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Analysis of Coupled Sequential Systems of Caputo Fractional Differential Equations under Two-Point Integral Boundary Constraints

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Abstract. In this paper, we investigate a coupled system of sequential Caputo fractional differential equations subject to two-point integral boundary conditions. Such systems arise naturally in the mathematical modeling of various phenomena in physics, biology, and engineering. By employing fixed point techniques based on the Banach contraction principle, sufficient conditions for the existence and uniqueness of solutions are established. Furthermore, we analyze the Hyers–Ulam stability of the proposed system and derive explicit criteria ensuring that small perturbations in the initial data lead to correspondingly small deviations in the solutions. To illustrate the applicability of the theoretical results, numerical examples are provided, which confirm the effectiveness and validity of the obtained analysis.

1. Introduction

Sequential Caputo fractional differential equations refer to systems in which derivatives of fractional order are applied successively, typically involving Caputo's definition due to its compatibility with initial conditions of classical type. These equations are crucial in modeling complex dynamical systems that exhibit memory and hereditary properties, such as viscoelastic materials, diffusion processes, and biological systems. The sequential aspect enables the capture of layered temporal effects by combining multiple fractional orders, often leading to more accurate and flexible representations of real-world phenomena. Unlike classical integer-order models, sequential Caputo systems offer refined control over system dynamics, making them valuable for theoretical analysis and applications where multi-scale temporal behavior is significant [1–6].

Coupled systems consist of two or more interdependent differential equations in which the solution of one equation directly influences the others. These systems naturally arise in modeling phenomena involving multiple interacting components, such as in mechanical structures, electrical

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circuits, or biological networks. In the context of fractional calculus, coupling adds complexity by integrating memory effects across interconnected variables. Studying such systems is essential for understanding the joint dynamics of processes in which the isolation of individual components fails to capture the full behavior of the system [7–13].

Ulam-Hyers stability is a concept in the qualitative theory of differential equations that addresses the stability of approximate solutions. Specifically, it ensures that if a function nearly satisfies a given differential equation, then there exists an exact solution nearby. This type of stability is particularly important in applied mathematics, where exact solutions are often inaccessible, and numerical or approximate methods are used. In the context of fractional differential equations, Ulam-Hyers stability provides a robust framework for assessing the reliability and resilience of solutions under small perturbations or modeling inaccuracies [14–17]. Let us now recall some works on closed boundary conditions. In [19], Bashir Ahmad studied mixed nonlinearities with nonlocal closed boundary conditions.

$$ {}^{C}\mathcal{D}^{\mu}_{\mathscr{T}^-}\vartheta_1(\varsigma) + \lambda I^{\rho}_{\mathscr{T}^-} I^{\sigma}_{0^+}\mathcal{K}_1(\varsigma, \vartheta_1(\varsigma)) = \mathcal{K}_2(\varsigma, \vartheta_1(\varsigma)), \quad \varsigma \in J := [0, \mathscr{T}], \tag{1.1} $$

$$ \vartheta_1(\varsigma) = p_1\vartheta_1(\xi) + p_2\mathscr{T}\vartheta_1'(\xi), \quad \mathscr{T}\vartheta_1'(\xi) = q_1\vartheta_1(\xi) + q_2\mathscr{T}\vartheta_1'(\xi), \quad 0 < \xi < \mathscr{T}, \tag{1.2} $$

here, ${}^{C}D^{\mu}_{\mathscr{T}^-}$ denotes the CFD of order $\mu \in (1, 2]$, while $I^{\rho}_{\mathscr{T}^-}$ and $I^{\sigma}_{0^+}$ stand for the right- and left-sided RL fractional integral operators of orders $\rho > 0$ and $\sigma > 0$, respectively. The mappings $\varphi_1, \varphi_2 : [0, \mathscr{T}] \times \mathbb{R} \to \mathcal{R}$ are assumed to be continuous, and $\lambda, p_1, p_2, q_1, q_2 \in \mathcal{R}$ are given constants. Moreover, the authors in [20] recently established existence and uniqueness results for a nonlinear fractional differential system subject to coupled closed boundary conditions, stated as follows:

$$ \begin{cases} ({}^{c}\mathcal{D}^{\varpi_1})\vartheta_1(\varsigma) = \mathcal{K}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)), \\ ({}^{c}\mathcal{D}^{\varrho_1})\Psi_1(\varsigma) = \mathcal{L}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)), \end{cases} $$

with closed boundary conditions;

$$ \begin{cases} \vartheta_1\mathscr{T} = \mathfrak{r}_1\Psi_1(0) + \mathfrak{r}_2\mathscr{T}\Psi_1'(0), & \mathscr{T}\vartheta_1'\mathscr{T} = \mathfrak{s}_1\Psi_1(0) + \mathfrak{s}_2\mathscr{T}\Psi_1'(0), \\ \Psi_1\mathscr{T} = \mathfrak{p}_1\vartheta_1(0) + \mathfrak{p}_2\mathscr{T}\vartheta_1'(0), & \mathscr{T}\Psi_1'\mathscr{T} = \mathfrak{q}_1\vartheta_1(0) + \mathfrak{q}_2\mathscr{T}\vartheta_1'(0), \end{cases} $$

where ${}^{c}\mathcal{D}^{\varpi_1}$ and ${}^{c}\mathcal{D}^{\varrho_1}$ CFD of order $\varpi_1, \varrho_1, 1 < \varpi_1, \varrho_1 < 2$, respectively, $\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2, \in \mathcal{R}, \mathscr{T} > 0$ and $\mathcal{K}, \mathcal{L} \in C(\mathcal{J} \times \mathcal{R} \times \mathcal{R} \times \mathcal{R})$. Motivated by the aforementioned study, we propose in this paper a nonlocal extension of closed boundary conditions and investigate a coupled sequential fractional differential system governed by a CFD. More precisely, we establish sufficient conditions for the existence and stability of solutions to the problem stated below.

We consider the following coupled system of nonlinear fractional differential equations subject to two-point integral-type coupled boundary conditions:

$$\begin{cases}({}^{C}D^{\varpi_1}+K_1{}^{C}D^{\varpi_1-1})\vartheta_1(\varsigma)=\mathfrak{U}(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma))\,, & \varsigma\in[0,\mathscr{T}], \quad 1<\varpi_1\leq 2,\\ ({}^{C}D^{\varrho_1}+K_2{}^{C}D^{\varrho_1-1})\Psi_1(\varsigma)=\mathfrak{H}(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma))\,, & \varsigma\in[0,\mathscr{T}], \quad 1<\varrho_1\leq 2,\end{cases} \tag{1.3}$$

where $\mathscr{T}>0$, and the system is equipped with the integral coupled boundary conditions of the form:

$$\begin{cases}\displaystyle\int_0^{\mathscr{T}}\vartheta_1{}'(\varsigma)\,d\varsigma=\zeta(\Psi_1{}'(\mu))\,, \quad \mu\in[0,\mathscr{T}],\\ \displaystyle\int_0^{\mathscr{T}}\Psi_1{}'(\varsigma)\,d\varsigma=\eta(\vartheta_1{}'(\rho))\,, \quad \rho\in[0,\mathscr{T}],\\ \vartheta_1(0)=0, \qquad \Psi_1(0)=0.\end{cases} \tag{1.4}$$

where ${}^{C}D^{i},{}^{C}D^{i-1}, i=\varpi_1,\varrho_1$ denotes the SCFD of orders $1<\varpi_1,\varrho_1<2$, $\mathfrak{U},\mathfrak{H}\in C([0,\mathscr{T}]\times\mathbb{R}^2,\mathbb{R})$ are continuous functions, and $\zeta,\eta\in\mathbb{R}$.

The remainder of this article is arranged as follows. In Section 2, we recall the fundamental concepts and auxiliary results required for the subsequent analysis. Section 3 is devoted to deriving an equivalent integral representation of the considered problem, which serves as the main tool for establishing existence and uniqueness of solutions. The Hyers–Ulam stability of the system is investigated in Section 4. Section 5 illustrates the theoretical findings through two representative examples. Finally, Section 6 summarizes the main conclusions and possible future directions.

2. Auxiliary Results

This section collects the definitions and preliminary lemmas used in the sequel.

Definition 2.1. *The RL fractional integral of order $\varpi_1>0$ is given by*

$$\mathcal{I}^{\varpi_1}\vartheta_1(\varsigma)=\frac{1}{\Gamma(\varpi_1)}\int_0^{\varsigma}(\varsigma-\mathfrak{s})^{\varpi_1-1}\vartheta_1 ds,\ \varsigma>0,\ \varpi_1>0,$$

provided that the right-hand side is point wise defined on (0,∞), where Γ is the gamma function, defined by $\Gamma(\mathfrak{p})=\int_0^{\infty}\varsigma^{\mathfrak{p}-1}e^{-\varsigma}d\varsigma$.

