

## Measurement of Groundwater Contaminant Concentration around a Landfill Flow through Heterogeneous Medium Using One-Dimensional Advection-Diffusion Equation

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**ABSTRACT.** Leachate from poorly managed landfills poses a substantial threat to groundwater, which is a vital resource for irrigation and drinking. Uncontrolled waste disposal leads to seriously impaired water quality. Such landfills contribute leachate to contaminated groundwater, which negatively impacts the physical and qualitative characteristics of the increased groundwater pollutant concentration. Monitoring every facet of transport distribution is impractical. Therefore, a prediction of groundwater pollution concentration needs to be modeled to assist in following and examining the contaminated area. This study suggested groundwater pollutant concentrations are transported via the inhomogeneous medium and unsteady flow systems using the one-dimensional advection-diffusion equation. The numerical forward time-centered space finite difference method drives the transport of both solutes as they flow through the two systems. The resulting transport equations are produced and simulated, which have a close value with the analytical solution.

### 1. Introduction

Currently, significant waste issues are plaguing nations all over the world, and the rapid rate of solid waste increase is entirely due to urbanization, economic and industrial development, and population growth; therefore, inappropriate solid waste disposal is a significant issue [1]. Landfilling, a waste management system, creates a structure either in or on top of the ground to contain trash, separating it from the surrounding area for solid waste disposal [2]. This method is low-cost and commonly used in developing countries; thus, if not managed properly, it can lead

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to pollution of air, soil, and water. Poor environmental management results in the production of landfill gases [3], which are generated when organic waste breaks down in a landfill [1] and are the primary cause of deteriorating air quality [3]. Landfill is one of the local sources of environmental contamination. Leachate is a liquid that drains from landfills, and its composition varies depending on the landfill's age and the types of waste it contains [4]. This liquid accumulates at the bottom of the landfill before gradually seeping into the soil, contaminating nearby surface water bodies [5], groundwater, and soil, which can lead to deterioration and result in aquagenic disorders. Heavy metals and persistent organic pollutants are among the several harmful substances found in this leachate [4]. Other contaminants include solvents and pesticides [3]. If the human body is exposed to leachate-polluted water through drinking or bathing, it results in health risks [4], as the groundwater quality was significantly impacted by leachate [2]. Thus, uncontrolled waste disposal, which leads to leachate leaching from dumpsites, can be a detrimental source of many environmental effects, such as air quality, the greenhouse effect, global warming, and groundwater and surface water quality, as well as public health concerns [3].

Mathematical modeling is one of the effective tools for examining the intricate relationships [6]. Partial differential equations, which offer a flexible mathematical framework for characterizing spatial and temporal dynamics for the behavior of systems under diverse conditions [7], have been widely applied among numerous techniques for modeling a variety of phenomena. A class of partial differential equations known as the advection-diffusion equation can be used to simulate natural processes [8] that occur in a variety of scientific and engineering issues [9]. Its applications are broad, legitimate, and most easily comprehended for describing transport phenomena. The advection-diffusion equation has two categories: advection, which is caused by materials moving from one area to another, and diffusion, which is caused by materials moving from a higher concentration to a lower concentration [8]. Many studies have been published about these equations, as in [10]-[18].

In solving these equations, the majority of differential equations are too challenging to solve analytically. In general, analytical answers can be obtained using simplifying assumptions, but actual solutions can be obtained without simplifying assumptions using numerical methods. As a result, there is a need to look at numerical approaches to the governing equations. The finite difference approach is used to solve the advection-diffusion equation [19], which converts the partial differential problem into a system and attempts to numerically resolve this system of difference equations in order to approximate the partial differential equation's solution. The finite difference method's benefits include ease of use, speedy computation, ease of implementation, and adaptability to a range of boundary conditions and physical challenges [20]. One explicit finite difference method is commonly used to solve the partial differential equations. The solution can be readily found by explicit finite-difference formulation. The Forward Time Centered Space

(FTCS) method is the available easy explicit finite-difference scheme [21]. These schemes are either first-order or second-order accurate and have advantages in programming and computation without losing much accuracy, and hence researchers are employed for many model applications [22], for which they have brought this method to solve various works such as in [16], they have used the standard forward time centered space finite difference technique to estimate groundwater pollution concentrations resulting from leachate in a heterogeneous soil model, which incorporates a vertically averaged two-dimensional advection-diffusion equation in the area surrounding a landfill. It is possible to forecast groundwater contamination for 1, 5, 10, 15, and 20 years. In [17], they have used Forward Time Centered Space to estimate the transport of fertilizer-water mixture through soil in subterranean water using the advection-dispersion equation. The findings show that as simulation time grows, fertilizer concentration rises as well, boosting plant development and helping regulate fertilizer dosage and plant growth. In [18], they have conducted approximate concentrations of groundwater pollutants using explicit forward time centered spaced finite difference techniques, which involve three mathematical models: a two-dimensional groundwater flow model, a velocity potential model, and a two-dimensional vertically averaged groundwater pollution dispersion model.

The current study examines the one-dimensional advection-diffusion equation of solute transports along the inhomogeneous medium and unsteady flow systems, driven by applications in groundwater pollution concentrations. Numerical methods (finite differences) are used to tackle the concentration transport problems. Forward time centered space (FTCS) is used to validate the computations. In this work, the numerical and analytical solutions were compared.

