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Chromatic Invariants and Complete Subgraph Structure in Commuting Graphs of Block-Diagonal Matrix Rings

Eman Almuhur[1], **Manal Al-Labadi**[2,*], **Wasim Audeh**[2], **Mohammad Esmael Samie**[3]

[1]*Applied Science Private University, Amman, Jordan*

[2]*Dep. of Mathematics, Faculty of Arts and Sciences, University of Petra, Amman, Jordan*

[3]*Dep. of Mathematics, Bu-Ali Sina University, Hamedan, Iran*

[*]*Corresponding author:* manal.allabadi@uop.edu.jo

Abstract. We investigate chromatic invariants within commuting graphs associated with block-diagonal matrix rings over finite commutative rings. For a finite commutative ring R with identity, we analyze the commuting graph $\Gamma(M(m \oplus m, R))$ whose vertex set consists of non-central block-diagonal matrices $A \oplus B$ with $A, B \in M(m, R)$, where edges represent commutativity relations. Our main contributions establish quantitative bounds for both chromatic as functions of the base ring cardinality. We prove that the commuting graph $\Gamma(M(m \oplus m, R))$ contains $|R|^{2m^2} - |R|^2$ vertices and derive the lower bound $\omega(\Gamma(M(m \oplus m, R))) \geq |R|^{2m} - |R|^2$ by constructing explicit maximal cliques from diagonal matrices. For rings of the form $\mathbb{Z}_{p^r}$, we finding the lower bound for the chromating number if R is a finite commutative ring with unity, then $\chi(\gamma(\Gamma(M(m \oplus m, R))) \geq 3$ also, investigate the chromating number through algebraic properties involving centralizers and construct novel families of maximal cliques using nilpotent elements in $\mathbb{Z}_{p^r}$. Our results demonstrate unbounded growth of chromatic numbers with increasing ring cardinality and illuminate deep connections between the algebraic structure of block-diagonal matrix rings and combinatorial properties of their associated commuting graphs. These findings extend classical results on commuting graphs of matrix rings to the block-diagonal setting and provide tools for analyzing commutativity patterns in structured matrix algebras.

1. Introduction

The interplay between algebraic structures and combinatorial graph theory has emerged as a vibrant area of contemporary mathematical research. By encoding algebraic objects as graphs, mathematicians have developed powerful techniques for investigating both structural and combinatorial properties of rings, groups, and related systems. The commuting graph construction exemplifies this fruitful synthesis, providing graph-theoretic insights into commutativity patterns within non-commutative algebraic structures.

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For a non-commutative ring R, the *commuting graph* $\Gamma(R)$ is defined by taking non-central elements as vertices, with adjacency determined by multiplicative commutativity. This graph-theoretic representation encodes fundamental information about the ring's commutativity structure and has been extensively studied across various ring families over the past two decades. Foundational work in this area includes the seminal contributions of Abdollahi, Akbari, and Maimani [1] on non-commuting graphs of groups, and Beck's pioneering work [26] on coloring of commutative rings.

Matrix rings over commutative base rings provide particularly rich territory for commuting graph analysis. Pioneering investigations by Akbari and colleagues [2–4] established foundational results concerning connectivity, diameter bounds, and structural characterizations. The study of simultaneous similarity of matrices by Friedland [28] and Gerstenhaber's work [29] on varieties of commuting matrices laid important groundwork for understanding the algebraic structure underlying these graphs. More recently, attention has shifted toward specialized classes of matrix rings, with block-diagonal structures attracting particular interest due to their theoretical significance and computational tractability, as evidenced by work on commuting graphs of linear groups and matrix rings [30].

We investigated some properties in special idealization rings [17] and zero-divisor graphs of idealization rings [13] and investigated additional graph invariants, particularly those governing vertex colorings and complete subgraph structures see [6]- [26].

Two classical graph invariants play central roles in this investigation. The *chromatic number* $\chi(G)$ quantifies the minimum number of colors required for proper vertex coloring—that is, coloring vertices so that adjacent vertices receive distinct colors. The *clique number* $\omega(G)$ measures the maximum size of a complete subgraph (clique). Both parameters are fundamental throughout graph theory [27, 32] and possess important applications in optimization, computer science, and discrete mathematics. Within the context of commuting graphs, these invariants encode significant information about collective commutativity behavior among ring elements, extending the classical theory of non-commutative rings [31].

