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Generalized Kernel Sets via Ideals and the $\widetilde{\Lambda}$-Operator

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Abstract. This work focuses on examining the properties of the co-local function set as defined in [1]. We define an operator $\widetilde{\Lambda}$ using co-local function set and investigate its various fundamental properties. Also, we introduce the notion of compatible kernel topology via ideal and obtain its characterizations along with several properties.

1. Introduction

In a topological space $(\mathfrak{P}, \mathfrak{T})$, given a subset $A \subseteq \mathfrak{P}$, a point $x \in \mathfrak{P}$ is contained in $\mathrm{cl}(A)$ precisely when each neighborhood of x intersects A. Analogous characterizations have been established for various generalized closure operators, each defined with respect to specific families of generalized open sets. In a similar manner, the topological kernel $\ker(A)$, of a set A can also be characterized in terms of neighborhood systems or related generalized open sets. The concept of the topological kernel was first brought into focus by Maki [2], in 1986, who introduced the notion of a kernel set ($\wedge$-set) within a topological space. All kernel sets, together with their complementary co-kernel sets ($\vee$-sets) have demonstrated significant utility in describing the T_1 separation property [2]. The study of generalized closed and open sets has seen significant developments over the past decades. The concepts of λ-closed and λ-open sets, formulated via kernel sets and closed sets, were presented by Arenas et al. [3], in 1997, with λ-closed sets serving to characterize the $T_{1/2}$ separation axiom. Khalimsky et al. [4], in 1990, analyzed the topological and geometric aspects of digital images, providing a fundamental example the digital line, or Khalimskys line of a space that satisfies the $T_{1/2}$ axiom but fails to satisfy the T_1 axiom. Subsequent studies expanded these ideas in different contexts; for instance, in [5] and [6], the kernel of topological approximations of fractals was determined based on their connecting points. Building on this, the authors in [7] introduced two new classes of sets, namely Λ^*-sets and $\vee^*$-sets, defined via kernel sets and ideal

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topological spaces. This line of research was further extended in [8] to generalized topological spaces, where the corresponding classes, Λ_μ-sets and $\vee_\mu$-sets, were systematically investigated. The study of topological structures in the presence of ideals has led to several important extensions of classical notions. Janković and Hamlett [1], in 1990, studied both the local and global aspects of topological spaces involving ideals and defined a Kuratowski-type closure operator, Cl^*, giving rise to a topology $\mathfrak{T}^*$ that is strictly finer than the original. Building on this line of research, Sanabria et al. [9] recently developed a generalized form of the kernel of a set, motivated by the analogous generalizations of the closure operator in [10], and defined the concept of the co-local function set, thereby providing a broader framework for studying kernel-based structures in topology. Building upon the foundational work on co-local function sets presented in [11], this paper aims to further explore their structural properties. The notions of ideals and kernal has been introduced in [12] and [13] and further investigated in [14] and [15]. We introduce a novel operator, denoted by $\widetilde{\Lambda}$, constructed via the co-local function set, and systematically investigate its fundamental characteristics. Moreover, we propose the notion of a compatible kernel topology defined through ideals, establishes its key characterizations, and analyze several related topological properties. This study thus contributes to the broader understanding of kernel-based structures and their interplay with ideal topologies.

2. Preliminaries

In this section, we introduce the fundamental definitions and preliminary concepts that will be used in the subsequent development of our work. Let $(\mathfrak{P}, \mathfrak{T})$ be a topological space, and let $\mathfrak{Ds}$ be a subset of $\mathfrak{P}$. We denote by $cl(\mathfrak{Ds})$ the closure of $\mathfrak{Ds}$ and by $Int(\mathfrak{Ds})$ its interior. For any point $t \in \mathfrak{P}$, the collection of open sets containing t is represented by:

$$\mathfrak{T}(t) = \{\mathfrak{Vs} \in \mathfrak{T} : t \in \mathfrak{Vs}\}.$$

A subset $U \subseteq \mathfrak{P}$ is said to be closed if it belongs to the family $\mathfrak{T}^c$, consisting of the complements of open sets in $\mathfrak{T}$.

An *ideal* $\widetilde{\mathbb{I}}$ on $\mathfrak{P}$ is a nonempty collection of subsets of $\mathfrak{P}$ satisfying two conditions: if $\mathfrak{Ds} \in \widetilde{\mathbb{I}}$ and $\mathfrak{Fs} \subseteq \mathfrak{Ds}$, then $\mathfrak{Fs} \in \widetilde{\mathbb{I}}$; and if $\mathfrak{Ds}, \mathfrak{Fs} \in \widetilde{\mathbb{I}}$, then $\mathfrak{Ds} \cup \mathfrak{Fs} \in \widetilde{\mathbb{I}}$. The triple $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ is called an *ideal space*, representing a topological space $(\mathfrak{P}, \mathfrak{T})$ equipped with the ideal $\widetilde{\mathbb{I}}$.

For an ideal space $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ and a subset $\mathfrak{Ds} \subseteq \mathfrak{P}$, the *local function* of $\mathfrak{Ds}$ relative to $\widetilde{\mathbb{I}}$ and $\mathfrak{T}$, denoted $\mathfrak{Ds}^*(\widetilde{\mathbb{I}}, \mathfrak{T})$ or simply $\mathfrak{Ds}^*$, is defined by

$$\mathfrak{Ds}^*(\widetilde{\mathbb{I}}, \mathfrak{T}) = \{t \in \mathfrak{P} : \mathfrak{Vs} \cap \mathfrak{Ds} \notin \widetilde{\mathbb{I}} \text{ for every } \mathfrak{Vs} \in \mathfrak{T}(t)\}.$$

From this definition, it is immediate that $\emptyset^* = \emptyset$, since the intersection of the empty set with any neighborhood is still empty and belongs to $\widetilde{\mathbb{I}}$. Moreover, $\mathfrak{P}^*$ is always a subset of $\mathfrak{P}$ and, in general, may not coincide with $\mathfrak{P}$.

The operator $\mathrm{Cl}^*(\mathfrak{Ds})$ [1], is introduced for any $\mathfrak{Ds} \subseteq \mathfrak{P}$ as

$$\mathrm{Cl}^*(\mathfrak{Ds}) = \mathfrak{Ds} \cup \mathfrak{Ds}^*.$$

In the special case where $\widetilde{\mathbb{I}} = \{\emptyset\}$, we have

$$\mathrm{Cl}^*(\mathfrak{D}\mathfrak{s}) = \mathfrak{D}\mathfrak{s} \cup cl(\mathfrak{D}\mathfrak{s}) = cl(\mathfrak{D}\mathfrak{s}).$$

It is well established that Cl^* satisfies the axioms of a Kuratowski closure operator [1]. The topology generated by Cl^*, denoted $\mathfrak{T}^*$ (or $\mathfrak{T}^*(\widetilde{\mathbb{I}})$), is defined by

$$\mathfrak{T}^* = \{\mathfrak{V}\mathfrak{s} \subseteq \mathfrak{P} : \mathrm{Cl}^*(\mathfrak{P} \setminus \mathfrak{V}\mathfrak{s}) = \mathfrak{P} \setminus \mathfrak{V}\mathfrak{s}\}.$$

The elements of $\mathfrak{T}^*$ are called $\mathfrak{T}^*$*-open sets*, and their complements are referred to as $\mathfrak{T}^*$*-closed sets*.

In what follows, we will consistently employ these notations and definitions.

Definition 2.1. *[2,16] Consider a subset $\mathfrak{D}\mathfrak{s}$ of a topological space $(\mathfrak{P}, \mathfrak{T})$, denoted here by TS. The* kernel *of $\mathfrak{D}\mathfrak{s}$, written as $ker(\mathfrak{D}\mathfrak{s})$ or sometimes $\Lambda(\mathfrak{D}\mathfrak{s})$, is defined by the intersection of all open sets in $\mathfrak{T}$ that include $\mathfrak{D}\mathfrak{s}$:*

$$ker(\mathfrak{D}\mathfrak{s}) = \bigcap\{\mathfrak{V}\mathfrak{s} \in \mathfrak{T} \mid \mathfrak{D}\mathfrak{s} \subseteq \mathfrak{V}\mathfrak{s}\}.$$

A subset $\mathfrak{D}\mathfrak{s}$ is called a kernel set *(or Λ-set) if it equals its kernel, that is,*

$$\mathfrak{D}\mathfrak{s} = ker(\mathfrak{D}\mathfrak{s})\,.$$

Definition 2.2. *[16] Consider a subset $\mathfrak{D}\mathfrak{s}$ of a TS $(\mathfrak{P}, \mathfrak{T})$. The* co-kernel *of $\mathfrak{D}\mathfrak{s}$, denoted co-ker$(\mathfrak{D}\mathfrak{s})$ or alternatively $\vee(\mathfrak{D}\mathfrak{s})$, is constructed by taking the union of all closed subsets that are contained in $\mathfrak{D}\mathfrak{s}$:*

$$co\text{-}ker(\mathfrak{D}\mathfrak{s}) = \bigcup\{\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{D}\mathfrak{s} : \mathfrak{F}\mathfrak{s} \text{ is closed}\}.$$

A subset $\mathfrak{D}\mathfrak{s}$ is referred to as a co-kernel set *(or $\vee$-set) if it coincides with this union, meaning*

$$\mathfrak{D}\mathfrak{s} = co\text{-}ker(\mathfrak{D}\mathfrak{s})\,.$$

Therefore, $A \subseteq \mathfrak{P}$ is a kernel set precisely when its complement $\mathfrak{P} \setminus A$ is a co-kernel set where appropriate, to characterize the T_1 axiom (see [2]).

The collection of all kernel sets in a *TS* $(\mathfrak{P}, \mathfrak{T})$ is denoted by $\mathfrak{T}^K$ and constitutes a topology on $\mathfrak{P}$. Similarly, the set of all co-kernel sets in *TS*, denoted by $\mathfrak{T}^\Lambda$, also forms a topology on $\mathfrak{P}$.

Lemma 2.1. *[2] Let $\mathfrak{D}\mathfrak{s}, \mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$ be subsets of a TS $(\mathfrak{P}, \mathfrak{T})$. Then the following statements are valid:*

(1) *A point $t \in \mathfrak{P}$ belongs to $ker(\mathfrak{D}\mathfrak{s})$ if and only if $\mathfrak{D}\mathfrak{s}$ intersects every closed set $\mathfrak{F}\mathfrak{s}$ containing t, i.e., $\mathfrak{D}\mathfrak{s} \cap \mathfrak{F}\mathfrak{s} \neq \emptyset$.*
(2) *The subset $\mathfrak{D}\mathfrak{s}$ is always contained in its kernel, $\mathfrak{D}\mathfrak{s} \subseteq ker(\mathfrak{D}\mathfrak{s})$, and if $\mathfrak{D}\mathfrak{s}$ is open in $\mathfrak{P}$, then $\mathfrak{D}\mathfrak{s} = ker(\mathfrak{D}\mathfrak{s})$.*

Lemma 2.2. *[2] The statement that follows holds for any subsets $\mathfrak{V}\mathfrak{s}$ and U of a TS $(\mathfrak{P}, \mathfrak{T})$:*

(1) *$ker(ker(\mathfrak{V}\mathfrak{s})) = ker(\mathfrak{V}\mathfrak{s})$,*
(2) *$ker(\mathfrak{V}\mathfrak{s})$ is itself a kernel set,*
(3) *Any open set $\mathfrak{V}\mathfrak{s}$ in the space is a kernel set,*
(4) *$ker(\cup_{i\in I}(\mathfrak{V}\mathfrak{s}_i)) = \cup_{i\in I} ker(\mathfrak{V}\mathfrak{s}_i)$*
(5) *$ker(\mathfrak{P} \setminus \mathfrak{V}\mathfrak{s}) = \mathfrak{P} \setminus co\text{-}ker(\mathfrak{V}\mathfrak{s})$,*

(6) $ker(\mathfrak{V}\mathfrak{s} \cap U) \subseteq ker(\mathfrak{V}\mathfrak{s}) \cap ker(U)$.

Lemma 2.3. *[2] Considering a TS $(\mathfrak{P}, \mathfrak{T})$, we may see the subsequent characteristics:*

(1) *Both the empty set $\emptyset$ and the entire space $\mathfrak{P}$ are kernel sets.*
(2) *The intersection of any collection of kernel sets is itself a kernel set.*
(3) *Similarly, the union of any collection of kernel sets also forms a kernel set.*

It is well established that when the collection of open sets in $(\mathfrak{P}, \mathfrak{T}^K)$ and $(\mathfrak{P}, \mathfrak{T}^\Lambda)$ are closed under arbitrary intersections. Thus $(\mathfrak{P}, \mathfrak{T}^K)$ and $(\mathfrak{P}, \mathfrak{T}^\Lambda)$ are Alexandroff spaces.