## 2. Governing equation

### 2.1 Groundwater pollution dispersion through inhomogeneous soil

In a groundwater quality model, the governing equation is a one-dimensional advection-diffusion partial differential equation [23],

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x} \left( D(x,t) \frac{\partial C(x,t)}{\partial x} - u(x,t)C(x,t) \right), \text{ for all } (x,t) \in [0,L] \times [0,T], \quad (2.1)$$

where  $C(x,t)$  is the groundwater pollutant concentration at position  $x$  along the longitudinal direction at time  $t$ ,  $D$  is the dispersion coefficient of the pollutant,  $u$  is a uniform flow velocity,  $L$  is the length of the considered area from the pollutant origin to the end point, and  $T$  is the simulation time. The inhomogeneity of the soil causes variations in the groundwater flow velocity, which tend to increase over time. They also assumed that the dispersion parameter and velocity parameter are given functions  $f_1(x,t)$  and  $f_2(x,t)$ . Equation (2.1) can be rewritten as

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x} \left( D_0 f_1(x,t) \frac{\partial C(x,t)}{\partial x} - u_0 f_2(x,t) C(x,t) \right). \quad (2.2)$$

Equation (2.2) can be written in the following form:

$$\begin{aligned} \frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x} \left( D_0 \frac{\partial f_1(x,t)}{\partial x} - u_0 f_2(x,t) \right) \frac{\partial C(x,t)}{\partial x} + D_0 f_1(x,t) \frac{\partial^2 C(x,t)}{\partial x^2} \\ - u_0 \frac{\partial f_2(x,t)}{\partial x} C(x,t). \end{aligned} \quad (2.3)$$

The dispersion occurs along an unsteady flow through an inhomogeneous medium. In the above equation,  $D_0$  and  $u_0$  are constants whose dimensions depend on the expressions  $f_1(x,t)$  and  $f_2(x,t)$ . The inhomogeneity of the soil causes variation in the flow velocity. According to [23], they have considered a variation in groundwater pollutant dispersion in heterogeneous soil as it increases, which also assumes a direct proportionality between the dispersion parameter and the square of the velocity. To predict the concentration along an unsteady flow through an inhomogeneous medium, it is assumed here that the inhomogeneity causes a linear increase in velocity defined by equation (2.3) and that dispersion is proportional to the square of the velocity [23]. Moreover, as flow varies with time, dispersion has a similar temporal dependence. Therefore, the following in equation (2.1) are considered in degenerate forms [23]. Thus, equation (2.2) considers

$$f_1(x,t) = f(mt) = (1+ax)^2 \text{ and } f_2(x,t) = f(mt) = 1+ax. \quad (2.4)$$

Where the parameter  $a$  with dimension of  $(\text{length})^{-1}$  accounts for the soil inhomogeneity in equation (2.3) becomes

$$\frac{\partial C(x,t)}{\partial t} = [(1+ax)(2aD_0 - u_0)] \frac{\partial C(x,t)}{\partial x} + D_0(1+ax)^2 \frac{\partial^2 C(x,t)}{\partial x^2} - u_0 a C(x,t), \quad (2.5)$$

## 2.2 Initial and boundary conditions

The initial state of the soil, with zero groundwater contamination concentration, implies the following initial condition:

$$C(x,0) = r(x), \quad 0 \leq x \leq L, 0 < t \leq T, \quad t = 0, \quad (2.6)$$

where  $r(x)$  is an initially assessed groundwater contaminant function. Because of a constant influx, a groundwater contaminant is released at the source, while the concentration gradient at the endpoint is determined by the mean rate of change of groundwater contaminant concentration in their vicinity, leading to the following boundary conditions:

$$C(0,t) = C_0, \quad t > 0, \quad (2.7)$$

$$\frac{\partial C(x,t)}{\partial x} = C_s, \quad x = L, \quad t > 0, \quad (2.8)$$

where  $C_0$  is a given averaged groundwater pollutant concentration at the considered landfill and

$C_s$  is the rate of change of the pollutant concentration around the far-field monitoring station.

### 3. Numerical Techniques

#### 3.1 Dispersion along unsteady flow through inhomogeneous medium

We assume an unsteady flow through an inhomogeneous medium, denoted as  $f_1(x, t) = f(mt) = 1 - \sin(mt)$ , for the linear and quadratic relations defined by using equation (2.3) [23,24]. Equation (2.3) can be written in its final-difference form as:

$$\begin{aligned} \frac{\partial C(x, t)}{\partial t} &= \left[ \frac{D_0 \partial f_1(x, t)}{\partial x} - u_0 f_2(x, t) \right] \frac{\partial C(x, t)}{\partial x} + D_0 f_1(x, t) \frac{\partial^2 C(x, t)}{\partial x^2} - \frac{u_0 \partial f_2(x, t)}{\partial x} C(x, t) \\ &= \left[ \frac{D_0 f_1(x, t) \partial f_1(x, t)}{\partial x} - u_0 f_1(x, t) f_2(x, t) \right] \frac{\partial C(x, t)}{\partial t} + D_0 f_1(x, t) f_1(x, t) \frac{\partial^2 C(x, t)}{\partial x^2} \\ &\quad - \frac{u_0 f_1(x, t) \partial f_2(x, t)}{\partial x} C(x, t) \end{aligned} \quad (3.1)$$

$$\begin{aligned} \frac{C_{i,j+1} - C_{i,j}}{\Delta t} &= [f_1(x, t) D_0 2a(1 + ax_i) - u_0 f_1(x, t)(1 + ax_i)] \frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \\ &\quad + [D_0 f_1(x, t) f_1(x, t)] \frac{C_{i+1,j} - 2C_{i,j} + C_{i-1,j}}{\Delta x^2} - \left[ u_0 f_1(x, t) \frac{\partial f_2(x, t)}{\partial x} \right] C_{i,j} \\ &= [(1 - \sin(mt_j)) 2aD_0(1 + ax_i) - (1 - \sin(mt_j)) u_0(1 + ax_i)] \\ &\quad \frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} + [D_0(1 + ax_i)^2(1 - \sin(mt_j))] \frac{C_{i+1,j} - 2C_{i,j} + C_{i-1,j}}{\Delta x^2} \\ &\quad - [u_0(1 - \sin(mt_j)) a] C_{i,j} \end{aligned} \quad (3.2)$$

$$\begin{aligned} \frac{C_{i,j+1} - C_{i,j}}{\Delta t} &= \frac{(2aD_0 - u_0)(1 + ax_i)(1 - \sin(mt_j))}{2\Delta x} C_{i+1,j} - \frac{(2aD_0 - u_0)(1 + ax_i)}{2\Delta x} \\ &\quad (1 - \sin(mt_j)) C_{i-1,j} + \frac{D_0(1 + ax_i)^2(1 - \sin(mt_j))}{\Delta x^2} C_{i+1,j} - \frac{2D_0(1 + ax_i)^2}{\Delta x^2} \\ &\quad (1 - \sin(mt_j)) C_{i,j} + \frac{D_0(1 + ax_i)^2(1 - \sin(mt_j))}{\Delta x^2} C_{i-1,j} \\ &\quad - u_0 a(1 - \sin(mt_j)) C_{i,j} \end{aligned} \quad (3.3)$$

