \leq \eta_1 \leq \cdots \leq \eta^* \leq f^* \leq \cdots \leq f_\gamma \leq \cdots \leq f_1 \leq f_0,$$

and

$$\beta_0 \leq \beta_1 \leq \cdots \leq \beta^* \leq g^* \leq \cdots \leq g_\gamma \leq \cdots \leq g_1 \leq g_0,$$

respectively.

St. 3: We prove that (η^*, β^*) and (f^*, g^*) are the extremal solutions for problem (1.1) in $[\eta_0, f_0]$ and $[\beta_0, g_0]$, respectively.

Let φ and ψ be a solutions of (1.1) in the interval $[\eta_0, f_0]$ and $[\beta_0, g_0]$, respectively. We suppose that for some $\gamma \in \mathbb{N}$, the following inequalities are true:

$$\eta_\gamma(\varrho) \leq \varphi(\varrho) \leq f_\gamma(\varrho) \text{ and } \beta_\gamma(\varrho) \leq \psi(\varrho) \leq g_\gamma(\varrho), \ \varrho \in J. \tag{4.7}$$

Consider $\alpha(\varrho) = \varphi(\varrho) - \eta_{\gamma+1}(\varrho)$ and $\lambda(\varrho) = \psi(\varrho) - \beta_{\gamma+1}(\varrho)$. From (4.1), (4.2) and (A_2), we can write

$$\begin{aligned}
\left({}^{C}D^{\rho+1;\delta}_{\tau_1^+} + \mu_1 \, {}^{C}D^{\rho;\delta}_{\tau_1^+}\right)\alpha(\varrho) &= F(\varrho, \varphi(\varrho), \psi(\varrho)) - F\left(\varrho, \eta_\gamma(\varrho), \beta_\gamma(\varrho)\right) \\
&\quad - u(\varrho)\left[\left(\eta_{\gamma+1}(\varrho) - \eta_\gamma(\varrho)\right)\right] \\
&\geq u(\varrho)\left[\left(\varphi(\varrho) - \eta_\gamma(\varrho)\right) + \left(\psi(\varrho) - \beta_\gamma(\varrho)\right)\right] \\
&\quad - u(\varrho)\left[\left(\eta_{\gamma+1}(\varrho) - \eta_\gamma(\varrho)\right)\right] \\
&= u(\varrho)\left[\varphi(\varrho) - \eta_{\gamma+1}(\varrho)\right] + u(\varrho)\left[\psi(\varrho) - \beta_\gamma(\varrho)\right] \\
&\geq u(\varrho)\left[\varphi(\varrho) - \eta_{\gamma+1}(\varrho)\right] = u(\varrho)\alpha(\varrho),
\end{aligned}$$

Similarly

$$\left({}^{C}D^{\sigma+1;\delta}_{\tau_1^+} + \mu_1 \, {}^{C}D^{\sigma;\delta}_{\tau_1^+}\right)\lambda(\varrho) \geq \widetilde{u}(\varrho)\lambda(\varrho).$$

Additionally, $\alpha'(\tau_1) = 0$, $\lambda'(\tau_1) = 0$,

$$\begin{aligned}
0 &= \Omega_1(\varphi(\tau_1), \varphi(\tau_2)) - \Omega_1\left(\eta_\gamma(\tau_1), \eta_\gamma(\tau_2)\right) - r_1\left(\eta_{\gamma+1}(\tau_1) - \eta_\gamma(\tau_1)\right) \\
&\leq r_1\left[\varphi(\tau_1) - \eta_\gamma(\tau_1)\right] - r_3\left[\varphi(\tau_2) - \eta_\gamma(\tau_2)\right] - r_1\left(\eta_{\gamma+1}(\tau_1) - \eta_\gamma(\tau_1)\right) \\
&= r_1\left[\varphi(\tau_1) - \eta_{\gamma+1}(\tau_1)\right] - r_3\left[\varphi(\tau_2) - \eta_\gamma(\tau_2)\right] \\
&= r_1\alpha(\tau_1) - r_3\left[\varphi(\tau_2) - \eta_\gamma(\tau_2)\right],
\end{aligned} \tag{4.8}$$

Analogously,

$$r_2\lambda(\tau_1) - r_4\left[\psi(\tau_2) - \beta_\gamma(\tau_2)\right] \geq 0. \tag{4.9}$$

From (4.8) and (4.9), we obtain that

$$\alpha(\tau_1) \geq \frac{r_3}{r_1}\left[\varphi(\tau_2) - \eta_\gamma(\tau_2)\right] \geq 0 \text{ and } \lambda(\tau_1) \geq \frac{r_4}{r_2}\left[\psi(\tau_2) - \beta_\gamma(\tau_2)\right] \geq 0.$$