Veličko [17] introduced the concept of θ-open sets in 1968. A subset A of a space is called θ-open [17] provided that each point of A admits an open neighborhood whose closure is entirely contained in A. The θ-interior of A, denoted by $Int_\theta(A)$, is defined as the union of all θ-open subsets contained in A. Consequently, the complement of a θ-open set is referred to as θ-closed. In [18], Al-Omari and Noiri introduced the local closure function as a generalization of the θ-closure and the local function in an ideal topological space. An equivalent characterization of θ-closure is given by

$$cl_\theta(A) = \{x \in \mathfrak{P} : cl(U) \cap A \neq \emptyset,\ U \in \mathfrak{T},\ x \in U\},$$

and a subset A is θ-closed precisely when $A = cl_\theta(A)$. The collection of all θ-open sets forms a topology on $\mathfrak{P}$, denoted by τ_θ, and this topology is always coarser than the original topology $\mathfrak{T}$, i.e., $\mathfrak{T}_\theta \subseteq \mathfrak{T}$. Furthermore, a topological space $(\mathfrak{P}, \mathfrak{T})$ is regular if and only if $\mathfrak{T}$ coincides with $\mathfrak{T}_\theta$.

3. Ideal kernel sets in *TS*

This section presents and examines the notion of an ideal kernel, viewed as a natural extension of the kernel of a set within a topological space.

Definition 3.1. *[9] Consider an ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$, hereafter referred to as ITS. For any subset $U \subseteq \mathfrak{P}$, we present the following definition for the co-local function set (ideal kernel) as: $\overline{\Lambda}(U)(\mathfrak{P}, \mathfrak{T}) = \{x \in \mathfrak{P} : U \cap \mathfrak{F}\mathfrak{s} \notin \widetilde{\mathbb{I}}$ for every closed set $\mathfrak{F}\mathfrak{s}$ containing $x\}$. To ensure there is no misunderstanding, $\overline{\Lambda}(U)(\widetilde{\mathbb{I}}, \mathfrak{T})$ is briefly denoted by $\overline{\Lambda}(U)$.*

We observe that one way to interpret the co-local function (also known as the ideal kernel) is as an operator $\overline{\Lambda}(.) : \mathcal{P}(\mathfrak{P}) \to \mathcal{P}(\mathfrak{P})$ defined by $U \subseteq \overline{\Lambda}(U)$. However, the co-local function cannot be regarded as a Kuratowski closure operator, since it generally fails to meet the condition $U \subseteq \overline{\Lambda}(U)$ for all subsets $U \subseteq \mathfrak{P}$. When the inclusion $U \subseteq \overline{\Lambda}(U)$ does hold, we say that U is Λ-dense in itself.

Theorem 3.1. *[9] Consider an ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. The conditions listed below are mutually equivalent:*

(1) $\mathfrak{P} = \overline{\Lambda}(\mathfrak{P})$.
(2) $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$, *where* $\mathfrak{T}^c = \{\mathfrak{D}\mathfrak{s} : \mathfrak{P} \setminus \mathfrak{D}\mathfrak{s} \in \mathfrak{T}\}$.
(3) *If* $I \in \widetilde{\mathbb{I}}$, $ker(I) = \emptyset$.
(4) $\mathfrak{D}\mathfrak{s} \subseteq \overline{\Lambda}(\mathfrak{D}\mathfrak{s})$ *for each closed set* $\mathfrak{D}\mathfrak{s}$.

Remark 3.1. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS. Considering an arbitrary subset $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, one obtains:*

(1) *If* $\widetilde{\mathbb{I}} = \{\emptyset\}$, *then* $\overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = ker(\mathfrak{D}\mathfrak{s})$.

(2) *If* $\widetilde{\mathbb{I}} = \mathcal{P}(\mathfrak{P})$, *then* $\overline{\Lambda}(\mathfrak{Ds}) = \emptyset$.

Theorem 3.2. *[9] Let* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ *be an ITS, then for all* $\mathfrak{Vs} \subseteq \mathfrak{P}$ *the ideal kernel* $\overline{\Lambda}(\mathfrak{Vs}) = \{x \in \mathfrak{P} : cl(\{x\}) \cap \mathfrak{Vs} \notin \widetilde{\mathbb{I}}\}$.

Corollary 3.1. *[9] Assume* $(\mathfrak{P}, \mathfrak{T})$ *is a TS, For any* $\mathfrak{Vs} \subseteq \mathfrak{P}$, *the kernel of* $\mathfrak{Vs}$ *is* $ker(\mathfrak{Vs}) = \{x \in \mathfrak{P} : cl(\{x\}) \cap \mathfrak{Vs} \neq \emptyset\}$.

Lemma 3.1. *[9] Consider a TS* $(\mathfrak{P}, \mathfrak{T})$ *with ideals* $\widetilde{\mathbb{I}}$ *and* $\mathcal{I}$ *defined on* $\mathfrak{P}$. *Let* $U, \mathfrak{Ds} \subseteq \mathfrak{P}$, *then the statements below are satisfied:*

(1) *If* $\mathfrak{Ds} \subseteq U$, *then* $\overline{\Lambda}(\mathfrak{Ds}) \subseteq \overline{\Lambda}(U)$.
(2) *If* $\widetilde{\mathbb{I}} \subseteq \mathcal{I}$, *then* $\overline{\Lambda}(\mathfrak{Ds})(\widetilde{\mathbb{I}}) \supseteq \overline{\Lambda}(\mathfrak{Ds})(\mathcal{I})$.
(3) $\overline{\Lambda}(\mathfrak{Ds}) = ker(\overline{\Lambda}(\mathfrak{Ds})) \subseteq ker(\mathfrak{Ds})$ *(i.e.* $\overline{\Lambda}(\mathfrak{Ds})$ *is a kernel set).*
(4) *If* $\mathfrak{Ds} \subseteq \overline{\Lambda}(\mathfrak{Ds})$, *then* $\overline{\Lambda}(\mathfrak{Ds}) = ker(\mathfrak{Ds})$.
(5) *If* $\mathfrak{Ds} \in \widetilde{\mathbb{I}}$, *then* $\overline{\Lambda}(\mathfrak{Ds}) = \emptyset$.
(6) $\overline{\Lambda}(\overline{\Lambda}(\mathfrak{Ds})) \subseteq \overline{\Lambda}(\mathfrak{Ds})$.
(7) $\overline{\Lambda}(\emptyset) = \emptyset$.
(8) $\overline{\Lambda}(U \cup \mathfrak{Ds}) = \overline{\Lambda}(U) \cup \overline{\Lambda}(\mathfrak{Ds})$.
(9) *For a closed set* $\mathfrak{Fs}$, $\mathfrak{Fs} \cap \overline{\Lambda}(U) = \mathfrak{Fs} \cap \overline{\Lambda}(\mathfrak{Fs} \cap U) \subseteq \overline{\Lambda}(\mathfrak{Fs} \cap U)$.
(10) $\overline{\Lambda}(U) \setminus \overline{\Lambda}(\mathfrak{Ds}) = \overline{\Lambda}(U \setminus \mathfrak{Ds}) \setminus \overline{\Lambda}(\mathfrak{Ds})$.
(11) *If* $\mathfrak{Ds} \in \widetilde{\mathbb{I}}$, *then* $\overline{\Lambda}(U \cup \mathfrak{Ds}) = \overline{\Lambda}(U) = \overline{\Lambda}(U \setminus \mathfrak{Ds})$.

Definition 3.2. *Let* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ *be an ITS. For any* $\mathfrak{Ds} \subseteq \mathfrak{P}$, *then* $cl_{\overline{\Lambda}}(\mathfrak{Ds}) = \mathfrak{Ds} \cup \overline{\Lambda}(\mathfrak{Ds})$.

Remark 3.2. *Consider the ITS* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. *For any subset* $\mathfrak{Ds} \subseteq \mathfrak{P}$, *one obtains:*

(1) *The operator* $cl_{\overline{\Lambda}}$ *defines a Kuratowski closure. The topology it generates is denoted by* $\mathfrak{T}^{\overline{\Lambda}}$, *which can be expressed as* $\mathfrak{T}^{\overline{\Lambda}} = \{\mathfrak{Ds} \subseteq \mathfrak{P} : cl_{\overline{\Lambda}}(\mathfrak{P} \setminus \mathfrak{Ds}) = \mathfrak{P} \setminus \mathfrak{Ds}\}$. *Members of the topology* $\mathfrak{T}^{\overline{\Lambda}}$ *are referred to as* $\overline{\Lambda}$*-open sets, while the complement of such a set is termed* $\overline{\Lambda}$*-closed. Consequently, a subset* $\mathfrak{Ds} \subseteq \mathfrak{P}$ *is* $\overline{\Lambda}$*-closed if and only if*

$$\overline{\Lambda}(\mathfrak{Ds}) \subseteq \mathfrak{Ds}.$$

(2) *If* $\widetilde{\mathbb{I}} = \{\emptyset\}$, *then* $cl_{\overline{\Lambda}}(\mathfrak{Ds}) = \overline{\Lambda}(\mathfrak{Ds}) \cup \mathfrak{Ds} = ker(\mathfrak{Ds}) \cup \mathfrak{Ds} = ker(\mathfrak{Ds})$.
(3) *If* $\widetilde{\mathbb{I}} = \mathcal{P}(\mathfrak{P})$, *then* $cl_{\overline{\Lambda}}(\mathfrak{Ds}) = \overline{\Lambda}(\mathfrak{Ds}) \cup \mathfrak{Ds} = \emptyset \cup \mathfrak{Ds} = \mathfrak{Ds}$ *and* $\mathfrak{T}^{\overline{\Lambda}}$ *is the discrete topology.*
(4) *Since* $\overline{\Lambda}(\mathfrak{Ds}) = ker(\overline{\Lambda}(\mathfrak{Ds})) \subseteq ker(\mathfrak{Ds})$, *then* $cl_{\overline{\Lambda}}(\mathfrak{Ds}) \subseteq ker(\mathfrak{Ds})$ *for each* $\mathfrak{Ds} \subseteq \mathfrak{P}$. *Thus, if* $\mathfrak{Ds}$ *is kernel sets, then* $\mathfrak{Ds}$ *is* $\overline{\Lambda}$*-closed. From this, we deduce that each co-kernel sets is* $\overline{\Lambda}$*-open that is* $\mathfrak{T}^{\Lambda} \subseteq \mathfrak{T}^{\overline{\Lambda}}$. *Also we have* $\mathfrak{T}_{\theta} \subseteq \mathfrak{T}^{\overline{\Lambda}}$ *and* $\mathfrak{T}_{\theta} \subseteq \mathfrak{T} \subseteq \mathfrak{T}^{*}$. *Moreover,* $\mathfrak{T}$ *and* $\mathfrak{T}^{\Lambda}$ *are independent topology.*

Theorem 3.3. *[9] Consider the ITS* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. *The collection* $\sigma = \{\mathfrak{Fs} \setminus I : \mathfrak{Fs}$ *is closed set and* $I \in \widetilde{\mathbb{I}}\}$ *is a base for the topology* $\mathfrak{T}^{\overline{\Lambda}}$.

The following diagram holds:

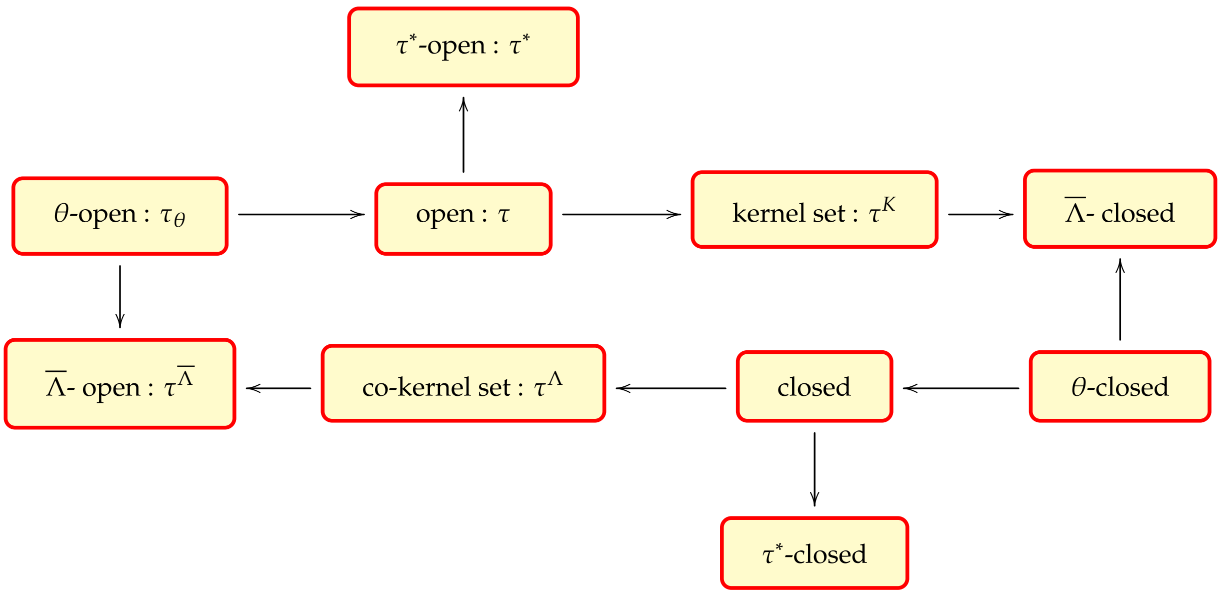

Figure 1. Schematic illustration of the $\widetilde{\Lambda}$-operator showing its relation in an ideal topological space $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$.

4. Characteristics of $\widetilde{\Lambda}$-operator

Definition 4.1. *Consider the ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. An operator $\widetilde{\Lambda}(.) : \mathcal{P}(\mathfrak{P}) \to \mathfrak{T}$ is formulated as: for every $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}) = \{x \in \mathfrak{P} : \exists$ a closed set $\mathfrak{F}\mathfrak{s}$ containing x and $\mathfrak{F}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}\}$.*

The following theorem presents several fundamental properties regarding the behavior of the operator $\widetilde{\Lambda}$.

