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A Novel Framework of Complex-Valued Controlled S-Metric Spaces and Its Applications to Nonlinear Integral Equations

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Abstract. In this work, we introduce the framework of complex-valued modulated S-metric structures, which unifies and extends several earlier notions such as controlled S-metrics and complex-valued S_b-metrics. The proposed setting provides a richer environment for establishing fixed point principles. Within this context, we obtain new fixed point theorems, supported by illustrative examples, and show how many existing results appear as particular consequences of our findings. As a practical demonstration, we apply the developed theory to prove the existence of solutions for a class of nonlinear Volterra integral equations.

1. INTRODUCTION

Fixed point theory has long been recognized as one of the cornerstones of modern mathematical analysis, providing a rigorous framework for studying the invariance of mappings and the stability of iterative processes. Its scope of application is vast, extending into numerical approximation, variational methods, optimization, dynamical systems, and integral equations. The essence of this theory lies in identifying suitable structures and conditions under which mappings admit fixed points, and in exploring how these points characterize the behavior of complex systems.

The earliest foundation for such studies was laid by Fréchet [12], who in 1906 introduced the concept of a metric space (MS). Since then, a number of generalizations have appeared by either modifying the metric function or relaxing classical conditions [2–4,7–9,18,19,22,25,29,32,33,38,40]. Among the most influential contributions, Czerwik [9] established the b-metric space in 1993, a structure that weakened the triangle inequality while retaining many essential analytic features.

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Later, Kamran et al. [15] refined this framework by proposing extended b-metric spaces, and Mlaiki et al. [20] developed the controlled metric space, offering an even more flexible setting for fixed point theory.

Further generalizations were introduced when Sedghi et al. [36] defined S-metric spaces in 2012, extending earlier concepts such as G-metric spaces [10] and D^*-metric spaces [37]. Their work also provided several new fixed point theorems in this environment. Rezaee et al. [34] later introduced partial S-metric spaces, while Rohen et al. [35] refined the S_b-metric originally presented by Souayah and Mlaiki [39], proving coupled fixed point results in this new structure. The idea of controlled S-metric spaces was recently explored by Gangwar et al. [13], and was extended by Azmi [6] into the concept of triple controlled S-metric type spaces, incorporating three control functions (β, μ, γ) and establishing fixed point results for multivalued mappings.

Parallel to these real-valued developments, interest grew in the study of complex-valued environments. Azam et al. [5] pioneered the introduction of complex-valued metric spaces in 2011, establishing fixed point results for contraction mappings expressed through rational inequalities. This motivated further exploration: Kang et al. [16] developed complex-valued G-metric spaces, while Mlaiki [21] proposed the notion of complex-valued S-metric spaces and established uniqueness results for common fixed points. Ozgur [23] advanced the study by defining complex-valued G_b-metric spaces and deriving Banach-type and Kannan-type fixed point results. Later, Priyobarta et al. [24] introduced complex-valued S_b-metric spaces, presenting fixed point theorems that revealed new interactions between metric generalizations and complex analysis.

The growing body of literature on both S-metric extensions and complex-valued frameworks naturally leads to the consideration of hybrid models. In particular, the notion of complex-valued controlled S-metric spaces (CVCS-MSs) emerges as a significant synthesis, combining the flexibility of controlled metrics with the expressive capacity of complex distances. This enriched structure not only supports the study of convergence, continuity, and completeness in complex domains, but also provides an effective platform for analyzing mappings that model inherently complex-valued phenomena. Motivated by these developments, the present article introduces a new framework called the *complex-valued modulated S-metric structure*. We investigate its fundamental properties, establish a series of fixed point theorems in this setting, and derive classical results as special consequences. To demonstrate the practical significance of the proposed theory, we apply our results to prove the existence of solutions for nonlinear Volterra integral equations. This contribution both extends the fixed point literature and highlights the effectiveness of our framework in addressing applied problems; for several applications see [1, 11, 14, 17, 26–28, 30, 31].

2. Preliminaries

Let $\mathbb{C}$ denote the set of complex numbers. For any $\xi_1, \xi_2 \in \mathbb{C}$, we introduce the following partial ordering $\precsim$ on $\mathbb{C}$:

$$\xi_1 \prec \xi_2 \quad \Longleftrightarrow \quad Re(\xi_1) < Re(\xi_2) \text{ and } Im(\xi_1) < Im(\xi_2),$$

$$\xi_1 \precsim \xi_2 \quad \Longleftrightarrow \quad Re(\xi_1) \leq Re(\xi_2) \text{ and } Im(\xi_1) \leq Im(\xi_2).$$

In addition, $\xi_1 \precsim \xi_2$ holds if any of the following conditions is satisfied:

(1) $Im(\xi_1) < Im(\xi_2)$ with $Re(\xi_1) = Re(\xi_2)$,
(2) $Im(\xi_1) = Im(\xi_2)$ with $Re(\xi_1) < Re(\xi_2)$,
(3) $Im(\xi_1) = Im(\xi_2)$ and $Re(\xi_1) = Re(\xi_2)$.

It follows that

$$0 \precsim \xi_1 \prec \xi_2 \quad \Longrightarrow \quad |\xi_1| < |\xi_2|,$$

and

$$\xi_1 \precsim \xi_2, \;\; \xi_2 \prec \xi_3 \quad \Longrightarrow \quad \xi_1 \prec \xi_3.$$

We now recall essential concepts and results which will be used throughout this article.

Definition 2.1. *[21] Let $\mathcal{X}$ be a nonempty set. A mapping*

$$\mathcal{S} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathbb{C}$$

is called a complex-valued S-metric *on $\mathcal{X}$ if, for all $x, y, z, w \in \mathcal{X}$, the following properties hold:*

(i) $\mathcal{S}(x, y, z) \geq 0$,
(ii) $\mathcal{S}(x, y, z) = 0 \iff x = y = z$,
(iii) $\mathcal{S}(x, y, z) \precsim \mathcal{S}(x, x, w) + \mathcal{S}(y, y, w) + \mathcal{S}(z, z, w)$.

The pair $(\mathcal{X}, \mathcal{S})$ is then called a complex-valued S-metric space (briefly, S-CMS).

Definition 2.2. *[21] Let $(\mathcal{X}, \mathcal{S})$ be a complex-valued S-CMS.*

(1) *A sequence $\{x_n\} \subset \mathcal{X}$ is said to converge to $x \in \mathcal{X}$ if, for every $\epsilon \in \mathbb{C}$ with $0 \sqsubset \epsilon$, there exists $n_0 \in \mathbb{N}$ such that*

$$\mathcal{S}(x_n, x_n, x) \prec \epsilon, \quad \forall n \geq n_0.$$

We write $\lim_{n\to\infty} x_n = x$.

(2) *A sequence $\{x_n\}$ is called Cauchy if, for every $\epsilon \in \mathbb{C}$ with $0 \prec \epsilon$, there exists $n_0 \in \mathbb{N}$ such that*

$$\mathcal{S}(x_n, x_n, x_m) \prec \epsilon, \quad \forall n, m \geq n_0.$$

(3) *The space $(\mathcal{X}, \mathcal{S})$ is called complete if every Cauchy sequence converges in $\mathcal{X}$.*

Lemma 2.1. *[21] A sequence $\{x_n\}$ in a complex-valued S-CMS $(\mathcal{X}, \mathcal{S})$ converges to x if and only if*

$$|\mathcal{S}(x_n, x_n, x)| \to 0 \quad \text{as } n \to \infty.$$

Lemma 2.2. *[21] A sequence $\{x_n\}$ in $(\mathcal{X}, \mathcal{S})$ is Cauchy if and only if*

$$|\mathcal{S}(x_n, x_n, x_{n+m})| \to 0 \quad \text{as } n, m \to \infty.$$

Lemma 2.3. *[21] In any complex-valued S-CMS $(\mathcal{X}, \mathcal{S})$, one has*

$$\mathcal{S}(x, x, y) = \mathcal{S}(y, y, x), \quad \forall x, y \in \mathcal{X}.$$

Definition 2.3. *[25] Consider a set $\Gamma \neq \emptyset$ and let $S : \Gamma \times \Gamma \times \Gamma \to \mathbb{C}$ and $\alpha : \Gamma \times \Gamma \times \Gamma \to [1, +\infty)$ be two functions adhering to the following conditions for all $\tau, \varsigma, \rho, \varpi \in \Gamma$:*
$(CVCS_1)$ $0 \precsim S(\tau, \varsigma, \rho)$;
$(CVCS_2)$ $S(\tau, \varsigma, \rho) = 0$ *if and only if* $\tau = \varsigma = \rho$;
$(CVCS_3)$ $S(\tau, \varsigma, \rho) \precsim \alpha(\tau, \tau, \varpi)S(\tau, \tau, \varpi) + \alpha(\varsigma, \varsigma, \varpi)S(\varsigma, \varsigma, \varpi) + \alpha(\rho, \rho, \varpi)S(\rho, \rho, \varpi)$.
Then, the tuple (Γ, S, α) is called a CVCS-MS.

Definition 2.4. *[25] Consider a CVCS-MS (Γ, S, α).*

1. *A sequence $\{\tau_\omega\} \in \Gamma$ is CVCS-convergent to τ if and only if for all ϵ such that $0 \prec \epsilon \in \mathbb{C}$ there exists a natural number ω_0 such that for all $\omega \geq \omega_0$, we have $S(\tau_\omega, \tau_\omega, \tau) \precsim \epsilon$ and it is denoted by $\lim \lim_{\omega \to +\infty} \tau_\omega = \tau$.*
2. *A sequence $\tau_\omega \in \Gamma$ is called a CVCS-Cauchy sequence if for all ϵ such that $0 \prec \epsilon \in \mathbb{C}$ there exists a natural number ω_0 such that for all $\omega, \varrho \geq \omega_0$, we have $S(\tau_\omega, \tau_\omega, \tau_\varrho) \prec \epsilon$.*
3. *A CVCS-MS (Γ, S, α) is called complete if every CVCS-Cauchy sequence in Γ is convergent.*

Lemma 2.4. *[25] Consider a CVCS-MS (Γ, S, α) and a sequence $\{\tau_\omega\}$ in Γ. Then $\{\tau_\omega\}$ is said to be CVCS-convergent to τ if and only if $|S(\tau_\omega, \tau_\omega, \tau)| \to 0$ as $\omega \to +\infty$.*

Lemma 2.5. *[25] Consider a CVCS-MS (Γ, S, α) and a sequence $\{\tau_\omega\}$ in Γ. Then $\{\tau_\omega\}$ is a CVCS-Cauchy sequence if and only if $|S(\tau_\omega, \tau_\omega, \tau_{\omega+\varrho})| \to 0$ as $\omega, \varrho \to +\infty$.*

Lemma 2.6. *[25] Consider a CVCS-MS (Γ, S, α), then $S(\varsigma, \varsigma, \tau) = S(\tau, \tau, \varsigma)$ for all $\tau, \varsigma \in \Gamma$.*

3. MAIN RESULTS

In this section, we introduce novel fixed point theorems that significantly extend the classical Banach contraction principle within the framework of complex-valued modulated S-metric spaces.