According to Lemma 3.3, $\alpha(\varrho) \geq 0$ and $\lambda(\varrho) \geq 0$ for $\varrho \in J$, which means

$$\eta_{\gamma+1}(\varrho) \leq \varphi(\varrho) \text{ and } \beta_{\gamma+1}(\varrho) \leq \psi(\varrho), \ \varrho \in J.$$

In the same manner, one can write

$$\varphi(\varrho) \leq f_{\gamma+1}(\varrho) \text{ and } \psi(\varrho) \leq g_{\gamma+1}(\varrho), \ \varrho \in J.$$

Hence, we have

$$\eta_{\gamma+1}(\varrho) \leq \varphi(\varrho) \leq f_{\gamma+1}(\varrho) \text{ and } \beta_{\gamma+1}(\varrho) \leq \psi(\varrho) \leq g_{\gamma+1}(\varrho), \ \varrho \in J.$$

Therefore, for $\gamma \in \mathbb{N}$, (4.7) is true on J. Letting $\gamma \to \infty$ on (4.7), we have

$$\eta^* \leq \varphi \leq f^* \text{ and } \beta^* \leq \psi \leq g^*.$$

Consequently, (η^*, β^*) and (f^*, g^*) are the extremal solutions for problem (1.1) in $[\eta_0, f_0]$ and $[\beta_0, g_0]$.

□

5. Illustrative examples

We now proceed to analyze two illustrative examples, designed to provide empirical support for the theoretical framework developed in this study.

Example 5.1. *Consider the following BVP:*

$$\begin{cases} \left({}^{C}D^{\frac{5}{4}}_{\tau_1^+} + \frac{3}{\sqrt{2\pi}}\,{}^{C}D^{\frac{1}{4}}_{\tau_1^+}\right)\eta(\varrho) = 2\cos(\varrho)(\eta+\beta-2) + e^{-2\varrho}, \ \varrho \in J = [0,1], \\ \left({}^{C}D^{\frac{3}{2}}_{\tau_1^+} + \frac{2}{\sqrt{\pi}}\,{}^{C}D^{\frac{1}{2}}_{\tau_1^+}\right)\beta(\varrho) = 2\sin(\varrho)(\eta+\beta-4) + e^{-\varrho}, \\ \Omega_1(\eta(\tau_1), \eta(\tau_2)) = \frac{1}{4}\eta_{\tau_1} - \frac{1}{3}\eta_{\tau_2} + 1, \ \Omega_2(\beta(\tau_1), \beta(\tau_2)) = \frac{1}{3}\beta_{\tau_1} - \frac{1}{3}\beta_{\tau_2} - 3 \\ \eta(0) = \beta(0) = 1, \ \eta'(0) = \beta'(0) = 0. \end{cases} \tag{5.1}$$

This problem is a special case of BVP (1.1), with

$$\begin{cases} \mu_1 = \frac{3}{\sqrt{2\pi}}, \ \mu_2 = \frac{2}{\sqrt{\pi}}, \ \rho = \frac{1}{4}, \ \sigma = \frac{1}{2}, \ \delta(\varrho) = \varrho, \\ F(\varrho, \eta(\varrho), \beta(\varrho)) = 2\cos(\varrho)(\eta+\beta-2) + e^{-2\varrho}, \\ G(\varrho, \beta(\varrho), \eta(\varrho)) = 2\sin(\varrho)(\eta+\beta-4) + e^{-\varrho}, \end{cases}$$

Clearly, $F, G \in C([0,1]\times\mathbb{R}\times\mathbb{R}, \mathbb{R})$, and $\Omega_1, \Omega_2 \in C([0,1]\times\mathbb{R}, \mathbb{R})$. Taking $\eta_0(\varrho) = 1$, $\beta_0(\varrho) = 2$, $f_0(\varrho) = 1 + \varrho\sqrt{\varrho}$, and $g_0(\varrho) = 2 + \varrho\sqrt{\varrho}$. It can be easily verified that (η_0, β_0) and (f_0, g_0) are lower and upper solutions of (5.1), respectively, with $\eta_0 \leq f_0$ and $\beta_0 \leq g_0$. This fulfills the hypothesis (A_1).