Theorem 4.1. *For an ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ and subsets $B, \mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, the next assertions are true:*

(1) $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}) = \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{D}\mathfrak{s})$.
(2) *If $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, then $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s})$ is a co-kernel set.*
(3) *If $\mathfrak{D}\mathfrak{s} \subseteq B$, then $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}) \subseteq \widetilde{\Lambda}(B)$.*
(4) *If $\mathfrak{D}\mathfrak{s}, B \subseteq \mathfrak{P}$, then $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s} \cap B) = \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}) \cap \widetilde{\Lambda}(B)$.*
(5) *If $\mathfrak{D}\mathfrak{s} \in \mathfrak{T}^{\Lambda}$, then $\mathfrak{D}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s})$.*
(6) *If $\mathfrak{D}\mathfrak{s}$ is θ-open, then $\mathfrak{D}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s})$.*
(7) *If $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, then necessarily $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}) \subseteq \widetilde{\Lambda}(\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}))$.*
(8) *Let $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, implies that $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}) = \widetilde{\Lambda}(\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}))$ is equivalent to $\overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{D}\mathfrak{s}) = \overline{\Lambda}(\overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{D}\mathfrak{s}))$.*
(9) *If $\mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$, then $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}) = \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P})$.*
(10) *If $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, then $\mathfrak{D}\mathfrak{s} \cap \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}) = Int_{\overline{\Lambda}}(\mathfrak{D}\mathfrak{s})$.*
(11) *If $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, $I \in \widetilde{\mathbb{I}}$, then $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s} \setminus I) = \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s})$.*
(12) *If $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, $I \in \widetilde{\mathbb{I}}$, then $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s} \cup I) = \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s})$.*

(13) *If* $(\mathfrak{Ds} \setminus B) \cup (B \setminus \mathfrak{Ds}) \in \widetilde{\mathbb{I}}$, *then* $\widetilde{\Lambda}(\mathfrak{Ds}) = \widetilde{\Lambda}(B)$.

Proof. (1) Let $x \in \widetilde{\Lambda}(\mathfrak{Ds})$. Then one can find a closed set $\mathfrak{Fs}$ containing x such that $\mathfrak{Fs} \setminus \mathfrak{Ds} = \mathfrak{Fs} \cap \mathfrak{Ds}^c \in \widetilde{\mathbb{I}}$. This implies that x is not an element of $\overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds})$, or equivalently, $x \in \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds})$. Conversely, if $x \in \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds})$, then a closed set $\mathfrak{Fs}$ containing x can be identified such that $\mathfrak{Fs} \cap \mathfrak{Ds}^c = \mathfrak{Fs} \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$. Thus, $x \in \widetilde{\Lambda}(\mathfrak{Ds})$. Therefore, we conclude that $\widetilde{\Lambda}(\mathfrak{Ds}) = \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds})$.

(2) This follows from (3) of Lemma 3.1.

(3) This follows from (1) of Lemma 3.1.

(4) It follows from (2) that $\widetilde{\Lambda}(\mathfrak{Ds} \cap B) \subseteq \widetilde{\Lambda}(\mathfrak{Ds})$ and $\widetilde{\Lambda}(\mathfrak{Ds} \cap B) \subseteq \widetilde{\Lambda}(B)$. Hence $\widetilde{\Lambda}(\mathfrak{Ds} \cap B) \subseteq \widetilde{\Lambda}(\mathfrak{Ds}) \cap \widetilde{\Lambda}(B)$. Now let $x \in \widetilde{\Lambda}(\mathfrak{Ds}) \cap \widetilde{\Lambda}(B)$. There are two closed sets, $\mathfrak{Fs}$ and $\mathfrak{Ds}$, both containing x, such that $\mathfrak{Fs} \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ and $\mathfrak{Ds} \setminus B \in \widetilde{\mathbb{I}}$. Let $G = \mathfrak{Fs} \cap \mathfrak{Ds} \in \mathfrak{T}^c(x)$ and we have $G \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ and $G \setminus B \in \widetilde{\mathbb{I}}$ by heredity. Thus $G \setminus (\mathfrak{Ds} \cap B) = (G \setminus \mathfrak{Ds}) \cup (G) \setminus B) \in \widetilde{\mathbb{I}}$ by additivity, and hence $x \in \widetilde{\Lambda}(\mathfrak{Ds} \cap B)$. Hence, $\widetilde{\Lambda}(\mathfrak{Ds}) \cap \widetilde{\Lambda}(B) \subseteq \widetilde{\Lambda}(\mathfrak{Ds} \cap B)$ and the proof is complete.

(5) If $\mathfrak{Ds} \in \mathfrak{T}^{\Lambda}$, then $\mathfrak{P} \setminus \mathfrak{Ds}$ is $\overline{\Lambda}$-closed which implies $\overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}) \subseteq \mathfrak{P} \setminus \mathfrak{Ds}$ and hence $\mathfrak{Ds} \subseteq \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}) = \widetilde{\Lambda}(\mathfrak{Ds})$.

(6) If $\mathfrak{Ds}$ is θ-open set, then $\mathfrak{P} \setminus \mathfrak{Ds}$ is θ-closed and hence it is $\overline{\Lambda}$-closed which implies $\overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}) \subseteq \mathfrak{P} \setminus \mathfrak{Ds}$ and hence $\mathfrak{Ds} \subseteq \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}) = \widetilde{\Lambda}(\mathfrak{Ds})$.

(7) This follows from (2) and (5).

(8) This follows from the facts:

(1) $\widetilde{\Lambda}(\mathfrak{Ds}) = \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds})$.

(2) $\widetilde{\Lambda}(\widetilde{\Lambda}(\mathfrak{Ds})) = \mathfrak{P} \setminus \overline{\Lambda}[\mathfrak{P} \setminus (\mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}))] = \mathfrak{P} \setminus \overline{\Lambda}(\overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}))$.

(9) By item (11) of Lemma 3.1, we obtain that $\overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}) = \overline{\Lambda}(\mathfrak{P})$ if $\mathfrak{Ds} \in \widetilde{\mathbb{I}}$.

(10) Let $x \in \mathfrak{Ds} \cap \widetilde{\Lambda}(\mathfrak{Ds})$, then $x \in \mathfrak{Ds}$ and there exists closed set $\mathfrak{Fs}_x$ containing x and $\mathfrak{Fs}_x \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$. According to Theorem 3.3, $\mathfrak{Fs}_x \setminus (\mathfrak{Fs}_x \setminus \mathfrak{Ds})$ is a $\overline{\Lambda}$-open neighborhood of x and $x \in Int_{\overline{\Lambda}}(\mathfrak{Ds})$. Alternatively, assuming that $x \in Int_{\overline{\Lambda}}(\mathfrak{Ds})$, there exists a basic $\overline{\Lambda}$-open neighborhood $K_x \setminus I$ of x, where K_x is closed set containing x and $I \in \widetilde{\mathbb{I}}$, such that $x \in K_x \setminus I \subseteq \mathfrak{Ds}$ which implies $K_x \setminus \mathfrak{Ds} \subseteq I$ and hence $K_x \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$. Hence, $x \in \mathfrak{Ds} \cap \widetilde{\Lambda}(\mathfrak{Ds})$.

(11) This follows from Lemma 3.1, and $\widetilde{\Lambda}(\mathfrak{Ds} \setminus I) = \mathfrak{P} \setminus \overline{\Lambda}[\mathfrak{P} \setminus (\mathfrak{Ds} \setminus I)] = \mathfrak{P} \setminus \overline{\Lambda}[(\mathfrak{P} \setminus \mathfrak{Ds}) \cup I] = \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}) = \widetilde{\Lambda}(\mathfrak{Ds})$.

(12) This follows from Lemma 3.1, and $\widetilde{\Lambda}(\mathfrak{Ds} \cup I) = \mathfrak{P} \setminus \overline{\Lambda}[\mathfrak{P} \setminus (\mathfrak{Ds} \cup I)] = \mathfrak{P} \setminus \overline{\Lambda}[(\mathfrak{P} \setminus \mathfrak{Ds}) \setminus I] = \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}) = \widetilde{\Lambda}(\mathfrak{Ds})$.

(13) Let $(\mathfrak{Ds} \setminus B) \cup (B \setminus \mathfrak{Ds}) \in \widetilde{\mathbb{I}}$. If $\mathfrak{Ds} \setminus B = I$ and $B \setminus \mathfrak{Ds} = J$. It is evident that $I, J \in \widetilde{\mathbb{I}}$ by heredity. Additionally, note that $B = (\mathfrak{Ds} \setminus I) \cup J$. Thus $\widetilde{\Lambda}(\mathfrak{Ds}) = \widetilde{\Lambda}(\mathfrak{Ds} \setminus I) = \widetilde{\Lambda}[(\mathfrak{Ds} \setminus I) \cup J] = \widetilde{\Lambda}(B)$ by (11) and (12). □

Theorem 4.2. *Consider a subset* $\mathfrak{Fs} \subseteq \mathfrak{P}$ *in the ITS* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. *It follows that* $Int_{\overline{\Lambda}}(\mathfrak{Fs}) \subseteq \widetilde{\Lambda}(\mathfrak{Fs})$.

Proof. Consider a subset $\mathfrak{Fs} \subseteq \mathfrak{P}$ and take any point $x \in \mathfrak{P}$ such that $x \notin \widetilde{\Lambda}(\mathfrak{Fs})$. By the definition of $\widetilde{\Lambda}$, this means

$$x \in \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Fs}).$$

Hence, every closed set $\mathfrak{D}\mathfrak{s}$ containing x satisfies

$$\mathfrak{D}\mathfrak{s} \cap (\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) \notin \widetilde{\mathbb{I}},$$

which implies $\mathfrak{D}\mathfrak{s} \cap (\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) \neq \emptyset$. Therefore,

$$cl_{\overline{\Lambda}}(\mathfrak{D}\mathfrak{s}) \cap (\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) \neq \emptyset,$$

and as a result, $cl_{\overline{\Lambda}}(\mathfrak{D}\mathfrak{s})$ cannot be entirely contained in $\mathfrak{F}\mathfrak{s}$. Consequently, $x \notin Int_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})$.

Since x was arbitrary, we conclude that

$$Int_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s}) \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$$

for every subset $\mathfrak{F}\mathfrak{s}$ of $\mathfrak{P}$. □

Corollary 4.1.

Consider an ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. For any $\overline{\Lambda}$-open subset $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$, we have $\mathfrak{F}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$.

Proof. We know that $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) = \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s})$. Now $\overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) \subseteq cl_{\overline{\Lambda}}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) = \mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}$, since $\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}$ is $\overline{\Lambda}$-closed. Therefore, $\mathfrak{F}\mathfrak{s} = \mathfrak{P} \setminus (\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) \subseteq \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) = \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$. □

Theorem 4.3. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS and $\mathfrak{F}\mathfrak{s}, W$ be a subsets of $\mathfrak{P}$. Then, co-ker$(\mathfrak{F}\mathfrak{s}) \cap \overline{\Lambda}(W) =$ co-ker$(\mathfrak{F}\mathfrak{s}) \cap \overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cap W) \subseteq \overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cap W)$.*

Proof. Let $x \in$ *co-ker*$(\mathfrak{F}\mathfrak{s}) \cap \overline{\Lambda}(W)$. So, $x \in$ *co-ker*$(\mathfrak{F}\mathfrak{s})$ and $x \in \overline{\Lambda}(W)$. Since $x \in$ *co-ker*$(\mathfrak{F}\mathfrak{s})$, there exists closed set $\mathfrak{D}\mathfrak{s}$ containing x and $x \in \mathfrak{D}\mathfrak{s} \subseteq \mathfrak{F}\mathfrak{s}$. For any closed set H containing x, $\mathfrak{D}\mathfrak{s} \cap H$ is closed set containing x. Since $x \in \overline{\Lambda}(W)$, $(\mathfrak{D}\mathfrak{s} \cap H) \cap W \notin \widetilde{\mathbb{I}}$. From the definition of ideal and $[H \cap (\mathfrak{D}\mathfrak{s} \cap W)] \subseteq [H \cap (\mathfrak{F}\mathfrak{s} \cap W)]$. Hence, $[H \cap (\mathfrak{F}\mathfrak{s} \cap W)] \notin \widetilde{\mathbb{I}}$ and $x \in \overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cap W)$. That is *co-ker*$(\mathfrak{F}\mathfrak{s}) \cap \overline{\Lambda}(W) \subseteq \overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cap W)$. Since $\overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cap W) \subseteq \overline{\Lambda}(W)$, *co-ker*$(\mathfrak{F}\mathfrak{s}) \cap \overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cap W) \subseteq$ *co-ker*$(\mathfrak{F}\mathfrak{s}) \cap \overline{\Lambda}(W)$. Therefore, *co-ker*$(\mathfrak{F}\mathfrak{s}) \cap \overline{\Lambda}(W) =$ *co-ker*$(\mathfrak{F}\mathfrak{s}) \cap \overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cap W) \subseteq \overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cap W)$. □

Corollary 4.2. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS, $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$ and if $W \subseteq Int(\overline{\Lambda}(W))$. Then, co-ker$(\mathfrak{F}\mathfrak{s}) \cap W \subseteq Int_{\overline{\Lambda}}(\overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cap W))$.*

Proof. Using Theorem 4.3,

$$\begin{aligned} co\text{-}ker(\mathfrak{F}\mathfrak{s}) \cap W \subseteq & co\text{-}ker(\mathfrak{F}\mathfrak{s}) \cap Int_{\overline{\Lambda}}(\overline{\Lambda}(W)) \\ = & Int_{\overline{\Lambda}}[co\text{-}ker(\mathfrak{F}\mathfrak{s}) \cap \overline{\Lambda}(W)] \\ \subseteq & Int_{\overline{\Lambda}}(\overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cap W)). \end{aligned}$$

□

Proposition 4.1. *Consider an ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ and $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$. Then the properties listed below are true:*

(1) $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) = \cup\{U : U$ *is closed set and* $U \setminus \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}\}$.