$$\begin{aligned} C_{i,j+1} - C_{i,j} &= \frac{(2aD_0 - u_0)(1 + ax_i)(1 - \sin(mt_j)) \Delta t}{2\Delta x} C_{i+1,j} - \frac{(2aD_0 - u_0)(1 + ax_i)}{2\Delta x} \\ &\quad (1 - \sin(mt_j)) \Delta t C_{i-1,j} + \frac{D_0(1 + ax_i)^2(1 - \sin(mt_j)) \Delta t}{\Delta x^2} C_{i+1,j} \\ &\quad - \frac{2D_0(1 + ax_i)^2(1 - \sin(mt_j)) \Delta t}{\Delta x^2} C_{i,j} + \frac{D_0(1 + ax_i)^2}{\Delta x^2} \\ &\quad (1 - \sin(mt_j)) \Delta t C_{i-1,j} - u_0 a(1 - \sin(mt_j)) \Delta t C_{i,j} \end{aligned} \quad (3.4)$$

$$C_{i,j+1} = \frac{(2aD_0 - u_0)(1 + ax_i)(1 - \sin(mt_j))\Delta t}{2\Delta x} C_{i+1,j} - \frac{(2aD_0 - u_0)(1 + ax_i)}{2\Delta x} (1 - \sin(mt_j))\Delta t C_{i-1,j} + \frac{D_0(1 + ax_i)^2(1 - \sin(mt_j))\Delta t}{\Delta x^2} C_{i+1,j} - \frac{2D_0(1 + ax_i)^2}{\Delta x^2} (1 - \sin(mt_j))\Delta t C_{i,j} + \frac{D_0(1 + ax_i)^2(1 - \sin(mt_j))\Delta t}{\Delta x^2} C_{i-1,j} - u_0 a(1 - \sin(mt_j))\Delta t C_{i,j} + C_{i,j} \quad (3.5)$$

$$C_{i,j+1} = \frac{D_0(1 + ax_i)^2(1 - \sin(mt_j))\Delta t}{\Delta x^2} C_{i-1,j} - \frac{(2aD_0 - u_0)(1 + ax_i)}{2\Delta x} (1 - \sin(mt_j))\Delta t C_{i,j} - \frac{2D_0(1 + ax_i)^2(1 - \sin(mt_j))\Delta t}{\Delta x^2} C_{i,j} - u_0 a(1 - \sin(mt_j))\Delta t C_{i,j} + \frac{D_0(1 + ax_i)^2(1 - \sin(mt_j))\Delta t}{\Delta x^2} C_{i+1,j} + \frac{(2aD_0 - u_0)(1 + ax_i)(1 - \sin(mt_j))\Delta t}{2\Delta x} C_{i+1,j} \quad (3.6)$$

$$C_{i,j+1} = \left[ \frac{D_0(1 + ax_i)^2(1 - \sin(mt_j))\Delta t}{\Delta x^2} - \frac{(2aD_0 - u_0)(1 + ax_i)(1 - \sin(mt_j))\Delta t}{2\Delta x} \right] C_{i-1,j} + \left[ 1 - \frac{2D_0(1 + ax_i)^2(1 - \sin(mt_j))\Delta t}{\Delta x^2} - u_0 a(1 - \sin(mt_j))\Delta t \right] C_{i,j} + \left[ \frac{D_0(1 + ax_i)^2(1 - \sin(mt_j))\Delta t}{\Delta x^2} + \frac{(2aD_0 - u_0)(1 + ax_i)(1 - \sin(mt_j))\Delta t}{2\Delta x} \right] C_{i+1,j} \quad (3.7)$$

$$\text{Thus } C_{i,j+1} = (J_{i,j} - K_{i,j})C_{i-1,j} + (1 - 2J_{i,j} - L_j)C_{i,j} + (J_{i,j} - K_{i,j})C_{i+1,j}, \quad (3.8)$$

where

$$J_{i,j} = \frac{D_0(1 + ax_i)^2(1 - \sin(mt_j))\Delta t}{\Delta x^2}, \quad (3.9)$$

$$K_{i,j} = \frac{(2aD_0 - u_0)(1 + ax_i)(1 - \sin(mt_j))\Delta t}{2\Delta x}, \quad (3.10)$$

$$L_j = u_0 a \sin(mt_j)\Delta t. \quad (3.11)$$

### 3.2 Temporally dependent dispersion along a uniform and steady flow

In the case of a temporally dependent solute dispersion from a continuous uniform point source along a uniform and steady flow in a longitudinal semi-infinite homogeneous and initially solute-free medium, the following form of the advection-diffusion equation (2.3) is obtained, considering that  $f_2(x,t) = f(mt) = \exp(mt)$  in equation (2.3) [23]:

$$\begin{aligned} \frac{C_{i,j+1} - C_{i,j}}{\Delta t} &= [f_1(x,t)D_0 2a(1+ax_i) - u_0 f_1(x,t)(1+ax_i)] \frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \\ &\quad + [D_0 f_1(x,t)(1+ax_i)^2] \frac{C_{i+1,j} - 2C_{i,j} + C_{i-1,j}}{\Delta x^2} - \left[ u_0 f_1(x,t) \frac{\partial f_2(x,t)}{\partial x} \right] C_{i,j} \\ &= [(1+mt_j)^{-1} 2aD_0(1+ax_i) - (1+mt_j)^{-1} u_0(1+ax_i)] \frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \\ &\quad + [(1+mt_j)^{-1} D_0(1+ax_i)^2] \frac{C_{i+1,j} - 2C_{i,j} + C_{i-1,j}}{\Delta x^2} - [(1+mt_j)^{-1} u_0 a] C_{i,j} \end{aligned} \tag{3.12}$$