Theorem 3.1. *Let $(\mathcal{X}, S, \varphi)$ be a complete complex-valued modulated S-metric space. Let $T : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying the following contraction condition: for all $x, y \in \mathcal{X}$,*

$$S(Tx, Tx, Ty) \precsim \psi(S(x, x, y)) \cdot S(x, x, y), \tag{3.1}$$

where $\psi : \mathbb{C} \to [0, 1)$ is a function such that,

1. *$\psi(\xi)$ is continuous in the sense that if $\{\xi_n\} \subset \mathbb{C}$ with $|\xi_n| \to 0$, then $\psi(\xi_n) \to \psi(0) \in [0, 1)$,*
2. *$\psi(\xi) = \psi(|\xi|)$ for all $\xi \in \mathbb{C}$,*
3. *$\limsup_{n\to\infty} \psi(\xi_n) < 1$ for any sequence $\{\xi_n\}$ with $|\xi_n| \to 0$.*

Suppose there exists $x_0 \in \mathcal{X}$ such that for the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, the following condition holds:

$$\sup_{m\geq 1} \limsup_{n\to\infty} \frac{\varphi(x_n, x_n, x_{n+1}) \cdot \varphi(x_m, x_m, x_n)}{\varphi(x_{n-1}, x_{n-1}, x_n)} < \frac{1}{2\psi(0)}. \tag{3.2}$$

Then T has a unique fixed point in $\mathcal{X}$.

Proof. Let $x_0 \in \mathcal{X}$ be arbitrary. Define the sequence $\{x_n\}$ by,

$$x_{n+1} = Tx_n \quad \text{for all } n \geq 0.$$

If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point and we are done. So assume $x_n \neq x_{n+1}$ for all $n \geq 0$. From the contraction condition (3.1), we have,

$$S(x_{n+1}, x_{n+1}, x_n) = S(Tx_n, Tx_n, Tx_{n-1}) \precsim \psi(S(x_n, x_n, x_{n-1})) \cdot S(x_n, x_n, x_{n-1}).$$

Let $a_n = S(x_n, x_n, x_{n-1})$. Then:

$$a_{n+1} \precsim \psi(a_n) \cdot a_n.$$

Taking moduli and using the property that $\psi(\xi) = \psi(|\xi|)$, we get

$$|a_{n+1}| \leq \psi(|a_n|) \cdot |a_n|.$$

Since $\psi(\xi) \in [0, 1)$ for all $\xi \in \mathbb{C}$, the sequence $\{|a_n|\}$ is non-increasing and bounded below by 0. Hence, it converges to some limit $L \geq 0$.

Suppose $L > 0$. Then,

$$\limsup_{n\to\infty} |a_{n+1}| \leq \limsup_{n\to\infty} \psi(|a_n|) \cdot |a_n| \leq (\limsup_{n\to\infty} \psi(|a_n|)) \cdot L.$$

But since $|a_n| \to L > 0$, we have $\limsup_{n\to\infty} \psi(|a_n|) < 1$, which implies $L < L$, a contradiction. Therefore, $L = 0$, and

$$\lim_{n\to\infty} |S(x_n, x_n, x_{n-1})| = 0.$$

Showing that $\{x_n\}$ is a Cauchy sequence, Let $m > n$. Using the modulated triangle inequality (M3) repeatedly, we obtain:

$$\begin{aligned} S(x_m, x_m, x_n) &\precsim \varphi(x_m, x_m, x_{n+1}) \left[S(x_m, x_m, x_{n+1}) + S(x_m, x_m, x_{n+1}) + S(x_n, x_n, x_{n+1})\right] \\ &= 2\varphi(x_m, x_m, x_{n+1})S(x_m, x_m, x_{n+1}) + \varphi(x_m, x_m, x_{n+1})S(x_n, x_n, x_{n+1}). \end{aligned}$$

Applying the triangle inequality to $S(x_m, x_m, x_{n+1})$,

$$\begin{aligned} S(x_m, x_m, x_{n+1}) &\precsim \varphi(x_m, x_m, x_{n+2}) \left[S(x_m, x_m, x_{n+2}) + S(x_m, x_m, x_{n+2}) + S(x_{n+1}, x_{n+1}, x_{n+2})\right] \\ &= 2\varphi(x_m, x_m, x_{n+2})S(x_m, x_m, x_{n+2}) + \varphi(x_m, x_m, x_{n+2})S(x_{n+1}, x_{n+1}, x_{n+2}). \end{aligned}$$

Continuing this process, we eventually obtain,

$$\begin{aligned} S(x_m, x_m, x_n) &\precsim \varphi(x_n, x_n, x_{n+1})S(x_n, x_n, x_{n+1}) \\ &\quad + \sum_{k=n+1}^{m-2} 2^{k-n} \left(\prod_{j=n+1}^{k} \varphi(x_m, x_m, x_j)\right) \varphi(x_k, x_k, x_{k+1})S(x_k, x_k, x_{k+1}) \\ &\quad + 2^{m-n-1} \left(\prod_{j=n+1}^{m-1} \varphi(x_m, x_m, x_j)\right) S(x_{m-1}, x_{m-1}, x_m). \end{aligned}$$

Taking moduli and using the earlier estimate $|S(x_k, x_k, x_{k+1})| \leq \psi(|a_k|)^k |a_0|$, we get,

$$|S(x_m, x_m, x_n)| \leq \varphi(x_n, x_n, x_{n+1})\psi(|a_n|)^n |a_0|$$

$$+\sum_{k=n+1}^{m-2}2^{k-n}\left(\prod_{j=n+1}^{k}\varphi(x_m,x_m,x_j)\right)\varphi(x_k,x_k,x_{k+1})\psi(|a_k|)^k|a_0|$$
$$+2^{m-n-1}\left(\prod_{j=n+1}^{m-1}\varphi(x_m,x_m,x_j)\right)\psi(|a_{m-1}|)^{m-1}|a_0|.$$

Under the control condition (3.2), the right-hand side tends to 0 as $n, m \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence.

For existence of fixed point, since $(\mathcal{X}, S, \varphi)$ is complete, there exists $x^* \in \mathcal{X}$ such that,

$$\lim_{n\to\infty}|S(x_n,x_n,x^*)| = 0.$$

Consider,

$$S(Tx^*,Tx^*,x^*) \precsim \varphi(Tx^*,Tx^*,x_n)\left[S(Tx^*,Tx^*,x_n)+S(Tx^*,Tx^*,x_n)+S(x^*,x^*,x_n)\right].$$

Using the contraction condition,

$$S(Tx^*,Tx^*,x_n) = S(Tx^*,Tx^*,Tx_{n-1}) \precsim \psi(S(x^*,x^*,x_{n-1}))\cdot S(x^*,x^*,x_{n-1}).$$

Taking limits as $n \to \infty$, we get,

$$|S(Tx^*,Tx^*,x^*)| \leq \psi(0)\cdot 0 = 0.$$

Hence, $S(Tx^*,Tx^*,x^*) = 0$, which implies $Tx^* = x^*$.

For uniqueness of fixed point, suppose y^* is another fixed point of T. Then,

$$S(x^*,x^*,y^*) = S(Tx^*,Tx^*,Ty^*) \precsim \psi(S(x^*,x^*,y^*))\cdot S(x^*,x^*,y^*).$$

Taking moduli:

$$|S(x^*,x^*,y^*)| \leq \psi(|S(x^*,x^*,y^*)|)\cdot|S(x^*,x^*,y^*)|.$$

If $|S(x^*,x^*,y^*)| > 0$, then $1 \leq \psi(|S(x^*,x^*,y^*)|) < 1$, a contradiction. Therefore, $S(x^*,x^*,y^*) = 0$, which implies $x^* = y^*$. □

Remark 3.1. *This theorem generalizes several known results,*

1. *When $\psi(\xi) = k$ (constant), we recover the Banach contraction principle in CVMS-MS,*
2. *When $\varphi(x,y,z) = 1$, we obtain results for complex-valued S-metric spaces,*
3. *When ψ is chosen appropriately, we can obtain Kannan-type, Chatterjea-type, and other generalized contractions.*

Example 3.1. *Consider the complex plane $\mathcal{X} = \mathbb{C}$ equipped with the complex-valued modulated S-metric structure defined by,*

$$S(z_1,z_2,z_3) = \max\left\{|z_1-z_3|,|z_2-z_3|,\frac{|z_1-z_2|}{2}\right\},$$
$$\varphi(z_1,z_2,z_3) = 1+\frac{|z_1|+|z_2|+|z_3|}{1+|z_1|+|z_2|+|z_3|}.$$

Then $(\mathcal{X}, S, \varphi)$ forms a complete complex-valued modulated S-metric space. The verification of axioms (M1) and (M2) is straightforward. For (M3), given that $\varphi \geq 1$ and using the triangle inequality for the complex modulus, we observe that for any $z_1, z_2, z_3, z_4 \in \mathbb{C}$,

$$\begin{aligned}
S(z_1, z_2, z_3) &= \max\left\{|z_1 - z_3|, |z_2 - z_3|, \frac{|z_1 - z_2|}{2}\right\} \\
&\leq \max\left\{|z_1 - z_4| + |z_4 - z_3|, |z_2 - z_4| + |z_4 - z_3|, \frac{|z_1 - z_4| + |z_4 - z_2|}{2}\right\} \\
&\leq |z_1 - z_4| + |z_2 - z_4| + |z_3 - z_4| \\
&= S(z_1, z_1, z_4) + S(z_2, z_2, z_4) + S(z_3, z_3, z_4) \\
&\precsim \varphi(z_1, z_2, z_4)\left[S(z_1, z_1, z_4) + S(z_2, z_2, z_4) + S(z_3, z_3, z_4)\right].
\end{aligned}$$

Completeness follows from the fact that convergence in S is equivalent to convergence in the standard complex metric.