(2) $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \supseteq \cup\{U : U$ *is closed set and* $(U \setminus \mathfrak{F}\mathfrak{s}) \cup (\mathfrak{F}\mathfrak{s} \setminus U) \in \widetilde{\mathbb{I}}\}$.

Proof. (1) It follows straightforwardly from the way the $\widetilde{\Lambda}$-operator is defined.

(2) Since $\widetilde{\mathbb{I}}$ is heredity, it is obvious that $\cup\{U : U$ is closed set and $(U \setminus \mathfrak{F}\mathfrak{s}) \cup (\mathfrak{F}\mathfrak{s} \setminus U) \in \widetilde{\mathbb{I}}\} \subseteq \cup\{U : U$ is closed set and $U \setminus \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}\} = \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$ for every $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$. □

Theorem 4.4. *Consider an ITS* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. *If* $\sigma = \{\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P} : \mathfrak{F}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})\}$. *Then* σ *is a topology for* $\mathfrak{P}$ *and* $\sigma = \mathfrak{T}^{\overline{\Lambda}}$.

Proof. Let $\sigma = \{\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P} : \mathfrak{F}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})\}$.We begin by proving that σ forms a topology. Note that $\emptyset \subseteq \widetilde{\Lambda}(\emptyset)$ and $\mathfrak{P} \subseteq \widetilde{\Lambda}(\mathfrak{P}) = \mathfrak{P}$, and thus $\emptyset$ and $X \in \sigma$. Now if $\mathfrak{F}\mathfrak{s}, K \in \sigma$, then by Theorem 4.1(4), $\mathfrak{F}\mathfrak{s} \cap K \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \cap \widetilde{\Lambda}(K) = \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s} \cap K)$ which implies that $\mathfrak{F}\mathfrak{s} \cap K \in \sigma$. If $\{\mathfrak{F}\mathfrak{s}_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $\mathfrak{F}\mathfrak{s}_\alpha \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}_\alpha) \subseteq \widetilde{\Lambda}(\cup \mathfrak{F}\mathfrak{s}_\alpha)$ for every $\alpha \in \Delta$ and hence $\cup \mathfrak{F}\mathfrak{s}_\alpha \subseteq \widetilde{\Lambda}(\cup \mathfrak{F}\mathfrak{s}_\alpha)$. This shows that σ is a topology. Consider the case where $\mathfrak{F}\mathfrak{s} \in \mathfrak{T}^{\overline{\Lambda}}$ according to Theorem 4.1(5), $\mathfrak{F}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$ and it has been demonstrated that $\mathfrak{T}^{\overline{\Lambda}} \subseteq \sigma$. Next, consider $\mathfrak{F}\mathfrak{s} \in \sigma$, then we have $\mathfrak{F}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$, that is, $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s})$ and $\overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) \subseteq \mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}$. This shows that $Cl_{\overline{\Lambda}}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) = (\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) \cup \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) = \mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}$ and hence $\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}$ is $\overline{\Lambda}$-closed and hence $\mathfrak{F}\mathfrak{s} \in \mathfrak{T}^{\overline{\Lambda}}$. Thus $\sigma \subseteq \mathfrak{T}^{\overline{\Lambda}}$ and hence $\sigma = \mathfrak{T}^{\overline{\Lambda}}$. □

Example 4.1. *Suppose that* $\mathfrak{P} = \{1, 2, 3\}$ *has topology* $\mathfrak{T}$. *The formula* $\mathfrak{T} = \{\emptyset, \mathfrak{P}, \{1, 3\}\}$ *and the ideal* $\widetilde{\mathbb{I}} = \{\emptyset, \{3\}\}$. *Evidently,* $\mathfrak{T}_\theta = \{\emptyset, \mathfrak{P}\}$. *If* $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$, *then we have the following table:*

$\mathfrak{F}\mathfrak{s}$	$ker(\mathfrak{F}\mathfrak{s})$	$co\text{-}ker(\mathfrak{F}\mathfrak{s})$	$\overline{\Lambda}(\mathfrak{F}\mathfrak{s})$	$\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$
$\{1\}$	$\{1, 3\}$	$\emptyset$	$\{1, 3\}$	$\emptyset$
$\{2\}$	$\mathfrak{P}$	$\{2\}$	$\mathfrak{P}$	$\{2\}$
$\{3\}$	$\{1, 3\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1, 2\}$	$\mathfrak{P}$	$\{2\}$	$\mathfrak{P}$	$\mathfrak{P}$
$\{1, 3\}$	$\{1, 3\}$	$\emptyset$	$\{1, 3\}$	$\emptyset$
$\{2, 3\}$	$\mathfrak{P}$	$\{2\}$	$\mathfrak{P}$	$\{2\}$

Table 1. Illustration of the relationship among the above concepts.

It is clear that $\mathfrak{T}$ and $\mathfrak{T}^{\Lambda}$ are independent topology since $\mathfrak{T}^{\overline{\Lambda}} = \{\emptyset, \mathfrak{P}, \{2\}, \{1, 2\}\}$.

Lemma 4.1. *Let* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ *be an ITS. A set* $\mathfrak{F}\mathfrak{s}$ *is closed in* $\sigma = \mathfrak{T}^{\overline{\Lambda}}$ *iff* $\overline{\Lambda}(\mathfrak{F}\mathfrak{s}) \subseteq \mathfrak{F}\mathfrak{s}$.

Proof. This result follows from the observation that, $\mathfrak{F}\mathfrak{s}$ is closed in $(\mathfrak{P}, \sigma)$ iff $\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}$ is open in σ; implies $\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) = \mathfrak{P} \setminus \overline{\Lambda}(A)$ iff $\overline{\Lambda}(\mathfrak{F}\mathfrak{s}) \subseteq \mathfrak{F}\mathfrak{s}$. □

Theorem 4.5. *Consider an ITS* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. *If for every* $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$ *we have* $\overline{\Lambda}(\overline{\Lambda}(\mathfrak{F}\mathfrak{s})) \subseteq \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$, *then* $cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s}) = \mathfrak{F}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$, *where* $cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})$ *is the closure of* $\mathfrak{F}\mathfrak{s}$ *in* $(\mathfrak{P}, \mathfrak{T}^{\overline{\Lambda}})$.

Proof. Since $\overline{\Lambda}(\mathfrak{F}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{F}\mathfrak{s})) = \overline{\Lambda}(\mathfrak{F}\mathfrak{s}) \cup \overline{\Lambda}(\overline{\Lambda}(\mathfrak{F}\mathfrak{s})) = \overline{\Lambda}(\mathfrak{F}\mathfrak{s}) \subseteq \mathfrak{F}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$, then by Lemma 4.1, $\mathfrak{F}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$ is a closed set in $(\mathfrak{P}, \mathfrak{T}^{\overline{\Lambda}})$ that contains $\mathfrak{F}\mathfrak{s}$, and hence $cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s}) \subseteq \mathfrak{F}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$. Now to show that $\mathfrak{F}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{F}\mathfrak{s}) \subseteq cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})$, let $x \in \overline{\Lambda}(\mathfrak{F}\mathfrak{s}) \cup \mathfrak{F}\mathfrak{s}$. If $x \in \mathfrak{F}\mathfrak{s}$, then $x \in cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})$. If $x \in \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$, then for every closed set $\mathfrak{D}\mathfrak{s}$ containing x, we have $\mathfrak{F}\mathfrak{s} \cap \mathfrak{D}\mathfrak{s} \notin \widetilde{\mathbb{I}}$. Since $\mathfrak{F}\mathfrak{s} \subseteq cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})]$, then $cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s}) \cap \mathfrak{D}\mathfrak{s} \notin \widetilde{\mathbb{I}}$ and hence $x \in \overline{\Lambda}(cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s}))$. Since $cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})$ is closed in $(\mathfrak{P}, \mathfrak{T}^{\overline{\Lambda}})$, then by Lemma

4.1, $\overline{\Lambda}[cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})] \subseteq cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})$, and so we have $x \in cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})$. Therefore, $cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s}) = \mathfrak{F}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$ for every $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$. □

Theorem 4.6. *Consider an ITS* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ *and* $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$. *If* $\mathfrak{F}\mathfrak{s} \subseteq \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$, *then* $cl_{\Lambda}(\mathfrak{F}\mathfrak{s}) = cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s}) = cl_{\Lambda}(\overline{\Lambda}(\mathfrak{F}\mathfrak{s})) = \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$.

Proof. Since $\mathfrak{T}^{\Lambda} \subseteq \mathfrak{T}^{\overline{\Lambda}}$, then $cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s}) \subseteq cl_{\Lambda}(\mathfrak{F}\mathfrak{s})$ for every $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$. Now suppose $x \notin cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})$, then there is closed set W and $S \in \widetilde{\mathbb{I}}$ with $x \in W \setminus S$ and $(W \setminus S) \cap \mathfrak{F}\mathfrak{s} = \emptyset$. It follows that $\overline{\Lambda}[(W \setminus S) \cap \mathfrak{F}\mathfrak{s}] = \emptyset$ and hence $\overline{\Lambda}[(W \cap \mathfrak{F}\mathfrak{s}) \setminus S] = \emptyset$. Thus by Theorem 3.1 (11), $\overline{\Lambda}(W \cap \mathfrak{F}\mathfrak{s}) = \emptyset$ and so by Theorem 3.1 (9), we get $W \cap \overline{\Lambda}(\mathfrak{F}\mathfrak{s}) = \emptyset$ and $W \cap \mathfrak{F}\mathfrak{s} = \emptyset$ (as $\mathfrak{F}\mathfrak{s} \subseteq \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$. Therefore, $x \notin cl_{\Lambda}(\mathfrak{F}\mathfrak{s})$ and hence $cl_{\Lambda}(\mathfrak{F}\mathfrak{s}) = cl_{\overline{\Lambda}}(\mathfrak{F}\mathfrak{s})$. Now, by Theorem 3.1 (4), $\overline{\Lambda}(\mathfrak{F}\mathfrak{s})$ is a kernel set (i.e. Λ-closed), then $\overline{\Lambda}(\mathfrak{F}\mathfrak{s}) = cl_{\Lambda}(\overline{\Lambda}(\mathfrak{F}\mathfrak{s}))$. Now, to prove that $\overline{\Lambda}(\mathfrak{F}\mathfrak{s}) \subseteq cl_{\Lambda}(\mathfrak{F}\mathfrak{s})$, let $x \notin cl_{\Lambda}(\mathfrak{F}\mathfrak{s})$. Then a closed set $\mathfrak{D}\mathfrak{s}$ exists that contains x with $\mathfrak{D}\mathfrak{s} \cap \mathfrak{F}\mathfrak{s} = \emptyset$ and so $\mathfrak{D}\mathfrak{s} \cap \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$. Hence $x \notin \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$. Again as $\overline{\Lambda}(\mathfrak{F}\mathfrak{s}) \subseteq cl_{\Lambda}(\mathfrak{F}\mathfrak{s})$, so we have $cl_{\Lambda}(\overline{\Lambda}(\mathfrak{F}\mathfrak{s})) \subseteq cl_{\Lambda}(cl_{\Lambda}(\mathfrak{F}\mathfrak{s})) = cl_{\Lambda}(\mathfrak{F}\mathfrak{s})$. Also since $\mathfrak{F}\mathfrak{s} \subseteq \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$, then $cl_{\Lambda}(\mathfrak{F}\mathfrak{s}) \subseteq cl_{\Lambda}(\overline{\Lambda}(\mathfrak{F}\mathfrak{s}))$. Therefore, $cl_{\Lambda}(\mathfrak{F}\mathfrak{s}) = cl_{\Lambda}(\overline{\Lambda}(\mathfrak{F}\mathfrak{s})) = \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$. □

5. Kernel compatible topology via ideal

Definition 5.1. *Let* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ *be an ITS. We say that the topology* $\mathfrak{T}$ *is kernel-compatible with the ideal* $\widetilde{\mathbb{I}}$, *and write* $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$, *if for every subset* $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, *the following condition is satisfied: If, for each point* $x \in \mathfrak{D}\mathfrak{s}$, *there exists a closed set* $\mathfrak{F}\mathfrak{s}$ *containing* x *such that* $\mathfrak{F}\mathfrak{s} \cap \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$, *then it must be that* $\mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$.

