

Conditional Stability and Uniqueness for Determining Coefficient in Some Multidimensional Consolidation Models

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Abstract. In this paper, we study an inverse problem of reconstruction spatially varying coefficient in a nonlinear Biot's consolidation model with the following observation data: both displacement and pressure in a subdomain $\omega \subset \Omega$. First the given problem is transformed into an optimization problem by using optimal control framework and we establish the existence of minimizer for the control functional. The solution of the optimization problem is based on a non-linear conjugate gradient method. Moreover, the well-posedness of the adjoint problem and the first order necessary optimality conditions are shown. The convergence proof of the adjoint problem is based on using a general compactness criterion. Based on the necessary optimality condition, we prove the Lipschitz stability and the uniqueness for the inverse problem under some *a priori* information.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega := \Gamma$ and denote by ν the outward unit normal of Ω on boundary Γ and $T > 0$ be a fixed time horizon. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. We consider the following hyperbolic-parabolic

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system,

$$\begin{cases} \rho(\mathbf{x})\partial_t^2 \mathbf{u} - \nabla(\lambda^*(\mathbf{x})\partial_t \operatorname{div} \mathbf{u}) - \nabla((\lambda(\mathbf{x}) + \mu(\mathbf{x}))\operatorname{div} \mathbf{u}) \\ \quad - \operatorname{div}(\mu(\mathbf{x})\nabla \mathbf{u}) + \alpha(\mathbf{x})\nabla p = \mathbf{f}(\mathbf{x}, t), & (\mathbf{x}, t) \in Q, \\ c_0(\mathbf{x})\partial_t p + \alpha(\mathbf{x})\operatorname{div} \partial_t \mathbf{u} - \operatorname{div}(\kappa(\mathbf{x})\nabla p) = h(\mathbf{x}, t), & (\mathbf{x}, t) \in Q, \end{cases} \quad (1.1)$$

where the bold notation is used for **vector** and the italic one for *scalar* and \cdot^t denotes the transpose of matrices. The notation t is the time variable, $\mathbf{x} = (x_1, x_2, x_3)$ is a generic point in \mathbb{R}^3 , and \mathbf{f} and h are given sources. Vector $\mathbf{u} = (u_1, u_2, u_3)$ denotes the displacement in the solid and p is the pressure of the fluid. We supplement the above two equations with the following physically motivated initial and boundary conditions:

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, \quad \partial_t \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_1, \quad p(\mathbf{x}, 0) = p_0, \quad \mathbf{x} \in \Omega \quad (1.2)$$

and

$$\mathbf{u}(\mathbf{x}, t) = 0, \quad p(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \Sigma. \quad (1.3)$$

We assume that there is a possibility to provide the additional information for the inverse problem; for instance, the additional measurements are on the arbitrary fixed subdomain $\omega \subset \Omega$,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_m(\mathbf{x}, t), \quad p(\mathbf{x}, t) = p_m(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \omega \times (0, T), \quad (1.4)$$

where the functions \mathbf{u}_m, p_m are known functions. The positive function ρ denotes the local density. The parameter λ^* is a positive function related to a viscosity term describing consolidation effects of the second type. Coefficients $\lambda, \mu \in C^2(\overline{\Omega})$ are the popular Lamé parameters involved in the linear theory of elasticity [14], satisfy

$$\mu(\mathbf{x}) > 0, \quad \lambda(\mathbf{x}) + 2\mu(\mathbf{x}) > 0, \quad \forall \mathbf{x} \in \overline{\Omega}. \quad (1.5)$$

The function α is the coupling deformation-pressure which implies the primary effects of consolidation, called the Biot-Willis coefficient. The non-negative function c_0 represent the combination of fluid compressibility and porosity. Finally, the viscosity effects of the fluid and the permeability of the medium are related to $\kappa > 0$. System (1.1) describes the motion in the solid as well as in the fluid. Its derivation follows from the combination of the Hooke law with the momentum balance equations in the structure and the combination of the Darcy law with the fluid mass conservation [5–9]. We refer to [15] for a detailed statement of Model (1.1). They are essentially concerned with the construction of a single solution. In [11], the existence-uniqueness theory is completed by the study of the long-time behaviour of \mathbf{u} when $\rho > 0$ and $\lambda^* = 0$ while [1] consider the complete system (1.1) and provides a comparison of different models through an asymptotic analysis based upon the local density and the secondary consolidation parameter.

The main subject of this paper is the inverse problem of determining the spatially varying density $\alpha(\mathbf{x})$ uniquely from observed data of displacement vector \mathbf{u} and the pressure p on a suitable subdomain $\omega \subset \Omega$. We use the parameter as control variable to minimize of the mismatch between the computed and the measured displacement \mathbf{u} and pressure p under the constraint of

the Biot system. Secondly, we use the optimal control framework to the Biot's model in order to establish a stability estimate for our inverse problem.

More precisely, let (\mathbf{u}, p) and $(\tilde{\mathbf{u}}, \tilde{p})$ be the solutions of the system (1.1) with corresponding spatially varying density. Then, for small T , there exists a constant $C > 0$ depending only on T and Ω satisfying

$$\|\alpha - \tilde{\alpha}\|_{H^1(\Omega)} \leq C \left(I_c + \|\mathbf{u}_m - \tilde{\mathbf{u}}_m\|_{L^2(0,T;L^2(\omega))} + \|p_m - \tilde{p}_m\|_{L^2(0,T;L^2(\omega))} \right),$$

where

$$I_c = \|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{H^2(\Omega)} + \|\mathbf{u}_1 - \tilde{\mathbf{u}}_1\|_{H_0^1(\Omega)} + \|p_0 - \tilde{p}_0\|_{H^2(\Omega)}. \quad (1.6)$$

In the framework of linear poroelasticity, the Biot–Willis coefficient $\alpha(x)$ is a fundamental parameter governing the coupling between the displacement field of the solid skeleton and the pore fluid pressure. It appears explicitly in both the momentum balance equation and the mass conservation law, thereby linking the divergence of the solid velocity $\partial_t \mathbf{u}$ to the evolution of the pressure p . This coupling plays a decisive role in the hyperbolic–parabolic nature of the system and significantly influences the well-posedness and stability properties of its solutions. From a physical viewpoint, $\alpha(x)$ measures the contribution of pore pressure to the effective stress in the porous medium and depends on intrinsic microstructural characteristics, such as porosity and compressibility [6, 10].

From the perspective of inverse problems, the spatial variability of $\alpha(x)$ raises substantial analytical difficulties, particularly with respect to observability and uniqueness from partial or internal measurements. The identification of $\alpha(x)$ therefore requires the development of suitable Carleman estimates for strongly coupled systems, which allow one to derive quantitative stability inequalities. Such estimates form the cornerstone for proving global uniqueness results and logarithmic or Hölder-type stability in the reconstruction of $\alpha(x)$, thereby justifying the central role of this coefficient in inverse poroelasticity problems [17].

However, to the best of our knowledge, the inverse problem identification for the Biot consolidation model has not been studied thoroughly yet using optimal control approach. In the literature, several works that are interested in stability results using Carleman's estimation. Bellassoued and Yamamoto [3] established Carleman estimates with second large parameter for a coupled parabolic-hyperbolic system, a thermoelastic plate system, and a thermoelasticity system with residual stress. According to the linear theory of thermoelasticity, Bellassoued and Yamamoto [4] consider a bounded and isotropic body whose mechanical behavior is described by the Lamé system coupled with the heat equation. Assuming the null surface displacement on the whole boundary, they prove a Hölder stability estimate for the inverse problem of determining the heat source only by observation of surface traction on a suitable subdomain along a sufficiently large time interval using Carleman estimate for thermoelasticity system. Then, in a thermoelastic model, B. Wu and J. Liu [16] study the inverse problem of determining two spatially varying coefficients with the following observation data: displacement in a subdomain ω satisfying $\omega \subset \Omega$ along a

sufficiently large time interval and both displacement and temperature at a suitable time over the whole spatial domain. Based on a Carleman estimate on the hyperbolic-parabolic system, B. Wu and J. Liu prove the Lipschitz stability and the uniqueness for this inverse problem under some a priori information. For Biot consolidation system in poro-elasticity, Bellassoued and Riahi [2] established a local Carleman estimate and proved uniqueness and Holder stability in determining one the one hand a physical parameter λ^* and on the other hand the spatially varying density $\alpha(\mathbf{x})$ by a single measurement of solution over $\omega \times (0, T)$, where $T > 0$ is a sufficiently large time. Obviously our result is not new but it presents an originality on two essential points. The first originality is that our result use an optimal regularity of the solution compared with the one used in [2]. Indeed, with only the L^2 regularity on the observations and at most H^2 on the initial conditions one could have our stability result, whereas in [2], they supposed rather strong regularities: a regularity H^7 on the displacement and H^5 on the pressure. The second essential point is that our result works even for a rather small time T that is not the case using Carleman's estimates. Our method for the stability estimate is based on the tool of optimal control approach developed by Gnanavel *et al.* in [13].