This paper focuses on chromatic properties and clique configurations within commuting graphs $\Gamma(M(r_1 \oplus r_2, R))$ associated with block-diagonal matrix rings. Our investigation addresses several natural questions:

- How does the chromatic number grow as a function of the base ring cardinality?
- What are the maximum sizes of complete subgraphs in these commuting graphs?
- Can we characterize maximal cliques through algebraic properties such as centralizers?
- What relationships exist between chromatic and clique numbers in this context?

Our principal contributions include:

(1) **Chromatic lower bounds:** We derive quantitative upper bounds relating chromatic numbers to base ring cardinality, demonstrating that $\chi(\Gamma(M(2 \oplus 2, R)))$ grows asymptotically as $|R|^8 - |R|^2$ (Proposition 2.1).

(2) **Exact clique numbers:** For finite fields $\mathbb{F}_q$, we compute precise lower bounds on the clique number $\omega(\Gamma(M(2 \oplus 2, \mathbb{F}_q)))$ (Theorem 3.1).
(3) **Structural characterizations:** We establish that maximal cliques correspond to specific classes of diagonal and near-diagonal matrices, providing explicit constructions via centralizer structures (Lemmas 3.1, 3.2, 3.3).
(4) **Enhanced bounds for $\mathbb{Z}_{p^r}$:** We prove improved lower bounds for chromatic and clique numbers when the base ring is $\mathbb{Z}_{p^r}$, with distinct formulas for odd and even values of r (Corollaries 3.1–3.4).
(5) **Unbounded growth:** We demonstrate that both chromatic and clique numbers grow without bound as ring cardinality increases, establishing the non-existence of uniform upper bounds.

These results illuminate how algebraic structure in block-diagonal matrix rings manifests through combinatorial properties of associated commuting graphs. The techniques employed combine ring-theoretic analysis of centralizers and commutative subrings with graph-theoretic methods for bounding chromatic and clique parameters. Our work complements recent investigations on related graph structures, including our studies on geodetic and domination numbers, circulant graphs, and idealization graphs.

Organization. The remainder of this paper proceeds as follows. Section 2 establishes preliminary concepts, notation, and foundational results regarding commuting graphs and matrix structures. Section 3 develops our main results on clique structures, including the construction of maximal cliques and lower bounds on clique and chromatic numbers. The paper concludes with a summary of findings and directions for future research.

Notation and Conventions. Throughout this paper, R denotes a finite commutative ring with multiplicative identity. We use $M(m, R)$ to denote the ring of $m \times m$ matrices with entries from R, and $Z(S)$ represents the center of any ring S. For positive integers r_1, r_2, the notation $M(r_1 \oplus r_2, R)$ denotes the collection of block-diagonal matrices with blocks of sizes $r_1 \times r_1$ and $r_2 \times r_2$. Specifically, elements have the form $\begin{pmatrix} T & 0 \\ 0 & N \end{pmatrix}$ where $T \in M(r_1, R)$ and $N \in M(r_2, R)$. For a matrix ring, $E_{i,j}$ denotes the elementary matrix with 1 in position (i, j) and 0 elsewhere. Graph-theoretic terminology follows standard conventions [27, 32].

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2. Foundational Concepts

We employ $M(r, R)$ to denote the collection of $r \times r$ matrices with entries from R, while $Z(R)$ represents the center of any ring R.

Definition 2.1. *Given positive integers r_1, r_2 and matrices $T \in M(r_1, R)$, $N \in M(r_2, R)$, the block-diagonal sum operation produces*

$$T \oplus N = \begin{pmatrix} T & 0 \\ 0 & N \end{pmatrix} \in M(r_1 + r_2, R).$$

The notation $M(r_1 \oplus r_2, R)$ denotes the collection of all such block-diagonal matrices.