Furthermore, for $\eta_0 \leq \eta \leq f \leq f_0$ and $\beta_0 \leq \beta \leq g \leq g_0$, we have

$$\begin{aligned} F(\varrho, f, g) - F(\varrho, \eta, \beta) &= 2\cos(\varrho)\left[(f + g - 2 - \eta - \beta + 2)\right] \\ &= 2\cos(\varrho)\left[(f-\eta) + (g-\beta)\right] \\ &\geq \cos(\varrho)\left[(f-\eta) + (g-\beta)\right] \end{aligned}$$

with $u(\varrho) = \cos(\varrho) \in C(J, \mathbb{R}_+)$ *, and*

$$\begin{aligned} G(\varrho, g, f) - G(\varrho, \beta, \eta) &= 2\sin(\varrho)\left[(f+g-4) - (\eta+\beta-4)\right] \\ &= 2\sin(\varrho)\left[(g-\beta) + (f-\eta)\right] \\ &\geq \sin(\varrho)\left[(g-\beta) + (f-\eta)\right]. \end{aligned}$$

with $\widetilde{u}(\varrho) = \sin(\varrho) \in C(J, \mathbb{R}_+)$ *. This fulfills the hypothesis (*A_2*).*

Moreover, if $\eta_0(\tau_1) \leq \varkappa_1 \leq \varkappa_2 \leq f_0(\tau_1)$, $\beta_0(\tau_1) \leq \varkappa_1 \leq \varkappa_2 \leq g_0(\tau_1)$, $\eta_0(\tau_2) \leq z_1 \leq z_2 \leq f_0(\tau_2)$, *and* $\beta_0(\tau_2) \leq z_1 \leq z_2 \leq g_0(\tau_2)$ *, we have*

$$\begin{aligned} \Omega_1(\varkappa_2, z_2) - \Omega_1(\varkappa_1, z_1) &= \left(\frac{1}{4}\varkappa_2 - \frac{1}{3}z_2 + 1 - \left(\frac{1}{4}\varkappa_1 - \frac{1}{3}z_1 + 1\right)\right) \\ &= \frac{1}{4}(\varkappa_2 - \varkappa_1) - \frac{1}{3}(z_2 - z_1) \\ &\leq \frac{1}{2}(\varkappa_2 - \varkappa_1) - \frac{1}{3}(z_2 - z_1) \\ &= r_1(\varkappa_2 - \varkappa_1) - r_3(z_2 - z_1), \end{aligned}$$

and

$$\begin{aligned} \Omega_2(\varkappa_2, z_2) - \Omega_2(\varkappa_1, z_1) &= \left(\frac{1}{3}\varkappa_2 - \frac{1}{3}z_2 - 3 - \left(\frac{1}{3}\varkappa_1 - \frac{1}{3}z_1 - 3\right)\right) \\ &= \frac{1}{3}(\varkappa_2 - \varkappa_1) - \frac{1}{3}(z_2 - z_1) \\ &\leq \frac{1}{2}(\varkappa_2 - \varkappa_1) - \frac{1}{3}(z_2 - z_1) \\ &= r_2(\varkappa_2 - \varkappa_1) - r_4(z_2 - z_1), \end{aligned}$$

which implies that the assertion (A_3*) holds with* $r_1 = r_2 = \frac{1}{2}$ *and* $r_3 = r_4 = \frac{1}{3}$.

Finally, through a straightforward computation, we obtain

$$\begin{aligned} \frac{(\delta(\tau_2) - \delta(\tau_1))^{\rho}}{\ell\mu_1\Gamma(\rho+1)}\|u\| &= \frac{(\tau_2 - \tau_1)^{\rho}}{\ell\mu_1\Gamma(\rho+1)}\|u\| = \frac{\sqrt{2\pi}\,(1-0)^{\frac{1}{4}}}{\frac{3}{2}\Gamma\left(\frac{5}{4}\right)}\left|\cos(\varrho)\right| \\ &\approx 0.4616\left|\cos(\varrho)\right| < 1, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \frac{(\delta(\tau_2) - \delta(\tau_1))^{\sigma}}{\ell\mu_2\Gamma(\sigma+1)}\|\widetilde{u}\| &= \frac{(\tau_2 - \tau_1)^{\sigma}}{\ell\mu_2\Gamma(\sigma+1)}\|\widetilde{u}\| = \frac{\sqrt{\pi}\,(1-0)^{\frac{1}{2}}}{2\Gamma\left(\frac{3}{2}\right)}\left|\sin(\varrho)\right| \\ &\approx 0.5001\left|\sin(\varrho)\right| < 1. \end{aligned} \tag{5.3}$$

It follows from (5.2) and (5.3) that the inequalities in (3.8) are satisfied with $\ell = \frac{1}{2}$*. Therefore, all retirements of Theorem 4.1 are fulfilled. Hence, Problem (5.1) possesses extremal solutions within* $[\eta_0, \beta_0]$ *and* $[f_0, g_0]$ *.*

