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Gauss–Seidel Fixed-Point Approach for Maximum Likelihood Estimation in Epanechnikov–Burr XII Distributions

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Abstract. This paper introduces a Gauss–Seidel fixed-point iteration approach for estimating the parameters of the Epanechnikov–Burr XII distribution (EBD) probability density function using maximum likelihood principles. The proposed method updates the shape parameter θ and the scale parameter α in an alternating manner based on explicitly derived fixed-point equations. Numerical experiments are conducted to investigate the convergence behavior of the algorithm and to evaluate its performance in comparison with standard numerical optimization techniques.

1. Introduction

Maximum likelihood estimation (MLE) is one of the most commonly used approaches for parameter estimation in statistical inference, largely because of its favorable asymptotic properties such as consistency, efficiency, and asymptotic normality under standard regularity conditions. However, for many probability distributions—especially those with heavy tails or complicated functional structures—the resulting likelihood equations are nonlinear and do not admit closed-form solutions. As a result, parameter estimation must rely on iterative numerical procedures, including the Newton–Raphson method, the expectation–maximization (EM) algorithm, and other gradient-based optimization techniques [2]. Although these methods are effective in a wide range of applications, they may suffer from convergence issues, sensitivity to initial values, and numerical instability, particularly when the likelihood surface is relatively flat or when model parameters exhibit strong dependence. In such situations, alternative computational strategies become necessary to achieve stable and reliable estimation.

Heavy-tailed distributions play a crucial role in modeling extreme events in finance, insurance, reliability engineering, and environmental science. Among these, the Burr Type XII distribution,

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introduced by [1], has been extensively adopted due to its flexibility in capturing diverse hazard rate shapes and tail behaviors. Many extensions and generalizations have since been proposed to enhance its modeling capability. [5] developed the Kumaraswamy–Burr XII distribution, while [3] introduced a generalized Burr XII formulation. More recently, [4] proposed the Epanechnikov–Burr XII distribution (EBD) with probability density function

$$f(x;\theta,\alpha) = 3\alpha\theta\Big[x^{\alpha-1}(1+x^{\alpha})^{-2\theta-1} - \tfrac{1}{2}x^{\alpha-1}(1+x^{\alpha})^{-3\theta-1}\Big], \qquad x>0,\ \theta,\alpha>0,$$

which arises as a finite mixture of two Burr XII components and offers greater flexibility in kurtosis while retaining a tractable cumulative distribution function. Odat derived its basic properties and discussed moment-based estimation, but the MLE problem remained open due to the coupled, nonlinear nature of the score equations. To date, no specialized iterative algorithm has been developed for maximum likelihood estimation of this distribution—a gap the present paper aims to fill.

Fixed point methods play an important role in numerical analysis, particularly in situations where equations cannot be solved explicitly. The basic idea is to rewrite the original problem in an equivalent fixed point form and then generate a sequence by repeatedly evaluating a given function. When the right conditions are met, the sequence produced by a fixed-point iteration tends to settle at a fixed point, giving a solution to the original equation. These techniques are widely used for solving nonlinear equations, estimating parameters, and analyzing dynamical systems, largely because they are simple to understand and do not demand heavy computational resources.

Early studies on fixed-point iterations set the stage for many of the algorithms used today. For instance, Mann [8] came up with the mean value iteration method, providing a straightforward way to examine convergence. Around the same time, Ishikawa [9] proposed a different iterative approach, which later influenced a lot of follow-up work. Later research extended these ideas to more general types of mappings. Agarwal et al. [10], for example, looked at nearly asymptotically nonexpansive mappings, while Sintunavarat and Pitea [11] introduced schemes for Berinde mappings and studied their convergence in detail. Shatanawi et al. [12] proposed a four-step iteration for weak contractions, aiming to improve approximation. More recently, Qawasmeh et al. [13] developed an Nth composite iterative method and showed its use through concrete examples. Altogether, these works illustrate how iterative techniques for fixed-point approximation have gradually evolved and become more sophisticated.