Definition 2.2. *The graph $\Gamma(M(r_1 \oplus r_2, R))$ has vertex set consisting of non-central elements*

$$V(\Gamma) = M(r_1 \oplus r_2, R) \setminus Z(M(r_1 \oplus r_2, R)),$$

where distinct vertices A, B share an edge precisely when $AB = BA$.

Remark 2.1. *Within $M(r_1 \oplus r_2, R)$, the center comprises exclusively scalar matrices cI with $c \in R$ and I representing the identity. Consequently, non-central elements exhibit genuine block structure or possess non-uniform diagonal entries.*

Definition 2.3. *[32] The **clique number** of a graph G, denoted $\omega(G)$, is the size of the largest clique in G:*

$$\omega(G) = \max\{|S| : S \text{ is a clique in } G\}.$$

We recall standard graph-theoretic terminology.

Definition 2.4. *[32] For an arbitrary graph G:*

(1) *The chromatic number $\chi(G)$ represents the minimum color count enabling proper vertex coloring.*
(2) *A clique within G constitutes a complete induced subgraph.*
(3) *The clique number $\omega(G)$ measures maximum clique cardinality.*
(4) *The independence number $\alpha(G)$ quantifies maximum independent set size.*

Classical relationships among these parameters prove valuable.

Lemma 2.1. *[32] Any graph G with n vertices satisfies:*

(1) $\chi(G) \geq \omega(G)$,
(2) $\chi(G) \cdot \alpha(G) \geq n$.
(3) $max\{\gamma(G), \chi(G)\} \leq \gamma(G) + \chi(G) - 1$.

Lemma 2.2. *[5] Let R be a finite commutative ring with unity. Then the domination number is*

$$\gamma(\Gamma(M(m \oplus m, R))) \geq 2. \tag{2.1}$$

Theorem 2.1. *Let R be a finite commutative ring with unity. Then the chromating number is $\chi(\gamma(\Gamma(M(m \oplus m, R))) \geq 3$.*

Proof. By Lemma 2.1 and inequality (2.1), then we give the result. □

Lemma 2.3. *The center of $M(m \oplus m, R)$ is $Z(M(m \oplus m, R)) = \{cI_m \oplus fI_m : c, f \in R\}$, and $|Z(M(m \oplus m, R))| = |R|^2$.*

Proof. Since R is commutative, any scalar matrix $cI_m \oplus fI_m$ commutes with all matrices in $M(m \oplus m, R)$. Conversely, if $A \oplus B = (a_{ij})_{i,j=1}^{m} \oplus (b_{i,j})_{i,j=1}^{m} \in Z(M(m \oplus m, R))$, then $A \oplus B$ commutes with all elementary matrices $E_{kl} \oplus E_{kl}$. The condition $(A \oplus B)(E_{12} \oplus E_{12}) = (E_{12} \oplus E_{12})(A \oplus B)$ implies $a_{ij} = 0, b_{ij} = 0$ for $i \neq j$ and $a_{11} = a_{22}, b_{11} = b_{22}$. Similarly, $(A \oplus B)(E_{23} \oplus E_{23}) = (E_{23} \oplus E_{23})(A \oplus B)$ implies $a_{22} = a_{33}, b_{22} = b_{33}$. Continuing this process shows $A \oplus B = a_{11}I_m \oplus b_{11}I_m$ for some $a_{11}, b_{11} \in R$. □

Proposition 2.1. *The commuting graph $\Gamma(M(m \oplus m, R))$ has $|R|^{2m^2} - |R|^2$ vertices.*

Proof. This follows immediately from $|M(m \oplus m, R)| = |R|^{2m^2}$ and Lemma 2.3. □

3. Structure of Maximal Cliques

We now establish our principal results on the clique number of matrix commuting graphs.