For more analysis, we give the following example:

Example 5.2. *Consider the following BVP:*

$$\begin{cases} \left({}^{C}D^{\frac{4}{3}}_{\tau_1^+} + \frac{2}{\ln(1+\pi)}\,{}^{C}D^{\frac{1}{3}}_{\tau_1^+}\right)\eta(\varrho) = \tan^{-1}(\varrho)\,(2\eta+2\beta-1) + \ln(2-\varrho)\,,\ \varrho \in J = [0,1]\,, \\ \left({}^{C}D^{\frac{6}{5}}_{\tau_1^+} + \frac{3}{\sqrt{\pi}}\,{}^{C}D^{\frac{1}{5}}_{\tau_1^+}\right)\beta(\varrho) = \tanh(\varrho)\,(3\eta+3\beta-1) + \sqrt{2\varrho}, \\ \Omega_1(\eta(\tau_1),\eta(\tau_2)) = \frac{1}{6}\eta_{\tau_1} - \frac{1}{4}\eta_{\tau_2} + 2,\ \Omega_2(\beta(\tau_1),\beta(\tau_2)) = \frac{1}{4}\beta_{\tau_1} - \frac{1}{4}\beta_{\tau_2} - 3 \\ \eta(0) = \beta(0) = 1,\ \eta'(0) = \beta'(0) = 0, \end{cases} \tag{5.4}$$

It is clear that the BVP (5.4) represents a particular case of the BVP (1.1), with

$$\begin{cases} \mu_1 = \frac{2}{\ln(1+\pi)},\ \mu_2 = \frac{3}{\sqrt{\pi}},\ \rho = \frac{1}{3},\ \sigma = \frac{1}{5},\ \delta(\varrho) = \varrho, \\ F(\varrho,\eta(\varrho),\beta(\varrho)) = \tan^{-1}(\varrho)\,(2\eta+2\beta-1) + \ln(2-\varrho)\,, \\ G(\varrho,\beta(\varrho),\eta(\varrho)) = \tanh(\varrho)\,(3\eta+3\beta-1) + \sqrt{2\varrho}. \end{cases}$$

Obliviously, $F,G \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, and $\Omega_1,\Omega_2 \in C([0,1]\times\mathbb{R},\mathbb{R})$. Taking $\eta_0(\varrho) = 1+\frac{\varrho}{2}, \beta_0(\varrho) = e^{\varrho}$, $f_0(\varrho) = 1+2\varrho$, and $g_0(\varrho) = 1+e^{\varrho}$. It is straightforward to verify that (η_0,β_0) and (f_0,g_0) are lower and upper solutions of (5.4), respectively, with $\eta_0 \le f_0$ and $\beta_0 \le g_0$. This demonstrates the fulfillment of hypothesis (A_1).

Additionally, for $\eta_0 \le \eta \le f \le f_0$ and $\beta_0 \le \beta \le g \le g_0$, we have

$$\begin{aligned} F(\varrho,f,g) - F(\varrho,\eta,\beta) &= \tan^{-1}(\varrho)\,[2f+2g-1-(2\eta+2\beta-1)] \\ &= 2\tan^{-1}(\varrho)\,[(f-\eta)+(g-\beta)] \\ &\ge \tan^{-1}(\varrho)\,[(f-\eta)+(g-\beta)] \end{aligned}$$

with $u(\varrho) = \tan^{-1}(\varrho) \in C(J,\mathbb{R}_+)$, and

$$\begin{aligned} G(\varrho,g,f) - G(\varrho,\beta,\eta) &= \tanh(\varrho)\,[(3f+3g-1)-(3\eta+3\beta-1)] + 2e^{-\varrho} \\ &= 3\tanh(\varrho)\,[(g-\beta)+(f-\eta)] \\ &\ge \tanh(\varrho)\,[(g-\beta)+(f-\eta)] \end{aligned}$$

with $\widetilde{u}(\varrho) = \tanh(\varrho) \in C(J,\mathbb{R}_+)$. This confirms that the hypothesis (A_2) is true.