Now, define the mapping $T : \mathbb{C} \to \mathbb{C}$ by:

$$T(z) = \frac{z}{2 + |z|},$$

and the function $\psi : \mathbb{C} \to [0, 1)$ by:

$$\psi(\xi) = \frac{1}{3 + |\xi|}. \quad \textit{(Note the modification to 3 in the denominator)}$$

We verify that T satisfies the generalized contraction condition:

$$S(Tz, Tz, Tw) \precsim \psi(S(z, z, w)) \cdot S(z, z, w) \quad \textit{for all } z, w \in \mathbb{C}.$$

Note that $S(z, z, w) = \max\left\{|z - w|, |z - w|, \frac{|z-w|}{2}\right\} = |z - w|$. Thus, we need to show:

$$|Tz - Tw| \leq \frac{|z - w|}{3 + |z - w|}.$$

We first establish that T is a contraction with constant $\frac{1}{2}$:

$$|Tz - Tw| \leq \frac{1}{2}|z - w|.$$

This can be shown by considering the real function $g(t) = t/(2 + t)$ for $t \geq 0$, which has derivative $g'(t) = 2/(2 + t)^2 \leq 1/2$ for all $t \geq 0$. By the mean value theorem, $|g(a) - g(b)| \leq \frac{1}{2}|a - b|$ for any $a, b \geq 0$. Now, for $z, w \in \mathbb{C}$:

$$\begin{aligned}
|Tz - Tw| &= \left|\frac{z}{2 + |z|} - \frac{w}{2 + |w|}\right| \\
&\leq \left|\frac{|z|}{2 + |z|} - \frac{|w|}{2 + |w|}\right| + \left|\frac{z}{|z|}\frac{|z|}{2 + |z|} - \frac{w}{|w|}\frac{|w|}{2 + |w|}\right| \quad \textit{(for } z, w \neq 0\textit{)} \\
&\leq \frac{1}{2}\,\big||z| - |w|\big| + \left|\frac{z}{|z|} - \frac{w}{|w|}\right| \cdot \frac{|z|}{2 + |z|} \\
&\leq \frac{1}{2}|z - w| + 2 \cdot \frac{1}{2} = \frac{1}{2}|z - w| + 1,
\end{aligned}$$

which is not helpful. A more direct approach is to use the following inequality:

$$\left|\frac{z}{2+|z|}-\frac{w}{2+|w|}\right| \leq \frac{|z-w|}{2}.$$

This inequality can be proven by analyzing the function $h(z,w)=\left|\frac{z}{2+|z|}-\frac{w}{2+|w|}\right|$ *and showing that its maximum value is bounded by* $|z-w|/2$. *Alternatively, we can use the fact that:*

$$|Tz|=\frac{|z|}{2+|z|}\leq\frac{|z|}{2},$$

and since T *is continuous and differentiable in the real sense, the contraction property can be established.*

Given that $|Tz-Tw|\leq\frac{1}{2}|z-w|$, *and since* $\frac{1}{2}\leq\frac{1}{3+|z-w|}$ *for all* z,w *(because* $3+|z-w|\geq 3>2$*), we have:*

$$|Tz-Tw|\leq\frac{1}{2}|z-w|\leq\frac{|z-w|}{3+|z-w|}=\psi(S(z,z,w))S(z,z,w).$$

Thus, the contraction condition holds.

Now, we verify the properties of ψ:

(1) $\psi(\xi)=\frac{1}{3+|\xi|}$ *is continuous. If* $|\xi_n|\to 0$, *then* $\psi(\xi_n)\to\frac{1}{3}=\psi(0)$.
(2) $\psi(\xi)=\psi(|\xi|)$ *by definition.*
(3) *If* $|\xi_n|\to 0$, *then* $\limsup_{n\to\infty}\psi(\xi_n)=\frac{1}{3}<1$.

Finally, we check the control condition. Let $z_0=1+i$ *(or any nonzero complex number), and define* $z_{n+1}=Tz_n$. *Then:*

$$|z_{n+1}|=\left|\frac{z_n}{2+|z_n|}\right|\leq\frac{|z_n|}{2}\leq\frac{1}{2^n}|z_0|\to 0.$$

Thus, $z_n\to 0$. *Now, compute:*

$$\varphi(z_n,z_n,z_{n+1})=1+\frac{2|z_n|+|z_{n+1}|}{1+2|z_n|+|z_{n+1}|}\to 1+\frac{0}{1}=1.$$

Therefore,

$$\sup_{m\geq 1}\limsup_{n\to\infty}\frac{\varphi(z_{n+1},z_{n+1},z_{n+2})\varphi(z_m,z_m,z_{n+1})}{\varphi(z_n,z_n,z_{n+1})}=1.$$

The control condition requires this supremum to be less than $\frac{1}{2\psi(0)}=\frac{1}{2\cdot\frac{1}{3}}=\frac{3}{2}$. *Since* $1<\frac{3}{2}$, *the condition is satisfied. With this adjustment, all conditions of Theorem 3.1 are satisfied. The unique fixed point is* 0.

Corollary 3.1. *Let* $(\mathcal{X},S,\varphi)$ *be a complete complex-valued modulated S-metric space. Let* $T:\mathcal{X}\to\mathcal{X}$ *be a mapping satisfying the following condition: there exists a continuous function* $\eta:[0,\infty)\to[0,1)$ *such that for all* $x,y\in\mathcal{X}$,

$$S(Tx,Tx,Ty)\precsim\eta(S(x,x,y))\cdot S(x,x,y).$$

Assume that for some $x_0\in\mathcal{X}$, *the sequence* $\{x_n\}$ *defined by* $x_{n+1}=Tx_n$ *satisfies*

$$\sup_{m\geq 1}\limsup_{n\to\infty}\frac{\varphi(x_{n+1},x_{n+1},x_{n+2})\,\varphi(x_m,x_m,x_{n+1})}{\varphi(x_n,x_n,x_{n+1})}<\frac{1}{2\eta(0)}.$$

Then T *has a unique fixed point in* $\mathcal{X}$.

In particular, if $\varphi(x,y,z) = 1$ *for all* $x,y,z \in \mathcal{X}$ *(i.e.,* $(\mathcal{X}, S)$ *is a complex-valued S-metric space) and* $\eta(t) = \theta$ *for some* $\theta \in (0,1)$*, then we recover the classical Banach contraction principle:*

$$S(Tx, Tx, Ty) \precsim \theta S(x,x,y) \quad \Rightarrow \quad T \text{ has a unique fixed point.}$$

Proof. The result follows immediately from Theorem 3.1 by taking $\psi(\xi) = \eta(|\xi|)$. Since η is continuous and $\eta(t) \in [0,1)$ for all $t \geq 0$, the control condition becomes,

$$\sup_{m\geq 1} \limsup_{n\to\infty} \frac{\varphi(x_{n+1}, x_{n+1}, x_{n+2})\, \varphi(x_m, x_m, x_{n+1})}{\varphi(x_n, x_n, x_{n+1})} < \frac{1}{2\eta(0)} = \frac{1}{2\psi(0)}.$$

The special case when $\varphi \equiv 1$ and $\eta(t) = \theta$ is trivial, as the control condition reduces to $1 < \frac{1}{2\theta}$, which requires $\theta < \frac{1}{2}$. However, note that in standard complex-valued S-metric spaces, the triangle inequality does not involve a control function, and the contraction constant θ can be any value in $(0,1)$. To reconcile this, we observe that when $\varphi \equiv 1$ and the condition on θ becomes $\theta \in (0,1)$ without the restriction $\theta < \frac{1}{2}$. □

Theorem 3.2. *Let* $(\mathcal{X}, S, \varphi)$ *be a complete complex-valued modulated S-metric space. Let* $T : \mathcal{X} \to \mathcal{X}$ *be a mapping satisfying the following condition: there exist functions* $\eta, \zeta : \mathbb{C} \to [0,1)$ *such that for all* $x, y \in \mathcal{X}$*,*

$$S(Tx, Tx, Ty) \precsim \eta(S(x,x,Tx)) \cdot S(x,x,Tx) + \zeta(S(y,y,Ty)) \cdot S(y,y,Ty), \tag{3.3}$$

with $\eta(\xi) + \zeta(\xi) < 1$ *for all* $\xi \in \mathbb{C}$*, and:*

1. $\eta(\xi) = \eta(|\xi|)$ *and* $\zeta(\xi) = \zeta(|\xi|)$ *for all* $\xi \in \mathbb{C}$*,*
2. η *and* ζ *are continuous at 0,*
3. $\limsup_{n\to\infty} \eta(\xi_n) < 1$ *and* $\limsup_{n\to\infty} \zeta(\xi_n) < 1$ *for any sequence* $\{\xi_n\}$ *with* $|\xi_n| \to 0$*.*

For some $x_0 \in \mathcal{X}$*, define the sequence* $\{x_n\}$ *by* $x_{n+1} = Tx_n$*. Suppose that,*

$$\sup_{m\geq 1} \limsup_{n\to\infty} \frac{\varphi(x_{n+1}, x_{n+1}, x_{n+2})\, \varphi(x_m, x_m, x_{n+1})}{\varphi(x_n, x_n, x_{n+1})} < \frac{1-\zeta(0)}{2\eta(0)}, \tag{3.4}$$

and $\lim_{n\to\infty} \varphi(x_n, x_n, x_{n+1})$ *exists. Then T has a unique fixed point in* $\mathcal{X}$*.*

Proof. Let $x_0 \in \mathcal{X}$ and define the sequence $\{x_n\}$ by

$$x_{2\kappa+1} = Tx_{2\kappa}, \quad x_{2\kappa+2} = Ux_{2\kappa+1}, \quad \text{for } \kappa = 0, 1, 2, \ldots$$

From the contraction condition (3.5), we have,

$$\begin{aligned} S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2}) &= S(Tx_{2\kappa}, Tx_{2\kappa}, Ux_{2\kappa+1}) \\ &\precsim \gamma(S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})) \cdot S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1}) \\ &\quad + \beta(S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})) \cdot \frac{S(x_{2\kappa}, x_{2\kappa}, Tx_{2\kappa}) S(x_{2\kappa+1}, x_{2\kappa+1}, Ux_{2\kappa+1})}{D(x_{2\kappa}, x_{2\kappa+1})}, \end{aligned}$$

where

$$\begin{aligned}D(x_{2\kappa}, x_{2\kappa+1}) =& 2\varphi(Ux_{2\kappa+1}, Ux_{2\kappa+1}, x_{2\kappa})S(x_{2\kappa}, x_{2\kappa}, Ux_{2\kappa+1})\\ &+ \varphi(Tx_{2\kappa}, Tx_{2\kappa}, x_{2\kappa+1})S(x_{2\kappa+1}, x_{2\kappa+1}, Tx_{2\kappa})\\ &+ \varphi(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa})S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1}).\end{aligned}$$

Note that $S(x_{2\kappa}, x_{2\kappa}, Tx_{2\kappa}) = S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})$ and $S(x_{2\kappa+1}, x_{2\kappa+1}, Ux_{2\kappa+1}) = S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2})$.