The paper is organized as follows: In Section 2, we present existence analysis of the direct problem and we establish some appropriate regularities and a priori estimates of the unique solution. In Section 3, we transform the given problem into an optimal control problem by using the optimization theory and prove the existence of minimizer and we derive the necessary optimality condition. Making use of the necessary conditions and some energy estimates, we complete the stability result in Section 4.

2. EXISTENCE ANALYSIS

Throughout this paper, we use some standard notations. For vector $\mathbf{v} = (v_1, v_2, v_3)^{tr}$ we set

$$\nabla \mathbf{v} = \left(\partial_{x_j} v_i \right)_{1 \leq i, j \leq 3} \quad (2.1)$$

and

$$|\mathbf{v}| = \left(\sum_{i=1}^3 |v_i|^2 \right)^{1/2}, \quad |\nabla \mathbf{v}| = \left(\sum_{i,j=1}^3 |\partial_{x_j} v_i|^2 \right)^{1/2}. \quad (2.2)$$

As we study time-dependent problems, we introduce the classical Banach spaces $L^p(0, T; V)$ where V is a Hilbert space. We use the bold notation for vector spaces. For example, $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^3$. At last, we suppose that all the physical data $\varrho(\mathbf{x})$, $\lambda^*(\mathbf{x})$, $\lambda(\mathbf{x})$, $\mu(\mathbf{x})$, $c_0(\mathbf{x})$, $\alpha(\mathbf{x})$, $\kappa(\mathbf{x})$ are in $L^\infty(\Omega)$ and that they are respectively greater than a strictly positive constant $\hat{\varrho}$, $\hat{\lambda}^*$, $\hat{\lambda}$, $\hat{\mu}$, \hat{c}_0 and $\hat{\kappa}$ until other condition is specified. The fonctionnel space $\mathcal{D}(0, T)$ is the set of C^∞ function with compact support in $[0, T]$.

Let $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$, $\mathbf{u}_1 \in L^2(\Omega)$ and $p_0 \in L^2(\Omega)$. We associate to the problem formely described by (1.1)-(1.3) the following variational formulation:

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}, p) \in L^\infty(0, T; \mathbf{H}_0^1(\Omega)) \times L^2(0, T; H_0^1(\Omega)) \text{ such that} \\ \partial_t \mathbf{u} \in L^2(0, T; \mathbf{L}^2(\Omega)), \quad \sqrt{\lambda^*} \operatorname{div} \partial_t \mathbf{u} \in L^2(0, T; \mathbf{L}^2(\Omega)), \\ \partial_t^2 \mathbf{u} \in L^2(0, T; \mathbf{H}^{-1}(\Omega)), \quad \partial_t(c_0(\mathbf{x})p) \in L^2(0, T; H^{-1}(\Omega)), \\ \text{verifying for a.e. } t \in]0, T[, \forall (v, q) \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega) : \\ \langle \rho \partial_t^2 \mathbf{u}, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_Q \lambda^* \operatorname{div} \partial_t \mathbf{u} \operatorname{div} v \, d\mathbf{x} \, dt + \int_Q (\lambda + \mu) \operatorname{div}(\mathbf{u}) \operatorname{div} v \, d\mathbf{x} \, dt \\ + \int_Q \mu \nabla \mathbf{u} \otimes \nabla v \, d\mathbf{x} \, dt - \int_Q \alpha p \operatorname{div} v \, d\mathbf{x} \, dt = \int_Q \mathbf{f} \cdot v \, d\mathbf{x} \, dt, \\ \langle c_0 \partial_t p, q \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_Q \alpha \operatorname{div}(\partial_t \mathbf{u}) q \, d\mathbf{x} \, dt + \int_Q \kappa \nabla p \cdot \nabla q \, d\mathbf{x} \, dt = \int_Q h q \, d\mathbf{x} \, dt, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad p(\mathbf{x}, 0) = p_0(\mathbf{x}), \quad \partial_t \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_1(\mathbf{x}), \end{array} \right. \quad (2.3)$$

The couple (\mathbf{u}, p) verify (2.3) is called weak solution of the Biot's model (1.1)-(1.3).

Now, we will establish a proposition dealing with the regularity of the Biot system solution. The aim is to improve the regularity results given in [1, thm 3], in order to satisfy the assumptions that would be taken in the stability result (Theorem 4.1).

Here we assume that f and h satisfy the following hypothesis:

$$\mathbf{f} \in W^{1,2}(0, T; \mathbf{L}^2(\Omega)), \quad h \in W^{1,2}(0, T; L^2(\Omega)). \quad (2.4)$$

First, we will make use of the following Gronwall lemma.

Lemma 2.1. (Gronwall's lemma)

Let $\gamma \in \mathbb{R}$, $\phi \in C^1([0, T], \mathbb{R})$ and $f \in C^0([0, T], \mathbb{R})$ with

$$\phi'(t) \leq \gamma \phi(t) + f(t),$$

then

$$\forall t \in [0, T], \quad \phi(t) \leq e^{\gamma t} \phi(0) + \int_0^t e^{\gamma(t-s)} f(s) \, ds. \quad (2.5)$$

Proposition 2.1 (A priori estimate). Let (\mathbf{u}, p) be the weak solution of equations system (2.3), with initial conditions $\mathbf{u}_0, \mathbf{u}_1$ and p_0 .

i) If $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$, $\mathbf{u}_1 \in \mathbf{L}^2(\Omega)$, $p_0 \in L^2(\Omega)$ and \mathbf{f}, h verify the regularity (2.4), then

$$\begin{aligned} \mathbf{u} &\in W^{1,\infty}(0, T; \mathbf{L}^2(\Omega)) \cap L^\infty(0, T; \mathbf{H}_0^1(\Omega)) \cap W^{1,2}(0, T; \mathbf{H}_0^1(\Omega)), \\ p &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)). \end{aligned} \quad (2.6)$$

Moreover, we have the following estimates:

$$\begin{aligned} &\max_{0 \leq t \leq T} \left\{ \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 \right\} \\ &\leq \max(1, C) e^{CT} \left(\Lambda_0 + \|\mathbf{f}\|_{L^2(0, T; \mathbf{L}^2(\Omega))}^2 + \|h\|_{L^2(0, T; L^2(\Omega))}^2 \right) \end{aligned} \quad (2.7)$$

and

$$\int_0^T \left(\|\operatorname{div}(\partial_t \mathbf{u})\|_{L^2(\Omega)}^2 + \|\nabla p\|_{L^2(\Omega)}^2 \right) dt \leq C \max(1, C)(1+T)e^{CT} \left(\Lambda_0 + \|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))}^2 + \|h\|_{L^2(0,T;L^2(\Omega))}^2 \right), \quad (2.8)$$

where C is a positive constant depend only the hypotheses on the physical parameters and

$$\Lambda_0 = \|\mathbf{u}_0\|_{\mathbf{H}_0^1(\Omega)}^2 + \|\mathbf{u}_1\|_{L^2(\Omega)}^2 + \|p_0\|_{L^2(\Omega)}^2. \quad (2.9)$$

ii) Moreover if $p_0 \in H_0^1(\Omega)$, then $p \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega))$ and $\partial_t p \in L^2(0, T; L^2(\Omega))$.