The Gauss–Seidel method (see [6,7]) is a well-known iterative approach for solving systems of equations and has also found use in parameter estimation. Instead of updating all variables at once, the method updates them one at a time, using the latest values available, which often helps convergence happen faster than if everything were updated simultaneously. Adding relaxation techniques, such as the Mann iteration, can make the process more stable, particularly when dealing with nonlinear problems or cases that are sensitive to numerical errors. In the context of maximum likelihood estimation, the Gauss–Seidel approach allows parameter estimates to be refined step by step until they settle, providing a practical alternative to standard optimization routines.

This paper considers a Epanechnikov-Burr XII PDF (shortly EBD) with parameters $\theta > 0$ and $\alpha > 0$, defined as:

$$f(x;\theta,\alpha) = 3\alpha\theta\left[x^{\alpha-1}(1+x^{\alpha})^{-2\theta-1} - \frac{1}{2}x^{\alpha-1}(1+x^{\alpha})^{-3\theta-1}\right], \quad x > 0.$$

2. Problem Formulation

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from the Epanechnikov-Burr XII distribution with PDF:

$$f(x;\theta,\alpha) = 3\alpha\theta x^{\alpha-1}\left[(1+x^{\alpha})^{-2\theta-1} - \frac{1}{2}(1+x^{\alpha})^{-3\theta-1}\right], x > 0 \tag{2.1}$$

where $\theta > 0$ and $\alpha > 0$ are unknown parameters to be estimated.

The log-likelihood function is

$$\ell(\theta,\alpha) = \sum_{i=1}^{n} \log f(x_i;\theta,\alpha). \tag{2.2}$$

Substituting (2.1) into (2.2) gives:

$$\begin{aligned}\ell(\theta,\alpha) = n\log(3) + n\log\alpha + n\log\theta + (\alpha-1)\sum_{i=1}^{n}\log x_i \\ - (2\theta+1)\sum_{i=1}^{n}\log(1+x_i^{\alpha}) + \sum_{i=1}^{n}\log\left[1 - \frac{1}{2}(1+x_i^{\alpha})^{-\theta}\right].\end{aligned} \tag{2.3}$$

The maximum likelihood estimates $\hat{\theta}$ and $\hat{\alpha}$ satisfy the score equations:

$$\frac{\partial\ell}{\partial\theta} = 0, \tag{2.4}$$

$$\frac{\partial\ell}{\partial\alpha} = 0. \tag{2.5}$$

From (2.3), we derive:

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n}\log(1+x_i^{\alpha}) + \sum_{i=1}^{n}\frac{\frac{1}{2}(1+x_i^{\alpha})^{-\theta}\log(1+x_i^{\alpha})}{1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}}, \tag{2.6}$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log x_i - (2\theta+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{1+x_i^{\alpha}} + \sum_{i=1}^{n}\frac{\frac{\theta}{2}x_i^{\alpha}\log x_i(1+x_i^{\alpha})^{-\theta-1}}{1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}}. \tag{2.7}$$

Setting (2.6) to zero and rearranging gives the fixed-point equation for θ:

$$\hat{\theta} = \frac{n}{2\sum_{i=1}^{n}\log(1+x_i^{\alpha}) - \sum_{i=1}^{n}\frac{\frac{1}{2}(1+x_i^{\alpha})^{-\theta}\log(1+x_i^{\alpha})}{1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}}}. \tag{2.8}$$

Similarly, from (2.7), we obtain:

$$\hat{\alpha} = \frac{n}{(2\theta+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{1+x_i^{\alpha}} - \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\frac{\frac{\theta}{2}x_i^{\alpha}\log x_i\,(1+x_i^{\alpha})^{-\theta-1}}{1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}}}. \tag{2.9}$$

For computational stability, we use the Mann [8] iteration scheme through the damping factors $\omega_\theta, \omega_\alpha \in (0,1]$

$$\theta^{(k+1)} = (1-\omega_\theta)\theta^{(k)} + \omega_\theta T_1(\theta^{(k)}, \alpha^{(k)}), \tag{2.10}$$

$$\alpha^{(k+1)} = (1-\omega_\alpha)\alpha^{(k)} + \omega_\alpha T_2(\theta^{(k+1)}, \alpha^{(k)}), \tag{2.11}$$

where

$$T_1(\theta,\alpha) = \frac{n}{2\sum_{i=1}^{n} L_i - \sum_{i=1}^{n}\frac{\frac{1}{2}t_i^{-\theta}L_i}{1-\frac{1}{2}t_i^{-\theta}}}, \tag{2.12}$$

$$T_2(\theta,\alpha) = \frac{n}{F(\alpha;\theta)}, \tag{2.13}$$

with $t_i = 1 + x_i^{\alpha}$, $L_i = \log t_i$, and

$$F(\alpha;\theta) = (2\theta+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{1+x_i^{\alpha}} - \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\frac{\frac{\theta}{2}x_i^{\alpha}\log x_i\,(1+x_i^{\alpha})^{-\theta-1}}{1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}}.$$