Definition 2.2. *For an $(n-1)$ times absolutely continuous function $\vartheta_1:[0,\infty)\to\mathcal{R}$, the Caputo derivative of fractional order ϖ_1 is defined as*

$${}^{c}\mathcal{D}^{\varpi_1}\vartheta_1(\varsigma)=\frac{1}{\Gamma(n-\varpi_1)}\int_0^{\varsigma}(\varsigma-\mathfrak{s})^{n-\varpi_1-1}\vartheta_1{}^{(n)}(\mathfrak{s})d\mathfrak{s},\ \varsigma>0, n-1<\varpi_1<n,\ n\in\mathcal{N}.$$

Lemma 2.1. *For $\mathcal{H}_1,\mathcal{H}_2\in C([0,\mathscr{T}],\mathcal{R})$, the solution of the problem is unique.*

$$\begin{cases}({}^{c}\mathcal{D}^{\varpi_1}+\lambda_1{}^{c}\mathcal{D}^{\varpi_1-1})\vartheta_1(\varsigma)=\mathcal{H}_1(\varsigma),\\ ({}^{c}\mathcal{D}^{\varrho_1}+\lambda_2{}^{c}\mathcal{D}^{\varrho_1-1})\Psi_1(\varsigma)=\mathcal{H}_2(\varsigma),\end{cases} \tag{2.1}$$

with boundary conditions.

$$\begin{cases} \int_0^{\mathscr{T}} \vartheta_1{}'(\varsigma)\, d\varsigma = \zeta(\Psi_1{}'(\mu)), \quad \mu \in [0, \mathscr{T}], \\ \int_0^{\mathscr{T}} \Psi_1{}'(\varsigma)\, d\varsigma = \eta(\vartheta_1{}'(\rho)), \quad \rho \in [0, \mathscr{T}], \\ \vartheta_1(0) = 0, \qquad \Psi_1(0) = 0. \end{cases} \tag{2.2}$$

$$\begin{aligned} \vartheta_1(\varsigma) =& \frac{(1 - e^{-\lambda_1 \varsigma})}{\Delta}\Bigg[\mathcal{A}_2\Big(- \eta\lambda_1 \int_0^{\rho} e^{-\lambda_1(\rho - \mathfrak{s})} \mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\mathfrak{s}) d\mathfrak{s} + \eta \mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\rho) \\ &+ \int_0^{\mathscr{T}} \Big[\lambda_2 \int_0^{\varsigma} e^{-\lambda_2(\varsigma - \mathfrak{s})} \mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\mathfrak{u}) d\mathfrak{u}\Big] d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\varsigma)\Big] d\varsigma\Big) \\ &+ \mathcal{B}_2\Big(- \zeta\lambda_2 \int_0^{\mu} e^{-\lambda_2(\mu - \mathfrak{s})} \mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\mathfrak{s}) d\mathfrak{s} + \zeta \mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\mu) \\ &+ \int_0^{\mathscr{T}} \Big[\lambda_1 \int_0^{\varsigma} e^{-\lambda_1(\varsigma - \mathfrak{s})} \mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\mathfrak{u}) d\mathfrak{u}\Big] d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\varsigma)\Big] d\varsigma\Big)\Bigg] \\ &+ \int_0^{\varsigma} e^{-\lambda_1(\varsigma - \mathfrak{s})} \mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\mathfrak{s}) d\mathfrak{s}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \Psi_1(\varsigma) =& \frac{(1 - e^{-\lambda_2 \varsigma})}{\Delta}\Bigg[\mathcal{B}_1\Big(- \zeta\lambda_2 \int_0^{\mu} e^{-\lambda_2(\mu - \mathfrak{s})} \mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\mathfrak{s}) d\mathfrak{s} + \zeta \mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\mu) \\ &+ \int_0^{\mathscr{T}} \Big[\lambda_1 \int_0^{\varsigma} e^{-\lambda_1(\varsigma - \mathfrak{s})} \mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\mathfrak{u}) d\mathfrak{u}\Big] d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\varsigma)\Big] d\varsigma\Big) \\ &+ \mathcal{A}_1\Big(- \eta\lambda_1 \int_0^{\rho} e^{-\lambda_1(\rho - \mathfrak{s})} \mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\mathfrak{s}) d\mathfrak{s} + \eta \mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\rho) \\ &+ \int_0^{\mathscr{T}} \Big[\lambda_2 \int_0^{\varsigma} e^{-\lambda_2(\varsigma - \mathfrak{s})} \mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\mathfrak{u}) d\mathfrak{u}\Big] d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\varsigma)\Big] d\varsigma\Big)\Bigg] \\ &+ \int_0^{\varsigma} e^{-\lambda_2(\varsigma - \mathfrak{s})} \mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\mathfrak{s}) d\mathfrak{s}, \end{aligned} \tag{2.4}$$

$$\begin{cases} \mathcal{I}_1 = & -\zeta\lambda_2 \int_0^{\mu} e^{-\lambda_2(\mu - \mathfrak{s})} \mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\mathfrak{s}) d\mathfrak{s} + \zeta \mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\mu) \\ & + \int_0^{\mathscr{T}} \Big[\lambda_1 \int_0^{\varsigma} e^{-\lambda_1(\varsigma - \mathfrak{s})} \mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\mathfrak{u}) d\mathfrak{u}\Big] d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\varsigma)\Big] d\varsigma, \\ \mathcal{I}_2 = & -\eta\lambda_1 \int_0^{\rho} e^{-\lambda_1(\rho - \mathfrak{s})} \mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\mathfrak{s}) d\mathfrak{s} + \eta \mathcal{I}^{\varpi_1 - 1}\mathcal{H}_1(\rho) \\ & + \int_0^{\mathscr{T}} \Big[\lambda_2 \int_0^{\varsigma} e^{-\lambda_2(\varsigma - \mathfrak{s})} \mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\mathfrak{u}) d\mathfrak{u}\Big] d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varrho_1 - 1}\mathcal{H}_2(\varsigma)\Big] d\varsigma, \\ & \mathcal{A}_1 = \int_0^{\mathscr{T}} e^{-\lambda_1 \varsigma} d\varsigma \\ & \mathcal{A}_2 = \zeta e^{-\lambda_1 \mu} \\ & \mathcal{B}_1 = \eta e^{-\lambda \rho} \\ & \mathcal{B}_2 = \int_0^{\mathscr{T}} e^{-\lambda_2 \varsigma} d\varsigma \\ \Delta = & \Big(- \mathcal{B}_1\mathcal{A}_2 + \mathcal{A}_1\mathcal{B}_2\Big). \end{cases} \tag{2.5}$$

Proof. The system (2.1) can be reformulated equivalently as the following system:

$$\begin{cases} ({}^c\mathcal{D}^{\varpi_1} + \lambda_1 {}^c\mathcal{D}^{\varpi_1-1})\vartheta_1(\varsigma) = \mathcal{H}_1(\varsigma), \\ ({}^c\mathcal{D}^{\varrho_1} + \lambda_2 {}^c\mathcal{D}^{\varrho_1-1})\Psi_1(\varsigma) = \mathcal{H}_2(\varsigma), \end{cases} \tag{2.6}$$

where ${}^c\mathcal{D}^{-1}$ represents the operator of fractional integration, denoted by $\mathcal{I}$. Accordingly, the general solution of (2.6) takes the form

$$\begin{cases} \vartheta_1(\varsigma) = \mathfrak{c}_0 e^{-\lambda_1\varsigma} + \frac{\mathfrak{c}_1}{\lambda_1}(1-e^{-\lambda_1\varsigma}) + \int_0^\varsigma e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}\mathcal{H}_1(\mathfrak{s})d\mathfrak{s}, \\ \Psi_1(\varsigma) = \mathfrak{d}_0 e^{-\lambda_2\varsigma} + \frac{\mathfrak{d}_1}{\lambda_2}(1-e^{-\lambda_2\varsigma}) + \int_0^\varsigma e^{-\lambda_2(\varsigma-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}\mathcal{H}_2(\mathfrak{s})d\mathfrak{s}, \end{cases} \tag{2.7}$$

Taking the derivative of system (2.7) yields

$$\begin{cases} \vartheta_1{}'(\varsigma) = \mathfrak{c}_1 e^{-\lambda_1\varsigma} - \lambda_1\int_0^\varsigma e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}\mathcal{H}_1(\mathfrak{s})d\mathfrak{s} + \mathcal{I}^{\varpi_1-1}\mathcal{H}_1(\varsigma), \\ \Psi_1{}'(\varsigma) = \mathfrak{d}_1 e^{-\lambda_2\varsigma} - \lambda_2\int_0^\varsigma e^{-\lambda_2(\varsigma-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}\mathcal{H}_2(\mathfrak{s})d\mathfrak{s} + \mathcal{I}^{\varrho_1-1}\mathcal{H}_2(\varsigma), \end{cases} \tag{2.8}$$

where $\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{d}_0$, and $\mathfrak{d}_1$ are unknown constants. Substituting (2.8) into the nonlocal closed boundary conditions

$$\int_0^{\mathscr{T}} \vartheta_1'(\varsigma), d\varsigma = \zeta\Big(\Psi_1'(\mu)\Big), \qquad \int_0^{\mathscr{T}} \Psi_1'(\varsigma), d\varsigma = \eta\Big(\vartheta_1'(\rho)\Big), \qquad \vartheta_1(0) = 0,\ \Psi_1(0) = 0,$$

we obtain a system of algebraic equations involving the constants $\mathfrak{c}_1$ and $\mathfrak{d}_1$:

$$\mathfrak{c}_1\mathcal{A}_1 - \mathfrak{d}_1\mathcal{A}_2 = \mathcal{I}_1,$$
$$-\mathfrak{c}_1\mathcal{B}_1 + \mathfrak{d}_1\mathcal{B}_2 = \mathcal{I}_2,$$

$$\begin{cases} \mathfrak{c}_1 = \frac{1}{\Delta}\Big(\mathcal{I}_2\mathcal{A}_2 + \mathcal{I}_1\mathcal{B}_2\Big) \\ \mathfrak{d}_1 = \frac{1}{\Delta}\Big(\mathcal{I}_1\mathcal{B}_1 + \mathcal{I}_2\mathcal{A}_1\Big). \end{cases} \tag{2.9}$$

Replacing $\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{d}_0$, and $\mathfrak{d}_1$ in (2.7) produces the solution, which completes the proof. □

3. Main Results

Define the space $\chi_1 = \vartheta_1 : [0, \mathscr{T}] \to \mathcal{R}, : \vartheta_1$ is continuous, equipped with the supremum norm

$$|\vartheta_1| = \sup_{\varsigma\in[0,\mathscr{T}]} |\vartheta_1(\varsigma)|.$$

Then $(\chi_1, |\cdot|)$ is a Banach space. Similarly, let

$$\chi_2 = \Psi_1 : [0, \mathscr{T}] \to \mathcal{R}, :, \Psi_1 \text{ is continuous},$$

with norm

$$|\Psi_1| = \sup_{\varsigma\in[0,\mathscr{T}]} |\Psi_1(\varsigma)|.$$

The product space $(\chi_1 \times \chi_2)$, endowed with the norm

$$|(\vartheta_1, \Psi_1)| = |\vartheta_1| + |\Psi_1|,$$

is also a Banach space.