Also we have the following estimate

$$\max_{0 \leq t \leq T} \left\{ \|p\|_{H_0^1(\Omega)}^2 \right\} + \|p\|_{L^2(0,T;H^2(\Omega))}^2 + \|\partial_t p\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \max(1, C)(1+T)e^{CT} \left(\Lambda_1 + \|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))}^2 + \|h\|_{L^2(0,T;L^2(\Omega))}^2 \right) \quad (2.10)$$

where

$$\Lambda_1 = \|\mathbf{u}_0\|_{\mathbf{H}_0^1(\Omega)}^2 + \|\mathbf{u}_1\|_{L^2(\Omega)}^2 + \|p_0\|_{H_0^1(\Omega)}^2. \quad (2.11)$$

iii) Suppose that the condition on \mathbf{f} and h (2.4) holds. For any initial data $(\mathbf{u}_0, \mathbf{u}_1, p_0) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times H^2(\Omega)$, then

$$p \in L^\infty(0, T; H^2(\Omega)), \quad \partial_t p \in L^\infty(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)), \quad \partial_t^2 p \in L^\infty(0, T; H^{-1}(\Omega)) \quad (2.12)$$

and

$$\max_{0 \leq t \leq T} \left\{ \|p\|_{H^2(\Omega)}^2 + \|\partial_t p\|_{L^2(\Omega)}^2 \right\} + \|\partial_t p\|_{L^2(0,T;H_0^1(\Omega))}^2 + \|\partial_t^2 p\|_{L^2(0,T;H^{-1}(\Omega))}^2 \leq C \max(1, C)(1+T)^2 e^{2CT} \left(\Lambda_2 + \|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))}^2 + \|h\|_{W^{1,2}(0,T;L^2(\Omega))}^2 \right), \quad (2.13)$$

where C is a positive constant depend only the hypotheses on the physical parameters and

$$\Lambda_2 = \|\mathbf{u}_0\|_{\mathbf{H}_0^1(\Omega)}^2 + \|\mathbf{u}_1\|_{\mathbf{H}_0^1(\Omega)}^2 + \|p_0\|_{H^2(\Omega)}^2 + \|h(\cdot, 0)\|_{L^2(\Omega)}^2. \quad (2.14)$$

iv) (Higher order regularity.) If $(\mathbf{u}_0, \mathbf{u}_1, p_0) \in \mathbf{H}^{m+2}(\Omega) \times \mathbf{H}^{m+1}(\Omega) \times \mathbf{H}^{m+2}(\Omega)$ and second members $\partial_t^j \mathbf{f} \in L^2(0, T; \mathbf{H}^{m-j+2}(\Omega))$, $\partial_t^j h \in L^2(0, T; H^{m-j+2}(\Omega))$, $j = 1, \dots, m$, then

$$\mathbf{u} \in \bigcap_{j=0}^{m+2} C^j(0, T; \mathbf{H}^{m+2-j}(\Omega)), \quad p \in \bigcap_{j=0}^{m+2} C^j(0, T; H^{m+2-j}(\Omega)) \quad (2.15)$$

and we have the following energy estimate

$$\max_{0 \leq t \leq T} \left\{ \sum_{j=0}^{m+2} \|\partial_t^j \mathbf{u}\|_{\mathbf{H}^{m+2-j}(\Omega)}^2 + \sum_{j=0}^{m+2} \|\partial_t^j p\|_{H^{m+2-j}(\Omega)}^2 \right\} \leq C(1+T)^{m+2} e^{2CT} \left(\Lambda_3 + \|\mathbf{f}\|_{W^{j,2}(0,T;\mathbf{H}^{m-j+2}(\Omega))}^2 + \|h\|_{W^{j,2}(0,T;H^{m-j+2}(\Omega))}^2 \right) \quad (2.16)$$

where

$$\begin{aligned} \Lambda_3 = & \|\mathbf{u}_0\|_{\mathbf{H}^{m+2}(\Omega)}^2 + \|\mathbf{u}_1\|_{\mathbf{H}^{m+1}(\Omega)}^2 + \|p_0\|_{\mathbf{H}^{m+2}(\Omega)}^2 \\ & + \sum_{j=0}^{m+2} \left\| \partial_t^j \mathbf{f}|_{t=0} \right\|_{\mathbf{H}^{m+1-j}}^2 + \sum_{j=0}^{m+2} \left\| \partial_t^j h|_{t=0} \right\|_{\mathbf{H}^{m+1-j}}^2. \end{aligned} \tag{2.17}$$

Proof. i) We multiply the first equation of (1.1) by $\partial_t \mathbf{u}$ and the second one by p , apply the divergence theorem, and use the Dirichlet boundary condition (1.3) and we integrate over Ω . Then by summing each term, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} & \left(\|\sqrt{\varrho} \partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + \|\sqrt{\lambda + \mu} \operatorname{div} \mathbf{u}\|_{L^2(\Omega)}^2 + \|\sqrt{\mu} \nabla \mathbf{u}\|_{L^2(\Omega)}^2 + \|\sqrt{c_0} p\|_{L^2(\Omega)}^2 \right) \\ & + \|\sqrt{\lambda^*} \operatorname{div} \partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + \|\sqrt{\kappa} \nabla p\|_{L^2(\Omega)}^2 = \int_{\Omega} \left(\nabla \alpha p \partial_t \mathbf{u} + \mathbf{f} \cdot \partial_t \mathbf{u} + h p \right) dx. \end{aligned} \tag{2.18}$$

Next, by using the Cauchy-Schwarz, Young and Poincaré inequalities and by assuming that the coefficient ϱ and c_0 are in $L^\infty(\Omega)$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} & \left(\|\sqrt{\varrho} \partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + \|\sqrt{\lambda + \mu} \operatorname{div} \mathbf{u}\|_{L^2(\Omega)}^2 + \|\sqrt{\mu} \nabla \mathbf{u}\|_{L^2(\Omega)}^2 + \|\sqrt{c_0} p\|_{L^2(\Omega)}^2 \right) \\ & + \|\sqrt{\lambda^*} \operatorname{div} \partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + \|\sqrt{\kappa} \nabla p\|_{L^2(\Omega)}^2 \\ & \leq C \left(\|\sqrt{\varrho} \partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + \|\sqrt{c_0} p\|_{L^2(\Omega)}^2 + \|\mathbf{f}\|_{L^2(\Omega)}^2 + \|h\|_{L^2(\Omega)}^2 \right) \end{aligned} \tag{2.19}$$

Applying Gronwall lemma and using hypothesis on all the physical data, we obtain have:

$$\begin{aligned} & \|\partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)}^2 + \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 \\ & \leq \max(1, C) e^{CT} \left(\|\mathbf{u}_1\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{u}_0\|_{L^2(\Omega)}^2 + \|\nabla \mathbf{u}_0\|_{L^2(\Omega)}^2 + \|p_0\|_{L^2(\Omega)}^2 \right. \\ & \quad \left. + \|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))}^2 + \|h\|_{L^2(0,T;L^2(\Omega))}^2 \right). \end{aligned} \tag{2.20}$$

Now, for $t \in (0, T)$, we integrate (2.19) over $(0, t)$, we obtain

$$\begin{aligned} & \|\partial_t \mathbf{u}\|_2^2 + \|\operatorname{div} \mathbf{u}\|_2^2 + \|\nabla \mathbf{u}\|_2^2 + \|p\|_2^2 + \int_0^t \left(\|\operatorname{div} \partial_t \mathbf{u}(s)\|_2^2 + \|\nabla p(s)\|_2^2 \right) ds \\ & \leq C \left(\|\mathbf{u}_1\|_2^2 + \|\operatorname{div} \mathbf{u}_0\|_2^2 + \|\nabla \mathbf{u}_0\|_2^2 + \|p_0\|_2^2 \right. \\ & \quad \left. + \int_0^t \left(\|\partial_t \mathbf{u}(s)\|_2^2 + \|p(s)\|_2^2 + \|\mathbf{f}(s)\|_2^2 + \|h(s)\|_2^2 \right) ds \right). \end{aligned} \tag{2.21}$$

From (2.21), we deduce the estimate (2.8).

ii) On the other hand, from the second equation of the system (1.1), we write

$$c_0 \partial_t p - \operatorname{div}(\kappa \nabla p) = -\alpha \operatorname{div}(\partial_t \mathbf{u}) + h \in L^2(0, T; L^2(\Omega)), \tag{2.22}$$

with $p(t = 0) = p_0 \in H_0^1(\Omega)$ and $p = 0$ on Σ . From (2.20), (2.21) and (2.3), we have $p \in L^2(0, T; H_0^1(\Omega))$ and $\partial_t p \in L^2(0, T; H^{-1}(\Omega))$, then we refer to [12, Thm 6 p 360], we deduce that $p \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega))$, $\partial_t p \in L^2(0, T; L^2(\Omega))$ and the desired

estimate (2.10).