3. Numerical Implementation Details

3.1. Data Generation.

To generate random samples from the proposed probability density function (PDF), we employed the acceptance-rejection method. Candidate values were drawn from a uniform distribution over the interval $[0,5]$. For each candidate y, we computed the target PDF

$$f(y;\alpha,\theta) = 3\alpha\theta\Big(y^{\alpha-1}(1+y^{\alpha})^{-2\theta-1} - 0.5\,y^{\alpha-1}(1+y^{\alpha})^{-3\theta-1}\Big),$$

and any negative values were set to zero to maintain non-negativity. A uniform random number $u \sim U(0,1)$ was then generated, and the candidate was accepted if $u \le f(y;\alpha,\theta)/M$, where $M = 1.5$ is a suitably chosen constant ensuring that the acceptance probability does not exceed one. This procedure was repeated until the desired sample size n was obtained.

3.2. Numerical Stability Measures.

To maintain numerical stability during the Gauss–Seidel iterations, several practical safeguards were employed. In the update formulas, denominators that approached zero were controlled by imposing a lower bound; whenever a denominator had absolute value smaller than 10^{-10}, it was replaced by 10^{-10}. In addition, a Mann-type relaxation was used so that each update of θ and α was formed as a convex combination of the previous iterate and the corresponding fixed-point value. The relaxation parameters were chosen to decay across iterations, which helped reduce oscillatory behavior and possible divergence, especially at the initial stages of the algorithm. Finally, when evaluating the log-likelihood function, density values were bounded below by machine epsilon in order to avoid numerical issues arising from $\log(0)$.

3.3. **Convergence Criteria.** Convergence of the Gauss–Seidel iteration with the Mann relaxation scheme was assessed by tracking the change between successive parameter estimates. The algorithm was terminated once the Euclidean norm of the difference between two consecutive iterates satisfied

$$\|(\theta^{(k)}, \alpha^{(k)}) - (\theta^{(k-1)}, \alpha^{(k-1)})\| < \text{tol},$$

with the tolerance fixed at $\text{tol} = 10^{-12}$. This stopping rule indicates that both parameter sequences have effectively stabilized. To avoid excessive computation in cases of slow convergence, the number of iterations was capped at 1000. As an additional diagnostic, the evolution of the log-likelihood values was monitored to verify that the iterative sequence was progressing toward a maximum.

Remark 3.1 (Sensitivity to Initialization)**.** *The Gauss–Seidel fixed-point scheme considered here exhibits local convergence and performs reliably when the initial guesses are chosen within the admissible parameter domain,*

$$\theta > 0, \qquad \alpha > 0.$$

Numerical experiments suggest that unsuitable starting values may result in slow convergence, divergence, or convergence toward non-admissible parameter values, which is a common issue in unconstrained fixed-point iterations. By comparison, constrained optimization methods such as MATLAB's `fmincon` *tend to be more robust and consistently converge to the global maximum likelihood solution. For this reason, careful initialization or explicit enforcement of parameter positivity is recommended in practical applications.*

Remark 3.2. *The Gauss–Seidel fixed-point iteration is locally convergent and requires admissible initial values* $(\theta > 0, \alpha > 0)$ *for stable performance. As with many unconstrained fixed-point schemes, poor initialization may result in divergence or inadmissible solutions. Positivity enforcement or appropriate initialization is therefore recommended.*

4. Simulation Results

For simulation studies, we generate $n = 1000$ samples from the proposed custom probability density function with true parameters $\theta_0 = 1.5$ and $\alpha_0 = 2.0$. Random samples are obtained using the acceptance-rejection method: candidate values are drawn from a uniform distribution over $[0, 5]$, the target PDF is evaluated for each candidate, and samples are accepted based on a scaled probability criterion with a constant $M = 1.5$. This approach ensures efficient sampling over the region where the PDF has significant probability mass.