In view of Lemma 2.1, we now introduce the operator $(\mathscr{T} : \chi_1 \times \chi_2 \to \chi_1 \times \chi_2)$ defined by

$$\mathscr{T}(\vartheta_1, \Psi_1)(\varsigma) = \begin{bmatrix} \mathscr{T}_1(\vartheta_1, \Psi_1)(\varsigma) \\ \mathscr{T}_2(\vartheta_1, \Psi_1)(\varsigma) \end{bmatrix},$$

where

$$\begin{aligned}
\mathscr{T}_1(\vartheta_1, \Psi_1)(\varsigma) = \frac{(1-e^{-\lambda_1 \varsigma})}{\Delta}\Bigg[&\mathcal{A}_2\Big(-\eta\lambda_1 \int_0^{\rho} e^{-\lambda_1(\rho-\mathfrak{s})} \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)d\mathfrak{s} + \eta \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)\Big)(\rho) \\
&+ \int_0^{\mathscr{T}} \Big[\lambda_2 \int_0^{\varsigma} e^{-\lambda_2(\varsigma-\mathfrak{s})} \mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)d\mathfrak{u}\Big]d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)\Big)(\varsigma)\Big]d\varsigma\Big) \\
&+ \mathcal{B}_2\Big(-\zeta\lambda_2 \int_0^{\mu} e^{-\lambda_2(\mu-\mathfrak{s})} \mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)d\mathfrak{s} + \zeta \mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)\Big)(\mu) \\
&+ \int_0^{\mathscr{T}} \Big[\lambda_1 \int_0^{\varsigma} e^{-\lambda_1(\varsigma-\mathfrak{s})} \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)d\mathfrak{u}\Big]d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)\Big)(\varsigma)\Big]d\varsigma\Big)\Bigg] \\
&+ \int_0^{\varsigma} e^{-\lambda_1(\varsigma-\mathfrak{s})} \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)d\mathfrak{s}, \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
\mathscr{T}_2(\vartheta_1, \Psi_1)(\varsigma) = \frac{(1-e^{-\lambda_2 \varsigma})}{\Delta}\Bigg[&\mathcal{B}_1\Big(-\zeta\lambda_2 \int_0^{\mu} e^{-\lambda_2(\mu-\mathfrak{s})} \mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)d\mathfrak{s} + \zeta \mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)\Big)(\mu) \\
&+ \int_0^{\mathscr{T}} \Big[\lambda_1 \int_0^{\varsigma} e^{-\lambda_1(\varsigma-\mathfrak{s})} \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)d\mathfrak{u}\Big]d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)\Big)\Big]d\varsigma\Big) \\
&+ \mathcal{A}_1\Big(-\eta\lambda_1 \int_0^{\rho} e^{-\lambda_1(\rho-\mathfrak{s})} \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)d\mathfrak{s} + \eta \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)\Big)(\rho) \\
&+ \int_0^{\mathscr{T}} \Big[\lambda_2 \int_0^{\varsigma} e^{-\lambda_2(\varsigma-\mathfrak{s})} \mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)d\mathfrak{u}\Big]d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)\Big)\Big]d\varsigma\Big)\Bigg] \\
&+ \int_0^{\varsigma} e^{-\lambda_2(\varsigma-\mathfrak{s})} \mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)d\mathfrak{s}. \qquad (3.2)
\end{aligned}$$

We need the following hypotheses in the sequel:

- $(\mathscr{H}_1)$ Assume that there exist real constant $\varkappa_i, \mathfrak{n}_i > 0, i = 1, 2$ and $\varkappa_0 > 0, \mathfrak{n}_0 > 0$ such that, for all $\varsigma \in [0, \mathscr{T}], \mathcal{U}_i \in \mathcal{R}, i = 1, 2,$

$$|\mathfrak{U}(\varsigma, \mathcal{U}_1, \mathcal{U}_2)| \leq \varkappa_0 + \varkappa_1|\mathcal{U}_1| + \varkappa_2|\mathcal{U}_2|,$$
$$|\mathfrak{H}(\varsigma, \mathcal{U}_1, \mathcal{U}_2)| \leq \mathfrak{n}_0 + \mathfrak{n}_1|\mathcal{U}_1| + \mathfrak{n}_2|\mathcal{U}_2|.$$

- $(\mathscr{H}_2)$ Let $(\mathfrak{U}, \mathfrak{H} : \mathcal{R}^2 \to \mathcal{R}$ be continuous mappings. Suppose that there exist positive constants $(\mathscr{W}_i)$ and $(\mathcal{N}_i)$ for (i=1,2) such that, for every $(\varsigma \in [0, \mathscr{T}])$ and for all $(\vartheta_{1i}, \Psi_{1i})((i = 1, 2)),$

$$|\mathfrak{U}(\varsigma, \vartheta_{11}, \vartheta_{12}) - \mathfrak{U}(\varsigma, \Psi_{11}, \Psi_{12})| \leq \mathscr{W}_1|\vartheta_{11} - \Psi_{11}| + \mathscr{W}_2|\vartheta_{12} - \Psi_{12}|$$
$$|\mathfrak{H}(\varsigma, \vartheta_{11}, \vartheta_{12}) - \mathfrak{H}(\varsigma, \Psi_{11}, \Psi_{12})| \leq \mathcal{N}_1|\vartheta_{11} - \Psi_{11}| + \mathcal{N}_2|\vartheta_{12} - \Psi_{12}|.$$

For convenience, we set

$$\begin{aligned}\mathfrak{Z}_1 =& \frac{1-e^{-\lambda_1\varsigma}}{\lambda_1\Delta}\Bigg\{\mathcal{A}_2\Bigg[-\eta\lambda_1\Bigg(\frac{\rho^{\varpi_1-1}}{\Gamma(\varpi_1)}\frac{(1-e^{-\lambda_1\rho})}{\lambda_1}\Bigg)+\eta\frac{\rho^{\varpi_1-1}}{\Gamma(\varpi_1)}\Bigg]+\mathcal{B}_2\Bigg[\lambda_1\Bigg(\frac{\mathscr{T}^{\varpi_1}}{\Gamma(\varpi_1+1)}\frac{(1-e^{-\lambda_1\mathscr{T}})}{\lambda_1}\Bigg)\\ &+\Bigg(\frac{\mathscr{T}^{\varpi_1}}{\Gamma(\varpi_1+1)}\Bigg)\Bigg]\Bigg\}+\Bigg(\frac{\varsigma^{\varpi_1-1}}{\Gamma(\varpi_1)}\frac{(1-e^{-\lambda_1\varsigma})}{\lambda_1}\Bigg). \qquad (3.3)\end{aligned}$$

$$\begin{aligned}\mathfrak{Z}_2 =& \frac{1-e^{-\lambda_1\varsigma}}{\lambda_1\Delta}\Bigg\{\mathcal{A}_2\Bigg[\lambda_2\Bigg(\frac{\mathscr{T}^{\varrho_1}}{\Gamma(\varrho_1+1)}\frac{(1-e^{-\lambda_2\mathscr{T}})}{\lambda_2}\Bigg)+\Bigg(\frac{\mathscr{T}^{\varrho_1}}{\Gamma(\varrho_1+1)}\Bigg)\Bigg]\\ &-\mathcal{B}_2\Bigg[\zeta\lambda_2\Bigg(\frac{\mu^{\varrho_1-1}}{\Gamma(\varrho_1)}\frac{(1-e^{-\lambda_2\mu})}{\lambda_2}\Bigg)+\zeta\Bigg(\frac{\mu^{\varrho_1-1}}{\Gamma(\varrho_1)}\Bigg)\Bigg]\Bigg\} \qquad (3.4)\end{aligned}$$

$$\begin{aligned}\hat{\mathfrak{Z}}_1 =& \frac{1-e^{-\lambda_2\varsigma}}{\lambda_2\Delta}\Bigg\{\mathcal{B}_1\Bigg[\lambda_1\Bigg(\frac{\mathscr{T}^{\varpi_1}}{\Gamma(\varpi_1+1)}\frac{(1-e^{-\lambda_1\mathscr{T}})}{\lambda_1}\Bigg)+\Bigg(\frac{\mathscr{T}^{\varpi_1}}{\Gamma(\varpi_1+1)}\Bigg)\Bigg]\\ &+\mathcal{A}_1\Bigg[-\eta\lambda_1\Bigg(\frac{\rho^{\varpi_1-1}}{\Gamma(\varpi_1)}\frac{(1-e^{-\lambda_1\rho})}{\lambda_1}\Bigg)+\eta\frac{\rho^{\varpi_1-1}}{\Gamma(\varpi_1)}\Bigg] \qquad (3.5)\end{aligned}$$

$$\begin{aligned}\hat{\mathfrak{Z}}_2 =& \frac{1-e^{-\lambda_2\varsigma}}{\lambda_2\Delta}\Bigg\{\mathcal{B}_1\Bigg[\zeta\lambda_2\Bigg(\frac{\mu^{\varrho_1-1}}{\Gamma(\varrho_1)}\frac{(1-e^{-\lambda_2\mu})}{\lambda_2}\Bigg)+\zeta\Bigg(\frac{\mu^{\varrho_1-1}}{\Gamma(\varrho_1)}\Bigg)\Bigg]\\ &+\mathcal{A}_1\Bigg[\lambda_2\Bigg(\frac{\mathscr{T}^{\varrho_1}}{\Gamma(\varrho_1+1)}\frac{(1-e^{-\lambda_2\mathscr{T}})}{\lambda_2}\Bigg)+\Bigg(\frac{\mathscr{T}^{\varrho_1}}{\Gamma(\varrho_1+1)}\Bigg)\Bigg]\\ &+\Bigg(\frac{\varsigma^{\varrho_1-1}}{\Gamma(\varrho_1)}\frac{(1-e^{-\lambda_2\varsigma})}{\lambda_2}\Bigg). \qquad (3.6)\end{aligned}$$

$$\Phi = \min\{1-[(\mathfrak{Z}_1+\hat{\mathfrak{Z}}_1)\kappa_1+(\mathfrak{Z}_2+\hat{\mathfrak{Z}}_2)\hat{\kappa}_1], 1-[(\mathfrak{Z}_1+\hat{\mathfrak{Z}}_1)\kappa_2+(\mathfrak{Z}_2+\hat{\mathfrak{Z}}_2)\hat{\kappa}_2]\}. \qquad (3.7)$$

We employ the following result to prove existence of solutions for (1.3) and (1.4).