Theorem 5.1. *Consider an ITS* $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. *The subsequent statements are mutually equivalent:*

(1) $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$;
(2) *If a subset* $\mathfrak{D}\mathfrak{s}$ *of* $\mathfrak{P}$ *has a cover of a closed set* $\mathfrak{F}\mathfrak{s}$ *containing* x *each of whose intersection with* $\mathfrak{D}\mathfrak{s}$ *is in* $\widetilde{\mathbb{I}}$, *then* $\mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$;
(3) *For all* $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, *the condition* $\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$ *implies that* $\mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$;
(4) *For all* $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, *we have* $\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \in \widetilde{\mathbb{I}}$;
(5) *If* $\mathfrak{D}\mathfrak{s}$ *includes no nonempty subset* B *such that* $B \subseteq \overline{\Lambda}(B)$, *then* $\mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$ *for every* $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$.

Proof. (1) ⇒ (2): The result is straightforward.
(2) ⇒ (3): Let $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$ and $x \in \mathfrak{D}\mathfrak{s}$. Then $x \notin \overline{\Lambda}(\mathfrak{D}\mathfrak{s})$ and there is a closed set $\cap\mathfrak{V}\mathfrak{s}_x$ containing x and $\cap\mathfrak{V}\mathfrak{s}_x \cap \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$. Therefore, we have $\mathfrak{D}\mathfrak{s} \subseteq \cup\{\cap\mathfrak{V}\mathfrak{s}_x : x \in \mathfrak{D}\mathfrak{s}\}$ and by (2) $\mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$.
(3) ⇒ (4): For any $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, $\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \subseteq \mathfrak{D}\mathfrak{s}$ and $(\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \subseteq (\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$. By (3), $\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \in \widetilde{\mathbb{I}}$.
(4) ⇒ (5): By (4), for every $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, $\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \in \widetilde{\mathbb{I}}$. Let $\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = J \in \widetilde{\mathbb{I}}$, then $\mathfrak{D}\mathfrak{s} = J \cup (\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}))$ and by Lemma 3.1, $\overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \overline{\Lambda}(J) \cup \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) = \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}))$. Therefore, we have $\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \subseteq \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}))$ and $\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \subseteq \mathfrak{D}\mathfrak{s}$. By the assumption $\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$ and hence $\mathfrak{D}\mathfrak{s} = \mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \in \widetilde{\mathbb{I}}$.
(5) ⇒ (1): Let $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$ and assume that for each $x \in \mathfrak{D}\mathfrak{s}$, there is a closed set U conataining x and

$U \cap \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$. Then $\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$. Since $(\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \subseteq (\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$, hence $B \nsubseteq \mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})$ and $B \subseteq \overline{\Lambda}(B)$. By (5), $\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \in \widetilde{\mathbb{I}}$ and so $\mathfrak{D}\mathfrak{s} = \mathfrak{D}\mathfrak{s} \cap (\mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) = \mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \in \widetilde{\mathbb{I}}$. Hence, $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$. □

Theorem 5.2. *Consider an ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. The subsequent statements are mutually equivalent:*

(1) $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$;
(2) *For every $\overline{\Lambda}$-closed subset $\mathfrak{D}\mathfrak{s}$, $\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \in \widetilde{\mathbb{I}}$.*

Proof. (1) ⇒ (2): It is clear by Theorem 5.1.
(2) ⇒ (1): Let $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$ and assume that for every $x \in \mathfrak{D}\mathfrak{s}$, there is a closed set U containing x, such that $U \cap \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$. Then $\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$. Since $cl_{\overline{\Lambda}}(\mathfrak{D}\mathfrak{s}) = \mathfrak{D}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{D}\mathfrak{s})$ is $\overline{\Lambda}$-closed, we have $(\mathfrak{D}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \in \widetilde{\mathbb{I}}$. Moreover, $(\mathfrak{D}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) = (\mathfrak{D}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \setminus (\overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \cup \overline{\Lambda}(\overline{\Lambda}(\mathfrak{D}\mathfrak{s}))) = (\mathfrak{D}\mathfrak{s} \cup \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \mathfrak{D}\mathfrak{s}$. Therefore $\mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$. Hence, $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$. □

Theorem 5.3. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS, and assume that $\mathfrak{T} \sim_\Lambda \mathcal{I}$. For all subsets $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, the statements below are mutually equivalent:*

(1) *If $\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$, then $\overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$.*
(2) $\overline{\Lambda}(\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) = \emptyset$.
(3) $\overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) = \overline{\Lambda}(\mathfrak{D}\mathfrak{s})$.

Proof. We begin by proving that (1) is valid under the assumption $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$. Let $\mathfrak{D}\mathfrak{s}$ be any subset of $\mathfrak{P}$ and $\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$. By Theorem 5.1, $\mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$ and by Lemma 3.1(5), $\overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$.
(1) ⇒ (2): Assume that for every $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, $\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$ implies that $\overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$. Let $B = \mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})$, then

$$\begin{aligned} B \cap \overline{\Lambda}(B) &= (\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \\ &= (\mathfrak{D}\mathfrak{s} \cap (\mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}))) \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cap (\mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}))) \\ &\subseteq \Big(\mathfrak{D}\mathfrak{s} \cap (\mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}))\Big) \cap \Big(\overline{\Lambda}(\mathfrak{D}\mathfrak{s}) \cap \overline{\Lambda}(\mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s}))\Big) = \emptyset. \end{aligned}$$

By (1), we have $\overline{\Lambda}(B) = \emptyset$. Hence $\overline{\Lambda}(\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) = \emptyset$.
(2) ⇒ (3): Assume for every $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$, $\overline{\Lambda}(\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) = \emptyset$.

$$\begin{aligned} D &= (\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \cup (\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \\ \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) &= \overline{\Lambda}\Big((\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \cup (\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}))\Big) \\ &= \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \setminus \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \cup \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) \\ &= \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s})). \end{aligned}$$

(3) ⇒ (1): Let $\mathfrak{D}\mathfrak{s} \subseteq \mathfrak{P}$ such that $\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$. Then by the assumption, $\overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) = \overline{\Lambda}(\mathfrak{D}\mathfrak{s})$. This implies that $\emptyset = \overline{\Lambda}(\emptyset) = \overline{\Lambda}(\mathfrak{D}\mathfrak{s} \cap \overline{\Lambda}(\mathfrak{D}\mathfrak{s})) = \overline{\Lambda}(\mathfrak{D}\mathfrak{s})$. □

Corollary 5.1. *Consider an ITS* $(\mathfrak{P},\mathfrak{T},\widetilde{\mathbb{I}})$. *If* $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$, *then* $\overline{\Lambda}(\bullet)$ *is an idempotent operator i.e.* $\overline{\Lambda}(\mathfrak{Ds}) = \overline{\Lambda}(\overline{\Lambda}(\mathfrak{Ds}))$ *for any* $\mathfrak{Ds} \subseteq \mathfrak{P}$.

Proof. By Theorem 5.3 and Lemma 3.1, we obtain $\overline{\Lambda}(\mathfrak{Ds}) = \overline{\Lambda}(\mathfrak{Ds} \cap \overline{\Lambda}(\mathfrak{Ds})) \subseteq \overline{\Lambda}(\mathfrak{Ds}) \cap \overline{\Lambda}(\overline{\Lambda}(\mathfrak{Ds})) = \overline{\Lambda}(\overline{\Lambda}(\mathfrak{Ds}))$ and by Lemma 3.1, we have $\overline{\Lambda}(\mathfrak{Ds}) = \overline{\Lambda}(\overline{\Lambda}(\mathfrak{Ds}))$ for any subset $\mathfrak{Ds}$ of $\mathfrak{P}$. □

Lemma 5.1. *Let* $(\mathfrak{P},\mathfrak{T},\widetilde{\mathbb{I}})$ *be an ITS such that* $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$ *and* $H \subseteq \mathfrak{P}$, *then* H *is a* $\overline{\Lambda}$*-closed iff* $H = K \cup I$ *such that* K *is a kernel set and* $I \in \widetilde{\mathbb{I}}$.

Proof. If H is a $\overline{\Lambda}$-closed set, then $\overline{\Lambda}(H) \subseteq H$. Hence $H = H \cup \overline{\Lambda}(H) = (H \setminus \overline{\Lambda}(H)) \cup \overline{\Lambda}(H) = K \cup I$. Then by Lemma 3.1 $K = \overline{\Lambda}(H)$ is a kernel set and by Theorem 5.2 $I = H \setminus \overline{\Lambda}(H) \in \widetilde{\mathbb{I}}$. Conversely, if $H = K \cup I$ such that K is a kernel set and $I \in \widetilde{\mathbb{I}}$, then by Lemma 3.1, we get that $\overline{\Lambda}(H) = \overline{\Lambda}(K \cup I) = \overline{\Lambda}(K) \cup \overline{\Lambda}(I) = \overline{\Lambda}(K) \subseteq ker(K) = K \subseteq H$. Implies that H is a $\overline{\Lambda}$-closed. □

Theorem 5.4. *Consider an ITS* $(\mathfrak{P},\mathfrak{T},\widetilde{\mathbb{I}})$. *Then* $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$ *iff* $\widetilde{\Lambda}(\mathfrak{Ds}) \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ *for every* $\mathfrak{Ds} \subseteq \mathfrak{P}$.

Proof. Necessity. Let $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$ and $\mathfrak{Ds} \subseteq \mathfrak{P}$. Consider the fact that $x \in \widetilde{\Lambda}(\mathfrak{Ds}) \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ by the definition of the $\widetilde{\Lambda}$-operator, this means that x does not belong to $\mathfrak{Ds}$ and $x \notin \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds})$ if and only if $x \notin \mathfrak{Ds}$ and there exists a closed set $\mathfrak{Fs}$ containing x such that $\mathfrak{Fs} \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ if and only if there exists a closed set $\mathfrak{Fs}$ containing x such that $x \in \mathfrak{Fs} \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$. Since, for all $x \in \widetilde{\Lambda}(\mathfrak{Ds}) \setminus \mathfrak{Ds}$ there exists a closed set $\mathfrak{Fs}$ containing x such that $F \cap (\widetilde{\Lambda}(\mathfrak{Ds}) \setminus \mathfrak{Ds}) \in \widetilde{\mathbb{I}}$ by heredity and hence $\widetilde{\Lambda}(\mathfrak{Ds}) \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ by assumption that $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$.

Sufficiency. Let $\mathfrak{Ds} \subseteq \mathfrak{P}$ and assume that for each $x \in \mathfrak{Ds}$ there exists a closed set $\mathfrak{Fs}$ containing x such that $\mathfrak{Fs} \cap \mathfrak{Ds} \in \widetilde{\mathbb{I}}$. Observe that $\widetilde{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}) = \{x$: there is a closed set $\mathfrak{Fs}$ containing x such that $x \in \mathfrak{Fs} \cap \mathfrak{Ds} \in \widetilde{\mathbb{I}}\}$. Thus $\mathfrak{Ds} \subseteq \widetilde{\Lambda}(\mathfrak{P} \setminus \mathfrak{Ds}) \setminus (\mathfrak{P} \setminus \mathfrak{Ds}) \in \widetilde{\mathbb{I}}$ and hence $\mathfrak{Ds} \in \widetilde{\mathbb{I}}$ by heredity of $\widetilde{\mathbb{I}}$. □

Proposition 5.1. *Let* $(\mathfrak{P},\mathfrak{T},\widetilde{\mathbb{I}})$ *be an ITS with* $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$, *and let* $\mathfrak{Ds} \subseteq \mathfrak{P}$. *Suppose* $\emptyset \neq \mathfrak{Fs}$ *is closed such that* $\mathfrak{Fs} \subseteq \overline{\Lambda}(\mathfrak{Ds}) \cap \widetilde{\Lambda}(\mathfrak{Ds})$, *then* $\mathfrak{Fs} \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ *and* $\mathfrak{Fs} \cap \mathfrak{Ds} \notin \widetilde{\mathbb{I}}$.

Proof. If $\mathfrak{Fs} \subseteq \overline{\Lambda}(\mathfrak{Ds}) \cap \widetilde{\Lambda}(\mathfrak{Ds})$, then $\mathfrak{Fs} \setminus \mathfrak{Ds} \subseteq \widetilde{\Lambda}(\mathfrak{Ds}) \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ by Theorem 5.4, and hence $\mathfrak{Fs} \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ by heredity. Since $\mathfrak{Fs}$ is a nonempty closed set and $\mathfrak{Fs} \subseteq \overline{\Lambda}(\mathfrak{Ds})$, we have $\mathfrak{Fs} \cap \mathfrak{Ds} \notin \widetilde{\mathbb{I}}$ by the definition of $\overline{\Lambda}(\mathfrak{Ds})$. □

As a consequence of the above, We obtain the following results.

Corollary 5.2. *Consider an ITS* $(\mathfrak{P},\mathfrak{T},\widetilde{\mathbb{I}})$ *such that* $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$. *For all* $\mathfrak{Ds} \subseteq \mathfrak{P}$ *we have* $\widetilde{\Lambda}(\widetilde{\Lambda}(\mathfrak{Ds})) = \widetilde{\Lambda}(\mathfrak{Ds})$.