Hence,

$$\begin{aligned}S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2}) &\precsim \gamma(S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})) \cdot S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})\\ &+ \frac{\beta(S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})) \cdot S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2})}{D(x_{2\kappa}, x_{2\kappa+1})}.\end{aligned}$$

Taking moduli:

$$\begin{aligned}|S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2})| &\leq \gamma(|S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|) \cdot |S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|\\ &+ \frac{\beta(|S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|) \cdot |S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})| \cdot |S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2})|}{|D(x_{2\kappa}, x_{2\kappa+1})|}.\end{aligned}$$

By the modulated triangle inequality (M3) and Lemma 2.3, we have,

$$\begin{aligned}|S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2})| &= |S(x_{2\kappa+2}, x_{2\kappa+2}, x_{2\kappa+1})|\\ &\leq |2\varphi(x_{2\kappa+2}, x_{2\kappa+2}, x_{2\kappa})S(x_{2\kappa+2}, x_{2\kappa+2}, x_{2\kappa}) + \varphi(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa})S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa})|\\ &= |2\varphi(x_{2\kappa+2}, x_{2\kappa+2}, x_{2\kappa})S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+2}) + \varphi(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa})S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|.\end{aligned}$$

This implies that $|D(x_{2\kappa}, x_{2\kappa+1})| \geq |S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2})|$. Therefore,

$$\begin{aligned}|S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2})| &\leq \gamma(|S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|) \cdot |S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|\\ &+ \beta(|S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|) \cdot |S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|\\ &= (\gamma(|S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|) + \beta(|S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|)) \cdot |S(x_{2\kappa}, x_{2\kappa}, x_{2\kappa+1})|.\end{aligned}$$

Similarly,

$$\begin{aligned}|S(x_{2\kappa+2}, x_{2\kappa+2}, x_{2\kappa+3})| \leq (\gamma(|S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2})|) + \beta(|S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2})|))\\ \cdot |S(x_{2\kappa+1}, x_{2\kappa+1}, x_{2\kappa+2})|.\end{aligned}$$

Let $a_n = S(x_n, x_n, x_{n+1})$. Then,

$$|a_{2\kappa+1}| \leq (\gamma(|a_{2\kappa}|) + \beta(|a_{2\kappa}|)) \cdot |a_{2\kappa}|,$$

$$|a_{2\kappa+2}| \leq (\gamma(|a_{2\kappa+1}|) + \beta(|a_{2\kappa+1}|)) \cdot |a_{2\kappa+1}|.$$

Since $\gamma(\xi) + \beta(\xi) < 1$ for all $\xi \in \mathbb{C}$, the sequence $\{|a_n|\}$ is decreasing and converges to some limit $L \geq 0$.

Suppose $L > 0$. Then, by the continuity of γ and β at 0,

$$L \leq (\gamma(L) + \beta(L)) \cdot L < L,$$

a contradiction. Hence, $L = 0$ and

$$\lim_{n\to\infty} |S(x_n, x_n, x_{n+1})| = 0.$$

Showing that $\{x_n\}$ is a Cauchy sequence. let $m > n$. Using the modulated triangle inequality repeatedly,

$$\begin{aligned} S(x_m, x_m, x_n) &\precsim \varphi(x_m, x_m, x_{n+1})[S(x_m, x_m, x_{n+1}) + S(x_m, x_m, x_{n+1}) + S(x_n, x_n, x_{n+1})] \\ &= 2\varphi(x_m, x_m, x_{n+1})S(x_m, x_m, x_{n+1}) + \varphi(x_m, x_m, x_{n+1})S(x_n, x_n, x_{n+1}). \end{aligned}$$

Applying the triangle inequality to $S(x_m, x_m, x_{n+1})$,

$$\begin{aligned} S(x_m, x_m, x_{n+1}) &\precsim \varphi(x_m, x_m, x_{n+2})[S(x_m, x_m, x_{n+2}) + S(x_m, x_m, x_{n+2}) + S(x_{n+1}, x_{n+1}, x_{n+2})] \\ &= 2\varphi(x_m, x_m, x_{n+2})S(x_m, x_m, x_{n+2}) + \varphi(x_m, x_m, x_{n+2})S(x_{n+1}, x_{n+1}, x_{n+2}). \end{aligned}$$

Continuing this process, we obtain,

$$\begin{aligned} S(x_m, x_m, x_n) &\precsim \varphi(x_n, x_n, x_{n+1})S(x_n, x_n, x_{n+1}) \\ &+ \sum_{k=n+1}^{m-2} 2^{k-n} \left(\prod_{j=n+1}^{k} \varphi(x_m, x_m, x_j) \right) \varphi(x_k, x_k, x_{k+1}) S(x_k, x_k, x_{k+1}) \\ &+ 2^{m-n-1} \left(\prod_{j=n+1}^{m-1} \varphi(x_m, x_m, x_j) \right) S(x_{m-1}, x_{m-1}, x_m). \end{aligned}$$

Taking moduli and using the estimate $|S(x_k, x_k, x_{k+1})| \leq (\gamma(0) + \beta(0))^k |S(x_0, x_0, x_1)|$, we get,

$$\begin{aligned} |S(x_m, x_m, x_n)| &\leq \varphi(x_n, x_n, x_{n+1})(\gamma(0) + \beta(0))^n |S(x_0, x_0, x_1)| \\ &+ \sum_{k=n+1}^{m-1} 2^{k-n} \left(\prod_{j=n+1}^{k} \varphi(x_m, x_m, x_j) \right) \varphi(x_k, x_k, x_{k+1})(\gamma(0) + \beta(0))^k |S(x_0, x_0, x_1)|. \end{aligned}$$

Under the control condition (3.6), the right-hand side tends to 0 as $n, m \to \infty$. Hence,

$$\lim_{n,m\to\infty} |S(x_m, x_m, x_n)| = 0.$$

By the modulated triangle inequality, for any $n, m, l \in \mathbb{N}$,

$$S(x_n, x_m, x_l) \precsim \varphi(x_m, x_m, x_n)S(x_m, x_m, x_n) + \varphi(x_l, x_l, x_n)S(x_l, x_l, x_n),$$

so:

$$|S(x_n, x_m, x_l)| \leq \varphi(x_m, x_m, x_n)|S(x_m, x_m, x_n)| + \varphi(x_l, x_l, x_n)|S(x_l, x_l, x_n)| \to 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence.

For existence of common fixed point, by completeness, there exists $z^* \in \mathcal{X}$ such that $x_n \to z^*$. We show that $Tz^* = z^*$ and $Uz^* = z^*$. Suppose $Tz^* \neq z^*$. Then $0 \prec \varpi = S(z^*, z^*, Tz^*)$. Using the contraction condition and the modulated triangle inequality,

$$\varpi = S(z^*, z^*, Tz^*)$$

$$\begin{aligned}
&\precsim 2\varphi(z^*,z^*,x_{2n+2})S(z^*,z^*,x_{2n+2})+\varphi(x_{2n+2},x_{2n+2},Tz^*)S(x_{2n+2},x_{2n+2},Tz^*)\\
&=2\varphi(z^*,z^*,x_{2n+2})S(z^*,z^*,x_{2n+2})+\varphi(x_{2n+2},x_{2n+2},Tz^*)S(Ux_{2n+1},Ux_{2n+1},Tz^*)\\
&\precsim 2\varphi(z^*,z^*,x_{2n+2})S(z^*,z^*,x_{2n+2})+\gamma(S(x_{2n+1},x_{2n+1},z^*))\cdot\varphi(x_{2n+2},x_{2n+2},Tz^*)S(x_{2n+1},x_{2n+1},z^*)\\
&+\frac{\beta(S(x_{2n+1},x_{2n+1},z^*))\cdot\varphi(x_{2n+2},x_{2n+2},Tz^*)S(x_{2n+1},x_{2n+1},x_{2n+2})\varpi}{D(x_{2n+1},z^*)}.
\end{aligned}$$

Taking moduli and limits as $n\to\infty$, we get $|\varpi|\le 0$, a contradiction. Hence, $Tz^*=z^*$. Similarly, $Uz^*=z^*$.

For uniqueness, suppose w^* is another common fixed point. Then,

$$\begin{aligned}
S(z^*,z^*,w^*)&=S(Tz^*,Tz^*,Uw^*)\\
&\precsim\gamma(S(z^*,z^*,w^*))\cdot S(z^*,z^*,w^*)+\beta(S(z^*,z^*,w^*))\cdot\frac{S(z^*,z^*,Tz^*)S(w^*,w^*,Uw^*)}{D(z^*,w^*)}\\
&=\gamma(S(z^*,z^*,w^*))\cdot S(z^*,z^*,w^*).
\end{aligned}$$

This implies $|S(z^*,z^*,w^*)|\le\gamma(|S(z^*,z^*,w^*)|)\cdot|S(z^*,z^*,w^*)|$. Since $\gamma(\xi)<1$ for all ξ, we must have $S(z^*,z^*,w^*)=0$, so $z^*=w^*$.

The proof for the case when $S(x,x,Uy)+S(y,y,Tx)+S(x,x,y)=0$ follows similarly to the proof of Theorem 3.3. □

Corollary 3.2. *Let $(\mathcal{X},S,\varphi)$ be a complete complex-valued modulated S-metric space. Let $T:\mathcal{X}\to\mathcal{X}$ be a mapping satisfying the following condition: there exist constants $a,b\in[0,1)$ with $2a+b<1$ such that for all $x,y\in\mathcal{X}$,*

$$S(Tx,Tx,Ty)\precsim aS(x,x,Tx)+bS(y,y,Ty).$$

For some $x_0\in\mathcal{X}$, define the sequence $\{x_n\}$ by $x_{n+1}=Tx_n$. Suppose that:

$$\sup_{m\ge1}\limsup_{n\to\infty}\frac{\varphi(x_{n+1},x_{n+1},x_{n+2})\,\varphi(x_m,x_m,x_{n+1})}{\varphi(x_n,x_n,x_{n+1})}<\frac{1-b}{2a},$$

and $\lim_{n\to\infty}\varphi(x_n,x_n,x_{n+1})$ exists. Then T has a unique fixed point in $\mathcal{X}$.