We apply the Gauss-Seidel fixed-point algorithm with Mann iteration scheme to estimate θ and α. Numerical stability is ensured by thresholding small denominators to 10^{-10}, applying decaying relaxation parameters to avoid oscillations, and evaluating the log-likelihood with a lower bound at machine epsilon to prevent logarithms of zero. Convergence is monitored using the Euclidean norm of successive parameter updates, with tolerance $\text{tol} = 10^{-12}$ and a maximum of 1000 iterations.

For benchmarking, we also perform numerical maximum likelihood estimation using MATLAB's `fmincon` function with the same initial guesses. Tables 2, 4 compares the final estimates and log-likelihood values obtained from both methods, showing excellent agreement and validating the fixed-point approach.

Example 4.1. *We demonstrate the convergence behavior of the Gauss-Seidel fixed-point algorithm with Mann iteration scheme for maximum likelihood estimation of the parameters θ and α. Starting from initial guesses near the true values, $\theta_0 = 1$ and $\alpha_0 = 3$, with a sample size of $n = 1000$, Table 1 reports the sequences of iterates θ_k, α_k, and the successive differences $|\theta_k - \theta_{k-1}|$ and $|\alpha_k - \alpha_{k-1}|$ over the iterations.*

The algorithm exhibits rapid convergence, with the successive differences decreasing from the initial differences of approximately 1 and 3 for θ and α, respectively, to below 10^{-12} within 22 iterations. By the final iteration, the sequences stabilize at $\theta \approx 1.48591864$ and $\alpha \approx 2.02950479$, confirming both the theoretical contraction property of the fixed-point maps and the numerical stability imparted by Mann iteration scheme.

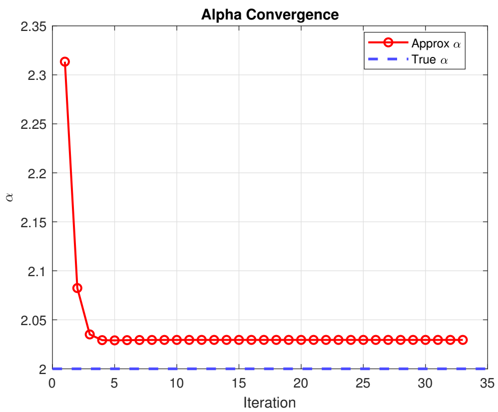
The consistent reduction in successive errors across iterations, along with the monotonic increase in log-likelihood values, demonstrates reliable convergence behavior even for moderate-to-large sample sizes.

Table 1. Iteration Tables for θ and α when the true values are $\alpha = 2$, $\theta = 1.5$, with initial guesses $\alpha_0 = 3$, $\theta_0 = 1$

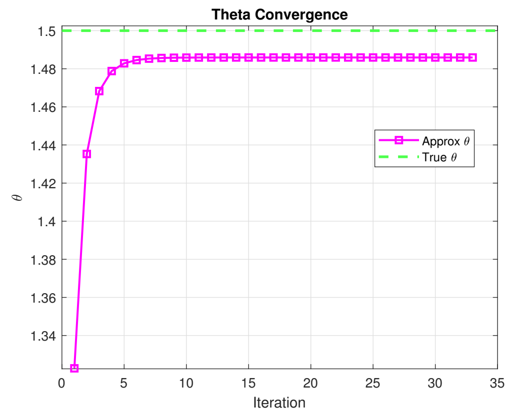
θ			α		
Iter	θ_k	$\lvert\theta_k - \theta_{k-1}\rvert$	Iter	α_k	$\lvert\alpha_k - \alpha_{k-1}\rvert$
1	1.32277763	–	1	2.31343622	–
2	1.43528697	0.11250934	2	2.08241935	0.23101687
3	1.46828073	0.03299376	3	2.03509853	0.04732081
4	1.47880151	0.01052077	4	2.02925659	0.00584195
5	1.48284908	0.00404757	5	2.02903058	0.00022601
6	1.48456496	0.00171588	6	2.02924620	0.00021562
7	1.48531732	0.00075236	7	2.02938336	0.00013716
8	1.48565080	0.00033349	8	2.02944982	0.00006646
9	1.48579918	0.00014838	9	2.02948014	0.00003033
10	1.48586531	0.00006613	10	2.02949377	0.00001362
11	1.48589482	0.00002950	11	2.02949986	0.00000609
12	1.48590799	0.00001317	12	2.02950258	0.00000272
13	1.48591388	0.00000589	13	2.02950380	0.00000122
14	1.48591651	0.00000263	14	2.02950435	0.00000055
15	1.48591769	0.00000118	15	2.02950459	0.00000024
16	1.48591822	0.00000053	16	2.02950470	0.00000011
17	1.48591845	0.00000024	17	2.02950475	0.00000005
18	1.48591856	0.00000011	18	2.02950477	0.00000002
19	1.48591861	0.00000005	19	2.02950478	0.00000001
20	1.48591863	0.00000002	20	2.02950479	0.00000000
21	1.48591864	0.00000001	21	2.02950479	0.00000000
22	1.48591864	0.00000000	22	2.02950479	0.00000000