iii) We take the time derivative of the second equation of the system (1.1)

$$c_0(\mathbf{x})\partial_t^2 p + \alpha(\mathbf{x})\operatorname{div}(\partial_t \mathbf{u}) - \operatorname{div}(\kappa(\mathbf{x})\nabla \partial_t p) = \partial_t h(\mathbf{x}, t), \quad (2.23)$$

with initial condition, for $\mathbf{x} \in \Omega$

$$\partial_t p(\mathbf{x}, 0) = -(c_0(\mathbf{x}))^{-1} \left(\alpha(\mathbf{x})\operatorname{div} \mathbf{u}_1 + \operatorname{div}(\kappa(\mathbf{x})\nabla p_0) + h(\mathbf{x}, 0) \right). \quad (2.24)$$

Using the hypothesis $\mathbf{u}_0 \in H_0^1(\Omega)$, $\mathbf{u}_1 \in H_0^1(\Omega)$ and $p_0 \in H^2(\Omega)$, we deduce that

$$\partial_t p(t=0) \in L^2(\Omega), \quad (2.25)$$

and we apply (i) of Proposition 2.1 to the system (2.23)-(2.24), we conclude (2.12) and (2.13).

iv) We assume that the coefficients $\varrho, \lambda, \mu, \lambda^*, \alpha, c_0, \kappa$ belong to $C^\infty(\overline{\Omega})$, that the source terms \mathbf{f} and h are smooth in space and time, and that the initial data $(\mathbf{u}_0, \mathbf{u}_1, p_0)$ satisfy the usual compatibility conditions up to the required order. We show that the solution (\mathbf{u}, p) inherits the same degree of regularity.

Let $k \geq 0$ be an integer. We formally differentiate the Biot system k times with respect to time. Setting

$$\mathbf{u}^{(k)} := \partial_t^k \mathbf{u}, \quad p^{(k)} := \partial_t^k p,$$

we obtain the system

$$\begin{cases} \varrho \partial_t^2 \mathbf{u}^{(k)} - \operatorname{div} \sigma(\mathbf{u}^{(k)}) - \nabla(\lambda^* \operatorname{div} \partial_t \mathbf{u}^{(k)}) + \alpha \nabla p^{(k)} = \partial_t^k \mathbf{f}, \\ c_0 \partial_t p^{(k)} + \alpha \operatorname{div}(\partial_t \mathbf{u}^{(k)}) - \operatorname{div}(\kappa \nabla p^{(k)}) = \partial_t^k h, \end{cases}$$

supplemented with homogeneous boundary conditions.

This system has exactly the same structure as the original one. Therefore, the energy estimates established in points (i)–(iii) apply to $(\mathbf{u}^{(k)}, p^{(k)})$ provided the right-hand sides are regular enough.

Assume that for some $m \geq 0$ the following regularity holds:

$$\mathbf{u} \in \bigcap_{j=0}^{m+1} C^j([0, T]; \mathbf{H}^{m+1-j}(\Omega)), \quad p \in \bigcap_{j=0}^{m+1} C^j([0, T]; H^{m+1-j}(\Omega)).$$

We show that this implies the same property with m replaced by $m + 1$.

Fix $t \in (0, T)$. The momentum equation can be rewritten as a Lamé-type elliptic problem:

$$-\operatorname{div} \sigma(\mathbf{u}(t)) = \varrho \partial_t^2 \mathbf{u}(t) + \nabla(\lambda^* \operatorname{div} \partial_t \mathbf{u}(t)) - \alpha \nabla p(t) + \mathbf{f}(t).$$

By the induction hypothesis,

$$\partial_t^2 \mathbf{u}(t) \in \mathbf{H}^{m-1}(\Omega), \quad \operatorname{div} \partial_t \mathbf{u}(t) \in H^m(\Omega), \quad \nabla p(t) \in \mathbf{H}^m(\Omega),$$

and $\mathbf{f}(t) \in \mathbf{H}^m(\Omega)$. Hence the right-hand side belongs to $\mathbf{H}^{m-1}(\Omega)$.

Since $\mathbf{u}(t) = 0$ on Γ and the Lamé operator is strongly elliptic, standard elliptic regularity yields

$$\mathbf{u}(t) \in \mathbf{H}^{m+1}(\Omega), \quad \|\mathbf{u}(t)\|_{\mathbf{H}^{m+1}} \leq C\|\text{RHS}(t)\|_{\mathbf{H}^{m-1}}.$$

Similarly, the fluid equation at fixed time reads

$$-\text{div}(\kappa \nabla p(t)) = -c_0 \partial_t p(t) - \alpha \text{div}(\partial_t \mathbf{u}(t)) + h(t).$$

By the induction hypothesis,

$$\partial_t p(t) \in H^m(\Omega), \quad \text{div}(\partial_t \mathbf{u}(t)) \in H^m(\Omega), \quad h(t) \in H^m(\Omega),$$

hence the right-hand side belongs to $H^m(\Omega)$. Using elliptic regularity with homogeneous Dirichlet boundary conditions, we infer

$$p(t) \in H^{m+2}(\Omega).$$

The time regularity follows by differentiating the equations once more and applying points (i)–(iii) to the differentiated system. Thus,

$$\mathbf{u} \in \bigcap_{j=0}^{m+2} C^j([0, T]; \mathbf{H}^{m+2-j}(\Omega)), \quad p \in \bigcap_{j=0}^{m+2} C^j([0, T]; H^{m+2-j}(\Omega)).$$

By induction on m , this completes the proof of higher-order regularity. □

3. VARIATIONAL FORMULATION OF THE INVERSE PROBLEM

In this section, the optimal control framework for the Biot’s equations is described. The existence of the control, the complete optimality system and the existence of the Lagrange multipliers are shown.

3.1. Existence of the control. The measurement functions satisfy the following assumption

$$\mathbf{u}_m(\mathbf{x}, t) \in L^2(0, T; L^2(\omega)), \quad p_m(\mathbf{x}, t) \in L^2(0, T; L^2(\omega)) \tag{3.1}$$

Now we define the admissible set

$$C_{ad} = \left\{ \alpha \in W^{1,\infty}(\Omega) : \alpha(\mathbf{x}) \in [m, M], \forall \mathbf{x} \in \Omega \right\}, \tag{3.2}$$

and the optimal control problem is stated as follows: Find α solving the minimizing problem

$$(\mathcal{P}) \begin{cases} \min_{\alpha \in C_{ad}} \mathcal{J}(\alpha), \\ \text{subject to equations (1.1)–(1.3),} \end{cases} \tag{3.3}$$

wher the cost function \mathcal{J} is given by the mismatch between the computed and the measured displacement and pressure over some sub-domain observation $\omega \subset \Omega$:

$$\mathcal{J}(\alpha) = \frac{1}{2} \int_{Q_\omega} \left(|\mathbf{u}(\alpha) - \mathbf{u}_m(\mathbf{x}, t)|^2 + |p(\alpha) - p_m(\mathbf{x}, t)|^2 \right) d\mathbf{x} dt + \frac{\epsilon}{2} \mathcal{R}(\alpha), \tag{3.4}$$

with $Q_\omega = \omega \times (0, T)$. Here ϵ is the regularization parameter. In the cost functional \mathcal{J} , the term \mathcal{R} denotes a Tikhonov-like regularization term used to weigh the impact of the regularization in the minimize procedure. In this work, \mathcal{R} could be taken as,

$$\mathcal{R}(\alpha) = \int_{\Omega} |\nabla \alpha(\mathbf{x})|^2 dx. \quad (3.5)$$

This reduced cost functional will be used in the following theorem concerning the existence of an optimal solution for (3.3).

Theorem 3.1 (Existence of minimizer). *Under the previous assumptions, for the regularization parameter $\epsilon \geq 0$, the control problem (3.3) has at least one solution.*

Proof. We aim to prove the existence of $\alpha^* \in C_{ad}$ such that

$$\mathcal{J}(\alpha^*) = \inf_{\alpha \in C_{ad}} \mathcal{J}(\alpha).$$

Since \mathcal{J} is nonnegative, there exists a minimizing sequence $\{\alpha^{(k)}\}_{k \geq 1} \subset C_{ad}$ such that

$$\mathcal{J}(\alpha^{(k)}) \rightarrow \inf_{\alpha \in C_{ad}} \mathcal{J}(\alpha).$$

By definition of C_{ad} , the sequence $\{\alpha^{(k)}\}$ is bounded in $W^{1,\infty}(\Omega)$. Hence, by the Banach–Alaoglu theorem, there exist a subsequence (not relabeled) and a function $\alpha^* \in W^{1,\infty}(\Omega)$ such that

$$\alpha^{(k)} \rightharpoonup^* \alpha^* \quad \text{in } W^{1,\infty}(\Omega). \quad (3.6)$$

Since C_{ad} is closed and convex in $W^{1,\infty}(\Omega)$, it is weakly-* closed, and therefore $\alpha^* \in C_{ad}$.