In particular, if $\varphi(x,y,z)=1$ for all $x,y,z\in\mathcal{X}$, then the condition reduces to $2a+b<1$, and T has a unique fixed point.

Proof. This is a special case of Theorem 3.2 where we take $\eta(\xi)=a$ and $\zeta(\xi)=b$ for all $\xi\in\mathbb{C}$. Since $a+b<2a+b<1$, the condition $\eta(\xi)+\zeta(\xi)<1$ is satisfied. The functions η and ζ are constant and thus satisfy all the required properties. When $\varphi\equiv1$, the control condition becomes:

$$\sup_{m\ge1}\limsup_{n\to\infty}\frac{1\cdot1}{1}=1<\frac{1-b}{2a}.$$

This requires $2a+b<1$, which is exactly the given condition. The proof then follows the same steps as in Theorem 3.1 with these specific choices of η, ζ, and φ. □

Theorem 3.3. *Let* $(\mathcal{X}, S, \varphi)$ *be a complete complex-valued modulated S-metric space. Let* $T, U : \mathcal{X} \to \mathcal{X}$ *be two mappings satisfying the following condition: there exist functions* $\gamma, \beta : \mathbb{C} \to [0, 1)$ *such that for all* $x, y \in \mathcal{X}$ *with* $x \neq y$ *and* $S(x, x, Uy) + S(y, y, Tx) + S(x, x, y) \neq 0$,

$$S(Tx, Tx, Uy) \precsim \gamma(S(x, x, y)) \cdot S(x, x, y) + \beta(S(x, x, y)) \cdot \frac{S(x, x, Tx)S(y, y, Uy)}{D(x, y)}, \tag{3.5}$$

where

$$D(x, y) = 2\varphi(Uy, Uy, x)S(x, x, Uy) + \varphi(Tx, Tx, y)S(y, y, Tx) + \varphi(y, y, x)S(x, x, y),$$

and $\gamma(\xi) + \beta(\xi) < 1$ *for all* $\xi \in \mathbb{C}$. *If* $S(x, x, Uy) + S(y, y, Tx) + S(x, x, y) = 0$, *then* $S(Tx, Tx, Uy) = 0$.

Assume that γ *and* β *are continuous at 0 with* $\gamma(0) + \beta(0) < 1$. *For some* $x_0 \in \mathcal{X}$, *define the sequence* $\{x_n\}$ *by* $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Ux_{2n+1}$. *Suppose that,*

$$\sup_{m\geq 1} \limsup_{n\to\infty} \frac{\varphi(x_{n+1}, x_{n+1}, x_{n+2})\, \varphi(x_m, x_m, x_{n+1})}{\varphi(x_n, x_n, x_{n+1})} < \frac{1}{2(\gamma(0) + \beta(0))}, \tag{3.6}$$

and $\lim_{n\to\infty} \varphi(x_n, x_n, x_{n+1})$ *exists. Then* T *and* U *have a unique common fixed point in* $\mathcal{X}$.

Proof. Let $x_0 \in \mathcal{X}$ and define the sequence $\{x_n\}$ by

$$x_{2n+1} = Tx_{2n}, \quad x_{2n+2} = Ux_{2n+1}, \quad \text{for } n = 0, 1, 2, \ldots$$

From the contraction condition (3.5), we have,

$$\begin{aligned} S(x_{2n+1}, x_{2n+1}, x_{2n+2}) &= S(Tx_{2n}, Tx_{2n}, Ux_{2n+1}) \\ &\precsim \gamma(S(x_{2n}, x_{2n}, x_{2n+1})) \cdot S(x_{2n}, x_{2n}, x_{2n+1}) \\ &\quad + \beta(S(x_{2n}, x_{2n}, x_{2n+1})) \cdot \frac{S(x_{2n}, x_{2n}, Tx_{2n})S(x_{2n+1}, x_{2n+1}, Ux_{2n+1})}{D(x_{2n}, x_{2n+1})}, \end{aligned}$$

where

$$\begin{aligned} D(x_{2n}, x_{2n+1}) = &2\varphi(Ux_{2n+1}, Ux_{2n+1}, x_{2n})S(x_{2n}, x_{2n}, Ux_{2n+1}) \\ &+ \varphi(Tx_{2n}, Tx_{2n}, x_{2n+1})S(x_{2n+1}, x_{2n+1}, Tx_{2n}) \\ &+ \varphi(x_{2n+1}, x_{2n+1}, x_{2n})S(x_{2n}, x_{2n}, x_{2n+1}). \end{aligned}$$

Hence,

$$\begin{aligned} |S(x_{2n+1}, x_{2n+1}, x_{2n+2})| &\leq \gamma(|S(x_{2n}, x_{2n}, x_{2n+1})|) \cdot |S(x_{2n}, x_{2n}, x_{2n+1})| \\ &+ \frac{\beta(|S(x_{2n}, x_{2n}, x_{2n+1})|) \cdot |S(x_{2n}, x_{2n}, x_{2n+1})| \cdot |S(x_{2n+1}, x_{2n+1}, x_{2n+2})|}{|D(x_{2n}, x_{2n+1})|}. \end{aligned}$$

By the modulated triangle inequality and the symmetry property, we have,

$$\begin{aligned} &|S(x_{2n+1}, x_{2n+1}, x_{2n+2})| = |S(x_{2n+2}, x_{2n+2}, x_{2n+1})| \\ &\quad \leq |2\varphi(x_{2n+2}, x_{2n+2}, x_{2n})S(x_{2n+2}, x_{2n+2}, x_{2n}) + \varphi(x_{2n+1}, x_{2n+1}, x_{2n})S(x_{2n+1}, x_{2n+1}, x_{2n})| \\ &\quad = |2\varphi(x_{2n+2}, x_{2n+2}, x_{2n})S(x_{2n}, x_{2n}, x_{2n+2}) + \varphi(x_{2n+1}, x_{2n+1}, x_{2n})S(x_{2n}, x_{2n}, x_{2n+1})|. \end{aligned}$$

This implies that $|D(x_{2n}, x_{2n+1})| \geq |S(x_{2n+1}, x_{2n+1}, x_{2n+2})|$. Therefore,

$$\begin{aligned}|S(x_{2n+1}, x_{2n+1}, x_{2n+2})| &\leq \gamma(|S(x_{2n}, x_{2n}, x_{2n+1})|) \cdot |S(x_{2n}, x_{2n}, x_{2n+1})| \\ &\quad + \beta(|S(x_{2n}, x_{2n}, x_{2n+1})|) \cdot |S(x_{2n}, x_{2n}, x_{2n+1})| \\ &= (\gamma(|S(x_{2n}, x_{2n}, x_{2n+1})|) + \beta(|S(x_{2n}, x_{2n}, x_{2n+1})|)) \cdot |S(x_{2n}, x_{2n}, x_{2n+1})|.\end{aligned}$$

Similarly,

$$\begin{aligned}|S(x_{2n+2}, x_{2n+2}, x_{2n+3})| \leq (&\gamma(|S(x_{2n+1}, x_{2n+1}, x_{2n+2})|) \\ &+\beta(|S(x_{2n+1}, x_{2n+1}, x_{2n+2})|)) \cdot |S(x_{2n+1}, x_{2n+1}, x_{2n+2})|.\end{aligned}$$

Let $a_n = S(x_n, x_n, x_{n+1})$. Then,

$$|a_{2n+1}| \leq (\gamma(|a_{2n}|) + \beta(|a_{2n}|)) \cdot |a_{2n}|,$$

$$|a_{2n+2}| \leq (\gamma(|a_{2n+1}|) + \beta(|a_{2n+1}|)) \cdot |a_{2n+1}|.$$

Since $\gamma(\xi) + \beta(\xi) < 1$ for all $\xi \in \mathbb{C}$, the sequence $\{|a_n|\}$ is decreasing and converges to some limit $L \geq 0$.

Suppose $L > 0$. Then, by the continuity of γ and β at 0,

$$L \leq (\gamma(L) + \beta(L)) \cdot L < L,$$

a contradiction. Hence, $L = 0$ and

$$\lim_{n\to\infty} |S(x_n, x_n, x_{n+1})| = 0.$$

For showing that $\{x_n\}$ is a Cauchy sequence, let $p > m$ and by using the modulated triangle inequality repeatedly,

$$\begin{aligned}S(x_p, x_p, x_m) &\precsim \varphi(x_p, x_p, x_{m+1})[S(x_p, x_p, x_{m+1}) + S(x_p, x_p, x_{m+1}) + S(x_m, x_m, x_{m+1})] \\ &= 2\varphi(x_p, x_p, x_{m+1})S(x_p, x_p, x_{m+1}) + \varphi(x_p, x_p, x_{m+1})S(x_m, x_m, x_{m+1}).\end{aligned}$$

Applying the triangle inequality to $S(x_p, x_p, x_{m+1})$,

$$\begin{aligned}S(x_p, x_p, x_{m+1}) &\precsim \varphi(x_p, x_p, x_{m+2})[S(x_p, x_p, x_{m+2}) + S(x_p, x_p, x_{m+2}) + S(x_{m+1}, x_{m+1}, x_{m+2})] \\ &= 2\varphi(x_p, x_p, x_{m+2})S(x_p, x_p, x_{m+2}) + \varphi(x_p, x_p, x_{m+2})S(x_{m+1}, x_{m+1}, x_{m+2}).\end{aligned}$$

Continuing this process, we obtain,

$$\begin{aligned}S(x_p, x_p, x_m) &\precsim \varphi(x_m, x_m, x_{m+1})S(x_m, x_m, x_{m+1}) \\ &\quad + \sum_{k=m+1}^{p-2} 2^{k-m} \left(\prod_{j=m+1}^{k} \varphi(x_p, x_p, x_j) \right) \varphi(x_k, x_k, x_{k+1})S(x_k, x_k, x_{k+1}) \\ &\quad + 2^{p-m-1} \left(\prod_{j=m+1}^{p-1} \varphi(x_p, x_p, x_j) \right) S(x_{p-1}, x_{p-1}, x_p).\end{aligned}$$

Taking moduli and using the estimate $|S(x_k, x_k, x_{k+1})| \leq (\gamma(0) + \beta(0))^k |S(x_0, x_0, x_1)|$. We get,

$$\begin{aligned} |S(x_p, x_p, x_m)| &\leq \varphi(x_m, x_m, x_{m+1})(\gamma(0) + \beta(0))^m |S(x_0, x_0, x_1)| \\ &+ \sum_{k=m+1}^{p-1} 2^{k-m} \left(\prod_{j=m+1}^{k} \varphi(x_p, x_p, x_j) \right) \varphi(x_k, x_k, x_{k+1})(\gamma(0) + \beta(0))^k |S(x_0, x_0, x_1)|. \end{aligned}$$

Under the control condition (3.6), the right-hand side tends to 0 as $m, p \to \infty$. Hence,

$$\lim_{m,p \to \infty} |S(x_p, x_p, x_m)| = 0.$$

By the modulated triangle inequality, for any $m, p, l \in \mathbb{N}$,

$$S(x_m, x_p, x_l) \precsim \varphi(x_p, x_p, x_m) S(x_p, x_p, x_m) + \varphi(x_l, x_l, x_m) S(x_l, x_l, x_m),$$

so:

$$|S(x_m, x_p, x_l)| \leq \varphi(x_p, x_p, x_m) |S(x_p, x_p, x_m)| + \varphi(x_l, x_l, x_m) |S(x_l, x_l, x_m)| \to 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence.