Proof. $\widetilde{\Lambda}(\mathfrak{Ds}) \subseteq \widetilde{\Lambda}(\widetilde{\Lambda}(\mathfrak{Ds}))$ follows from Theorem 4.1 (7). Since $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$, as a consequence of Theorem 5.4, we have that $\widetilde{\Lambda}(\mathfrak{Ds}) \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ implies $\widetilde{\Lambda}(\mathfrak{Ds}) \subseteq \mathfrak{Ds} \cup I$ for some $I \in \widetilde{\mathbb{I}}$ and hence $\widetilde{\Lambda}(\widetilde{\Lambda}(\mathfrak{Ds})) = \widetilde{\Lambda}(\mathfrak{Ds})$ by Theorem 4.1 (12). □

Theorem 5.5. *Let* $(\mathfrak{P},\mathfrak{T},\widetilde{\mathbb{I}})$ *be an ITS with* $\mathfrak{T} \sim_{\Lambda} \widetilde{\mathbb{I}}$. *Then* $\widetilde{\Lambda}(\mathfrak{Ds}) = \cup\{\widetilde{\Lambda}(\mathfrak{Fs})$: *F is a closed set and* $\widetilde{\Lambda}(\mathfrak{Fs}) \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}\}$.

Proof. Let $\sigma(\mathfrak{D}\mathfrak{s}) = \cup\{\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) : \mathfrak{F}\mathfrak{s}$ is a closed set and $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \setminus \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}\}$. Let $x \in \sigma(\mathfrak{D}\mathfrak{s})$. so $\exists$ a closed set $\mathfrak{F}\mathfrak{s}$ such that $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \setminus \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$ and $x \in \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$. Therefore, there exists a closed set $\mathfrak{V}\mathfrak{s}$ containing x such that $\mathfrak{V}\mathfrak{s} \setminus \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$. Since $\mathfrak{F}\mathfrak{s}$ is also $\overline{\Lambda}$-open by Corollary 4.6, $\mathfrak{F}\mathfrak{s} \subset \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$ and $\mathfrak{F}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \setminus \mathfrak{D}\mathfrak{s}$ and hence $\mathfrak{F}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$. Therefore, $\mathfrak{V}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s} \subseteq (\mathfrak{V}\mathfrak{s} \setminus \mathfrak{F}\mathfrak{s}) \cup (\mathfrak{F}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s}) \in \widetilde{\mathbb{I}}$ and hence $\mathfrak{V}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$. Since $\mathfrak{V}\mathfrak{s}$ is a closed set containing x and $\mathfrak{V}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$, thus $x \in \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s})$. Hence, $\sigma(\mathfrak{D}\mathfrak{s}) \subseteq \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s})$. Now let $x \in \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s})$. Next, $\exists$ a closed set $\mathfrak{F}\mathfrak{s}$ that contains x in such a way that $\mathfrak{F}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$. By Corollary 4.6, $\mathfrak{F}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$ and $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \setminus \mathfrak{D}\mathfrak{s} \subseteq [\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \setminus \mathfrak{F}\mathfrak{s}] \cup [\mathfrak{F}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s}]$. By Theorem 5.4, $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \setminus \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$ and hence $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \setminus \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$. Hence $x \in \sigma(\mathfrak{D}\mathfrak{s})$ and $\sigma(\mathfrak{D}\mathfrak{s}) \supseteq \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s})$. Consequently, we obtain $\sigma(\mathfrak{D}\mathfrak{s}) = \widetilde{\Lambda}(\mathfrak{D}\mathfrak{s})$. □

In [19], Newcomb defines $\mathfrak{F}\mathfrak{s} = K$ [mod $\widetilde{\mathbb{I}}$] if $(\mathfrak{F}\mathfrak{s} \setminus K) \cup (K \setminus \mathfrak{F}\mathfrak{s}) \in \widetilde{\mathbb{I}}$ and observes that = [mod $\widetilde{\mathbb{I}}$] is an equivalence relation. According to Theorem 4.1 (13), we have that if $\mathfrak{F}\mathfrak{s} = K$ [mod $\widetilde{\mathbb{I}}$], then $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) = \widetilde{\Lambda}(K)$.

Definition 5.2. *Consider the ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. If $\exists$ a closed set K such that $F = K$ [mod $\widetilde{\mathbb{I}}$], then $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$ is a c-baire set with regard to $\mathfrak{T}$ and $\widetilde{\mathbb{I}}$, denoted $\mathfrak{F}\mathfrak{s} \in B_c(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$.*

Lemma 5.2. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS with $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$. If $\mathfrak{F}\mathfrak{s}$ and K are closed sets and $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) = \widetilde{\Lambda}(K)$, then $\mathfrak{F}\mathfrak{s} = K$ [mod $\widetilde{\mathbb{I}}$].*

Proof. Since $\mathfrak{F}\mathfrak{s}$ is closed set, we have $\mathfrak{F}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$ and hence $\mathfrak{F}\mathfrak{s} \setminus K \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \setminus K = \widetilde{\Lambda}(K) \setminus K \in \widetilde{\mathbb{I}}$ by Theorem 5.4. Therefore, $\mathfrak{F}\mathfrak{s} \setminus K \in \widetilde{\mathbb{I}}$. Similarly $K \setminus H \in \widetilde{\mathbb{I}}$. Now $(\mathfrak{F}\mathfrak{s} \setminus K) \cup (K \setminus \mathfrak{F}\mathfrak{s}) \in \widetilde{\mathbb{I}}$ by additivity. Hence, $\mathfrak{F}\mathfrak{s} = K$ [mod $\widetilde{\mathbb{I}}$]. □

Theorem 5.6. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS with $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$. If $\mathfrak{F}\mathfrak{s}$, $K \in B_c(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$, and $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) = \widetilde{\Lambda}(K)$, then $F = K$ [mod $\widetilde{\mathbb{I}}$].*

Proof. Let $U, \mathfrak{V}\mathfrak{s}$ be closed sets such that $\mathfrak{F}\mathfrak{s} = U$ [mod $\widetilde{\mathbb{I}}$] and $K = \mathfrak{V}\mathfrak{s}$ [mod $\widetilde{\mathbb{I}}$]. Now $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) = \widetilde{\Lambda}(U)$ and $\widetilde{\Lambda}(K) = \widetilde{\Lambda}(\mathfrak{V}\mathfrak{s})$ by Theorem 4.1(13). Since $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) = \widetilde{\Lambda}(K)$ implies that $\widetilde{\Lambda}(U) = \widetilde{\Lambda}(\mathfrak{V}\mathfrak{s})$, $U = \mathfrak{V}\mathfrak{s}$ [mod $\widetilde{\mathbb{I}}$] by Lemma 5.3. Hence $\mathfrak{F}\mathfrak{s} = K$ [mod $\widetilde{\mathbb{I}}$] by transitivity. □

Definition 5.3. *Consider the ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. If there is a nonempty closed set $\mathfrak{D}\mathfrak{s}$ with $(\mathfrak{F}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s}) \cup (\mathfrak{D}\mathfrak{s} \setminus \mathfrak{F}\mathfrak{s}) \in \widetilde{\mathbb{I}}$, then a subset $\mathfrak{F}\mathfrak{s}$ of $\mathfrak{P}$ is a baire closed set with regard to $\mathfrak{T}$ and $\widetilde{\mathbb{I}}$.*

Example 5.1. *In Example 4.1 we have*

(1) $\mathfrak{F}\mathfrak{s} = \{2, 3\}$, *is baire closed set because there exist closed set $\mathfrak{D}\mathfrak{s} = \{2\}$ such that $(\mathfrak{F}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s}) \cup (\mathfrak{D}\mathfrak{s} \setminus \mathfrak{F}\mathfrak{s}) \in \widetilde{\mathbb{I}}$.*
(2) $\mathfrak{F}\mathfrak{s} = \{1, 3\}$, *is not baire closed set because for all closed sets $\mathfrak{D}\mathfrak{s} = \{2\}$, $\mathfrak{D}\mathfrak{s} = \emptyset$ and $\mathfrak{D}\mathfrak{s} = \mathfrak{P}$ we obtain that $(\mathfrak{F}\mathfrak{s} \setminus \mathfrak{D}\mathfrak{s}) \cup (\mathfrak{D}\mathfrak{s} \setminus \mathfrak{F}\mathfrak{s}) \notin \widetilde{\mathbb{I}}$.*

Lemma 5.3. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS with $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$. If $\mathfrak{F}\mathfrak{s}$ and S are closed set and $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) = \widetilde{\Lambda}(S)$, then $\mathfrak{F}\mathfrak{s}$ and S are baire closed sets.*

Proof. Since $\mathfrak{Fs}$ is closed set then it is a $co\Lambda$-set, then by Theorem 4.1 $\mathfrak{Fs} \subseteq \widetilde{\Lambda}(\mathfrak{Fs})$ and hence by Theorem 5.4 $\widetilde{\Lambda}(\mathfrak{Fs}) \setminus S = \widetilde{\Lambda}(S) \setminus S \in \widetilde{\mathbb{I}}$. Since $\mathfrak{Fs} \setminus S \subseteq \widetilde{\Lambda}(S) \setminus S$, then $\mathfrak{Fs} \setminus S \in \widetilde{\mathbb{I}}$. Similarly, $S \setminus \mathfrak{Fs} \in \widetilde{\mathbb{I}}$. Therefore, $(\mathfrak{Fs} \setminus S) \cup (S \setminus \mathfrak{Fs}) \in \widetilde{\mathbb{I}}$ by additivity. Hence, $\mathfrak{Fs}$ and S are baire closed sets □

Theorem 5.7. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS with $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$. If $\mathfrak{Fs}$ and S are baire closed sets and $\widetilde{\Lambda}(\mathfrak{Fs}) = \widetilde{\Lambda}(S)$, then $(\mathfrak{Fs} \setminus S) \cup (S \setminus \mathfrak{Fs}) \in \widetilde{\mathbb{I}}$.*

Proof. Since $\mathfrak{Fs}$ and S are baire closed sets, then their are two closed sets $\mathfrak{Ds}$ and W with $(\mathfrak{Fs} \setminus \mathfrak{Ds}) \cup (\mathfrak{Ds} \setminus \mathfrak{Fs}) \in \widetilde{\mathbb{I}}$ and $(W \setminus S) \cup (S \setminus W) \in \widetilde{\mathbb{I}}$. By part (13) of Theorem 4.1 we get $\widetilde{\Lambda}(\mathfrak{Fs}) = \widetilde{\Lambda}(\mathfrak{Ds})$ and $\widetilde{\Lambda}(S) = \widetilde{\Lambda}(W)$. Since $\widetilde{\Lambda}(\mathfrak{Fs}) = \widetilde{\Lambda}(S)$, then $\widetilde{\Lambda}(\mathfrak{Ds}) = \widetilde{\Lambda}(W)$, and hence by Lemma 5.3, $(\mathfrak{Ds} \setminus W) \cup (W \setminus \mathfrak{Ds}) \in \widetilde{\mathbb{I}}$. It follows that $(\mathfrak{Fs} \setminus \mathfrak{Ds}) \cup (\mathfrak{Ds} \setminus \mathfrak{Fs}) \cup (W \setminus S) \cup (S \setminus W) \cup (\mathfrak{Ds} \setminus W) \cup (W \setminus \mathfrak{Ds}) \in \widetilde{\mathbb{I}}$. Therefore, $(\mathfrak{Fs} \setminus S) \cup (S \setminus \mathfrak{Fs}) \in \widetilde{\mathbb{I}}$. □

Theorem 5.8. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS with $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$ and $S \subseteq \mathfrak{P}$, then $\overline{\Lambda}(\mathfrak{Fs} \cap S) = \overline{\Lambda}(\mathfrak{Fs} \cap \overline{\Lambda}(S)) = cl_\Lambda(\mathfrak{Fs} \cap \overline{\Lambda}(S))$ for every closed set $\mathfrak{Fs}$.*

Proof. Assume that $\mathfrak{Fs}$ is closed set. First, we will show that $\overline{\Lambda}(\mathfrak{Fs} \cap S) = \overline{\Lambda}(\mathfrak{Fs} \cap \overline{\Lambda}(S))$. By Lemma 3.1 (9), $\mathfrak{Fs} \cap \overline{\Lambda}(S) \subseteq \overline{\Lambda}(\mathfrak{Fs} \cap S)$ and thus by Lemma 3.1 (1) and by Corollary 5.1 we get $\overline{\Lambda}(\mathfrak{Fs} \cap \overline{\Lambda}(S)) \subseteq \overline{\Lambda}[\overline{\Lambda}(\mathfrak{Fs} \cap S)] = \overline{\Lambda}(\mathfrak{Fs} \cap S)$. Now by Lemma 3.1 (1), and by Theorem 5.3, we get $\overline{\Lambda}[\mathfrak{Fs} \cap (S \setminus \overline{\Lambda}(S))] \subseteq \overline{\Lambda}(S \setminus \overline{\Lambda}(S)) = \emptyset$. Also, by Lemma 3.1 (10), $\overline{\Lambda}(\mathfrak{Fs} \cap S) \setminus \overline{\Lambda}(\mathfrak{Fs} \cap \overline{\Lambda}(S)) \subseteq \overline{\Lambda}[(\mathfrak{Fs} \cap S) \setminus (\mathfrak{Fs} \cap \overline{\Lambda}(S))] = \overline{\Lambda}[\mathfrak{Fs} \cap (S \setminus \overline{\Lambda}(S))] = \emptyset$. Hence, $\overline{\Lambda}(\mathfrak{Fs} \cap S) \subseteq \overline{\Lambda}(\mathfrak{Fs} \cap \overline{\Lambda}(S))$ and so we get $\overline{\Lambda}(\mathfrak{Fs} \cap S) = \overline{\Lambda}(\mathfrak{Fs} \cap \overline{\Lambda}(S))$.