Let $(\mathbf{u}^{(k)}, p^{(k)})$ denote the solution of the state system associated with $\alpha^{(k)}$. By the a priori estimates stated in Proposition 2.1, the sequence $\{(\mathbf{u}^{(k)}, p^{(k)})\}$ is bounded in

$$L^\infty(0, T; H_0^1(\Omega)) \times L^2(0, T; H_0^1(\Omega)),$$

with $\partial_t \mathbf{u}^{(k)}$ bounded in $L^\infty(0, T; L^2(\Omega))$. Therefore, up to a subsequence, there exist (\mathbf{u}^*, p^*) such that

$$\begin{aligned} \mathbf{u}^{(k)} &\rightharpoonup^* \mathbf{u}^* \quad \text{in } L^\infty(0, T; H_0^1(\Omega)), \\ \partial_t \mathbf{u}^{(k)} &\rightharpoonup^* \partial_t \mathbf{u}^* \quad \text{in } L^\infty(0, T; L^2(\Omega)), \\ p^{(k)} &\rightharpoonup p^* \quad \text{in } L^2(0, T; H_0^1(\Omega)). \end{aligned}$$

Moreover, by the Aubin–Lions compactness lemma, we have the strong convergences

$$\mathbf{u}^{(k)} \rightarrow \mathbf{u}^* \quad \text{in } L^2(Q), \quad p^{(k)} \rightarrow p^* \quad \text{in } L^2(Q). \quad (3.7)$$

The weak-* convergence (3.6) together with the weak convergence of $p^{(k)}$ ensures that

$$\alpha^{(k)} p^{(k)} \rightharpoonup \alpha^* p^* \quad \text{in } L^2(Q),$$

which allows passing to the limit in all coupling terms of the weak formulation. Consequently, (\mathbf{u}^*, p^*) solves the state system corresponding to α^* .

The strong convergence (3.7) yields

$$\int_{Q_\omega} (|\mathbf{u}^* - \mathbf{u}_m|^2 + |p^* - p_m|^2) \, dx dt = \lim_{k \rightarrow \infty} \int_{Q_\omega} (|\mathbf{u}^{(k)} - \mathbf{u}_m|^2 + |p^{(k)} - p_m|^2) \, dx dt.$$

Moreover, the weak lower semicontinuity of the $H^1(\Omega)$ -seminorm implies

$$\int_{\Omega} |\nabla \alpha^*|^2 \, dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla \alpha^{(k)}|^2 \, dx.$$

Hence,

$$\mathcal{J}(\alpha^*) \leq \liminf_{k \rightarrow \infty} \mathcal{J}(\alpha^{(k)}) = \inf_{\alpha \in C_{ad}} \mathcal{J}(\alpha).$$

Therefore, α^* is a minimizer of the optimal control problem (3.3). □

3.2. Optimal conditions and dual problem. In this subsection, we derive the first-order optimality conditions for the control problem using the Lagrangian framework. We introduce the augmented Lagrangian functional

$$\mathcal{L}(\mathbf{u}, p, \alpha, \boldsymbol{\theta}) = \mathcal{J}(\alpha) - \langle \boldsymbol{\theta}, \mathcal{S}(\mathbf{u}, p, \alpha) \rangle, \tag{3.8}$$

where $\boldsymbol{\theta} = (\mathbf{v}, q)$ denotes the adjoint variable (Lagrange multiplier) and $\mathcal{S}(\mathbf{u}, p, \alpha) = 0$ represents the state equations. Explicitly,

$$\begin{aligned} \mathcal{L}(\mathbf{u}, p, \alpha, \mathbf{v}, q) &= \mathcal{J}(\alpha) - \int_Q \mathbf{v} \cdot \left(\rho \partial_t^2 \mathbf{u} - \nabla(\lambda^* \partial_t \operatorname{div} \mathbf{u}) \right. \\ &\quad \left. - \nabla((\lambda + \mu) \operatorname{div} \mathbf{u}) - \operatorname{div}(\mu \nabla \mathbf{u}) + \alpha \nabla p - \mathbf{f} \right) \, dx dt \\ &\quad - \int_Q q \left(c_0 \partial_t p + \alpha \operatorname{div}(\partial_t \mathbf{u}) - \operatorname{div}(\kappa \nabla p) - h \right) \, dx dt. \end{aligned} \tag{3.9}$$

Applying Green's formula in space and integration by parts in time, the Lagrangian can be reformulated as

$$\begin{aligned} \mathcal{L}(\mathbf{u}, p, \alpha, \boldsymbol{\theta}) &= \mathcal{J}(\alpha) \\ &\quad - \left[\int_Q \mathbf{u} \left(\rho \partial_t^2 \mathbf{v} + \nabla(\lambda^* \operatorname{div} \partial_t \mathbf{v}) - \nabla((\lambda + \mu) \operatorname{div} \mathbf{v}) - \operatorname{div}(\mu \nabla \mathbf{v}) \right) \, dx dt \right. \\ &\quad \left. - \int_Q \operatorname{div}(\alpha \mathbf{v}) p \, dx dt - \int_Q \mathbf{f} \mathbf{v} \, dx dt \right. \\ &\quad \left. + \int_Q p \left(-c_0 \partial_t q - \operatorname{div}(\kappa \nabla q) \right) \, dx dt + \int_Q \nabla(\alpha \partial_t q) \cdot \mathbf{u} \, dx dt - \int h q \, dx dt \right. \\ &\quad \left. + \int_{\Omega} \left[\varrho \mathbf{u} \partial_t \mathbf{v} - \varrho \partial_t \mathbf{u} \mathbf{v} - \lambda^* \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} + c_0 p q + \alpha \operatorname{div} \mathbf{u} q \right]_{t=0}^{t=T} \, dx \right] \end{aligned} \tag{3.10}$$

The first-order optimality system is given by the Karush–Kuhn–Tucker (KKT) conditions, obtained by setting the partial derivatives of \mathcal{L} with respect to the state variables \mathbf{u} and p to zero.

Computing the Gâteaux derivative with respect to \mathbf{u} yields

$$\begin{aligned} \left\langle \frac{\partial \mathcal{L}}{\partial \mathbf{u}}, \delta \mathbf{u} \right\rangle &= \int_{Q_\omega} (\mathbf{u}(\alpha) - \mathbf{u}_m) \cdot \delta \mathbf{u} \, dx \, dt \\ &\quad - \int_Q \delta \mathbf{u} \cdot \left(\rho \partial_t^2 \mathbf{v} + \nabla(\lambda^* \operatorname{div} \partial_t \mathbf{v}) \right. \\ &\quad \left. - \nabla((\lambda + \mu) \operatorname{div} \mathbf{v}) - \operatorname{div}(\mu \nabla \mathbf{v}) + \nabla(\alpha \partial_t q) \right) dx \, dt, \end{aligned} \quad (3.11)$$

supplemented with the boundary condition

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Sigma, \quad (3.12)$$

and the terminal conditions

$$\mathbf{v}(\mathbf{x}, T) = \mathbf{0}, \quad \partial_t \mathbf{v}(\mathbf{x}, T) = \mathbf{0} \quad \text{in } \Omega. \quad (3.13)$$

Similarly, the derivative with respect to p gives

$$\begin{aligned} \left\langle \frac{\partial \mathcal{L}}{\partial p}, \delta p \right\rangle &= \int_{Q_\omega} (p(\alpha) - p_m) \delta p \, dx \, dt \\ &\quad + \int_Q \delta p \left(\operatorname{div}(\alpha \mathbf{v}) + c_0 \partial_t q + \operatorname{div}(\kappa \nabla q) \right) dx \, dt, \end{aligned} \quad (3.14)$$

with the boundary condition

$$q = 0 \quad \text{on } \Sigma, \quad (3.15)$$

and the terminal condition

$$q(\mathbf{x}, T) = 0 \quad \text{in } \Omega. \quad (3.16)$$

Enforcing $\partial \mathcal{L} / \partial \mathbf{u} = 0$ and $\partial \mathcal{L} / \partial p = 0$ leads to the following adjoint system:

$$\begin{cases} \rho \partial_t^2 \mathbf{v} + \nabla(\lambda^* \operatorname{div} \partial_t \mathbf{v}) - \nabla((\lambda + \mu) \operatorname{div} \mathbf{v}) \\ \quad - \operatorname{div}(\mu \nabla \mathbf{v}) + \nabla(\alpha \partial_t q) = (\mathbf{u}(\alpha) - \mathbf{u}_m) \chi_{Q_\omega} & \text{in } Q, \\ -c_0 \partial_t q - \operatorname{div}(\kappa \nabla q) - \operatorname{div}(\alpha \mathbf{v}) = (p(\alpha) - p_m) \chi_{Q_\omega} & \text{in } Q, \\ \mathbf{v} = 0, \quad p = 0 & \text{on } \Sigma \\ \mathbf{v}(\mathbf{x}, T) = 0, \quad \partial_t \mathbf{v}(\mathbf{x}, T) = 0, \quad p(\mathbf{x}, T) = 0 & \mathbf{x} \in \Omega. \end{cases} \quad (3.17)$$