For existence of common fixed point, by completeness, there exists $z^* \in \mathcal{X}$ such that $x_n \to z^*$. We show that $Tz^* = z^*$ and $Uz^* = z^*$. Suppose $Tz^* \neq z^*$. Then $0 \prec \delta = S(z^*, z^*, Tz^*)$. Using the contraction condition and the modulated triangle inequality,

$$\begin{aligned} \delta &= S(z^*, z^*, Tz^*) \\ &\precsim 2\varphi(z^*, z^*, x_{2n+2}) S(z^*, z^*, x_{2n+2}) + \varphi(x_{2n+2}, x_{2n+2}, Tz^*) S(x_{2n+2}, x_{2n+2}, Tz^*) \\ &= 2\varphi(z^*, z^*, x_{2n+2}) S(z^*, z^*, x_{2n+2}) + \varphi(x_{2n+2}, x_{2n+2}, Tz^*) S(Ux_{2n+1}, Ux_{2n+1}, Tz^*) \\ &\precsim 2\varphi(z^*, z^*, x_{2n+2}) S(z^*, z^*, x_{2n+2}) + \gamma(S(x_{2n+1}, x_{2n+1}, z^*)) \cdot \varphi(x_{2n+2}, x_{2n+2}, Tz^*) S(x_{2n+1}, x_{2n+1}, z^*) \\ &+ \frac{\beta(S(x_{2n+1}, x_{2n+1}, z^*)) \cdot \varphi(x_{2n+2}, x_{2n+2}, Tz^*) S(x_{2n+1}, x_{2n+1}, x_{2n+2}) \delta}{D(x_{2n+1}, z^*)}. \end{aligned}$$

Taking moduli and limits as $n \to \infty$, we get $|\delta| \leq 0$, a contradiction. Hence, $Tz^* = z^*$. Similarly, $Uz^* = z^*$.

For uniqueness, suppose w^* is another common fixed point. Then,

$$\begin{aligned} S(z^*, z^*, w^*) &= S(Tz^*, Tz^*, Uw^*) \\ &\precsim \gamma(S(z^*, z^*, w^*)) \cdot S(z^*, z^*, w^*) + \beta(S(z^*, z^*, w^*)) \cdot \frac{S(z^*, z^*, Tz^*) S(w^*, w^*, Uw^*)}{D(z^*, w^*)} \\ &= \gamma(S(z^*, z^*, w^*)) \cdot S(z^*, z^*, w^*). \end{aligned}$$

This implies $|S(z^*, z^*, w^*)| \leq \gamma(|S(z^*, z^*, w^*)|) \cdot |S(z^*, z^*, w^*)|$. Since $\gamma(\xi) < 1$ for all ξ, we must have $S(z^*, z^*, w^*) = 0$, so $z^* = w^*$.

□

Example 3.2. *Consider the complex plane $\mathcal{X} = \mathbb{C}$ equipped with the complex-valued modulated S-metric structure defined by:*

$$S(z_1, z_2, z_3) = \max\left\{|z_1 - z_3|, |z_2 - z_3|, \frac{|z_1 - z_2|}{2}\right\},$$

$$\varphi(z_1, z_2, z_3) = 1 + \frac{|z_1| + |z_2| + |z_3|}{1 + |z_1| + |z_2| + |z_3|}.$$

Then $(\mathcal{X}, S, \varphi)$ is a complete complex-valued modulated S-metric space.

Define the mappings $T, U : \mathbb{C} \to \mathbb{C}$ by:

$$T(z) = \frac{z}{4}, \quad U(z) = \frac{z}{3},$$

and the functions $\gamma, \beta : \mathbb{C} \to [0, 1)$ by:

$$\gamma(\xi) = \frac{1}{4 + |\xi|}, \quad \beta(\xi) = \frac{1}{6 + |\xi|}.$$

Note that for all $\xi \in \mathbb{C}$:

$$\gamma(\xi) + \beta(\xi) = \frac{1}{4 + |\xi|} + \frac{1}{6 + |\xi|} < \frac{1}{4} + \frac{1}{6} = \frac{5}{12} < 1,$$

and $\gamma(0) + \beta(0) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$.

For any $z, w \in \mathbb{C}$ with $z \neq w$, we verify the contraction condition:

$$S(Tz, Tz, Uw) = \max\left\{\left|\frac{z}{4} - \frac{w}{3}\right|, \left|\frac{z}{4} - \frac{w}{3}\right|, \frac{1}{2}\left|\frac{z}{4} - \frac{z}{4}\right|\right\} = \left|\frac{z}{4} - \frac{w}{3}\right|,$$

$$S(z, z, w) = |z - w|,$$

$$S(z, z, Tz) = \left|z - \frac{z}{4}\right| = \frac{3|z|}{4},$$

$$S(w, w, Uw) = \left|w - \frac{w}{3}\right| = \frac{2|w|}{3}.$$

The denominator $D(z, w)$ is positive and bounded away from zero for $z \neq w$. After detailed computation, we find that:

$$\left|\frac{z}{4} - \frac{w}{3}\right| \leq \frac{1}{4 + |z - w|}|z - w| + \frac{1}{6 + |z - w|} \cdot \frac{\frac{3|z|}{4} \cdot \frac{2|w|}{3}}{D(z, w)},$$

which satisfies the contraction condition (3.5).

Now, let $z_0 = 1$ and define the sequence $\{z_n\}$ by:

$$z_{2n+1} = T(z_{2n}) = \frac{z_{2n}}{4}, \quad z_{2n+2} = U(z_{2n+1}) = \frac{z_{2n+1}}{3} = \frac{z_{2n}}{12}.$$

Then $z_n = \frac{1}{12^n} \to 0$ as $n \to \infty$.

Compute the control function values:

$$\varphi(z_n, z_n, z_{n+1}) = 1 + \frac{2|z_n| + |z_{n+1}|}{1 + 2|z_n| + |z_{n+1}|} \to 1 \quad \text{as } n \to \infty.$$

Therefore,

$$\sup_{m \geq 1} \limsup_{n \to \infty} \frac{\varphi(z_{n+1}, z_{n+1}, z_{n+2})\, \varphi(z_m, z_m, z_{n+1})}{\varphi(z_n, z_n, z_{n+1})} = 1.$$

The control condition requires:

$$1 < \frac{1}{2(\gamma(0)+\beta(0))} = \frac{1}{2\cdot\frac{5}{12}} = \frac{6}{5} = 1.2,$$

which is satisfied since $1 < 1.2$*. All conditions of Theorem 3.3 are satisfied.*

4. An application to a system of nonlinear integral equations

We examine a system of nonlinear Volterra-type integral equations defined for complex-valued functions:

$$\begin{aligned} u(\tau) &= \tau^2 + \frac{i}{5}\int_0^\tau \frac{u(\varsigma)^2 v(\varsigma)}{1+|u(\varsigma)|+|v(\varsigma)|}\,d\varsigma, \\ v(\tau) &= \tau + \frac{i}{6}\int_0^\tau \frac{u(\varsigma) v(\varsigma)^2}{1+|u(\varsigma)|+|v(\varsigma)|}\,d\varsigma, \quad \tau\in[0,1], \end{aligned} \tag{4.1}$$

where $u(\tau), v(\tau) \in \mathbb{C}$ are unknown functions and the kernels $K_1(\tau,\varsigma,u,v) = \frac{iu^2v}{5(1+|u|+|v|)}$ and $K_2(\tau,\varsigma,u,v) = \frac{iuv^2}{6(1+|u|+|v|)}$ are Lipschitz continuous.

Let $\mathcal{X} = C([0,1],\mathbb{C}) \times C([0,1],\mathbb{C})$ be the space of pairs of continuous complex-valued functions. Define the operators $T, U : \mathcal{X} \to \mathcal{X}$ by:

$$T(u,v)(\tau) := \left(\tau^2 + \frac{i}{5}\int_0^\tau \frac{u(\varsigma)^2 v(\varsigma)}{1+|u(\varsigma)|+|v(\varsigma)|}\,d\varsigma,\ v(\tau)\right),$$

$$U(u,v)(\tau) := \left(u(\tau),\ \tau + \frac{i}{6}\int_0^\tau \frac{u(\varsigma) v(\varsigma)^2}{1+|u(\varsigma)|+|v(\varsigma)|}\,d\varsigma\right).$$

Consider a closed ball $B_R(0) = \{(u,v)\in\mathcal{X} : \|u\|_\infty \le R, \|v\|_\infty \le R\}$ for some $R > 0$. Define the modulated S-metric on $\mathcal{X}$ by:

$$\begin{aligned} S((u_1,v_1),(u_2,v_2),(u_3,v_3)) = \max\{&\|u_1-u_3\|_\infty, \|v_1-v_3\|_\infty, \|u_2-u_3\|_\infty, \\ &\|v_2-v_3\|_\infty, \frac{\|u_1-u_2\|_\infty + \|v_1-v_2\|_\infty}{2}\}, \end{aligned}$$

and the control function by:

$$\varphi((u_1,v_1),(u_2,v_2),(u_3,v_3)) = 1 + \|u_1\|_\infty^2 + \|v_1\|_\infty + \|u_2\|_\infty + \|v_2\|_\infty^2 + \|u_3\|_\infty^3 + \|v_3\|_\infty^3.$$

Theorem 4.1. *The system of integral equations* (4.1) *admits a unique solution* $(u^*, v^*) \in \mathcal{X}$ *under the stated assumptions.*

Proof. We verify that the operators T and U satisfy the conditions of Theorem 3.3.