Again $\overline{\Lambda}(\mathfrak{Fs} \cap S) = \overline{\Lambda}(\mathfrak{Fs} \cap \overline{\Lambda}(S)) \subseteq cl_{\overline{\Lambda}}((\mathfrak{Fs} \cap \overline{\Lambda}(S)) \subseteq cl_\Lambda(\mathfrak{Fs} \cap \overline{\Lambda}(S))$, since $\mathfrak{T}^\Lambda \subseteq \mathfrak{T}^{\overline{\Lambda}}$. Now, by using Lemma 3.1 (9), $\mathfrak{Fs} \cap \overline{\Lambda}(S) \subseteq \overline{\Lambda}(\mathfrak{Fs} \cap S)$, and hence $cl_\Lambda(\mathfrak{Fs} \cap \overline{\Lambda}(S)) \subseteq cl_\Lambda(\overline{\Lambda}(\mathfrak{Fs} \cap S)) = \overline{\Lambda}(\mathfrak{Fs} \cap S)$ by Lemma 3.1 (3). Therefore, $\overline{\Lambda}(\mathfrak{Fs} \cap S) = cl_\Lambda(\mathfrak{Fs} \cap \overline{\Lambda}(S))$. □

Corollary 5.3. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS such that $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$. If $\mathfrak{Fs}$ is closed set and $\mathfrak{Fs} \in \widetilde{\mathbb{I}}$, then $\mathfrak{Fs} \subseteq \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P})$.*

Proof. Set $S = \mathfrak{P}$ in Theorem 5.8, then $\overline{\Lambda}(\mathfrak{Fs}) = \overline{\Lambda}(\mathfrak{Fs} \cap \mathfrak{P}) = cl_\Lambda(\mathfrak{Fs} \cap \overline{\Lambda}(\mathfrak{P}))$. Since $\mathfrak{Fs} \in \widetilde{\mathbb{I}}$, then $\overline{\Lambda}(\mathfrak{Fs}) = \emptyset$ and hence $\emptyset = \overline{\Lambda}(\mathfrak{Fs}) = \overline{\Lambda}(\mathfrak{Fs} \cap \mathfrak{P}) = cl_\Lambda(\mathfrak{Fs} \cap \overline{\Lambda}(\mathfrak{P}))$. Thus, $\mathfrak{Fs} \cap \overline{\Lambda}(\mathfrak{P}) = \emptyset$ and hence $\mathfrak{Fs} \subseteq \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P})$. □

6. On $\overline{\Lambda}$-dense sets and $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$

Theorem 6.1. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ denote an ITS. Then, the following conditions are equivalent:*

(1) $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$,
(2) $\widetilde{\Lambda}(\emptyset) = \emptyset$,
(3) *if $\mathfrak{Fs} \in \mathfrak{T}$, then $\widetilde{\Lambda}(\mathfrak{Fs}) \setminus \mathfrak{Fs} = \emptyset$, (i.e. $\mathfrak{Fs} \in \mathfrak{T}^c$, then $\mathfrak{Fs} \subseteq \overline{\Lambda}(\mathfrak{Fs})$),*
(4) *if $K \in \widetilde{\mathbb{I}}$, then $\widetilde{\Lambda}(K) = \emptyset$.*

Proof. (1) ⇒ (2) Since $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$, then by Proposition 4.1, $\widetilde{\Lambda}(\emptyset) = \cup\{\mathfrak{Ds} : \mathfrak{Ds}$ is closed set and $\mathfrak{Ds} \in \widetilde{\mathbb{I}}\} = \emptyset$.

(2) ⇒ (3) Assume that $x \in \widetilde{\Lambda}(\mathfrak{Fs}) \setminus \mathfrak{Fs}$. Consequently, one can find a closed set S that contains x

and satisfies $S \setminus \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$ and $S \setminus \mathfrak{F}\mathfrak{s}$ is closed set containing x. But $S \setminus \mathfrak{F}\mathfrak{s} \subseteq \cup\{\mathfrak{D}\mathfrak{s} : \mathfrak{D}\mathfrak{s}$ is closed set and $\mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}\} = \widetilde{\Lambda}(\emptyset)$ which implies that $\widetilde{\Lambda}(\emptyset) \neq \emptyset$. Hence $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \setminus \mathfrak{F}\mathfrak{s} = \emptyset$.

(3) ⇒ (4) Let $K \in \widetilde{\mathbb{I}}$ and since $\emptyset$ open set, then $\widetilde{\Lambda}(K) = \widetilde{\Lambda}(K \cup \emptyset) = \widetilde{\Lambda}(\emptyset) = \emptyset$.

(4) ⇒ (1) Suppose that $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} \neq \{\emptyset\}$. Then there is a nonempty closed set $\mathfrak{F}\mathfrak{s}$ and $\mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$ by item (4), $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) = \emptyset$. Since $\mathfrak{F}\mathfrak{s}$ is closed set implies that $\mathfrak{F}\mathfrak{s}$ is $\overline{\Lambda}$-open, by Corollary 4.1, we have $\mathfrak{F}\mathfrak{s} \subseteq \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) = \emptyset$. This is a contradiction. Hence, $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$. □

Theorem 6.2. *Assume $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ is an ITS satisfying $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$, then for every subset $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$ we have $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \subseteq \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$.*

Proof. Let $x \in \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$ and suppose that $x \notin \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$. Consequently, one can find a closed set $\mathfrak{D}\mathfrak{s}$ that contains x and satisfies $\mathfrak{D}\mathfrak{s} \cap \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$. Since $x \in \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s})$, then by Proposition 4.1, $x \in \cup\{\mathfrak{D}\mathfrak{s} : \mathfrak{D}\mathfrak{s}$ is closed set and $\mathfrak{D}\mathfrak{s} \setminus \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}\}$ and hence there is a closed set S containing x such that $S \setminus \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$. It follows that $\mathfrak{D}\mathfrak{s} \cap S$ is closed set containing x such that $\mathfrak{D}\mathfrak{s} \cap S \cap \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$ and $(\mathfrak{D}\mathfrak{s} \cap S) \setminus \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$ by heredity. Now, by finite additivity $[\mathfrak{D}\mathfrak{s} \cap S] = [\mathfrak{D}\mathfrak{s} \cap S \cap \mathfrak{F}\mathfrak{s}] \cup [(\mathfrak{D}\mathfrak{s} \cap S)\setminus\mathfrak{F}\mathfrak{s}] \in \widetilde{\mathbb{I}}$ this leads to a contradiction because $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$. Thus, $x \in \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$ and hence $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \subseteq \overline{\Lambda}(\mathfrak{F}\mathfrak{s})$. □

Corollary 6.1. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ denote an ITS. If $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$, then $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \subseteq cl_{\Lambda}(\overline{\Lambda}(\mathfrak{F}\mathfrak{s}))$ for every $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$.*

Theorem 6.3. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ denote an ITS. If $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$, then $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \cap \widetilde{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) = \emptyset$ for every $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$.*

Proof. Assume that $x \in \widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \cap \widetilde{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s})$ for some $x \in \mathfrak{P}$. Then there are are two closed sets S and $\mathfrak{D}\mathfrak{s}$ containing x such that $S \setminus \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$ and $\mathfrak{D}\mathfrak{s} \cap \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$ respectively. Thus, $[(S \cap \mathfrak{D}\mathfrak{s}) \setminus \mathfrak{F}\mathfrak{s}] \in \widetilde{\mathbb{I}}$ and $(S \cap \mathfrak{D}\mathfrak{s}) \cap \mathfrak{F}\mathfrak{s} \in \widetilde{\mathbb{I}}$ and hence $S \cap \mathfrak{D}\mathfrak{s} \in \widetilde{\mathbb{I}}$ such that $S \cap \mathfrak{D}\mathfrak{s}$ is closed set. Now, since $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$, then $S \cap \mathfrak{D}\mathfrak{s} = \emptyset$, That is contradictory. Therefore, $\widetilde{\Lambda}(\mathfrak{F}\mathfrak{s}) \cap \widetilde{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) = \emptyset$. □

Corollary 6.2. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ be an ITS. If $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$, then $\overline{\Lambda}(\mathfrak{F}\mathfrak{s}) \cup \overline{\Lambda}(\mathfrak{P} \setminus \mathfrak{F}\mathfrak{s}) = \mathfrak{P}$ for every subset $\mathfrak{F}\mathfrak{s}$ of $\mathfrak{P}$.*

A subset $\mathfrak{F}\mathfrak{s}$ in ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ iscalled $\overline{\Lambda}$-dense if $\overline{\Lambda}(\mathfrak{F}\mathfrak{s}) = \mathfrak{P}$.

Proposition 6.1. *Consider an ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. Then, for every $x \in \mathfrak{P}$, the complement of $\{x\}$ is $\overline{\Lambda}$–dense if and only if $\widetilde{\Lambda}(\{x\})$ is empty.*

Proof. This outcome is an immediate consequence of the definition of $\overline{\Lambda}$-dense sets, given that $\widetilde{\Lambda}(\{x\}) = \mathfrak{P} \setminus \overline{\Lambda}(\mathfrak{P} \setminus \{x\}) = \emptyset$ iff $\mathfrak{P} = \overline{\Lambda}(\mathfrak{P} \setminus \{x\})$. □

Theorem 6.4. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ denote an ITS with the property that $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$ and $\mathfrak{F}\mathfrak{s} \subseteq \mathfrak{P}$, then $\mathfrak{F}\mathfrak{s}$ is $\overline{\Lambda}$-dense iff $\mathfrak{F}\mathfrak{s}$ is dense in $\mathfrak{T}^{\overline{\Lambda}}$.*

Proof. Suppose that $\mathfrak{Fs}$ is $\overline{\Lambda}$-dense, then $\overline{\Lambda}(\mathfrak{Fs}) = \mathfrak{P}$ and $cl_{\overline{\Lambda}}(\mathfrak{Fs}) = \mathfrak{Fs} \cup \overline{\Lambda}(\mathfrak{Fs}) = \mathfrak{P}$. Therefore, $\mathfrak{Fs}$ is dense in $\mathfrak{T}^{\overline{\Lambda}}$.

Conversely, let $\mathfrak{Fs}$ is dense in $\mathfrak{T}^{\overline{\Lambda}}$. Then $cl_{\overline{\Lambda}}(\mathfrak{Fs}) = \mathfrak{Fs} \cup \overline{\Lambda}(\mathfrak{Fs}) = \mathfrak{P}$. To prove that $\overline{\Lambda}(\mathfrak{Fs}) = \mathfrak{P}$, let $x \in \mathfrak{P}$ such that $x \notin \overline{\Lambda}(\mathfrak{Fs})$. Then $\exists$ a closed set W containing x such that $W \cap \mathfrak{Fs} \in \widetilde{\mathbb{I}}$. Since $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$, then $W \notin \widetilde{\mathbb{I}}$. Now we want to show $W \setminus \mathfrak{Fs} \notin \widetilde{\mathbb{I}}$. Suppose $W \setminus \mathfrak{Fs} \in \widetilde{\mathbb{I}}$, then $[W \cap \mathfrak{Fs}] \cup [W \setminus \mathfrak{Fs}] = W \in \widetilde{\mathbb{I}}$, which is a contradiction. Therefore, $W \cap \mathfrak{Fs}^c \notin \widetilde{\mathbb{I}}$ and hence $W \cap \mathfrak{Fs}^c \neq \emptyset$. Let $x \in W \cap \mathfrak{Fs}^c$. Then, $x \notin \mathfrak{Fs}$ and also $x \notin \overline{\Lambda}(\mathfrak{Fs})$. Because if $x \in \overline{\Lambda}(\mathfrak{Fs})$ implies that $W \cap \mathfrak{Fs} \notin \widetilde{\mathbb{I}}$ which is a contrary to $W \cap \mathfrak{Fs} \in \widetilde{\mathbb{I}}$. Thus, $x \notin \mathfrak{Fs} \cup \overline{\Lambda}(\mathfrak{Fs}) = cl_{\overline{\Lambda}}(\mathfrak{Fs}) = \mathfrak{P}$, This leads to a contradiction. Thus, we get $\overline{\Lambda}(\mathfrak{Fs}) = \mathfrak{P}$. Hence, $\mathfrak{Fs}$ is $\overline{\Lambda}$-dense. □

Proposition 6.2. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ denote an ITS with the property that $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$. Then, $\widetilde{\Lambda}(\mathfrak{Fs}) \neq \emptyset$ iff $\mathfrak{Fs}$ contains the nonempty $\overline{\Lambda}$-interior.*

Proof. Let $\widetilde{\Lambda}(\mathfrak{Fs}) \neq \emptyset$. By Proposition 4.1, $\widetilde{\Lambda}(\mathfrak{Fs}) = \cup\{\mathfrak{Ds} : \mathfrak{Ds}$ is closed and $\mathfrak{Ds} \setminus \mathfrak{Fs} \in \widetilde{\mathbb{I}}\}$. Let $\mathfrak{Ds} \setminus \mathfrak{Fs} = K$, then $K \in \widetilde{\mathbb{I}}$. Now $\mathfrak{Ds} \setminus K \subseteq \mathfrak{Fs}$ and hence by Theorem 3.3, $\mathfrak{Ds} \setminus K \in \mathfrak{T}^{\overline{\Lambda}}$ and so $\mathfrak{Fs}$ contains the nonempty $\overline{\Lambda}$-interior.