$$\begin{aligned} \frac{C_{i,j+1} - C_{i,j}}{\Delta t} &= \frac{(1+mt_i)^{-1} (2aD_0 - u_0)(1+ax_i)}{2\Delta x} C_{i+1,j} - \frac{(1+mt_j)^{-1} (2aD_0 - u_0)(1+ax_i)}{2\Delta x} \\ &\quad C_{i-1,j} + \frac{(1+mt_i)^{-1} D_0(1+ax_i)^2}{\Delta x^2} C_{i+1,j} - \frac{2(1+mt_j)^{-1} D_0(2a - u_0)(1+ax_i)^2}{2\Delta x} \\ &\quad C_{i,j} + \frac{(1+mt_i)^{-1} D_0(1+ax_i)^2}{\Delta x^2} C_{i-1,j} - (1+mt_j)^{-1} u_0 a C_{i,j} \end{aligned} \tag{3.13}$$

$$\begin{aligned} C_{i,j+1} - C_{i,j} &= \frac{(1+mt_i)^{-1} (2aD_0 - u_0)(1+ax_i)\Delta t}{2\Delta x} C_{i+1,j} - \frac{(1+mt_j)^{-1} (2aD_0 - u_0)(1+ax_i)}{2\Delta x} \\ &\quad \Delta t C_{i-1,j} + \frac{(1+mt_i)^{-1} D_0(1+ax_i)^2 \Delta t}{\Delta x^2} C_{i+1,j} - \frac{2(1+mt_j)^{-1} D_0(2a - u_0)}{2\Delta x} \\ &\quad (1+ax_i)^2 \Delta t C_{i,j} + \frac{(1+mt_i)^{-1} D_0(1+ax_i)^2 \Delta t}{\Delta x^2} C_{i-1,j} - (1+mt_j)^{-1} u_0 a \Delta t C_{i,j} \\ &\quad + C_{i,j} \end{aligned} \tag{3.14}$$

$$\begin{aligned} C_{i,j+1} &= \frac{D_0(1+ax_i)^2 \Delta t}{(1+mt_j)\Delta x^2} C_{i-1,j} - \frac{(2aD_0 - u_0)(1+ax_i)\Delta t}{(1+mt_j)2\Delta x} C_{i-1,j} + C_{i,j} - \frac{2D_0(1+ax_i)^2 \Delta t}{(1+mt_j)\Delta x^2} \\ &\quad C_{i,j} - \frac{u_0 a \Delta t}{(1+mt_j)} C_{i,j} + \frac{D_0(1+ax_i)^2 \Delta t}{(1+mt_j)\Delta x^2} C_{i-1,j} - \frac{(2aD_0 - u_0)(1+ax_i)\Delta t}{(1+mt_j)2\Delta x} C_{i+1,j} \end{aligned} \tag{3.15}$$

$$\begin{aligned} C_{i,j+1} &= \left[ \frac{D_0(1+ax_i)^2 \Delta t}{(1+mt_j)\Delta x^2} - \frac{(2aD_0 - u_0)(1+ax_i)\Delta t}{(1+mt_j)2\Delta x} \right] C_{i-1,j} \\ &\quad + \left[ 1 - \frac{2D_0(1+ax_i)^2 \Delta t}{(1+mt_i)\Delta x^2} - \frac{u_0 a \Delta t}{(1+mt_i)} \right] C_{i,j} \\ &\quad + \left[ \frac{D_0(1+ax_i)^2 \Delta t}{(1+mt_j)\Delta x^2} + \frac{(2aD_0 - u_0)(1+ax_i)\Delta t}{(1+mt_j)2\Delta x} \right] C_{i+1,j}. \end{aligned} \tag{3.16}$$

$$\text{Thus } C_{i,j+1} = (M_{i,j} - N_{i,j})C_{i-1,j} + (1 - 2M_{i,j} - P_j)C_{i,j} + (M_{i,j} - N_{i,j})C_{i+1,j}, \tag{3.17}$$

where

$$M_{i,j} = \frac{D_0(1+ax_i)^2 \Delta t}{(1+mt_j)\Delta x^2}, \quad (3.18)$$

$$N_{i,j} = \frac{(2aD_0 - u_0)(1+ax_i)\Delta t}{(1+mt_j)\Delta x^2}, \quad (3.19)$$

$$P_j = \frac{u_0 a \Delta t}{(1+mt_j)}. \quad (3.20)$$

#### 4. The initial condition and boundary conditions

The truncation error for the difference equation (2.6) is  $O(\Delta t, \Delta x^2)$ . Using small enough values of  $\Delta x$  and  $\Delta t$ , the truncation error can be reduced until the achieved accuracy is within the error tolerance [25]. The initial condition in equation (2.6) for equation (2.5) can be expressed in finite difference form as

$$C_i^0 = 0, \quad x \geq L, \quad t = 0. \quad (4.1)$$

The boundary condition equation (2.7) can be written in finite difference form as

$$C_0^n = C_0, \quad x = 0, \quad t = 0. \quad (4.2)$$

If we employ the forward space method in equation (2.8) to the right boundary condition, we have

$$C_N^n = C_{N-1}^n + \Delta x C_s, \quad x = L, \quad t \geq 0, \quad (4.3)$$

where  $N = x_\infty / \Delta x$  is the grid dimension in the  $x$  direction and  $x_\infty$  is the distance in direction  $x$

at which  $\frac{\partial C}{\partial x} = 0$ ,  $x_\infty$  replaces  $x \rightarrow \infty$  in equation (2.8).