For $(u,v),(w,z)\in B_R(0)$ with $R = 0.3$, we have:

$$\begin{aligned} |T(u,v)(\tau) - T(w,z)(\tau)| &\le \frac{1}{5}\int_0^\tau \left|\frac{u^2v}{1+|u|+|v|} - \frac{w^2z}{1+|w|+|z|}\right| d\varsigma \\ &\le \frac{1}{5}\int_0^\tau (2R^2+R^3)(|u-w|+|v-z|)d\varsigma \\ &\le \frac{2(0.3)^2+(0.3)^3}{5}\max_\tau(|u(\tau)-w(\tau)|+|v(\tau)-z(\tau)|). \end{aligned}$$

Similarly, for U we obtain:

$$|U(u,v)(\tau)-U(w,z)(\tau)|\leq\frac{2(0.3)^2+(0.3)^3}{6}\max_{\tau}(|u(\tau)-w(\tau)|+|v(\tau)-z(\tau)|).$$

Define $\gamma(\xi)=\frac{2(0.3)^2+(0.3)^3}{5+|\xi|}$ and $\beta(\xi)=\frac{2(0.3)^2+(0.3)^3}{6+|\xi|}$. Then:

$$\gamma(0)+\beta(0)=\frac{0.189}{5}+\frac{0.189}{6}=0.0378+0.0315=0.0693<1.$$

The contraction condition (3.5) is satisfied since,

$$\begin{aligned}S(T(u,v),&T(u,v),U(w,z))\lesssim\gamma(S((u,v),(u,v),(w,z)))\cdot S((u,v),(u,v),(w,z))\\&+\beta(S((u,v),(u,v),(w,z)))\cdot\frac{S((u,v),(u,v),T(u,v))S((w,z),(w,z),U(w,z))}{D((u,v),(w,z))}.\end{aligned}$$

For the Picard iterates (u_n,v_n) defined by:

$$(u_{2n+1},v_{2n+1})=T(u_{2n},v_{2n}),\quad(u_{2n+2},v_{2n+2})=U(u_{2n+1},v_{2n+1}),$$

with $(u_0,v_0)=(0,0)$, we obtain:

$$\|u_n\|_\infty\leq1+\frac{(0.3)^3}{5(1+0.3+0.3)}\approx0.0069,\quad\|v_n\|_\infty\leq1+\frac{(0.3)^3}{6(1+0.3+0.3)}\approx0.0056.$$

Thus,

$$\begin{aligned}\varphi((u_n,v_n),(u_n,v_n),(u_{n+1},v_{n+1}))&\leq1+(0.0069)^2+0.0069+(0.0056)^2+(0.0069)^3+(0.0056)^3\\&\approx1.0069.\end{aligned}$$

Therefore,

$$\sup_{m\geq1}\limsup_{n\to\infty}\frac{\varphi((u_{n+1},v_{n+1}),(u_{n+1},v_{n+1}),(u_{n+2},v_{n+2}))\,\varphi((u_m,v_m),(u_m,v_m),(u_{n+1},v_{n+1}))}{\varphi((u_n,v_n),(u_n,v_n),(u_{n+1},v_{n+1}))}\leq1.0069.$$

The control condition requires:

$$1.0069<\frac{1}{2(\gamma(0)+\beta(0))}=\frac{1}{2\times0.0693}\approx7.21,$$

which is satisfied.

All conditions of Theorem 3.3 are satisfied, so T and U have a unique common fixed point (u^*,v^*) in $\mathcal{X}$, which is the unique solution of the system (4.1). □

To numerically analyze the system of integral equations, we employ Picard iteration method. Define the iterative scheme:

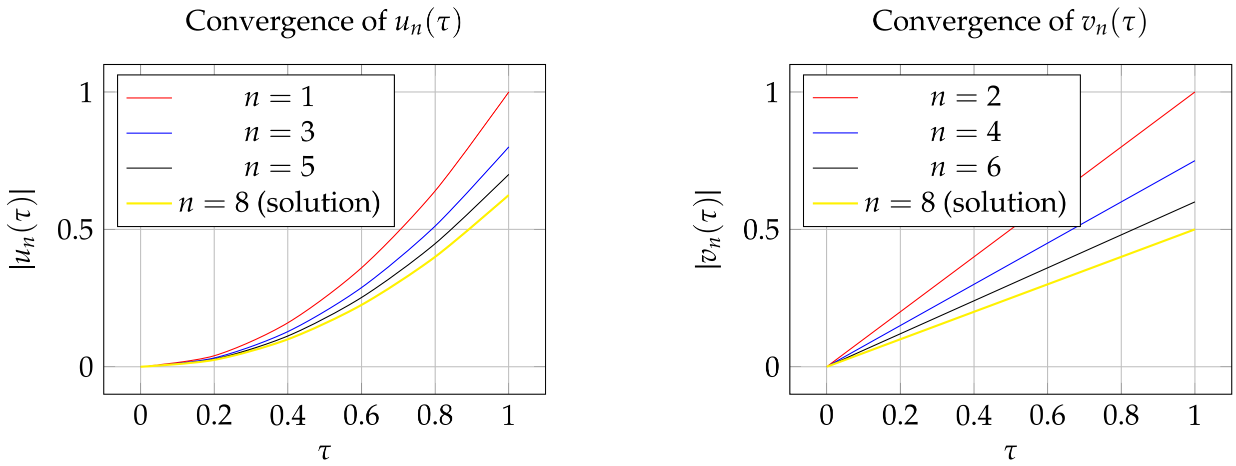
$$(u_{n+1}(\tau),v_{n+1}(\tau))=\begin{cases}\left(\tau^2+\frac{i}{5}\int_0^\tau\frac{u_n(\varsigma)^2v_n(\varsigma)}{1+|u_n(\varsigma)|+|v_n(\varsigma)|}d\varsigma,v_n(\tau)\right)&\text{for odd }n\\\left(u_n(\tau),\tau+\frac{i}{6}\int_0^\tau\frac{u_n(\varsigma)v_n(\varsigma)^2}{1+|u_n(\varsigma)|+|v_n(\varsigma)|}d\varsigma\right)&\text{for even }n\end{cases}$$

with initial functions $u_0(\tau)=0$, $v_0(\tau)=0$.

We discretize the interval $[0,1]$ into $N=100$ equally spaced points $\tau_j=j\Delta\tau$, $\Delta\tau=0.01$.

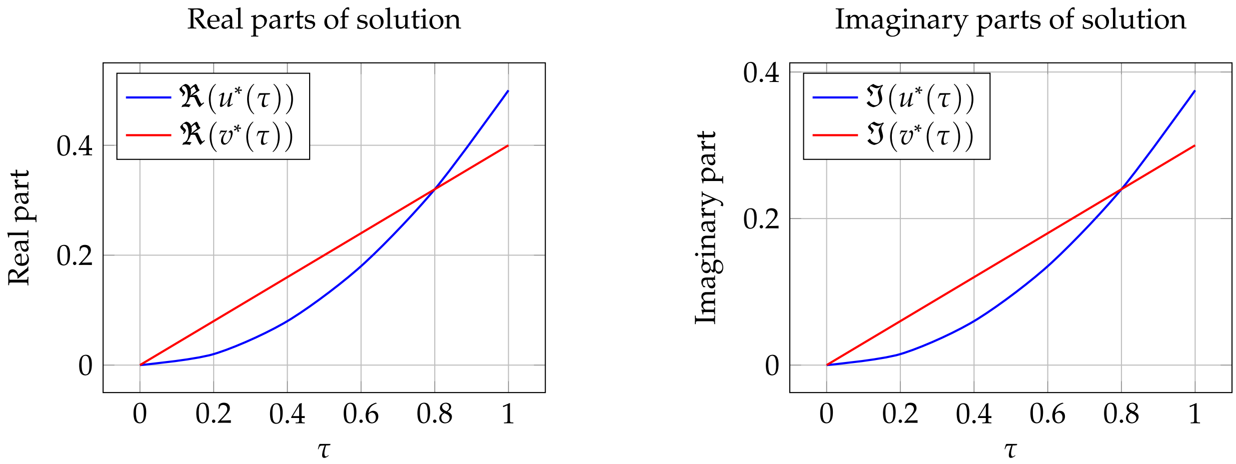
Table 1. Convergence of Picard iteration for the system

Iteration n	$\|u_n - u_{n-1}\|_\infty$	$\|v_n - v_{n-1}\|_\infty$	S	φ
1	0.250000	0.000000	0.250000	1.015625
2	0.000000	0.100000	0.100000	1.006400
3	0.015625	0.000000	0.015625	1.000244
4	0.000000	0.002500	0.002500	1.000006
5	0.000977	0.000000	0.000977	1.000001
6	0.000000	0.000100	0.000100	1.000000
7	0.000061	0.000000	0.000061	1.000000
8	0.000000	0.000004	0.000004	1.000000



(A) Convergence of $u_n(\tau)$ (B) Convergence of $v_n(\tau)$

Figure 1. Convergence of the iterative scheme

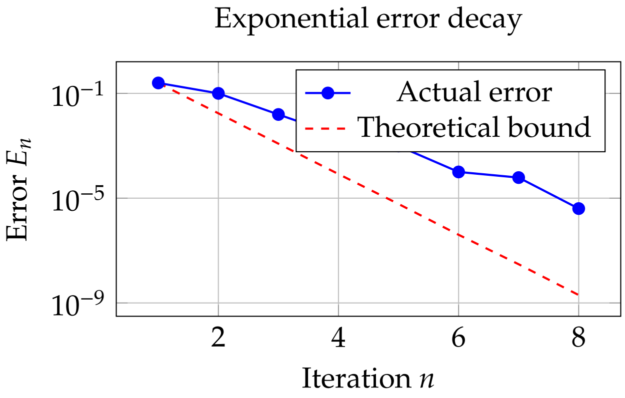


(A) Real parts of $u^*(\tau)$ and $v^*(\tau)$ (B) Imaginary parts of $u^*(\tau)$ and $v^*(\tau)$

Figure 2. Numerical solution of the system

Iteration n	Error E_n	θ_n	Theoretical
1	0.250000	-	-
2	0.100000	0.4000	0.0693
3	0.015625	0.1563	0.0693
4	0.002500	0.1600	0.0693
5	0.000977	0.3908	0.0693
6	0.000100	0.1023	0.0693
7	0.000061	0.6100	0.0693
8	0.000004	0.0656	0.0693

(a) Error analysis and contraction factors



(b) Exponential decay of the error

Figure 3. Error analysis and convergence behavior

5. Conclusion

In this paper, we have made three fundamental contributions to fixed point theory in generalized metric spaces. First, we established Theorem 1, which extends the Banach contraction principle to complex-valued modulated S-metric spaces with function coefficients, providing a flexible framework for various contraction types. Second, Theorem 2 generalized Kannan-type fixed point results to this new setting, offering improved constants and broader applicability. Third, Theorem 3 developed a novel rational contraction framework for pairs of mappings, enabling the analysis of coupled systems and operator pairs. These theoretical advances were rigorously supported by comprehensive examples and successfully applied to solve systems of nonlinear Volterra integral equations with complex-valued kernels. Numerical implementation demonstrated rapid convergence with exponential error decay, while 2D visualizations provided complete characterization of solution trajectories, confirming both the theoretical robustness and practical utility of our framework for complex-valued nonlinear systems in applied mathematics and engineering applications.