3.1. **Existence Results via Leray–Schauder Alternative.** This subsection is devoted to proving the existence of solutions by means of the Leray–Schauder alternative principle.

Theorem 3.1. *Assume that* $(\mathscr{H}_1)$ *holds. In addition, it is assumed that*

$$(\mathfrak{Z}_1+\mathfrak{Z}_2)\kappa_1+(\hat{\mathfrak{Z}}_1+\hat{\mathfrak{Z}}_2)\hat{\kappa}_1<1, \qquad (3.8)$$

$$(\mathfrak{Z}_1+\mathfrak{Z}_2)\kappa_2+(\hat{\mathfrak{Z}}_1+\hat{\mathfrak{Z}}_2)\hat{\kappa}_2<1, \qquad (3.9)$$

then system (1.3) and (1.4) has at least one solution on $[0,\mathscr{T}]$.

Proof. **First**, we show that $\mathscr{T}$ is continuous, continuity of functions ϑ_1 and Ψ_1. Let $\Omega\subset\chi_1\times\chi_2$ be bounded. Then there exist positive constants $\mathscr{L}_1$ and $\mathscr{L}_2$ such that

$$|\mathfrak{U}(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma))|\le\mathscr{L}_1, \quad |\mathfrak{H}(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma))|\le\mathscr{L}_2,$$

$\forall(\vartheta_1,\Psi_1)\in\Omega$. Then for any $(\vartheta_1,\Psi_1)\in\Omega$, we have

$$\begin{aligned}&|\mathscr{T}_1(\vartheta_1,\Psi_1)(\varsigma)|\\ \le&\left|\frac{(1-e^{-\lambda_1\varsigma})}{\Delta}\right.\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^\rho e^{-\lambda_1(\rho-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})\Big)d\mathfrak{s}+\eta\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\Big)(\rho)\end{aligned}$$

$$+\int_0^{\mathscr{T}}\Big[\lambda_2\int_0^{\varsigma}e^{-\lambda_2(\varsigma-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})\Big)d\mathfrak{u}\Big]d\mathfrak{s}+\int_0^{\mathscr{T}}\Big[\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\Big)(\varsigma)\Big]d\varsigma\Big)$$
$$+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})\Big)d\mathfrak{s}+\zeta\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\Big)(\mu)$$
$$+\int_0^{\mathscr{T}}\Big[\lambda_1\int_0^{\varsigma}e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})\Big)d\mathfrak{u}\Big]d\mathfrak{s}+\int_0^{\mathscr{T}}\Big[\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\Big)(\varsigma)\Big]d\varsigma\Big)\Big]$$
$$+\int_0^{\varsigma}e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})\Big)d\mathfrak{s}\Big|, \tag{3.10}$$
$$\leq\frac{1-e^{-\lambda_1\varsigma}}{\lambda_1\Delta}\Bigg\{\Bigg\{\mathcal{A}_2\Bigg[-\eta\lambda_1\Bigg(\frac{\rho^{\varpi_1-1}}{\Gamma(\varpi_1)}\frac{(1-e^{-\lambda_1\rho})}{\lambda_1}\Bigg)+\eta\frac{\rho^{\varpi_1-1}}{\Gamma(\varpi_1)}\Bigg]+\mathcal{B}_2\Bigg[\lambda_1\Bigg(\frac{\mathscr{T}^{\varpi_1}}{\Gamma(\varpi_1+1)}\frac{(1-e^{-\lambda_1\mathscr{T}})}{\lambda_1}\Bigg)$$
$$+\Bigg(\frac{\mathscr{T}^{\varpi_1}}{\Gamma(\varpi_1+1)}\Bigg)\Bigg]\Bigg\}+\Bigg(\frac{\varsigma^{\varpi_1-1}}{\Gamma(\varpi_1)}\frac{(1-e^{-\lambda_1\varsigma})}{\lambda_1}\Bigg)\Bigg\}\mathscr{L}_1$$
$$+\frac{1-e^{-\lambda_1\varsigma}}{\lambda_1\Delta}\Bigg\{\mathcal{A}_2\Bigg[\lambda_2\Bigg(\frac{\mathscr{T}^{\varrho_1}}{\Gamma(\varrho_1+1)}\frac{(1-e^{-\lambda_2\mathscr{T}})}{\lambda_2}\Bigg)+\Bigg(\frac{\mathscr{T}^{\varrho_1}}{\Gamma(\varrho_1+1)}\Bigg)\Bigg]$$
$$+\mathcal{B}_2\Bigg[\zeta\lambda_2\Bigg(\frac{\mu^{\varrho_1-1}}{\Gamma(\varrho_1)}\frac{(1-e^{-\lambda_2\mu})}{\lambda_2}\Bigg)+\zeta\Bigg(\frac{\mu^{\varrho_1-1}}{\Gamma(\varrho_1)}\Bigg)\Bigg]\Bigg\}\mathscr{L}_2,$$

which implies that

$$|\mathscr{T}_1(\vartheta_1,\Psi_1)(\varsigma)|=\mathfrak{Z}_1\mathscr{L}_1+\mathfrak{Z}_2\mathscr{L}_2.$$

Similarly, we get

$$|\mathscr{T}_2(\vartheta_1,\Psi_1)(\varsigma)|=\mathscr{L}_1\mathfrak{Z}_3+\mathscr{L}_2\mathfrak{Z}_4,$$

which, in view of the notation (3.3) and (3.4) yields

$$\|\mathscr{T}_1(\vartheta_1,\Psi_1)\|\leq\mathfrak{Z}_1\mathscr{L}_1+\mathfrak{Z}_2\mathscr{L}_2. \tag{3.11}$$

Likewise, using the notation (3.5) and (3.6), we have

$$\|\mathscr{T}_2(\vartheta_1,\Psi_1)\|\leq\hat{\mathfrak{Z}}_1\mathscr{L}_1+\hat{\mathfrak{Z}}_2\mathscr{L}_2. \tag{3.12}$$

Then it follows from (3.11) and (3.12) that, from the above inequalities, we conclude that the operator $\mathscr{T}$ is uniformly bounded, since

$$\|\mathscr{T}(\vartheta_1,\Psi_1)\|\leq(\mathfrak{Z}_1+\hat{\mathfrak{Z}}_1)\mathscr{L}_1+(\mathfrak{Z}_2+\hat{\mathfrak{Z}}_2)\mathscr{L}_2, \tag{3.13}$$

which demonstrates that the operator $\mathscr{T}$ is uniformly bounded.

Next, we establish the equi-continuity of $\mathscr{T}$. Let $(\varsigma_1,\varsigma_2\in[0,\mathscr{T}])$ with $\varsigma_1<\varsigma_2$. Then, it follows that

$$|\mathscr{T}_1(\vartheta_1(\varsigma_2),\vartheta_1(\varsigma_2))-\mathscr{T}_1(\vartheta_1(\varsigma_1),\vartheta_1(\varsigma_1))|$$
$$\leq\Bigg|\frac{(1-e^{-\lambda_1\varsigma})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})\Big)d\mathfrak{s}+\eta\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\Big)(\rho)$$
$$+\int_0^{\mathscr{T}}\Big[\lambda_2\int_0^{\varsigma}e^{-\lambda_2(\varsigma-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})\Big)d\mathfrak{u}\Big]d\mathfrak{s}-\int_0^{\mathscr{T}}\Big[\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\Big)(\varsigma)\Big]d\varsigma\Big)$$
$$+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})\Big)d\mathfrak{s}+\zeta\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\Big)(\mu)$$

$$+\int_0^{\mathscr{T}}\Big[\lambda_1\int_0^{\varsigma}e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})\Big)d\mathfrak{u}\Big]d\mathfrak{s}-\int_0^{\mathscr{T}}\Big[\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\Big)(\varsigma)\Big]d\varsigma\Big)\Big\|$$
$$+\left|\int_0^{\varsigma}[e^{-\lambda_1(\varsigma_2-\mathfrak{s})}-e^{-\lambda_1(\varsigma_1-\mathfrak{s})}]\mathcal{I}^{\varpi_1-1}(\mathfrak{U}(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})))d\mathfrak{s}\right|+\left|\int_{\varsigma_2}^{\varsigma_1}e^{-\lambda_1(\varsigma_2-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}(\mathfrak{U}(\mathfrak{s},\vartheta_1(\mathfrak{s}),\Psi_1(\mathfrak{s})))d\mathfrak{s}\right|$$
$$\to 0.$$

Clearly,

$$\Big|\mathscr{T}_1(\vartheta_1(\varsigma_2),\Psi_1(\varsigma_2))-\mathscr{T}_1(\vartheta_1(\varsigma_1),\Psi_1(\varsigma_1))\Big|\to 0,\quad\text{and}\quad\Big|\mathscr{T}_2(\vartheta_1(\varsigma_2),\Psi_1(\varsigma_2))-\mathscr{T}_2(\vartheta_1(\varsigma_1),\Psi_1(\varsigma_1))\Big|\to 0$$

as $\varsigma_2\to\varsigma_1$, uniformly for all $(\vartheta_1,\Psi_1)\in\Omega$. Consequently, the operator $\mathscr{T}(\vartheta_1,\Psi_1)$ is equi-continuous. Therefore, by the Arzelà–Ascoli theorem, $\mathscr{T}(\vartheta_1,\Psi_1)$ is completely continuous.