#### 5. Numerical Experiments

Imagine that the concentration of groundwater pollutants  $C$  underneath a landfill and its surrounding area is being evaluated. The studied area is aligned over a longitudinal span, measuring a total of 1.0 km in length ( $C_0 = 1.0$  kg/l,  $D_0 = 1.71$  km<sup>2</sup>/year,  $u_0 = 1.60$  km/year, and  $m = 0.1$  year<sup>-1</sup>). In the numerical test, both space and time are divided into discrete units by  $\Delta x = 0.1$  km and  $\Delta t = 0.00001$  year. The groundwater concentration is estimated by applying the conventional forward time centered space method (FTCS). An analytical solution for an ideal advection-diffusion equation is presented in [26],

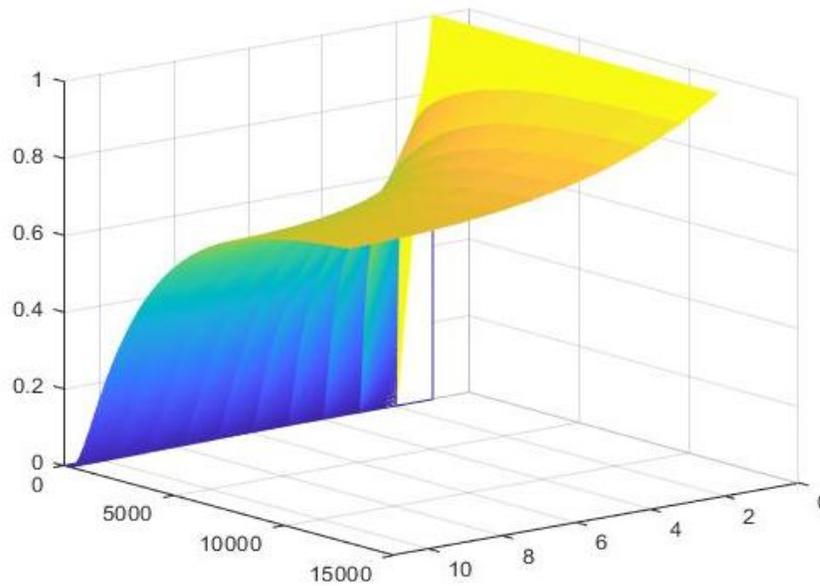
$$\tilde{C}(x,t) = \frac{C_0}{2} \left( \operatorname{erfc} \left( \frac{\frac{x}{f(mt)} - u_0 T}{2\sqrt{D_0 T}} \right) + \exp \left( \frac{u_0 x}{D_0 f(mt)} \right) \operatorname{erfc} \left( \frac{\frac{x}{f(mt)} - u_0 T}{2\sqrt{D_0 T}} \right) \right), \quad (5.1)$$

where  $T$  can be written in terms of  $t$  using the following transformation for an expression of

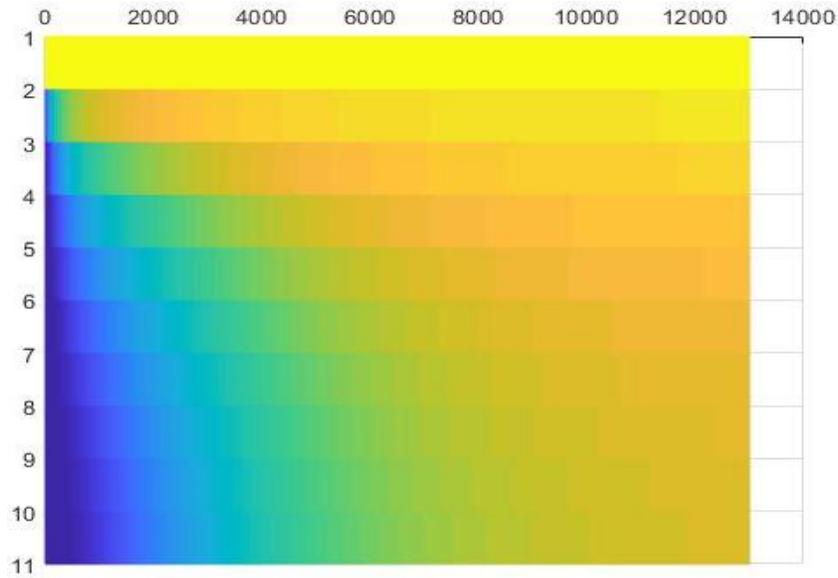
$f(mt)$  as

$$T = \int_0^t \frac{dt}{f(mt)}. \tag{5.2}$$

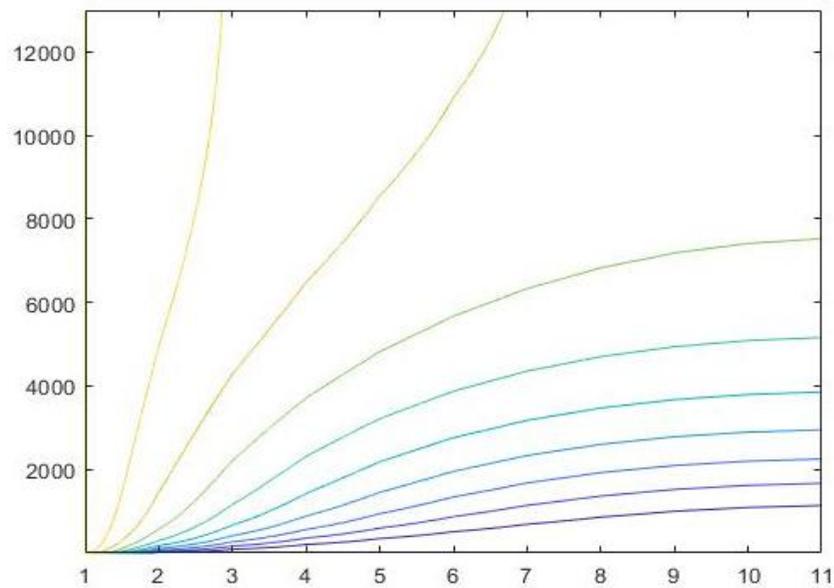
By utilizing the conventional finite difference method, we can solve realistic problems involving unsteady flow in an inhomogeneous medium, as described by equation (3.1) to equation (3.8). Assuming that this medium varies with time according to  $f_1(x,t) = f(mt) = 1 - \sin(mt)$  and considering the linear and quadratic relations defined by equation (2.4), we notice that the estimated groundwater pollution along the path is illustrated in Figure 1 and Table 1. We have employed the conventional finite difference approach by applying  $f_2(x,t) = f(mt) = \exp(mt)$  to solve the problem in this work. The approximated values for unsteady flow through an inhomogeneous medium are illustrated in Figure 5 and Table 2. Both functions are shown in Table 3 and Table 4. The accuracy of both approximations is tested by using the analytical solution and the absolute error.