To obtain the gradient of the reduced cost functional $\mathcal{J}(\alpha)$ with respect to the control α , we employ the adjoint approach. Let $\mathbf{y} = (\mathbf{u}, p)$ denote the state variable. The state equation $\mathcal{S}(\mathbf{y}, \alpha) = 0$ implies the sensitivity equation

$$\frac{\partial \mathcal{S}}{\partial \alpha} + \frac{\partial \mathcal{S}}{\partial \mathbf{y}} \frac{d\mathbf{y}}{d\alpha} = \mathbf{0}. \quad (3.18)$$

Applying the chain rule to $\mathcal{J}(\alpha) = \mathcal{J}(\mathbf{y}(\alpha), \alpha)$ gives

$$\frac{D\mathcal{J}}{D\alpha} = \frac{\partial \mathcal{J}}{\partial \alpha} + \frac{\partial \mathcal{J}}{\partial \mathbf{y}} \frac{d\mathbf{y}}{d\alpha}. \quad (3.19)$$

Using (3.18) and the adjoint equation $\partial\mathcal{L}/\partial\mathbf{y} = 0$, which is equivalent to

$$\frac{\partial\mathcal{J}}{\partial\mathbf{y}} = \boldsymbol{\theta} \frac{\partial\mathcal{S}}{\partial\mathbf{y}'},$$

we obtain the gradient expression

$$\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\alpha} = \frac{\partial\mathcal{J}}{\partial\alpha} - \boldsymbol{\theta} \frac{\partial\mathcal{S}}{\partial\alpha} = \frac{\partial\mathcal{L}}{\partial\alpha}. \tag{3.20}$$

Computing this derivative yields

$$\left\langle \frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\alpha}, \delta\alpha \right\rangle = \epsilon \int_{\Omega} (-\Delta\alpha) \delta\alpha \, d\mathbf{x} + \int_Q \partial_t q \operatorname{div} \mathbf{u} \delta\alpha \, d\mathbf{x} \, dt - \int_Q \mathbf{v} \cdot \nabla p \delta\alpha \, d\mathbf{x} \, dt. \tag{3.21}$$

Thus, the first-order necessary optimality condition for the optimal control problem (3.3) is

$$\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\alpha} = 0 \implies -\epsilon\Delta\alpha + \int_0^T (\partial_t q \operatorname{div} \mathbf{u} - \mathbf{v} \cdot \nabla p) \, dt = 0 \quad \text{in } \Omega, \tag{3.22}$$

where (\mathbf{v}, q) is the unique solution of the adjoint system (3.17).

Theorem 3.2 (First-order necessary optimality conditions). *Let (\mathbf{u}, p) be a local solution of the optimal control problem governed by the Biot system (1.1). Then there exists a unique Lagrange multiplier $\boldsymbol{\theta} = (\mathbf{v}, q)$ with regularity*

$$\mathbf{v} \in W^{1,\infty}(0, T; \mathbf{L}^2(\Omega)) \cap L^\infty(0, T; \mathbf{H}_0^1(\Omega)), \tag{3.23}$$

$$q \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \tag{3.24}$$

such that (\mathbf{v}, q) is a weak solution of the adjoint equations (3.17). Moreover, the optimality condition (3.22) holds for almost every $t \in [0, T]$ and almost every $\mathbf{x} \in \Omega$.

The proof of this theorem follows from standard arguments in optimal control of PDEs and relies on the existence and uniqueness theory for the weak solution of the primal and adjoint systems (see, e.g., [1]).

4. STABILITY RESULT

In this section, we establish the stability and uniqueness results for the inverse problem of retrieving smooth coefficient $\alpha(\mathbf{x})$ in the given Biots system. The optimal control problem established in the previous section will be the key ingredient in the proof of such stability estimate. Even though the optimization technique provides the classical solution to the inverse problem without uniqueness, one can establish local uniqueness and stability results for the same system for small final time.

Theorem 4.1. *Let $\alpha, \tilde{\alpha} \in C_{ad}$. Suppose that $\alpha(\mathbf{x}) = \tilde{\alpha}(\mathbf{x})$ along the boundary $\partial\Omega$. For any initial data $(\mathbf{u}_0, \mathbf{u}_1, p_0) \in \mathbf{H}^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times H^2(\Omega)$, there exists an instant of time $T_0 \ll 1$ such that, for $T \geq T_0$, there exists a constant $C > 0$, independant of the lower bound m given in C_{ad} , satisfying the following estimate*

$$\|\alpha - \tilde{\alpha}\|_{H^1(\Omega)} \leq C \left(I_c + \|\mathbf{u}_m - \tilde{\mathbf{u}}_m\|_{L^2(0, T; L^2(\omega))} + \|p_m - \tilde{p}_m\|_{L^2(0, T; L^2(\omega))} \right), \tag{4.1}$$

where C is te generic constant depending on Ω and T . Here, (\mathbf{u}, p) and $(\tilde{\mathbf{u}}, \tilde{p})$ are two solutions to (1.1)–(1.3) corresponding α and $\tilde{\alpha}$, and

$$I_c = \|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_{H^2(\Omega)} + \|\mathbf{u}_1 - \tilde{\mathbf{u}}_1\|_{H^1_0(\Omega)} + \|p_0 - \tilde{p}_0\|_{H^2(\Omega)}. \tag{4.2}$$

To prove the Theorem 4.1 we need some preliminary lemmas.

4.1. Preliminary lemmas. Let $(\tilde{\mathbf{u}}, \tilde{p})$ the solution of the system (1.1)–(1.3) with the coefficient $\tilde{\alpha}$ and the initial data $\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_1, \tilde{p}_0$. Then by setting $\mathbf{U} = \mathbf{u} - \tilde{\mathbf{u}}, P = p - \tilde{p}$, we obtain the following system

$$\begin{cases} \rho \partial_t^2 \mathbf{U} - \nabla(\lambda^* \partial_t \operatorname{div} \mathbf{U}) - \nabla((\lambda + \mu) \operatorname{div} \mathbf{U}) \\ \qquad \qquad \qquad - \operatorname{div}(\mu \nabla \mathbf{U}) + \alpha \nabla P = -\Xi \nabla \tilde{p}, \\ c_0 \partial_t P + \alpha \operatorname{div} \partial_t \mathbf{U} - \operatorname{div}(\kappa \nabla P) = \Xi \operatorname{div} \partial_t \tilde{\mathbf{u}}, \\ \mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0, \quad \partial_t \mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_1, \quad P(\mathbf{x}, 0) = P_0, \quad \mathbf{x} \in \Omega, \end{cases} \tag{4.3}$$

where we denote $\mathbf{U}_0 = \mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{U}_1 = \mathbf{u}_1 - \tilde{\mathbf{u}}_1, P_0 = p_0 - \tilde{p}_0$ and $\Xi = \alpha - \tilde{\alpha}$.

We consider the following hypothesis verify by $(\mathbf{U}_0, \mathbf{U}_1, P_0)$: $\mathbf{U}_0 \in H^2(\Omega), \mathbf{U}_1 \in H^1_0(\Omega)$ and $P_0 \in H^2(\Omega)$.

Lemma 4.1. *Let (\mathbf{U}, P) be the solution of the system (4.3). Then we have the following estimate: for all $t \in (0, T)$,*

$$\begin{aligned} \max_{0 \leq t \leq T} \left\{ \|\mathbf{U}\|_{L^2(\Omega)}^2 + \|P\|_{L^2(\Omega)}^2 + \|\operatorname{div} \partial_t \mathbf{U}\|_{H^1(\Omega)}^2 + \|\nabla P\|_{H^1(\Omega)}^2 \right\} \\ \leq C(1 + T)^2 e^{2CT} \left(\|\mathbf{U}_0\|_{H^2(\Omega)}^2 + \|\mathbf{U}_1\|_{H^1(\Omega)}^2 + \|P_0\|_{H^2(\Omega)}^2 + \|\Xi\|_{L^2(\Omega)}^2 \right), \end{aligned} \tag{4.4}$$

where C is a positive contant depend only the hypotheses on the physical parameters which are $L^\infty(\Omega)$.