Finally, we define the set

$$\mathcal{W}=\Big\{(\vartheta_1,\Psi_1)\in\chi_1\times\chi_2\;\Big|\;(\vartheta_1,\Psi_1)=\sigma,\mathscr{T}(\vartheta_1,\Psi_1),\ 0<\sigma<1\Big\},$$

and show that it is bounded. Let $((\vartheta_1,\Psi_1)\in\mathcal{W})$, so that

$$(\vartheta_1,\Psi_1)=\sigma\mathscr{T}(\vartheta_1,\Psi_1).$$

Then, for any $(\varsigma\in[0,\mathscr{T}])$, we have

$$(\vartheta_1,\Psi_1)(\varsigma)=\Big[\sigma\mathscr{T}_1(\vartheta_1,\Psi_1)(\varsigma),\ \sigma\mathscr{T}_2(\vartheta_1,\Psi_1)(\varsigma)\Big].$$

Using the hypotheses on $(\mathscr{T}_1)$ and $(\mathscr{T}_2)$, it follows that

$$\begin{aligned}|\mathscr{T}_1(\vartheta_1,\Psi_1)(\varsigma)|\le&\frac{(1-e^{-\lambda_1\varsigma})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}(\varkappa_0+\varkappa_1\|\mathcal{U}_1\|+\varkappa_2\|\mathcal{U}_2\|)(\mathfrak{s})d\mathfrak{s}\\&+\eta\mathcal{I}^{\varpi_1-1}(\varkappa_0+\varkappa_1\|\mathcal{U}_1\|+\varkappa_2\|\mathcal{U}_2\|)(\rho)\\&+\int_0^{\mathscr{T}}\Big[\lambda_2\int_0^{\varsigma}e^{-\lambda_2(\varsigma-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}(\mathfrak{n}_0+\mathfrak{n}_1\|\mathcal{U}_1\|+\mathfrak{n}_2\|\mathcal{U}_2\|)(\mathfrak{u})d\mathfrak{u}\Big]d\mathfrak{s}\\&-\int_0^{\mathscr{T}}\Big[\mathcal{I}^{\varrho_1-1}(\mathfrak{n}_0+\mathfrak{n}_1\|\mathcal{U}_1\|+\mathfrak{n}_2\|\mathcal{U}_2\|)(\varsigma)\Big]d\varsigma\Big)\\&+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}(\mathfrak{n}_0+\mathfrak{n}_1\|\mathcal{U}_1\|+\mathfrak{n}_2\|\mathcal{U}_2\|)(\mathfrak{s})d\mathfrak{s}\\&+\zeta\mathcal{I}^{\varrho_1-1}(\mathfrak{n}_0+\mathfrak{n}_1\|\mathcal{U}_1\|+\mathfrak{n}_2\|\mathcal{U}_2\|)(\mu)\\&+\int_0^{\mathscr{T}}\Big[\lambda_1\int_0^{\varsigma}e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}(\varkappa_0+\varkappa_1\|\mathcal{U}_1\|+\varkappa_2\|\mathcal{U}_2\|)(\mathfrak{u})d\mathfrak{u}\Big]d\mathfrak{s}\\&-\int_0^{\mathscr{T}}\Big[\mathcal{I}^{\varpi_1-1}(\varkappa_0+\varkappa_1\|\mathcal{U}_1\|+\varkappa_2\|\mathcal{U}_2\|)(\varsigma)\Big]d\varsigma\Big)\Bigg]\\&+\int_0^{\varsigma}e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}(\varkappa_0+\varkappa_1\|\mathcal{U}_1\|+\varkappa_2\|\mathcal{U}_2\|)(\mathfrak{s})d\mathfrak{s}.\end{aligned}$$

Hence, we have

$$\|\vartheta_1\|\le(\varkappa_0+\varkappa_1\|\mathcal{U}_1\|+\varkappa_2\|\mathcal{U}_2\|)\mathfrak{Z}_1+(\mathfrak{n}_0+\mathfrak{n}_1\|\mathcal{U}_1\|+\mathfrak{n}_2\|\mathcal{U}_2\|)\mathfrak{Z}_2.\tag{3.14}$$

Similarly, one can obtain

$$\|\Psi_1\| \leq (\varkappa_0 + \varkappa_1\|\mathcal{U}_1\| + \varkappa_2\|\mathcal{U}_2\|)\mathfrak{Z}_3 + (\mathfrak{n}_0 + \mathfrak{n}_1\|\mathcal{U}_1\| + \mathfrak{n}_2\|\mathcal{U}_2\|)\mathfrak{Z}_4. \tag{3.15}$$

From (3.14) and (3.15), we obtain

$$\begin{aligned}\|\vartheta_1\| + \|\Psi_1\| \leq & (\varkappa_0 + \varkappa_1\|\mathcal{U}_1\| + \varkappa_2\|\mathcal{U}_2\|)\mathfrak{Z}_1 + (\mathfrak{n}_0 + \mathfrak{n}_1\|\mathcal{U}_1\| + \mathfrak{n}_2\|\mathcal{U}_2\|)\mathfrak{Z}_2 \\ & + (\varkappa_0 + \varkappa_1\|\mathcal{U}_1\| + \varkappa_2\|\mathcal{U}_2\|)\mathfrak{Z}_3 + (\mathfrak{n}_0 + \mathfrak{n}_1\|\mathcal{U}_1\| + \mathfrak{n}_2\|\mathcal{U}_2\|)\mathfrak{Z}_4.\end{aligned}$$

Consequently, we obtain

$$|(\vartheta_1, \Psi_1)| \leq \frac{\varkappa_0(\mathfrak{Z}_1 + \mathfrak{Z}_3) + \mathfrak{n}_0(\mathfrak{Z}_2 + \mathfrak{Z}_4)}{\Phi},$$

which demonstrates that the set $(\mathcal{W})$ is bounded. Therefore, the operator $\mathscr{T}$ admits at least one fixed point. As a result, the problem $(1.3) - (1.4)$ possesses at least one solution on $[0, \mathscr{T}]$. This completes the proof. □

Next, we employ Banach's contraction principle to establish both the existence and uniqueness of solutions for the problem (1.3)–(1.4).

Theorem 3.2. *Assume that $\vartheta_1, \Psi_1 : [0, \mathscr{T}] \times \mathcal{R}^2 \to \mathcal{R}$ are continuous functions and there exist constants $\mathscr{W}_i, \mathcal{N}_i = 1, 2$ such that for all $\varsigma \in [0, \mathscr{T}]$ and $\vartheta_{1i}, \Psi_{1i} \in \mathcal{R}, i = 1, 2.$*

$$|\mathfrak{U}(\varsigma, \vartheta_{11}, \vartheta_{12}) - \mathfrak{U}(\varsigma, \Psi_{11}, \Psi_{12})| \leq \mathscr{W}_1|\vartheta_{11} - \Psi_{11}| + \mathscr{W}_2|\vartheta_{12} - \vartheta_{12}|$$
$$|\mathfrak{H}(\varsigma, \vartheta_{11}, \vartheta_{12}) - \mathfrak{H}(\varsigma, \Psi_{11}, \Psi_{12})| \leq \mathcal{N}_1|\vartheta_{11} - \Psi_{11}| + \mathcal{N}_2|\vartheta_{12} - \Psi_{12}|.$$

In addition, assume that

$$(\mathfrak{Z}_1 + \mathfrak{Z}_3)(\mathscr{W}_1 + \mathscr{W}_2) + (\mathfrak{Z}_2 + \mathfrak{Z}_4)(\mathcal{N}_1 + \mathcal{N}_2) < 1,$$

where $(\mathfrak{Z}_1) - (\mathfrak{Z}_4)$ are given by (3.3)-(3.6). Then the BVP (1.4) has a unique solution.