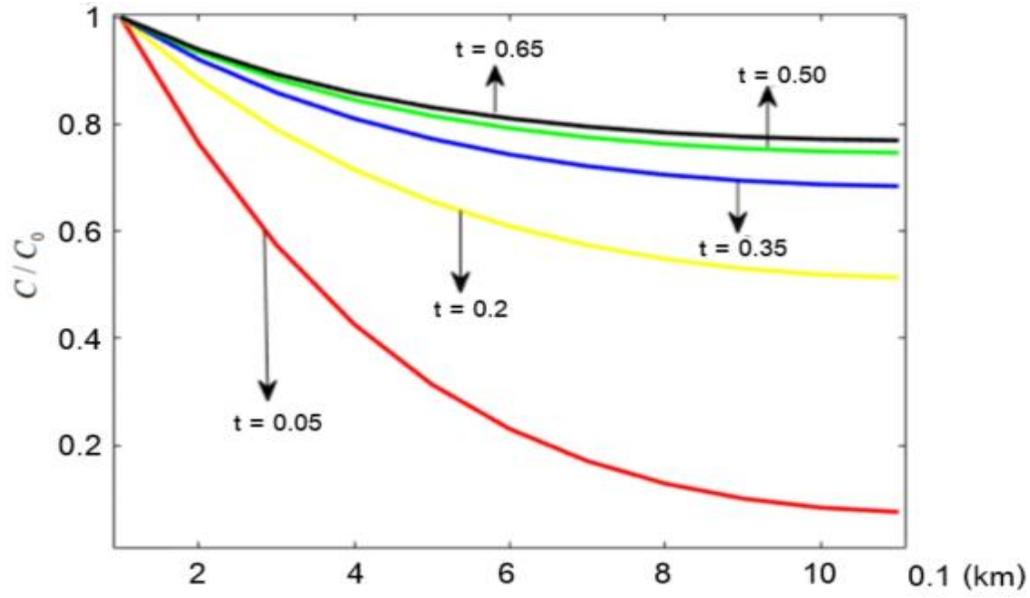
**Figure 1.** Estimated groundwater contaminants were derived using the FTCS technique under unsteady flow conditions ( $f_1(x,t) = f(mt) = 1 - \sin(mt)$ ) for  $m = 10.0 \text{ year}^{-1}$ , in an inhomogeneous medium. Solid squares represent analytical solutions.



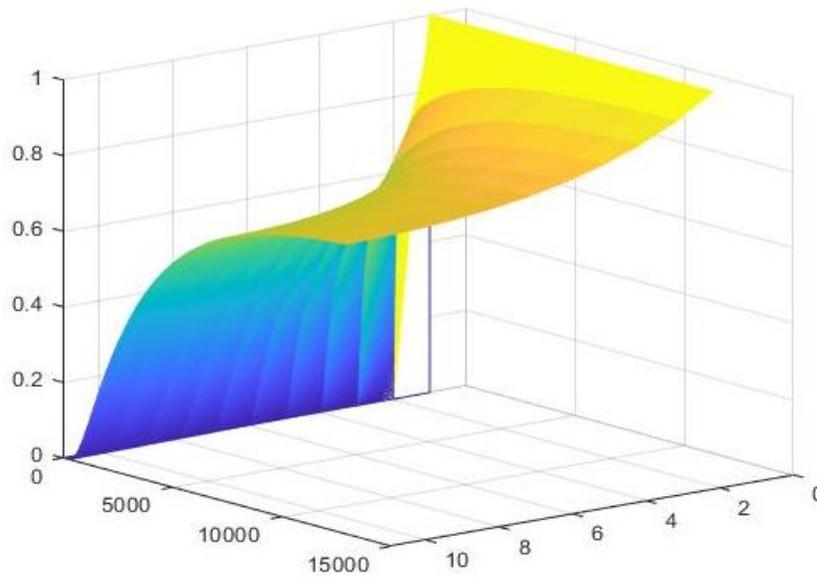
**Figure 2** Top view of estimated groundwater contaminants derived through the FTCS technique, unsteady flow  $f_1(x, t) = f(mt) = 1 - \sin(mt)$ , for  $m = 10.0 \text{ year}^{-1}$ , through inhomogeneous medium. Solid squares represent analytical solutions.



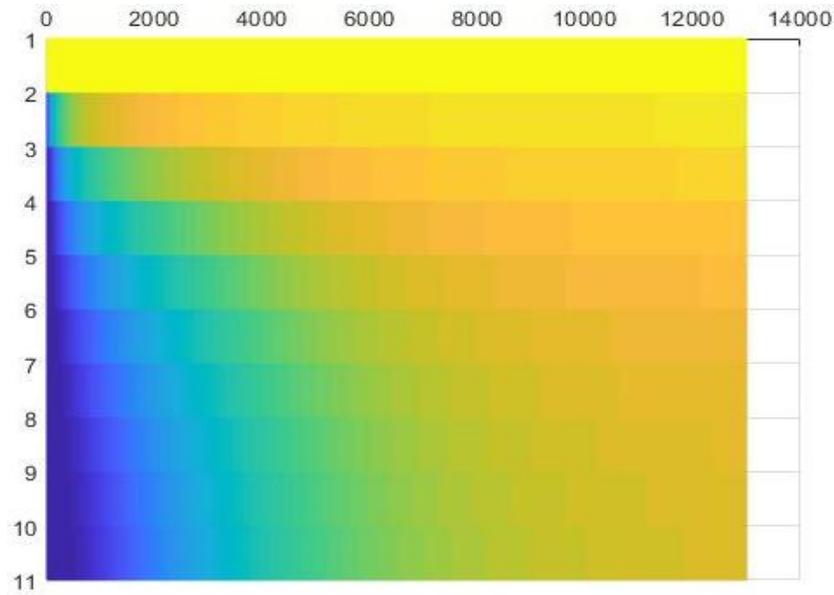
**Figure 3** The contour plot estimated groundwater contaminants derived through the FTCS technique, unsteady flow  $f_1(x, t) = f(mt) = 1 - \sin(mt)$ , for  $m = 10.0 \text{ year}^{-1}$ , through inhomogeneous medium. Solid squares represent analytical solutions.