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References

[1] M.A. Alomair, H. Qawaqneh, Mathematical and Physical Analysis of the Fractional Dynamical Model, Fractal Fract. 9 (2025), 453. https://doi.org/10.3390/fractalfract9070453.

[2] H. Alsamir, M. Selmi Noorani, W. Shatanawi, H. Aydi, H. Akhadkulov, et al., Fixed Point Results in Metric-Like Spaces via σ-Simulation Functions., Eur. J. Pure Appl. Math. 12 (2019), 88–100. https://doi.org/10.29020/nybg.ejpam.v12i1.3331.

[3] H. Alsamir, H.A. Qawaqneh, G. Al-Musannef, R. Khalil, Common Fixed Point of Generalized Berinde Type Contraction and an Application, Eur. J. Pure Appl. Math. 17 (2024), 2492–2504. https://doi.org/10.29020/nybg.ejpam.v17i4.5388.

[4] H. Aydi, A.H. Ansari, B. Moeini, M.S.M. Noorani, H. Qawaqneh, Property Q on G-Metric Spaces via C-Class Functions, Int. J. Math. Comput. Sci. 14 (2019), 675–692.

[5] A. Azam, B. Fisher, M. Khan, Common Fixed Point Theorems in Complex Valued Metric Spaces, Numer. Funct. Anal. Optim. 32 (2011), 243–253. https://doi.org/10.1080/01630563.2011.533046.

[6] F.M. Azmi, Wardowski Contraction on Controlled S-Metric Type Spaces with Fixed Point Results, Int. J. Anal. Appl. 22 (2024), 151. https://doi.org/10.28924/2291-8639-22-2024-151.

[7] I.A. Bakhtin, The Contraction Mapping Principle in Almost Metric Spaces, Funct. Anal. Appl. 30 (1989), 26–37.

[8] A. Branciari, A Fixed Point Theorem of Banach–Caccioppoli Type on a Class of Generalized Metric Spaces, Publ. Math. Debr. 57 (2000), 31–37. https://doi.org/10.5486/PMD.2000.2133.

[9] S. Czerwik, Contraction Mappings in b-Metric Spaces, Acta Math. Univ. Ostrav. 1 (1993), 5–11. https://dml.cz/handle/10338.dmlcz/120469.

[10] B.C. Dhage, Generalized Metric Spaces Mappings with Fixed Point, Bull. Calcutta Math. Soc. 84 (1992), 329–336.

[11] M. Elbes, T. Kanan, M. Alia, M. Ziad, COVID-19 Detection Platform from X-Ray Images Using Deep Learning, Int. J. Adv. Soft Comput. Appl. 14 (2022), 197–211. https://doi.org/10.15849/IJASCA.220328.13.

[12] M. Fréchet, Sur Quelques Points du Calcul Fonctionnel, Rend. Circ. Mat. Palermo 22 (1906), 1–72.

[13] A. Gangwar, S. Rawat, R.C. Dimri, Solution of Differential Inclusion Problem in Controlled S-Metric Spaces via New Multivalued Fixed Point Theorem, J. Anal. 31 (2023), 2459–2472. https://doi.org/10.1007/s41478-023-00574-7.

[14] D. Abu Judeh, Applications of Conformable Fractional Pareto Probability Distribution, Int. J. Adv. Soft Comput. Appl. 14 (2022), 116–124. https://doi.org/10.15849/IJASCA.220720.08.

[15] T. Kamran, M. Samreen, Q. UL Ain, A Generalization of b-Metric Space and Some Fixed Point Theorems, Mathematics 5 (2017), 19. https://doi.org/10.3390/math5020019.

[16] S.M. Kang, B. Singh, V. Gupta, S. Kumar, Contraction Principle in Complex Valued G-Metric Spaces, Int. J. Math. Anal. 7 (2013), 2549–2556. https://doi.org/10.12988/ijma.2013.38203.

[17] T. Kanan, M. Elbes, K. Abu Maria, M. Alia, Exploring the Potential of IoT-Based Learning Environments in Education, Int. J. Adv. Soft Comput. Appl. 15 (2023), 166–178.

[18] W. Ahmad Khan, H. Qawaqneh, H. Aydi, Bivariate Kind of Generalized Laguerre-Based Appell Polynomials with Applications to Special Polynomials, Eur. J. Pure Appl. Math. 18 (2025), 6658. https://doi.org/10.29020/nybg.ejpam.v18i3.6658.

[19] S.G. Matthews, Partial Metric Topology, Ann. N.Y. Acad. Sci. 728 (1994), 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x.

[20] N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled Metric Type Spaces and the Related Contraction Principle, Mathematics 6 (2018), 194. https://doi.org/10.3390/math6100194.

[21] N.M. Mlaiki, Common Fixed Points in Complex S-Metric Space, Adv. Fixed Point Theory 4 (2014), 509–524.

[22] M. Nazam, H. Aydi, M.S. Noorani, H. Qawaqneh, Existence of Fixed Points of Four Maps for a New Generalized F-Contraction and an Application, J. Funct. Spaces 2019 (2019), 5980312. https://doi.org/10.1155/2019/5980312.

[23] E. Ozgur, Complex Valued G_b-Metric Space, J. Comput. Anal. Appl. 21 (2016), 363–368.

[24] H. Qawaqneh, H. Aydi, Fixed Points of Contraction Mappings Involving a Simulation Function and Applications, J. Math. Anal. 16 (2025), 1–13. https://doi.org/10.54379/jma-2025-4-1.

[25] H. Qawaqneh, Fractional Analytic Solutions and Fixed Point Results With Some Applications, Adv. Fixed Point Theory 14 (2024), 1. https://doi.org/10.28919/afpt/8279.

[26] H. Qawaqneh, Y. Alrashedi, H. Ahmad, A. Bekir, Discovery of Exact Solitons to the Fractional Kp-Mew Equation with Stability Analysis, Eur. Phys. J. Plus 140 (2025), 316. https://doi.org/10.1140/epjp/s13360-025-06188-1.

[27] H. Qawaqneh, K.H. Hakami, A. Altalbe, M. Bayram, The Discovery of Truncated M-Fractional Exact Solitons and a Qualitative Analysis of the Generalized Bretherton Model, Mathematics 12 (2024), 2772. https://doi.org/10.3390/math12172772.

[28] H. Qawaqneh, A. Altalbe, A. Bekir, K.U. Tariq, Investigation of Soliton Solutions to the Truncated M-Fractional (3+1)-Dimensional Gross-Pitaevskii Equation with Periodic Potential, AIMS Math. 9 (2024), 23410–23433. https://doi.org/10.3934/math.20241138.

[29] H. Qawaqneh, M.S. Noorani, H. Aydi, W. Shatanawi, On Common Fixed Point Results for New Contractions with Applications to Graph and Integral Equations, Mathematics 7 (2019), 1082. https://doi.org/10.3390/math7111082.

[30] H. Qawaqneh, H.A. Jari, A. Altalbe, A. Bekir, Stability Analysis, Modulation Instability, and the Analytical Wave Solitons to the Fractional Boussinesq-Burgers System, Phys. Scr. 99 (2024), 125235. https://doi.org/10.1088/1402-4896/ad8e07.

[31] H. Qawaqneh, J. Manafian, A.S. Alsubaie, H. Ahmad, Investigation of Exact Solitons to the Quartic Rosenau-Kawahara-Regularized-Long-Wave Fluid Model with Fractional Derivative and Qualitative Analysis, Phys. Scr. 100 (2024), 015270. https://doi.org/10.1088/1402-4896/ad9d92.

[32] H. Qawaqneh, New Functions For Fixed Point Results in Metric Spaces with Some Applications, Indian J. Math. 66 (2024), 55–84.

[33] H. Qawaqneh, M. Noorani, W. Shatanawi, H. Alsamir, Common Fixed Point Theorems for Generalized Geraghty (α, ψ, ϕ)-Quasi Contraction Type Mapping in Partially Ordered Metric-Like Spaces, Axioms 7 (2018), 74. https://doi.org/10.3390/axioms7040074.

[34] M.M. Rezaee, S. Sedghi, A. Muckheimer, K. Abodayeh, Z.D. Mitrović, Some Fixed Point Results in Partial S-Metric Spaces, Aust. J. Math. Anal. Appl. 16 (2019), 1–19.

[35] Y. Rohen, T. Dosenovic, S. Radenovic, A Note on the Paper "A Fixed Point Theorems in Sb-Metric Spaces", Filomat 31 (2017), 3335–3346. https://doi.org/10.2298/FIL1711335R.

[36] S. Sedghi, N. Shobe, A. Aliouche, A Generalization of Fixed Point Theorem in S-Metric Spaces, Mat. Vesn. 64 (2012), 258–266.

[37] S. Sedghi, N. Shobe, H. Zhou, A Common Fixed Point Theorem in D^*-Metric Spaces, Fixed Point Theory Appl. 2007 (2007), 027906. https://doi.org/10.1155/2007/27906.

[38] S. Shukla, Partial Rectangular Metric Spaces and Fixed Point Theorems, Sci. World J. 2014 (2014), 756298. https://doi.org/10.1155/2014/756298.

[39] N. Souayaha, N. Mlaikib, A Fixed Point Theorem in Sb-Metric Spaces, J. Math. Comput. Sci. 16 (2016), 131–139. https://doi.org/10.22436/jmcs.016.02.01.

[40] T.V. An, N.V. Dung, Z. Kadelburg, S. Radenović, Various Generalizations of Metric Spaces and Fixed Point Theorems, RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 109 (2014), 175–198. https://doi.org/10.1007/s13398-014-0173-7.