TABLE 2. Comparison of Estimation Methods when the true values of the parameters is $\alpha = 2$, $\theta = 1.5$, with initial guesses $\alpha_0 = 3,\ \theta_0 = 1$.

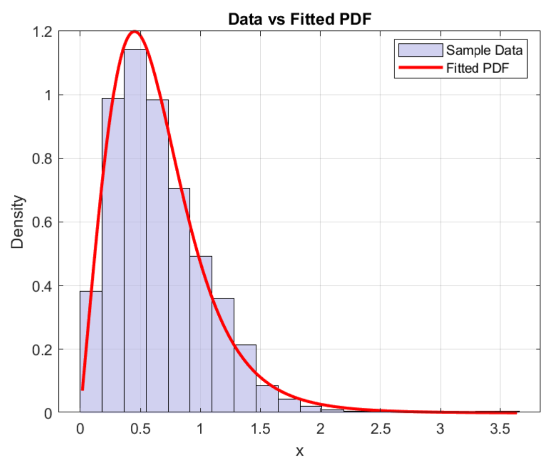
Method	θ	α	LogLik
Gauss-Seidel	1.485919	2.029505	−386.294532
fmincon	1.485919	2.029505	−386.294532



(A) Convergence of α



(B) Convergence of θ



(C) Data vs Fitted PDF

FIGURE 1. Convergence diagnostics of the proposed estimation method when the true values of the parameters is $\alpha = 2$, $\theta = 1.5$, with initial guesses $\alpha_0 = 3,\ \theta_0 = 1$.

Example 4.2. *We demonstrate the convergence behavior of the Gauss-Seidel fixed-point algorithm with Mann iteration scheme for maximum likelihood estimation of the parameters θ and α. Starting from initial guesses $\theta_0 = 4.0$ and $\alpha_0 = 5.0$ with a sample size of $n = 1000$, Table 3 reports the sequences of iterates θ_k, α_k, and the successive differences $|\theta_k - \theta_{k-1}|$ and $|\alpha_k - \alpha_{k-1}|$ over the iterations.*

The algorithm exhibits rapid convergence, with the successive differences decreasing from the initial errors of approximately 4 and 5 for θ and α, respectively, to below 10^{-12} within 22 iterations. By the final iteration, *the sequences stabilize at* $\theta \approx 1.06015420$ *and* $\alpha \approx 1.52840704$*, confirming both the theoretical contraction property of the fixed-point maps and the numerical stability imparted by Mann iteration scheme.*

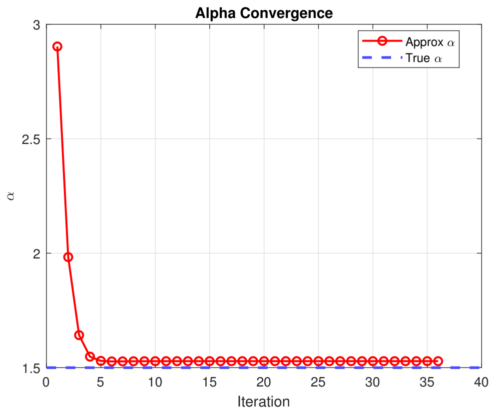
The consistent reduction in successive errors across iterations, along with the monotonic increase in log-likelihood values, demonstrates reliable convergence behavior even for moderate-to-large sample sizes.