Proof. Fixing $\sup_{\varsigma\in[0,\mathscr{T}]} \mathfrak{U}(\varsigma, 0, 0) = \mathcal{Q}_1 < \infty$ and $\sup_{\varsigma\in[0,\mathscr{T}]} \mathfrak{H}(\varsigma, 0, 0) = \mathcal{Q}_2 < \infty$ and using the assumption $(\mathscr{H}_2)$, we obtain the following.

$$|\mathfrak{U}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma))| = |\mathfrak{U}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)) - \mathfrak{U}(\varsigma, 0, 0) + \mathfrak{U}(\varsigma, 0, 0)| \leq \mathscr{W}_1\|\vartheta_1\| + \mathscr{W}_2\|\Psi_1\| + \mathcal{Q}_1,$$
$$|\mathfrak{H}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma))| = |\mathfrak{H}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)) - \mathfrak{H}(\varsigma, 0, 0) + \mathfrak{H}(\varsigma, 0, 0)| \leq \mathcal{N}_1\|\vartheta_1\| + \mathcal{N}_2\|\Psi_1\| + \mathcal{Q}_2.$$

Now, we consider the closed ball $\mathcal{B}_r = \{(\vartheta_1, \Psi_1) \in \chi_1 \times \chi_2 : \|(\vartheta_1, \Psi_1)\| \leq r\}$ and show that $\mathscr{T}(\mathcal{B}_r) \subset \mathcal{B}_r$, where

$$\frac{\mathcal{Q}_1(\mathfrak{Z}_1 + \mathfrak{Z}_3) + \mathcal{Q}_2(\mathfrak{Z}_2 + \mathfrak{Z}_4)}{1 - [(\mathfrak{Z}_1 + \mathfrak{Z}_3)(\mathscr{W}_1 + \mathscr{W}_2) + (\mathfrak{Z}_2 + \mathfrak{Z}_4)(\mathcal{N}_1 + \mathcal{N}_2)]} < r.$$

Let $(\vartheta_1, \Psi_1) \in \mathcal{B}_r$. We obtain,

$$\begin{aligned}|\mathscr{T}_1(\vartheta_1, \Psi_1)(\varsigma)| \leq & \frac{(1 - e^{-\lambda_1\varsigma})}{\Delta}\Bigg[\mathcal{A}_2\Big(- \eta\lambda_1 \int_0^{\rho} e^{-\lambda_1(\rho-\mathfrak{s})} \mathcal{I}^{\varpi_1-1}(\mathscr{W}_1\|\vartheta_1\| + \mathscr{W}_2\|\Psi_1\| + \mathcal{Q}_1)(\mathfrak{s})d\mathfrak{s} \\ & + \eta\mathcal{I}^{\varpi_1-1}(\mathscr{W}_1\|\vartheta_1\| + \mathscr{W}_2\|\Psi_1\| + \mathcal{Q}_1)(\rho) \\ & + \int_0^{\mathscr{T}} \Big[\lambda_2 \int_0^{\varsigma} e^{-\lambda_2(\varsigma-\mathfrak{s})} \mathcal{I}^{\varrho_1-1}(\mathcal{N}_1\|\vartheta_1\| + \mathcal{N}_2\|\Psi_1\| + \mathcal{Q}_2)(\mathfrak{u})d\mathfrak{u}\Big]d\mathfrak{s} \\ & - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varrho_1-1}(\mathcal{N}_1\|\vartheta_1\| + \mathcal{N}_2\|\Psi_1\| + \mathcal{Q}_2)(\varsigma)\Big]d\varsigma\Big)\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu} e^{-\lambda_2(\mu-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}(\mathcal{N}_1\|\vartheta_1\|+\mathcal{N}_2\|\Psi_1\|+\mathcal{Q}_2)(\mathfrak{s})d\mathfrak{s}\\
&+\zeta\mathcal{I}^{\varrho_1-1}(\mathcal{N}_1\|\vartheta_1\|+\mathcal{N}_2\|\Psi_1\|+\mathcal{Q}_2)(\mu)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\varsigma} e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}(\mathscr{W}_1\|\vartheta_1\|+\mathscr{W}_2\|\Psi_1\|+\mathcal{Q}_1)(\mathfrak{u})d\mathfrak{u}\Big]d\mathfrak{s}\\
&-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\varpi_1-1}(\mathscr{W}_1\|\vartheta_1\|+\mathscr{W}_2\|\Psi_1\|+\mathcal{Q}_1)(\varsigma)\Big]d\varsigma\Big)\Big]\\
&+\int_0^{\varsigma} e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}(\mathscr{W}_1\|\vartheta_1\|+\mathscr{W}_2\|\Psi_1\|+\mathcal{Q}_1)(\mathfrak{s})d\mathfrak{s},
\end{aligned}
$$

which, on taking the norm for $\varsigma\in[0,\mathcal{T}]$, yields

$$\|(\mathcal{T}_1(\vartheta_1,\Psi_1))\|\le[(\mathscr{W}_1+\mathscr{W}_2)r+\mathcal{Q}_1]\mathfrak{Z}_1+[(\mathcal{N}_1+\mathcal{N}_2)r+\mathcal{Q}_2]\mathfrak{Z}_2.$$

In the same way, we obtain

$$\|(\mathcal{T}_2(\vartheta_1,\Psi_1))\|\le[(\mathscr{W}_1+\mathscr{W}_2)r+\mathcal{Q}_1]\mathfrak{Z}_3+[(\mathcal{N}_1+\mathcal{N}_2)r+\mathcal{Q}_2]\mathfrak{Z}_4.$$

Consequently, it follows that $\|\mathcal{T}(\vartheta_1,\Psi_1)\|\le r$, that is, $(\vartheta_1,\Psi_1)\in\mathcal{B}_r$. Hence, $\mathcal{T}(\mathcal{B}_r)\subset\mathcal{B}_r$. **Next,** we show that the operator $\mathcal{T}$ is a contraction. For that, let $(\vartheta_{11},\Psi_{11}),(\vartheta_{12},\Psi_{12})\in\chi_1\times\chi_2$. Then, for any $\varsigma\in[0,\mathcal{T}]$, we get

$$
\begin{aligned}
&\|\mathcal{T}_1(\vartheta_{12},\Psi_{12})-\mathcal{T}_1(\vartheta_{11},\Psi_{11})\|\\
&\le\frac{(1-e^{-\lambda_1\varsigma})}{\Delta}\Big[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^{\rho} e^{-\lambda_1(\rho-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}|\mathfrak{U}(\varsigma,\vartheta_{12}(\varsigma),\Psi_{12}(\varsigma))-\mathfrak{U}(\varsigma,\vartheta_{11}(\varsigma),\Psi_{11}(\varsigma))|(\mathfrak{s})d\mathfrak{s}\\
&+\eta\mathcal{I}^{\varpi_1-1}|\mathfrak{U}(\varsigma,\vartheta_{12}(\varsigma),\Psi_{12}(\varsigma))-\mathfrak{U}(\varsigma,\vartheta_{11}(\varsigma),\Psi_{11}(\varsigma))|(\rho)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\varsigma} e^{-\lambda_2(\varsigma-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}|\mathfrak{H}(\varsigma,\vartheta_{12}(\varsigma),\Psi_{12}(\varsigma))-\mathfrak{H}(\varsigma,\vartheta_{11}(\varsigma),\Psi_{11}(\varsigma))|(\mathfrak{u})d\mathfrak{u}\Big]d\mathfrak{s}\\
&-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\varrho_1-1}|\mathfrak{H}(\varsigma,\vartheta_{12}(\varsigma),\Psi_{12}(\varsigma))-\mathfrak{H}(\varsigma,\vartheta_{11}(\varsigma),\Psi_{11}(\varsigma))|(\varsigma)\Big]d\varsigma\Big)\\
&+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu} e^{-\lambda_2(\mu-\mathfrak{s})}\mathcal{I}^{\varrho_1-1}|\mathfrak{H}(\varsigma,\vartheta_{12}(\varsigma),\Psi_{12}(\varsigma))-\mathfrak{H}(\varsigma,\vartheta_{11}(\varsigma),\Psi_{11}(\varsigma))|(\mathfrak{s})d\mathfrak{s}\\
&+\zeta\mathcal{I}^{\varrho_1-1}|\mathfrak{H}(\varsigma,\vartheta_{12}(\varsigma),\Psi_{12}(\varsigma))-\mathfrak{H}(\varsigma,\vartheta_{11}(\varsigma),\Psi_{11}(\varsigma))|(\mu)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\varsigma} e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}|\mathfrak{U}(\varsigma,\vartheta_{12}(\varsigma),\Psi_{12}(\varsigma))-\mathfrak{U}(\varsigma,\vartheta_{11}(\varsigma),\Psi_{11}(\varsigma))|(\mathfrak{u})d\mathfrak{u}\Big]d\mathfrak{s}\\
&-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\varpi_1-1}|\mathfrak{U}(\varsigma,\vartheta_{12}(\varsigma),\Psi_{12}(\varsigma))-\mathfrak{U}(\varsigma,\vartheta_{11}(\varsigma),\Psi_{11}(\varsigma))|(\varsigma)\Big]d\varsigma\Big)\Big]\\
&+\int_0^{\varsigma} e^{-\lambda_1(\varsigma-\mathfrak{s})}\mathcal{I}^{\varpi_1-1}|\mathfrak{U}(\varsigma,\vartheta_{12}(\varsigma),\Psi_{12}(\varsigma))-\mathfrak{U}(\varsigma,\vartheta_{11}(\varsigma),\Psi_{11}(\varsigma))|(\mathfrak{s})d\mathfrak{s},\\
&\le(\mathscr{W}_1\|\vartheta_{12}-\vartheta_{11}\|+\mathscr{W}_2\|\Psi_{12}-\Psi_{11}\|)\mathfrak{Z}_1+(\mathcal{N}_1\|\vartheta_{12}-\vartheta_{11}\|+\mathcal{N}_2\|\Psi_{12}-\Psi_{11}\|)\mathfrak{Z}_2.
\end{aligned}
$$

In consequence, we get

$$\|\mathcal{T}_1(\vartheta_{12},\Psi_{12})(\varsigma)-\mathcal{T}_1(\vartheta_{11},\Psi_{11})(\varsigma)\|\le[(\mathscr{W}_1+\mathscr{W}_2)\mathfrak{Z}_1+(\mathcal{N}_1+\mathcal{N}_2)\mathfrak{Z}_2](\|\vartheta_{12}-\vartheta_{11}\|+\|\Psi_{12}-\Psi_{11}\|).\quad(3.16)$$

Similarly,

$$\|\mathscr{T}_2(\vartheta_{12},\Psi_{12})(\varsigma)-\mathscr{T}_2(\vartheta_{11},\Psi_{11})(\varsigma)\|\leq[(\mathscr{W}_1+\mathscr{W}_2)\mathfrak{Z}_3+(\mathcal{N}_1+\mathcal{N}_2)\mathfrak{Z}_4](\|\vartheta_{12}-\vartheta_{11}\|+\|\Psi_{12}-\Psi_{11}\|). \quad (3.17)$$