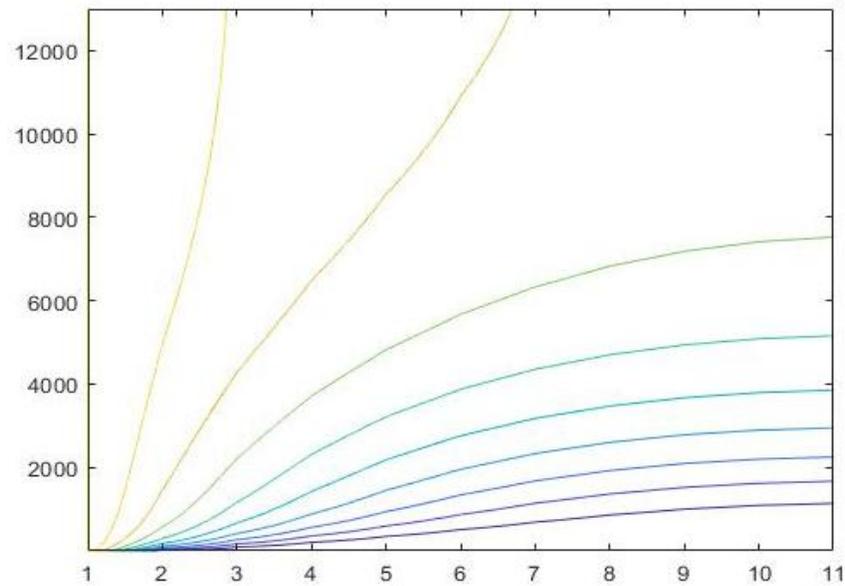
**Figure 4** Concentration distribution pattern of uniform continuous input along a sinusoidally varying, unsteady flow  $f_1(x, t) = f(mt) = 1 - \sin(mt)$ , for  $m = 10.0 \text{ year}^{-1}$ , through inhomogeneous medium. Solid squares represent analytical solutions.



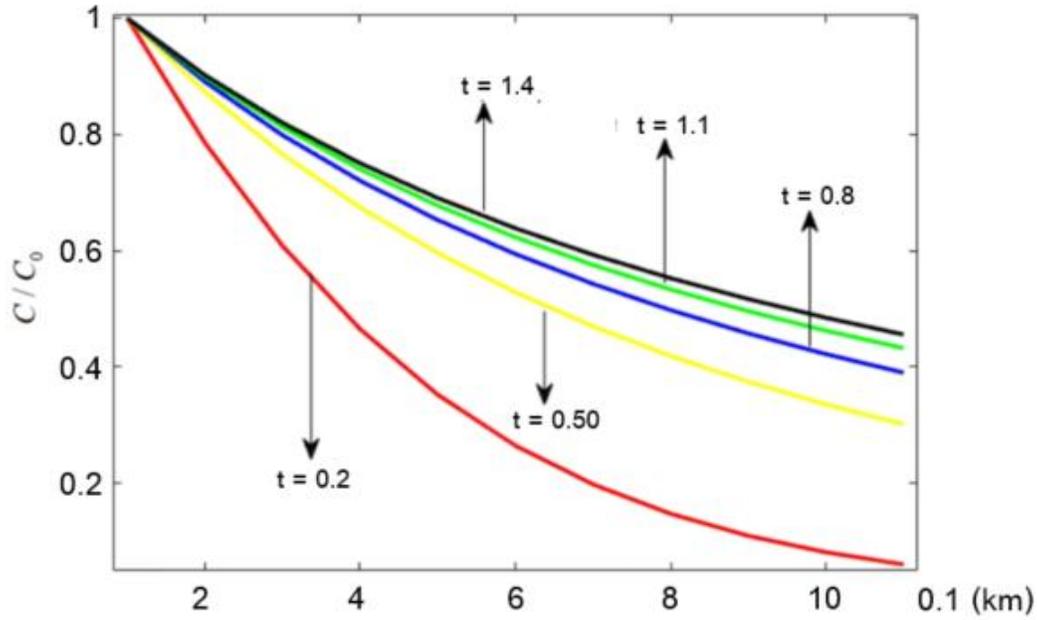
**Figure 5**. Estimated groundwater contaminants were derived using the FTCS technique, with a uniform input and flow for  $f_2(x, t) = f(mt) = \exp(mt)$ , for  $m = 0.1 \text{ year}^{-1}$ . Solid squares represent analytical solutions.



**Figure 6.** Top view of estimated groundwater contaminants derived from the FTCS technique, with a uniform input and flow for  $f_2(x, t) = f(mt) = \exp(mt)$ , where  $m = 0.1 \text{ year}^{-1}$ . Solid squares represent analytical solutions.



**Figure 7.** The contour plot estimated groundwater contaminants derived through the FTCS technique, a uniform input and along a uniform flow for  $f_2(x, t) = f(mt) = \exp(mt)$ , for  $m = 0.1 \text{ year}^{-1}$ . Solid squares represent analytical solutions.



**Figure 8.** Numerical solution for the temporally dependent dispersion from a uniform input and along a uniform flow for  $f_2(x,t) = f(mt) = \exp(mt)$ , for  $m = 0.1 \text{ year}^{-1}$ . Solid squares represent analytical solutions.

**Table 1.** The groundwater pollutant concentration was approximated using the FTCS method over a considered area, spanning from 0.05 to 0.65 years, which is the concentration distribution pattern of a uniform continuous input along a sinusoidally varying, unsteady flow  $f_1(x,t) = f(mt) = 1 - \sin(mt)$  through an inhomogeneous medium. Solid squares represent analytical solutions.