Proof. We apply the Proposition 2.1 (iv) where we remplace \mathbf{f} by $-\Xi \nabla \tilde{p}$ and h by $\Xi \operatorname{div} \partial_t \tilde{\mathbf{u}}$, and using $\operatorname{div} \partial_t \tilde{\mathbf{u}}, \nabla \tilde{p} \in L^\infty(Q)$. □

Lemma 4.2. *Let α be the solution of the optimal control problem (3.3), then there exists a set of functions $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t), \xi(\mathbf{x}, t), \eta(\mathbf{x}, t))$ satisfying*

$$\int_{Q_\omega} \left((\mathbf{u}(\alpha) - \mathbf{u}_m) \xi + (p(\alpha) - p_m) \eta \right) d\mathbf{x} dt - \epsilon \int_{\Omega} \nabla \alpha \cdot \nabla (\alpha - h) d\mathbf{x} \geq 0 \tag{4.5}$$

for any $h \in C_{ad}$.

Proof. For any $h \in C_{ad}$ and $0 \leq \delta \leq 1$, set

$$\alpha_\delta = (1 - \delta)\alpha + \delta h.$$

Let $(\mathbf{u}_\delta, p_\delta)$ be the solution of the system (1.1) with the coefficient $\alpha = \alpha_\delta$, which satisfying

$$J_\delta = \mathcal{J}(\alpha_\delta) = \frac{1}{2} \int_{Q_\omega} |\mathbf{u}(\alpha_\delta) - \mathbf{u}_m(\mathbf{x}, t)|^2 + |p(\alpha_\delta) - p_m(\mathbf{x}, t)|^2 d\mathbf{x} dt + \frac{\epsilon}{2} \mathcal{R}(\alpha_\delta). \tag{4.6}$$

Now taking the fréchet derivative of $\mathcal{J}(\alpha_\delta)$ with respect to δ , we have

$$\begin{aligned} \frac{dJ_\delta}{d\delta} \Big|_{\delta=0} &= \int_{Q_\omega} (\mathbf{u}(\alpha) - \mathbf{u}_m(\mathbf{x}, t)) \frac{\partial \mathbf{u}(\alpha_\delta)}{\partial \delta} \Big|_{\delta=0} \\ &\quad + (p(\alpha_\delta) - p_m(\mathbf{x}, t)) \frac{\partial p(\alpha_\delta)}{\partial \delta} \Big|_{\delta=0} d\mathbf{x} dt + \epsilon \int_{\Omega} \nabla \alpha \cdot \nabla (\alpha - h) d\mathbf{x}. \end{aligned} \tag{4.7}$$

Moreover, α is the optimal solution and therefore

$$\frac{dJ_\delta}{d\delta} \Big|_{\delta=0} \geq 0. \tag{4.8}$$

Let $(\bar{\mathbf{u}}_\delta, \bar{p}_\delta) = (\frac{\partial \mathbf{u}(\alpha_\delta)}{\partial \delta}, \frac{\partial p(\alpha_\delta)}{\partial \delta})$ the solution of system (1.1) with the coefficient α_δ . If we take $\xi = \frac{\partial \mathbf{u}(\alpha_\delta)}{\partial \delta} \Big|_{\delta=0}$ and $\eta = \frac{\partial p(\alpha_\delta)}{\partial \delta} \Big|_{\delta=0}$, then (ξ, η) is solution of the following system with the coefficient α :

$$\begin{cases} \rho(\mathbf{x}) \partial_t^2 \xi - \nabla(\lambda^*(\mathbf{x}) \partial_t \operatorname{div} \xi) - \nabla((\lambda(x) + \mu(x)) \operatorname{div} \xi) \\ \quad - \operatorname{div}(\mu(\mathbf{x}) \nabla \xi) + \alpha(\mathbf{x}) \nabla \eta + (h - \alpha) \nabla p = 0, \\ c_0(\mathbf{x}) \partial_t \eta + \alpha(\mathbf{x}) \operatorname{div} \partial_t \xi - \operatorname{div}(\kappa(\mathbf{x}) \nabla \eta) + (h - \alpha) \operatorname{div} \partial_t \mathbf{u} = 0, \\ \xi(\mathbf{x}, 0) = 0, \quad \partial_t \xi(\mathbf{x}, 0) = 0, \quad \eta(\mathbf{x}, 0) = 0, \end{cases} \tag{4.9}$$

Using (4.9) in (4.8), it's easy to conclude the proof of Lemma 4.2. □

Lemma 4.3. *We have the following estimation for (ξ, η) :*

$$\max_{0 \leq t \leq T} \left\{ \|\xi\|_{L^2(\Omega)}^2 + \|\eta\|_{L^2(\Omega)}^2 + \|\operatorname{div} \partial_t \xi\|^2 + \|\nabla \tilde{\eta}\|^2 \right\} \leq CT(1 + T)^2 e^{2CT} \|\Xi\|_{L^2(\Omega)}^2, \tag{4.10}$$

Proof. We proceed as in Proposition 2.1 for the system (4.9) where we remplace \mathbf{f} by $\Xi \nabla p$ and h by $\Xi \operatorname{div} \partial_t \mathbf{u}$. □

Lemma 4.4. *Suppose $(\tilde{\xi}, \tilde{\eta})$ is solution of the system (4.9). Then by setting $\mathbf{E} = \xi + \tilde{\xi}$, $F = \eta + \tilde{\eta}$, we obtain the following estimate:*

$$\begin{aligned} &\max_{0 \leq t \leq T} \left\{ \|\mathbf{E}\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)} \right\} \\ &\leq C(1 + T)^4 e^{2CT} \left(\|\mathbf{U}_0\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{U}_1\|_{\mathbf{H}_0^1(\Omega)}^2 + \|P_0\|_{H^2(\Omega)}^2 + \|\nabla \Xi\|_{L^2(\Omega)}^2 \right). \end{aligned} \tag{4.11}$$

Proof. We take $h = \tilde{\alpha}$ in the first equation of system (4.12) and $h = \alpha$ in the second one. Also, in system verify by $(\tilde{\xi}, \tilde{\eta})$, we take $h = \alpha$ in the first equation and $h = \tilde{\alpha}$ in the second one. Then, (\mathbf{E}, F) is solution of the following system:

$$\begin{cases} \rho(\mathbf{x}) \partial_t^2 \mathbf{E} - \nabla(\lambda^*(\mathbf{x}) \partial_t \operatorname{div} \mathbf{E}) - \nabla((\lambda(x) + \mu(x)) \operatorname{div} \mathbf{E}) \\ \quad - \operatorname{div}(\mu(\mathbf{x}) \nabla \mathbf{E}) + \alpha(\mathbf{x}) \nabla F = \Xi(\nabla \tilde{\eta} + \nabla P), \\ c_0(\mathbf{x}) \partial_t F + \alpha(\mathbf{x}) \operatorname{div} \partial_t \mathbf{E} - \operatorname{div}(\kappa(\mathbf{x}) \nabla F) = \Xi(\operatorname{div} \partial_t \tilde{\xi} + \operatorname{div} \partial_t \mathbf{U}), \\ \mathbf{E}(\mathbf{x}, 0) = 0, \quad \partial_t \mathbf{E}(\mathbf{x}, 0) = 0, \quad F(\mathbf{x}, 0) = 0, \end{cases} \tag{4.12}$$

We apply the Proposition 2.1, we can deduce

$$\begin{aligned} \|\mathbf{E}\|_{L^2(\Omega)}^2 + \|F\|_{L^2(\Omega)}^2 &\leq C e^{CT} \left[\int_Q |\Xi|^2 \left(|\operatorname{div} \partial_t \mathbf{U}|^2 + |\nabla P|^2 \right) dx dt \right. \\ &\quad \left. + \int_Q |\Xi|^2 \left(|\operatorname{div} \partial_t \tilde{\xi}|^2 + |\nabla \tilde{\eta}|^2 \right) dx dt \right]. \end{aligned} \quad (4.13)$$

Let $I_1 = \int_Q |\Xi|^2 \left(|\operatorname{div} \partial_t \mathbf{U}|^2 + |\nabla P|^2 \right) dx dt$ and $I_2 = \int_Q |\Xi|^2 \left(|\operatorname{div} \partial_t \tilde{\xi}|^2 + |\nabla \tilde{\eta}|^2 \right) dx dt$.