It follows from (3.16) and (3.17) that:

$$\begin{aligned}\|\mathscr{T}(\vartheta_{12},\Psi_{12})(\varsigma)-&\mathscr{T}(\vartheta_{11},\Psi_{11})(\varsigma)\|\\ &\leq[(\mathscr{W}_1+\mathscr{W}_2)(\mathfrak{Z}_1+\mathfrak{Z}_3)+(\mathcal{N}_1+\mathcal{N}_2)(\mathfrak{Z}_2+\mathfrak{Z}_4)](\|\vartheta_{12}-\vartheta_{11}\|+\|\Psi_{12}-\Psi_{11}\|).\end{aligned}$$

From (3.7), it is evident that the operator $\mathscr{T}$ satisfies the contraction property. Therefore, by Banach's contraction mapping principle, $\mathscr{T}$ admits a unique fixed point, which corresponds to the unique solution of the problem (1.3)–(1.4). This completes the proof. □

4. Hyers–Ulam Stability of the System

This section is devoted to investigating the Hyers–Ulam (HU) stability of the proposed coupled system. Consider the following inequality:

$$\begin{cases}\left({}^{c}\mathcal{D}^{\varpi_1}+\lambda_1{}^{c}\mathcal{D}^{\varpi_1-1}\right)\vartheta_1(\varsigma)-\mathfrak{U}\left(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\right)\leq\epsilon_1, & \varsigma\in[0,\mathscr{T}],\\ \left({}^{c}\mathcal{D}^{\varrho_1}+\lambda_2{}^{c}\mathcal{D}^{\varrho_1-1}\right)\Psi_1(\varsigma)-\mathfrak{H}\left(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\right)\leq\epsilon_2, & \varsigma\in[0,\mathscr{T}],\end{cases} \quad (4.1)$$

where ($\epsilon_1,\epsilon_2>0$) are given constants.

Definition 4.1. *The problem* (1.3) *is said to be Hyers–Ulam stable if there exist positive constants* ($\mathfrak{Z}_1-\mathfrak{Z}_4$) *such that, for each solution* ($(\vartheta_1,\Psi_1)\in C([0,\mathscr{T}]\times\mathcal{R}^2,\mathcal{R})$) *of inequality* (4.1)*, there exists a true solution* (ϑ_1^*,Ψ_1^*) *of* (1.3) *satisfying*

$$\begin{cases}|\vartheta_1(\varsigma)-\vartheta_1^*(\varsigma)|\leq\mathfrak{Z}_1\epsilon_1+\mathfrak{Z}_2\epsilon_2, & \varsigma\in[0,\mathscr{T}],\\ |\Psi_1(\varsigma)-\Psi_1^*(\varsigma)|\leq\mathfrak{Z}_3\epsilon_1+\mathfrak{Z}_4\epsilon_2, & \varsigma\in[0,\mathscr{T}].\end{cases}$$

Remark 4.1. *A pair* (ϑ_1,Ψ_1) *is a solution of inequality (4.1) if there exist functions* ($\mathcal{Z}_1,\mathcal{Z}_2\in C([0,\mathscr{T}],\mathcal{R})$)*, depending on* ($\vartheta_1$) *and* ($\Psi_1$) *respectively, such that*

$$\begin{cases}\left({}^{c}\mathcal{D}^{\varpi_1}+\lambda_1{}^{c}\mathcal{D}^{\varpi_1-1}\right)\vartheta_1(\varsigma)=\mathfrak{U}\left(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\right)+\mathcal{Z}_1(\varsigma),\\ \left({}^{c}\mathcal{D}^{\varrho_1}+\lambda_2{}^{c}\mathcal{D}^{\varrho_1-1}\right)\Psi_1(\varsigma)=\mathfrak{H}\left(\varsigma,\vartheta_1(\varsigma),\Psi_1(\varsigma)\right)+\mathcal{Z}_2(\varsigma),\end{cases}$$

with

$$|\mathcal{Z}_1(\varsigma)|\leq\epsilon_1,\quad |\mathcal{Z}_2(\varsigma)|\leq\epsilon_2,\quad \varsigma\in[0,\mathscr{T}].$$

Remark 4.2. *If* (ϑ_1,Ψ_1) *is a solution of inequality* (4.1)*, then it satisfies the Hyers–Ulam bounds*

$$\begin{cases}|\vartheta_1(\varsigma)-\vartheta_1^*(\varsigma)|\leq\mathfrak{Z}_1\epsilon_1+\mathfrak{Z}_2\epsilon_2,\\ |\Psi_1(\varsigma)-\Psi_1^*(\varsigma)|\leq\mathfrak{Z}_3\epsilon_1+\mathfrak{Z}_4\epsilon_2,\end{cases} \quad \varsigma\in[0,\mathscr{T}],$$

as a direct consequence of Remark 4.1 and Definition 4.1.

$$
\begin{aligned}
&|\vartheta_1(\varsigma) - \vartheta_1{}^*(\varsigma)| \\
&\leq \left| \frac{(1-e^{-\lambda_1\varsigma})}{\Delta} \left[\mathcal{A}_2\Big(-\eta\lambda_1 \int_0^\rho e^{-\lambda_1(\rho-\mathfrak{s})} \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big) d\mathfrak{s} + \eta \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)(\rho) \right.\right. \\
&+ \int_0^{\mathscr{T}} \Big[\lambda_2 \int_0^{\varsigma} e^{-\lambda_2(\varsigma-\mathfrak{s})} \mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big) d\mathfrak{u} \Big] d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)(\mathfrak{s}) \Big] d\mathfrak{s} \Big) \\
&+ \mathcal{B}_2\Big(-\zeta\lambda_2 \int_0^{\mu} e^{-\lambda_2(\mu-\mathfrak{s})} \mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big) d\mathfrak{s} + \zeta \mathcal{I}^{\varrho_1-1}\mathfrak{H}\Big(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)\Big)(\mu) \\
&+ \int_0^{\mathscr{T}} \Big[\lambda_1 \int_0^{\varsigma} e^{-\lambda_1(\varsigma-\mathfrak{s})} \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big) d\mathfrak{u} \Big] d\mathfrak{s} - \int_0^{\mathscr{T}} \Big[\mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big)(\mathfrak{s}) \Big] d\mathfrak{s} \Big) \Big] \\
&\left. + \int_0^{\varsigma} e^{-\lambda_1(\varsigma-\mathfrak{s})} \mathcal{I}^{\varpi_1-1}\mathfrak{U}\Big(\mathfrak{s}, \vartheta_1(\mathfrak{s}), \Psi_1(\mathfrak{s})\Big) d\mathfrak{s} \right|, \\
&\leq \frac{1-e^{-\lambda_1\varsigma}}{\lambda_1\Delta} \Bigg\{ \Bigg\{ \mathcal{A}_2\Bigg[\eta\lambda_1\Bigg(\frac{\rho^{\varpi_1-1}}{\Gamma(\varpi_1)} \frac{(1-e^{-\lambda_1\rho})}{\lambda_1} \Bigg) + \eta \frac{\rho^{\varpi_1-1}}{\Gamma(\varpi_1)} \Bigg] + \mathcal{B}_2\Bigg[\lambda_1\Bigg(\frac{\mathscr{T}^{\varpi_1}}{\Gamma(\varpi_1+1)} \frac{(1-e^{-\lambda_1\mathscr{T}})}{\lambda_1} \Bigg) \\
&+ \Bigg(\frac{\mathscr{T}^{\varpi_1}}{\Gamma(\varpi_1+1)} \Bigg) \Bigg] \Bigg\} + \Bigg(\frac{\varsigma^{\varpi_1-1}}{\Gamma(\varpi_1)} \frac{(1-e^{-\lambda_1\varsigma})}{\lambda_1} \Bigg) \Bigg\} \epsilon_1 \\
&+ \frac{1-e^{-\lambda_1\varsigma}}{\lambda_1\Delta} \Bigg\{ \mathcal{A}_2\Bigg[\lambda_2\Bigg(\frac{\mathscr{T}^{\varrho_1}}{\Gamma(\varrho_1+1)} \frac{(1-e^{-\lambda_2\mathscr{T}})}{\lambda_2} \Bigg) + \Bigg(\frac{\mathscr{T}^{\varrho_1}}{\Gamma(\varrho_1+1)} \Bigg) \Bigg] \\
&+ \mathcal{B}_2\Bigg[\zeta\lambda_2\Bigg(\frac{\mu^{\varrho_1-1}}{\Gamma(\varrho_1)} \frac{(1-e^{-\lambda_2\mu})}{\lambda_2} \Bigg) + \zeta\Bigg(\frac{\mu^{\varrho_1-1}}{\Gamma(\varrho_1)} \Bigg) \Bigg] \Bigg\} \epsilon_2,
\end{aligned}
$$

$$
|\vartheta_1(\varsigma) - \vartheta_1{}^*(\varsigma)| \leq \mathfrak{Z}_1\epsilon_1 + \mathfrak{Z}_2\epsilon_2. \tag{4.2}
$$

Using the same arguments, we also obtain

$$
|\Psi_1(\varsigma) - \Psi_1^*(\varsigma)| \leq \mathfrak{Z}_3\epsilon_1 + \mathfrak{Z}_4\epsilon_2, \quad \varsigma \in [0, \mathscr{T}], \tag{4.3}
$$

where the constants $(\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3,)$ and $(\mathfrak{Z}_4)$ are defined in (3.3)–(3.6). Therefore, in view of inequalities (4.2) and (4.3), Remark 4.2 is satisfied. Consequently, the considered nonlinear sequential coupled system of Caputo fractional differential equations is Hyers–Ulam stable, and hence problem (1.3) enjoys Hyers–Ulam stability.