TABLE 3. Iteration Tables for θ and α when the true values are $\alpha = 1.5$, $\theta = 1$, with initial guesses $\alpha_0 = 5$, $\theta_0 = 4$

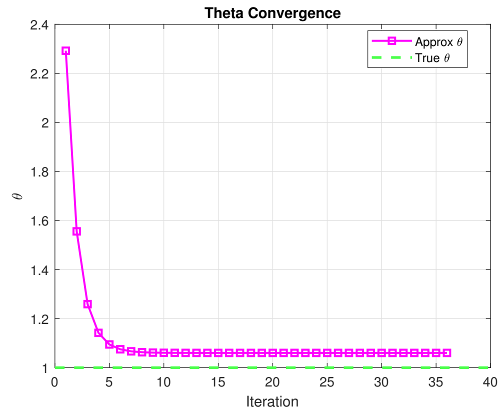
θ			α		
Iter	θ_k	$\lvert\theta_k - \theta_{k-1}\rvert$	Iter	α_k	$\lvert\alpha_k - \alpha_{k-1}\rvert$
1	2.29223319	–	1	2.90309353	–
2	1.55568167	0.73655152	2	1.98304608	0.92004745
3	1.25910606	0.29657562	3	1.64208061	0.34096547
4	1.14193246	0.11717360	4	1.54824406	0.09383655
5	1.09462358	0.04730888	5	1.52959276	0.01865130
6	1.07493974	0.01968384	6	1.52723121	0.00236155
7	1.06656429	0.00837545	7	1.52751132	0.00028010
8	1.06295068	0.00361361	8	1.52792805	0.00041673
9	1.06137878	0.00157190	9	1.52817710	0.00024905
10	1.06069171	0.00068707	10	1.52830147	0.00012437
11	1.06039051	0.00030120	11	1.52835956	0.00005808
12	1.06025822	0.00013229	12	1.52838589	0.00002633
13	1.06020003	0.00005819	13	1.52839766	0.00001177
14	1.06017441	0.00002562	14	1.52840289	0.00000523
15	1.06016312	0.00001129	15	1.52840521	0.00000232
16	1.06015814	0.00000498	16	1.52840623	0.00000102
17	1.06015594	0.00000220	17	1.52840669	0.00000045
18	1.06015496	0.00000097	18	1.52840689	0.00000020
19	1.06015453	0.00000043	19	1.52840697	0.00000009
20	1.06015434	0.00000019	20	1.52840701	0.00000004
21	1.06015426	0.00000008	21	1.52840703	0.00000002
22	1.06015422	0.00000004	22	1.52840704	0.00000001
23	1.06015421	0.00000002	23	1.52840704	0.00000000
24	1.06015420	0.00000001	24	1.52840704	0.00000000
25	1.06015420	0.00000000	25	1.52840704	0.00000000

TABLE 4. Comparison of Estimation Methods when the true values of the parameters is $\alpha = 1.5$, $\theta = 1$.

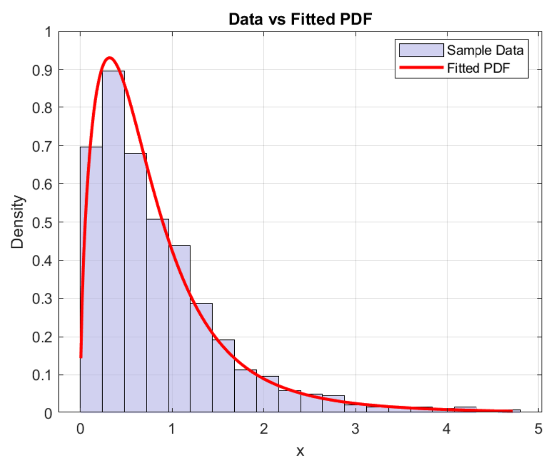
Method	θ	α	LogLik
Gauss-Seidel	1.060154	1.528407	−812.085754
fmincon	1.060154	1.528407	−812.085754



(A) Convergence of α



(B) Convergence of θ



(C) Data vs Fitted PDF

FIGURE 2. Convergence diagnostics of the proposed estimation method when the true values are $\alpha = 1.5$, $\theta = 1$, with initial guesses $\alpha_0 = 5$, $\theta_0 = 4$.