We have

$$\begin{aligned} I_1 &\leq \int_0^T \|\Xi\|_{L^4(\Omega)}^2 \left\| |\operatorname{div} \partial_t \mathbf{U}|^2 + |\nabla P|^2 \right\|_{L^2(\Omega)} dt \\ &\leq \int_0^T \|\Xi\|_{L^4(\Omega)}^2 \left(\|\operatorname{div} \partial_t \mathbf{U}\|_{L^4(\Omega)}^2 + \|\nabla P\|_{L^4(\Omega)}^2 \right) dt \\ &\leq C \left[T \|\Xi\|_{L^4(\Omega)}^4 + \int_0^T \left(\|\operatorname{div} \partial_t \mathbf{U}\|_{L^4(\Omega)}^4 + \|\nabla P\|_{L^4(\Omega)}^4 \right) dt \right]. \end{aligned} \quad (4.14)$$

Here we have used Young inequality.

Note that, by Sobolev-Gagliardo-Nirenberg Theorem [18], we have

$$H^1(\Omega) \hookrightarrow L^4(\Omega).$$

Then, from Lemma 4.1 we obtain

$$\begin{aligned} \|\operatorname{div} \partial_t \mathbf{U}\|_{L^4(\Omega)}^4 + \|\nabla P\|_{L^4(\Omega)}^4 &\leq \|\operatorname{div} \partial_t \mathbf{U}\|_{\mathbf{H}^1(\Omega)}^4 + \|\nabla P\|_{H^1(\Omega)}^4 \\ &\leq C(1+T)^4 e^{4CT} \left[\left(\|\mathbf{U}_0\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{U}_1\|_{\mathbf{H}^1(\Omega)}^2 \right. \right. \\ &\quad \left. \left. + \|P_0\|_{H^2(\Omega)}^2 \right)^2 + \|\Xi\|_{L^2(\Omega)}^4 \right] \end{aligned} \quad (4.15)$$

Since $\Xi = 0$ along the boundary Γ , then from Poincaré inequality [18], we have $\|\Xi\|_{L^4(\Omega)} \leq \|\nabla \Xi\|_{L^2(\Omega)}$. We deduce

$$I_1 \leq CT(1+T)^4 e^{4CT} \left[\left(\|\mathbf{U}_0\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{U}_1\|_{\mathbf{H}^1(\Omega)}^2 + \|P_0\|_{H^2(\Omega)}^2 \right)^2 + \|\nabla \Xi\|_{L^2(\Omega)}^4 \right], \quad (4.16)$$

Similarly, from Lemma 4.3, we have

$$I_2 \leq CT^2(1+T)^4 e^{4CT} \|\nabla \Xi\|_{L^2(\Omega)}^4. \quad (4.17)$$

Now coupling the above estimates, we obtain:

$$\begin{aligned} \|\mathbf{E}\|_{L^2(\Omega)}^2 + \|F\|_{L^2(\Omega)}^2 &\leq C(1+T)^8 e^{4CT} \left[\left(\|\mathbf{U}_0\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{U}_1\|_{\mathbf{H}^1(\Omega)}^2 + \|P_0\|_{H^2(\Omega)}^2 \right)^2 \right. \\ &\quad \left. + \|\nabla \Xi\|_{L^2(\Omega)}^4 \right]. \end{aligned} \quad (4.18)$$

and one can conclude the proof of the Lemma 4.4. \square

4.2. Proof of stability Theorem 4.1. We start by taking $h = \tilde{\alpha}$ in (4.2). We have

$$\int_{Q_\omega} \left((\mathbf{u} - \mathbf{u}_m)\xi + (p - p_m)\eta \right) d\mathbf{x} dt - \epsilon \int_{\Omega} \nabla \alpha \cdot \nabla (\Xi) d\mathbf{x} \geq 0. \quad (4.19)$$

Similarly, for the choice of $h = \alpha$ when $\alpha = \tilde{\alpha}$, we also have

$$\int_{Q_\omega} \left((\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_m)\tilde{\xi} + (\tilde{p} - \tilde{p}_m)\tilde{\eta} \right) d\mathbf{x} dt + \epsilon \int_{\Omega} \nabla \tilde{\alpha} \cdot \nabla (\Xi) d\mathbf{x} \geq 0. \quad (4.20)$$

where (\mathbf{u}, p) , $(\tilde{\mathbf{u}}, \tilde{p})$ are the solutions of the system (1.1) with the coefficients α and $\tilde{\alpha}$ respectively. Now, from (4.19) and (4.20), we get

$$\epsilon \int_{\Omega} |\nabla(\Xi)|^2 d\mathbf{x} \leq \int_{Q_\omega} \left((\mathbf{u} - \mathbf{u}_m)\xi + (\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_m)\tilde{\xi} + (p - p_m)\eta + (\tilde{p} - \tilde{p}_m)\tilde{\eta} \right) d\mathbf{x} dt. \quad (4.21)$$

This equation (4.21) is also reformulated in terms of $\mathbf{U} = \mathbf{u} - \tilde{\mathbf{u}}$ and $P = p - \tilde{p}$ as

$$\begin{aligned} \epsilon \int_{\Omega} |\nabla(\Xi)|^2 d\mathbf{x} &\leq \int_{Q_\omega} \left(\mathbf{U}\xi + P\eta + (\tilde{\mathbf{u}}_m - \mathbf{u}_m)\xi + (\tilde{p}_m - p_m)\eta \right. \\ &\quad \left. + (\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_m)(\xi + \tilde{\xi}) + (\tilde{p} - \tilde{p}_m)(\eta + \tilde{\eta}) \right) d\mathbf{x} dt \\ &\leq CT \left(\max_{0 \leq t \leq T} (\|\mathbf{U}\|_{L^2(\Omega)}^2 + \|P\|_{L^2(\Omega)}^2) + \max_{0 \leq t \leq T} (\|\xi\|_{L^2(\Omega)}^2 + \|\eta\|_{L^2(\Omega)}^2) \right) \\ &\quad + \max_{0 \leq t \leq T} (\|\mathbf{E}\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}) + \|\tilde{\mathbf{u}}_m - \mathbf{u}_m\|_{L^2(\Omega)}^2 + \|\tilde{p}_m - p_m\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.22)$$

Finally, from Lemmas 4.1, 4.3 and 4.4, one obtain

$$\begin{aligned} \epsilon \int_{\Omega} |\nabla(\Xi)|^2 d\mathbf{x} &\leq CT(1+T)^4 e^{2CT} \left(\|\mathbf{U}_0\|_{\mathbf{H}^2(\Omega)}^2 + \|\mathbf{U}_1\|_{\mathbf{H}^1(\Omega)}^2 + \|P_0\|_{H^2(\Omega)}^2 \right. \\ &\quad \left. + \|\nabla \Xi\|_2^2 + \|\mathbf{u}_m - \tilde{\mathbf{u}}_m\|_{L^2(0,T;L^2(\omega))}^2 + \|p_m - \tilde{p}_m\|_{L^2(0,T;L^2(\omega))}^2 \right). \end{aligned} \quad (4.23)$$

For fixed ϵ , we can choose $T_0 \ll 1$ such that

$$CT_0(1+T_0)^4 e^{2CT_0} \leq \epsilon. \quad (4.24)$$

From (4.23) and (4.24), we deduce the stability inequality (4.1).

5. DISCUSSION AND CONCLUSIONS

The development of mathematical tools in optimal control applications is a necessary step towards the identification of some parameters arising in Biot's consolidation models from a set of observed data. The optimal control problem is considered as a PDE constrained optimization problem. This approach is based on minimizing a properly chosen cost functional depending on the spatially varying densities $\alpha(\mathbf{x})$ as input, where the displacement \mathbf{u} and the pressure p are considered as one of the state variables. We signal here that the identification process is also valid when we add the identification of the initial data \mathbf{u}_0 , \mathbf{u}_1 and p_0 . The optimal control approach is used in order to establish a stability result for the parameter identification α under some restrictive conditions and for small final time T , and thus, for solving the parameter identification stability

problem. In future works, our aim is to try to solve numerically the inverse problem by the optimal control approach.

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