5. Examples

Consider the coupled fractional differential equations given by

$$
\begin{cases}
({}^c\mathcal{D}^{\varpi_1} + \lambda_1{}^c\mathcal{D}^{\varpi_1-1})\vartheta_1(\varsigma) = \mathfrak{U}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)) \\
({}^c\mathcal{D}^{\varrho_1} + \lambda_2{}^c\mathcal{D}^{\varrho_1-1})\Psi_1(\varsigma) = \mathfrak{H}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)),
\end{cases} \tag{5.1}
$$

with boundary conditions.

$$\begin{cases} \displaystyle\int_0^{\mathscr{T}} \vartheta_1{}'(\varsigma)\, d\varsigma = \zeta(\Psi_1{}'(\mu))\,, \quad \mu \in [0, \mathscr{T}], \\ \displaystyle\int_0^{\mathscr{T}} \Psi_1{}'(\varsigma)\, d\varsigma = \eta(\vartheta_1{}'(\rho))\,, \quad \rho \in [0, \mathscr{T}], \\ \vartheta_1(0) = 0, \qquad \Psi_1(0) = 0. \end{cases} \tag{5.2}$$

we have

$$\varpi_1 = 1.5, \varrho_1 = 1.7, \lambda_1 = \lambda_2 = 0.5, \varsigma = 0.9, \rho = 0.9, \mu = 0.7, \mathscr{T} = 2, \eta = 0.5, \zeta = 0.6,$$
$$\mathcal{A}_1 \approx 1.2642, \mathcal{A}_2 \approx 0.4228, \mathcal{B}_1 \approx 0.3188, \mathcal{B}_2 \approx 1.2642, \Delta \approx 1.4633$$

Example 5.1. *We illustrate Theorem 3.1 by choosing*

$$\mathfrak{U}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)) = \frac{1}{3(\varsigma^2+7)} \cdot \frac{|\vartheta_1(\varsigma)||\vartheta_1(\varsigma)|}{1+|\vartheta_1(\varsigma)|} + \frac{1}{27}\sin \Psi_1(\varsigma) + \frac{1}{8\sqrt{2}+9},$$
$$\mathfrak{H}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)) = \frac{1}{30(1+|\Psi_1(\varsigma)|)} \cdot \frac{|\vartheta_1(\varsigma)||\vartheta_1(\varsigma)|}{(9+\varsigma^2)^2} + \frac{\Psi_1(\varsigma)\cos\vartheta_1(\varsigma)}{2(\varsigma+4)^2}. \tag{5.3}$$

From (5.3), it is easy to find that

$$\kappa_0 = \frac{1}{24}, \quad \kappa_1 = \frac{1}{21}, \quad \kappa_2 = \frac{1}{27}, \quad \hat{\kappa}_0 = \frac{1}{32}, \quad \hat{\kappa}_1 = \frac{1}{30}, \quad \hat{\kappa}_2 = \frac{1}{81}.$$

Moreover, using the numerical values obtained for the solution bounds $\mathfrak{Z}_1$, $\mathfrak{Z}_2$, $\hat{\mathfrak{Z}}_1$, and $\hat{\mathfrak{Z}}_2$, we compute

$$\mathfrak{Z}_1 + \mathfrak{Z}_2 = 3.268, \qquad \hat{\mathfrak{Z}}_1 + \hat{\mathfrak{Z}}_2 = 3.659.$$

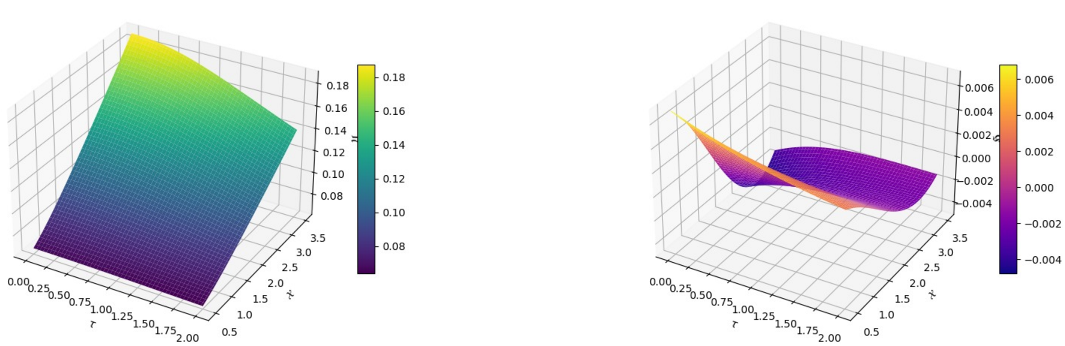
For the Lipschitz constants $k_1 = \frac{1}{21}$, $k_2 = \frac{1}{27}$, $\hat{k}_1 = \frac{1}{30}$, and $\hat{k}_2 = \frac{1}{81}$, we obtain

$$(\mathfrak{Z}_1 + \mathfrak{Z}_2)k_1 + (\hat{\mathfrak{Z}}_1 + \hat{\mathfrak{Z}}_2)\hat{k}_1 \approx 0.2775 < 1,$$

and

$$(\mathfrak{Z}_1 + \mathfrak{Z}_2)k_2 + (\hat{\mathfrak{Z}}_1 + \hat{\mathfrak{Z}}_2)\hat{k}_2 \approx 0.1660 < 1.$$

Hence, all the required contractive-type conditions of Theorem 3.1 are satisfied. Consequently, the considered fractional system admits a unique solution and enjoys Ulam–Hyers stability on the interval $\varsigma \in [0, 2]$.



(A) Plot of $\mathfrak{U}(\varsigma)$ exhibiting mild oscillatory behavior. (B) Plot of $\mathfrak{H}(\varsigma)$ showing a decreasing periodic trend.

Figure 1 illustrates the three-dimensional surface plot of $\mathfrak{U}(\varsigma, \vartheta_1, \Psi_1)$ *with respect to* ς *and* ϑ_1. *Figure 2 depicts the corresponding surface of* $\mathfrak{H}(\varsigma, \vartheta_1, \Psi_1)$. *The smoothness and boundedness of both surfaces further confirm the stability and well-posedness of the proposed fractional model.*

Remark 5.1. *The numerical simulations of the functions* $\mathfrak{U}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma))$ *and* $\mathfrak{H}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma))$ *are illustrated in Figures 1-2 for* $\varsigma \in [0, 2]$. *The graphical results show that both functions remain bounded and continuous over the entire interval. Moreover, the numerical evaluation of the contractive expressions confirms that they are strictly less than one, providing strong numerical evidence for the existence, uniqueness, and Ulam–Hyers stability of the proposed fractional system. Hence, the numerical simulations fully support the theoretical results established in Theorem 3.1.*

Example 5.2. *To illustrate the applicability of Theorem 3.2, consider the following nonlinear functions:*

$$\begin{cases} \mathfrak{U}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)) &= \dfrac{e^{-\varsigma^2}}{\varsigma^2 + 40} \tan^{-1}\Big(\vartheta_1(\varsigma)\Big) + \dfrac{1}{(35 + \varsigma^3)(1 + |\Psi_1(\varsigma)|)} + \dfrac{\cos \varsigma}{8\sqrt{\varsigma^2 + 1}}, \\ \mathfrak{H}(\varsigma, \vartheta_1(\varsigma), \Psi_1(\varsigma)) &= \dfrac{1}{(\varsigma^4 + 36)(1 + |\vartheta_1(\varsigma)|)} + \dfrac{|\vartheta_1(\varsigma)|}{\sqrt{\varsigma^2 + 625}} \cos\Big(\Psi_1(\varsigma)\Big) + \dfrac{e^{\varsigma}}{2(\sin^2 \varsigma + 3)}. \end{cases} \tag{5.4}$$

From (5.4), *we can choose the constants as*

$$\mathscr{W}_1 = \frac{1}{40}, \quad \mathscr{W}_2 = \frac{1}{35}, \quad \mathcal{N}_1 = \frac{1}{36}, \quad \mathcal{N}_2 = \frac{1}{25},$$

and consequently,

$$(\mathfrak{z}_1 + \mathfrak{z}_3)(\mathscr{W}_1 + \mathscr{W}_2) + (\mathfrak{z}_2 + \mathfrak{z}_4)(\mathcal{N}_1 + \mathcal{N}_2) \approx 0.2428 < 1.$$

Thus, all the hypotheses of Theorem 3.2 are fulfilled. Therefore, problem (5.1) *admits a unique solution corresponding to the nonlinearities defined in* (5.4) *on the interval* $[0, 2]$.

6. Conclusion

In this work, we have carried out a comprehensive analysis of a coupled system of nonlinear sequential Caputo fractional differential equations supplemented with two-point integral boundary conditions. Such boundary conditions significantly enrich the mathematical structure of the problem and enhance its applicability in modeling memory-dependent processes arising in physics, biology, and engineering sciences. By reformulating the problem into an equivalent operator equation in an appropriate Banach space, we established sufficient conditions for the existence and uniqueness of solutions via the Banach contraction principle.

In addition, the Hyers–Ulam stability of the considered system was investigated, and explicit stability bounds were derived, demonstrating that the solutions depend continuously on perturbations in the system data. This stability analysis ensures the robustness of the model and highlights its reliability for practical applications. Numerical examples were presented to illustrate the theoretical findings, confirming the validity and effectiveness of the proposed analytical approach.

The results obtained in this paper extend and complement several existing works in the literature on fractional differential systems with nonlocal boundary conditions. The developed framework can be further generalized to include other classes of fractional operators, such as Hilfer, Hadamard, or Atangana–Baleanu derivatives, as well as more complex boundary conditions and control mechanisms, which will be the subject of future investigations.

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