Conversely, let us assume that $\mathfrak{Fs}$ contains a nonempty $\overline{\Lambda}$-interior. It follows that there exists a closed set $\mathfrak{Ds}$ and $K \in \widetilde{\mathbb{I}}$ such that $\mathfrak{Ds} \setminus K \subseteq \mathfrak{Fs}$, then $\mathfrak{Ds} \setminus \mathfrak{Fs} \subseteq K$. Since $\mathfrak{Ds} \setminus \mathfrak{Fs} \subseteq K \in \widetilde{\mathbb{I}}$. Hence, $\cup\{\mathfrak{Ds} : \mathfrak{Ds}$ is closed set and $\mathfrak{Ds} \setminus \mathfrak{Fs} \in \widetilde{\mathbb{I}}\} = \widetilde{\Lambda}(\mathfrak{Fs}) \neq \emptyset$. □

Theorem 6.5. *Let $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ denote an ITS with the property that $\mathfrak{T} \sim_\Lambda \widetilde{\mathbb{I}}$ and $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$. If W is $\overline{\Lambda}$-open set such that $W = \mathfrak{Ds} \setminus \mathfrak{Fs}$ with $\mathfrak{Ds}$ is closed set and $\mathfrak{Fs} \in \widetilde{\mathbb{I}}$, then $cl_\Lambda(W) = cl_{\overline{\Lambda}}(W) = \overline{\Lambda}(W) = \overline{\Lambda}(\mathfrak{Ds}) = cl_\Lambda(\mathfrak{Ds}) = cl_{\overline{\Lambda}}(\mathfrak{Ds})$.*

Proof. Let $W = \mathfrak{Ds} \setminus \mathfrak{Fs}$ such that $\mathfrak{Ds}$ is closed set and $\mathfrak{Fs} \in \widetilde{\mathbb{I}}$. Since $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$, then by Theorem 6.1 $\mathfrak{Ds} \subseteq \overline{\Lambda}(\mathfrak{Ds})$. Thus, by Theorem 4.6, $\overline{\Lambda}(\mathfrak{Ds}) = cl_\Lambda(\mathfrak{Ds}) = cl_{\overline{\Lambda}}(\mathfrak{Ds})$. Now, W is $\overline{\Lambda}$-open. We claim that $W \subseteq \overline{\Lambda}(W)$. Since $cl_{\overline{\Lambda}}(\mathfrak{P} \setminus W) = \mathfrak{P} \setminus W$, then $\overline{\Lambda}(\mathfrak{P} \setminus W) \subseteq \mathfrak{P} \setminus W$. By Theorem 3.1, $\overline{\Lambda}(\mathfrak{P}) = \mathfrak{P}$, then $\mathfrak{P} \setminus \overline{\Lambda}(W) = \overline{\Lambda}(\mathfrak{P}) \setminus \overline{\Lambda}(W) \subseteq \overline{\Lambda}(\mathfrak{P} \setminus W) \subseteq \mathfrak{P} \setminus W$. Thus, $W \subseteq \overline{\Lambda}(W)$ and hence by Theorem 4.6, $\overline{\Lambda}(W) = cl_\Lambda(W) = cl_{\overline{\Lambda}}(W)$.

Moreover, $W \subseteq \mathfrak{Ds}$ and so we have $\overline{\Lambda}(W) \subseteq \overline{\Lambda}(\mathfrak{Ds})$. Also, using Theorem 3.1, we get $\overline{\Lambda}(W) = \overline{\Lambda}(\mathfrak{Ds} \setminus \mathfrak{Fs}) \supseteq \overline{\Lambda}(\mathfrak{Ds}) \setminus \overline{\Lambda}(\mathfrak{Fs}) = \overline{\Lambda}(\mathfrak{Ds})$ since $\mathfrak{Fs} \in \widetilde{\mathbb{I}}$, $\overline{\Lambda}(\mathfrak{Fs}) = \emptyset$. Therefore, $\overline{\Lambda}(\mathfrak{Ds}) = \overline{\Lambda}(W)$ and hence $cl_\Lambda(W) = cl_{\overline{\Lambda}}(W) = \overline{\Lambda}(W) = \overline{\Lambda}(\mathfrak{Ds}) = cl_\Lambda(\mathfrak{Ds}) = cl_{\overline{\Lambda}}(\mathfrak{Ds})$. □

Proposition 6.3. *Consider an ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$. Then:*

1. *If S is baire closed and does not belong to $\widetilde{\mathbb{I}}$, a nonempty closed set $\mathfrak{Fs}$ exists with $(\mathfrak{Fs} \setminus S) \cup (S \setminus \mathfrak{Fs}) \in \widetilde{\mathbb{I}}$.*
2. *a set S is a baire closed with $S \notin \widetilde{\mathbb{I}}$ iff $\exists$ a nonempty closed $\mathfrak{Fs}$ with $(\mathfrak{Fs} \setminus S) \cup (S \setminus \mathfrak{Fs}) \in \widetilde{\mathbb{I}}$, whenever $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$.*

Proof. (1) Assume that S is a baire closed with $S \notin \widetilde{\mathbb{I}}$. So, $\exists$ a closed set $\mathfrak{Fs}$ with $(\mathfrak{Fs} \setminus S) \cup (S \setminus \mathfrak{Fs}) \in \widetilde{\mathbb{I}}$ and hence $S \setminus \mathfrak{Fs} \in \widetilde{\mathbb{I}}$. Note that, if $\mathfrak{Fs} = \emptyset$, then $S \in \widetilde{\mathbb{I}}$ hence, a contradiction is obtained.

(2) Let us assume that a nonempty closed set $\mathfrak{Fs}$ exists with $(\mathfrak{Fs} \setminus S) \cup (S \setminus \mathfrak{Fs}) \in \widetilde{\mathbb{I}}$. Then $\mathfrak{Fs} = (S \setminus L) \cup K$, where $L = S \setminus \mathfrak{Fs} \in \widetilde{\mathbb{I}}$ and $K = \mathfrak{Fs} \setminus S \in \widetilde{\mathbb{I}}$. Now, suppose that $S \in \widetilde{\mathbb{I}}$, then $S \setminus L \in \widetilde{\mathbb{I}}$ and so $\mathfrak{Fs} \in \widetilde{\mathbb{I}}$ by additivity, which leads to a contradiction since $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$. Therefore, S is a baire closed set and $S \notin \widetilde{\mathbb{I}}$. □

Proposition 6.4. *Consider an ITS $(\mathfrak{P}, \mathfrak{T}, \widetilde{\mathbb{I}})$ satisfying $\mathfrak{T}^c \cap \widetilde{\mathbb{I}} = \{\emptyset\}$. If $\mathfrak{Ds}$ is a Baire closed set and $\mathfrak{Ds}$ lies in $\widetilde{\mathbb{I}}$, then $\widetilde{\Lambda}(\mathfrak{Ds}) \cap \overline{\Lambda}(\mathfrak{Ds}) \neq \emptyset$.*

Proof. Let $\mathfrak{Ds}$ be a baire closed set and $\mathfrak{Ds} \notin \widetilde{\mathbb{I}}$. Then by Proposition 6.3 item (1) there is a nonempty closed set $\mathfrak{Fs}$ with $(\mathfrak{Fs} \setminus \mathfrak{Ds}) \cup (\mathfrak{Ds} \setminus \mathfrak{Fs}) \in \widetilde{\mathbb{I}}$. It follows that by Theorem 3.1, $\mathfrak{Fs} \subseteq \overline{\Lambda}(\mathfrak{Fs}) = \overline{\Lambda}((\mathfrak{Ds} \setminus K) \cup L) = \overline{\Lambda}(\mathfrak{Ds})$, where $K = \mathfrak{Ds} \setminus \mathfrak{Fs} \in \widetilde{\mathbb{I}}$ and $L = \mathfrak{Fs} \setminus \mathfrak{Ds} \in \widetilde{\mathbb{I}}$ by using Lemma 3.1. Hence, $\mathfrak{Fs} \subseteq \overline{\Lambda}(\mathfrak{Ds})$. Also, since $\mathfrak{Fs}$ is closed set imples that $\mathfrak{Fs}$ is $\overline{\Lambda}$-open, then $\mathfrak{Fs} \subseteq \widetilde{\Lambda}(\mathfrak{Fs}) = \widetilde{\Lambda}(\mathfrak{Ds})$ by Corollary 4.1 and Theorem 4.1 item (13), we get $\mathfrak{Fs} \subseteq \widetilde{\Lambda}(\mathfrak{Ds}) \cap \overline{\Lambda}(\mathfrak{Ds})$. Hence, $\widetilde{\Lambda}(\mathfrak{Ds}) \cap \overline{\Lambda}(\mathfrak{Ds}) \neq \emptyset$. □

The converse of Proposition 6.4 is not true, as shown by the following remarks:

(1) In Example 4.1, if $\mathfrak{Ds} = \{2, 3\}$, then $\widetilde{\Lambda}(\mathfrak{Ds}) \cap \overline{\Lambda}(\mathfrak{Ds}) \neq \emptyset$ and $\mathfrak{Ds} = \{2, 3\}$ is baire closed set but $\mathfrak{Ds} = \{2, 3\} \notin \widetilde{\mathbb{I}}$.
(2) In Example 6.1, if $\mathfrak{Ds} = [0, \infty)$, then $\widetilde{\Lambda}(\mathfrak{Ds}) \cap \overline{\Lambda}(\mathfrak{Ds}) \neq \emptyset$ and $\mathfrak{Ds} = [0, \infty)$ is baire closed set but $\mathfrak{Ds} = [0, \infty) \notin \widetilde{\mathbb{I}}$.

Example 6.1. *Let $\mathfrak{T}_l = \{(-\infty, d) : d \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ be the left ray topology on real numbers $\mathbb{R}$. Let $\widetilde{\mathbb{I}}_f$ be the ideal of all finite subsets of $\mathbb{R}$. Then,*

(1) *if $\mathfrak{Ds}$ be open set in left ray topology, then*
  (a) $ker(\mathfrak{Ds}) = \mathfrak{Ds}$.
  (b) $co\text{-}ker(\mathfrak{Ds}) = \emptyset$.
  (c) $\overline{\Lambda}(\mathfrak{Ds}) = \mathfrak{Ds}$.
  (d) $\widetilde{\Lambda}(\mathfrak{Ds}) = \emptyset$.
(2) *if $\mathfrak{Ds}$ be closed set in left ray topology, then*
  (a) $ker(\mathfrak{Ds}) = \mathbb{R}$.
  (b) $co\text{-}ker(\mathfrak{Ds}) = \mathfrak{Ds}$.
  (c) $\overline{\Lambda}(\mathfrak{Ds}) = \mathbb{R}$.
  (d) $\widetilde{\Lambda}(\mathfrak{Ds}) = \mathbb{R} \setminus \mathfrak{Ds}$.
(3) *if $\mathfrak{Ds}$ be a finite set, then*
  (a) $\overline{\Lambda}(\mathfrak{Ds}) = \emptyset$.
  (b) $\widetilde{\Lambda}(\mathfrak{Ds}) = \emptyset$.
(4) *if the complement of $\mathfrak{Ds}$ is finite set, then*
  (a) $\overline{\Lambda}(\mathfrak{Ds}) = \mathbb{R}$.
  (b) $\widetilde{\Lambda}(\mathfrak{Ds}) = \mathbb{R}$.
(5) *consider $\mathfrak{Ds}$ as the set of all rational numbers; it follows that*
  (a) $ker(\mathfrak{Ds}) = \mathbb{R}$.

(b) *co-ker*$(\mathfrak{D}\mathfrak{s}) = \emptyset$.
(c) $\overline{\Lambda}(\mathfrak{D}\mathfrak{s}) = \mathbb{R}$.
(d) $\widetilde{\Lambda}(\mathfrak{D}\mathfrak{s}) = \emptyset$.

Notes:

(1) *The set* $\mathbb{R} \setminus \{0\}$ *is* $\overline{\Lambda}$*-open set but not co-ker set because*
$$co\text{-}ker(\mathbb{R} \setminus \{0\}) \neq \mathbb{R} \setminus \{0\}.$$
(2) $\mathfrak{T}_l$ *is a compatible kernel with the ideal* $\widetilde{\mathbb{I}}_{\mathfrak{f}}$.

Example 6.2. *Let* $\mathbb{R}$ *be a set of real numbers, equipped with the co-finite topology* $\mathfrak{T}_{co} = \{A \subseteq \mathbb{R} : \mathbb{R} \setminus A$ *is finite* $\} \cup \{\emptyset\}$. *Let* $\widetilde{\mathbb{I}}_f$ *be the ideal of all finite subsets of* $\mathbb{R}$. *Then, for all* $A \subseteq \mathbb{R}$, *we have*

(1) $\overline{\Lambda}(A) = \emptyset$.
(2) $\widetilde{\Lambda}(A) = \mathbb{R}$.

Notes:

(1) *For all* $A \subseteq \mathbb{R}$, *we have* A *is* $\overline{\Lambda}$*-open set, hence* $\mathfrak{T}^{\overline{\Lambda}}$ *is a discrete topology.*
(2) $\mathfrak{T}_{co}$ *is not a compatible kernel with the ideal* $\widetilde{\mathbb{I}}_{\mathfrak{f}}$.

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