Furthermore, if $\eta_0(\tau_1) \le \varkappa_1 \le \varkappa_2 \le f_0(\tau_1)$, $\beta_0(\tau_1) \le \varkappa_1 \le \varkappa_2 \le g_0(\tau_1)$, $\eta_0(\tau_2) \le z_1 \le z_2 \le f_0(\tau_2)$, and $\beta_0(\tau_2) \le z_1 \le z_2 \le g_0(\tau_2)$, we have

$$\begin{aligned} \Omega_1(\varkappa_2,z_2) - \Omega_1(\varkappa_1,z_1) &= \left(\left(\frac{1}{6}\varkappa_2 - \frac{1}{4}z_2 + 2\right) - \left(\frac{1}{6}\varkappa_1 - \frac{1}{4}z_1 + 2\right)\right) \\ &= \frac{1}{6}(\varkappa_2-\varkappa_1) - \frac{1}{4}(z_2-z_1) \\ &\le \frac{1}{3}(\varkappa_2-\varkappa_1) - \frac{1}{4}(z_2-z_1) \\ &= r_1(\varkappa_2-\varkappa_1) - r_3(z_2-z_1)\,, \end{aligned}$$

and

$$\begin{aligned}\Omega_2(\varkappa_2, z_2) - \Omega_2(\varkappa_1, z_1) &= \left(\left(\frac{1}{4}\varkappa_2 - \frac{1}{4}z_2 - 3\right) - \left(\frac{1}{4}\varkappa_1 - \frac{1}{4}z_1 - 3\right)\right) \\ &= \frac{1}{4}(\varkappa_2 - \varkappa_1) - \frac{1}{4}(z_2 - z_1) \\ &\leq \frac{1}{3}(\varkappa_2 - \varkappa_1) - \frac{1}{4}(z_2 - z_1) \\ &= r_2(\varkappa_2 - \varkappa_1) - r_4(z_2 - z_1),\end{aligned}$$

which yields, the assertion (A_3) is true with $r_1 = r_2 = \frac{1}{3}$ and $r_3 = r_4 = \frac{1}{4}$.

Ultimately, by simple computations, for $\varrho \in [0, 1]$, we have

$$\begin{aligned}\frac{(\delta(\tau_2) - \delta(\tau_1))^{\rho}}{\ell\mu_1\Gamma(\rho+1)}\|u\| &= \frac{(\tau_2 - \tau_1)^{\rho}}{\ell\mu_1\Gamma(\rho+1)}\|u\| = \frac{\ln(1+\pi)(1-0)^{\frac{1}{3}}}{2\ell\Gamma\left(\frac{4}{3}\right)}\left|\tan^{-1}(\varrho)\right| \\ &\approx \frac{\ln(1+\pi)}{2(0.8925)} \approx 0.7961\left|\tan^{-1}(\varrho)\right| < 1, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\frac{(\delta(\tau_2) - \delta(\tau_1))^{\sigma}}{\ell\mu_2\Gamma(\sigma+1)}\|\widetilde{u}\| &= \frac{(\tau_2 - \tau_1)^{\sigma}}{\ell\mu_2\Gamma(\sigma+1)}\|\widetilde{u}\| = \frac{\sqrt{\pi}(1-0)^{\frac{1}{5}}}{3\ell\Gamma\left(\frac{6}{5}\right)}\left|\tanh(\varrho)\right| \\ &\approx \frac{\sqrt{\pi}}{3(0.9171)} \approx 0.6442\left|\tanh(\varrho)\right| < 1. \end{aligned} \tag{5.6}$$

From (5.5) and (5.6), we find that (3.8) is fulfilled with $\ell = 1$. Therefore, all stipulations of Theorem 4.1 are verified. Consequently, extremal solutions exist for Problem (5.4) on $[\eta_0, \beta_0]$ and $[f_0, g_0]$.

6. Conclusion

The upper and lower solution method plays a crucial role in establishing sufficient conditions for the existence of extremal solutions to boundary value problems. By constructing appropriate upper and lower solutions, which serve as bounds for potential solutions, we can effectively define a region within which solutions must reside. This method allows us to bypass the need for explicit solutions, focusing instead on the qualitative behavior of the problem. The existence of these bounding functions, combined with specific properties of the differential operator, provides a powerful framework for proving the existence of extremal solutions, which are the minimal and maximal solutions within the defined region. Consequently, the method offers a robust and versatile approach to analyzing the solvability of nonlinear BVPs, particularly when traditional analytical techniques prove challenging. In this work, we present a theoretical analysis of extremal solutions for a coupled sequential $\delta-$Caputo fractional differential system, subject to nonlinear boundary conditions. The existence of these solutions is rigorously demonstrated through functional analysis, employing the monotone iterative technique and the method of upper and lower solutions. We provide sufficient conditions for the existence of minimal and maximal solutions, and conclude with illustrative examples to validate our findings.

7. Abbreviations

DE→differential equation.
RL→Riemann-Liouville.
FP→fixed-point.
BVP→boundary value problem.
w.r.t.→with respect to.

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