		$C(x,t)$									
$t \text{ (yr)}$	$x \text{ (km)}$										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.05	1.00	0.68092	0.43972	0.27274	0.16421	0.09677	0.05625	0.03255	0.01912	0.01196	0.00881
0.20	1.00	0.82673	0.68455	0.56954	0.47798	0.40646	0.35201	0.31206	0.28439	0.26717	0.26717
0.35	1.00	0.87321	0.77033	0.6876	0.62188	0.57056	0.53145	0.50271	0.48279	0.47037	0.46434
0.50	1.00	0.90058	0.82109	0.75786	0.70804	0.66937	0.64004	0.61856	0.6037	0.59445	0.58997
0.65	1.00	0.91715	0.8518	0.80038	0.76019	0.72919	0.70579	0.68872	0.67693	0.66961	0.66606

**Table 2.** The groundwater pollutant concentration was approximated using the FTCS method over a considered area spanning from 0.20 to 1.30 years. The numerical solution calculates the temporally dependent dispersion from a uniform input and along a uniform flow

$$f_2(x,t) = f(mt) = \exp(mt). \text{ Solid squares represent analytical solutions.}$$

$C(x,t)$											
$t$ (yr)	$x$ (km)										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.20	1.00	0.82673	0.68455	0.56954	0.47798	0.40646	0.35201	0.31206	0.28439	0.26717	0.2588
0.50	1.00	0.90058	0.82109	0.75786	0.70804	0.66937	0.64004	0.61856	0.6037	0.59445	0.58997
0.80	1.00	0.92718	0.8704	0.82612	0.79177	0.76542	0.74561	0.7312	0.72128	0.71512	0.71214
1.10	1.00	0.93693	0.88848	0.85115	0.82247	0.80064	0.78432	0.7725	0.76439	0.75937	0.75694
1.30	1.00	0.93968	0.89359	0.85822	0.83114	0.81058	0.79525	0.78416	0.77657	0.77186	0.76959

**Table 3.** The absolute error of an FTCS method approximation. Concentration distribution pattern of uniform continuous input along a sinusoidally varying, unsteady flow for  $f_1(x,t) = f(mt) = 1 - \sin(mt)$  and  $m = 10.0 \text{ year}^{-1}$  through an inhomogeneous medium.

$$\text{Solid squares represent analytical solutions where } e(x,t) = |C(x,t) - \tilde{C}(x,t)|.$$

$e(x,t)$											
$t$ (yr)	$x$ (km)										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.05	0.00	0.000844	0.001278	0.001334	0.0011677	0.000919	0.000678	0.000481	0.00034	0.000252	0.000209
0.20	0.00	0.000125	0.000228	0.000311	0.0003745	0.000422	0.000456	0.000481	0.000497	0.000507	0.000511
0.35	0.00	0.000006	0.000129	0.000178	0.0002194	0.000251	0.000276	0.000294	0.000307	0.000315	0.000319
0.50	0.00	0.000004	0.000007	0.000108	0.0001329	0.000152	0.000167	0.000178	0.000186	0.000191	0.000193
0.65	0.00	0.000002	0.000004	0.000006	0.000008	0.000009	0.000101	0.000107	0.000112	0.000115	0.000117

**Table 4.** The absolute error of an FTCS method approximation. The numerical solution for the temporally dependent dispersion from a uniform input and along a uniform flow for  $f_1(x,t) = f(mt) = \exp(mt)$  and  $m = 0.1 \text{ year}^{-1}$  through an inhomogeneous medium.

$$\text{Solid squares represent analytical solutions where } e(x,t) = |C(x,t) - \tilde{C}(x,t)|.$$

$e(x,t)$											
$t$ (yr)	$x$ (km)										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.20	0.00	0.000125	0.00023	0.000311	0.00037	0.000422	0.0004565	0.000481	0.000497	0.000507	0.000511
0.50	0.00	0.000042	0.000078	0.000108	0.0001325	0.000152	0.000167	0.000178	0.000186	0.000191	0.000193
0.80	0.00	0.000015	0.000029	0.00004	0.000049	0.000056	0.000061	0.000065	0.000068	0.00007	0.000071
1.10	0.00	0.000007	0.000015	0.00002	0.000025	0.000029	0.000031	0.000034	0.000035	0.000036	0.000037
1.30	0.00	0.00	0.00	0.00	0.00	0.000001	0.0000015	0.0000023	0.0000028	0.0000032	0.0000033

## 6. Discussion

The realistic problems are solved by employing the one-dimensional advection-diffusion equation. The realistic problems are solved by employing the one-dimensional advection-diffusion equation, forward time centered space (FTCS). This approach yields approximate groundwater pollutant concentrations that closely align with the ideal case. In fact, the influence of the sinusoidal unsteadiness in the solute transport along the inhomogeneous medium results in only a slight delay in the development of the solute concentration profiles in comparison to the case without this unsteadiness. This process is illustrated in Figure 1 to Figure 4. Shows the numerical solution for the realistic problem obtained by solving of equation (3.8) using explicit finite difference method (EFDM), where  $f_1(x,t) = f(mt) = 1 - \sin(mt)$  and  $m = 10.0 \text{ year}^{-1}$ . This is illustrated in Figure 5 to Figure 8. Shows the numerical solution for the realistic problem obtained by solving of equation (3.17) using EFDM, where  $f_2(x,t) = f(mt) = \exp(mt)$  and  $m = 0.1 \text{ year}^{-1}$ . It can be observed from the respective figures and tables. In both cases, the groundwater pollutant measurement was simulated for a long period of time, around 1.3 years, as shown in Table 1 to Table 2 and the absolute error values in Table 3 to Table 4. The proposed numerical techniques provide an accurate approximate solution.

## 7. Conclusion

The long-term behavior of groundwater contamination was simulated in heterogeneous soil. An updated model of groundwater quality was applied over a long period of time. The pollutant concentration was approximated using a numerical technique, and the concentration of groundwater pollutants at their monitoring stations was assumed to be the model's initial and boundary conditions. The model solution was approximated using a finite difference method, specifically an explicit finite difference method. We considered two solutions: equation (3.8) for temporally dependent solute dispersion along uniform flow through a homogenous medium, and equation (3.17) for solute dispersion along temporally dependent unsteady flow through an inhomogeneous medium. The continuous point source of uniform nature is considered the origin of the medium. Results are compared to analytical solutions, and satisfactory agreement was found. This work employs the explicit finite difference method to solve the advection-diffusion equation with variable coefficients in semi-infinite media, allowing for arbitrary initial and boundary conditions, as well as variations in dispersion and velocity. The proposed model can be used to provide warnings about the future behavior of groundwater contamination.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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