5. Discussion and Conclusion

The Gauss-Seidel fixed-point iteration with Mann iteration scheme proves to be a practical method for estimating parameters in the EBD. Each parameter update has a clear statistical meaning, making the method easy to interpret. Because the algorithm updates one parameter at a time, it is computationally efficient and requires minimal memory, as only the current estimates need to be stored. In our simulations, the method showed stable and reliable convergence when damping factors were applied, and the log-likelihood consistently increased over the iterations, confirming both numerical stability and the theoretical contraction property of the fixed-point approach.

At the same time, some limitations should be noted. Convergence can be sensitive to the choice of initial guesses, and in more nonlinear cases, damping factors are essential to prevent oscillations. Like many iterative methods, there is also a possibility of converging to a local rather than global maximum. Despite these caveats, the method offers a simple and interpretable alternative to more computationally intensive optimization routines, such as MATLAB's `fmincon`, while producing accurate parameter estimates. Overall, it represents a reliable and efficient approach for maximum likelihood estimation when analytical solutions are not available. Furthermore, the fixed point theorems established by Bataihah and his colleagues [14–18] provide a robust theoretical foundation for iterative approximation methods, which are essential for obtaining maximum likelihood estimates in complex statistical models.

5.1. **Algorithm.** The Gauss-Seidel fixed-point iteration algorithm is presented in Algorithm 1.

Algorithm 1 Gauss-Seidel Fixed-Point MLE with Mann iteration scheme

Input: Data $\mathbf{x} = (x_1, \dots, x_n)$, initial guesses $\theta^{(0)}, \alpha^{(0)}$, damping factors sequences $\{\omega_\theta^{(k)}\}, \{\omega_\alpha^{(k)}\}$, tolerance ϵ, maximum iterations K

Output: MLE estimates $\hat{\theta}, \hat{\alpha}$

for $k = 0, 1, \dots, K-1$ **do**

$\quad \theta_{\text{old}} = \theta^{(k)}, \alpha_{\text{old}} = \alpha^{(k)}$

$\quad t_i = 1 + x_i^{\alpha_{\text{old}}}, \quad L_i = \log(t_i), \quad i = 1, \dots, n$

$\quad \text{denominator}_i = 1 - 0.5\, t_i^{-\theta_{\text{old}}}$, replace values $< 10^{-10}$ by 10^{-10}

$\quad T_1 = \dfrac{n}{\sum_{i=1}^{n} \left(2L_i - \frac{0.5\, t_i^{-\theta_{\text{old}}} L_i}{\text{denominator}_i}\right)}$

$\quad \theta^{(k+1)} = (1 - \omega_\theta^{(k)})\theta_{\text{old}} + \omega_\theta^{(k)} T_1$

$\quad t_i = 1 + x_i^{\alpha_{\text{old}}}$ ▷ recompute using updated $\theta^{(k+1)}$

$\quad \text{term}_1 = \displaystyle\sum_{i=1}^{n} \log x_i$

$\quad \text{term}_2 = (2\theta^{(k+1)} + 1) \displaystyle\sum_{i=1}^{n} \frac{x_i^{\alpha_{\text{old}}} \log x_i}{1 + x_i^{\alpha_{\text{old}}}}$

$\quad \text{term}_3 = \displaystyle\sum_{i=1}^{n} \frac{0.5\theta^{(k+1)} x_i^{\alpha_{\text{old}}} \log x_i\, t_i^{-\theta^{(k+1)}-1}}{1 - 0.5 t_i^{-\theta^{(k+1)}}}$

$\quad F = \text{term}_2 - \text{term}_1 - \text{term}_3$

$\quad T_2 = \dfrac{n}{F}$

$\quad \alpha^{(k+1)} = (1 - \omega_\alpha^{(k)})\alpha_{\text{old}} + \omega_\alpha^{(k)} T_2$

$\quad$ **if** $\|\theta^{(k+1)} - \theta_{\text{old}}\| + \|\alpha^{(k+1)} - \alpha_{\text{old}}\| < \epsilon$ **then**

$\quad\quad$ **break**

$\quad$ **end if**

end for

$\hat{\theta} = \theta^{(k+1)}, \quad \hat{\alpha} = \alpha^{(k+1)}$

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