Lemma 3.1. *Let $D(m \oplus m, R) = \{diag(d_1, d_2, \ldots, d_m) \oplus diag(x_1, x_2, \ldots, x_m) : d_i, x_i \in R\} \setminus Z(M(m \oplus m, R))$ be the set of non-central diagonal matrices. Then $D(m \oplus m, R)$ is a maximal cliques in $\Gamma(M(m \oplus m, R))$.*

Proof. Any two diagonal matrices commute since

$$\begin{aligned}
& (diag(a_1, \ldots, a_m) \oplus diag(x_1, x_2, \ldots, x_m))(diag(b_1, \ldots, b_m) \oplus diag(y_1, y_2, \ldots, y_m)) \\
= \; & (diag(a_1b_1, \ldots, a_mb_m) \oplus diag(x_1y_1, \ldots, x_my_m)) \\
= \; & (diag(b_1a_1, \ldots, b_ma_m) \oplus diag(y_1x_1, \ldots, y_mx_m)) \\
& \text{(since } a_i, b_i, x_i, y_i \in R, R \text{ is a commutative ring)} \\
= \; & (diag(b_1, \ldots, b_m) \oplus diag(y_1, \ldots, y_m))(diag(a_1, \ldots, a_m) \oplus diag(x_1, \ldots, x_m)).
\end{aligned}$$

Any matrix commute with all diagonal matrices must be a diagonal matrix, therefore $D(m \oplus m, R) = \{diag(d_1, d_2, \ldots, d_m) \oplus diag(x_1, x_2, \ldots, x_m) : d_i, x_i \in R\} \setminus Z(M(m \oplus m, R))$ is a maximal cliques in $\Gamma(M(m \oplus m, R))$. □

Theorem 3.1. *For any finite commutative ring R with unity and positive integer $m \geq 3$,*

$$\omega(\Gamma(M(m, R))) \geq |R|^{2m} - |R|^2. \tag{3.1}$$

Proof. By Lemma 3.1, the diagonal matrices form a clique. The number of diagonal matrices is $|R|^{2m}$, and the number of scalar matrices (which are central) is $|R|$. Therefore, $|D(m \oplus m, R)| = |R|^{2m} - |R|^2$, giving the desired lower bound. □

Now, we give other examples of maximal cliques in $\Gamma(M(m \oplus m, Z_{p^r}))$ when $r \geq 3$ is odd. We start with the following remark.

Remark 3.1. *Any matrix* $A \oplus B = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}$ *commutes with the matrix*

$$(A + c_1 I) \oplus (B + c_2 I) = \begin{pmatrix} x_{1,1} + c_1 & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} + c_1 & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} + c_1 \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} + c_2 & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} + c_2 & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} + c_2 \end{pmatrix}.$$

Let $S_1 = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r} \text{ and } t \geq \frac{r+1}{2}\}$ and $O_1 = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r} \text{ and } t \geq \frac{t-1}{2}\}$. Let L be the set of all matrices $A \oplus B$ of the form

$$\begin{pmatrix} x_{1,1} + c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2} + c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2} & \cdots & x_{m-1,m-1} + c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} + c_2 & y_{1,2} & \cdots & y_{1,m-1} & y_{1,m} \\ y_{2,1} & y_{2,2} + c_2 & \cdots & y_{2,m-1} & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{m-1,1} & y_{m-1,2} & \cdots & y_{m-1,m-1} + c_2 & y_{m-1,m} \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m-1} & c_2 \end{pmatrix},$$

such that $x_{1,1}, \ldots, x_{m-1,m-1}, y_{1,1}, \ldots, y_{m-1,m-1} \in O_1$, $c_1, c_2 \in Z_{p^r}$, and $x_{i,j}, y_{i,j} \in S_1$, $i \neq j$.

Lemma 3.2. *Suppose that r is an odd number, L and S_1 are defined as above. Then L induces a maximal clique in $\Gamma(M(m \oplus m, Z_{p^r}))$.*

Proof. Let X and Y be any two matrices in the set L. Then

$$X = \begin{pmatrix} a_{1,1} + c_1 & a_{1,2} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{2,1} & a_{2,2} + c_1 & \cdots & a_{2,m-1} & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & \vdots & \cdots & a_{m-1,m-1} + c_1 & a_{m-1,m} \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m-1} & c_1 \end{pmatrix} \oplus \begin{pmatrix} f_{1,1} + c_2 & f_{1,2} & \cdots & f_{1,m-1} & f_{1,m} \\ f_{2,1} & f_{2,2} + c_2 & \cdots & f_{2,m-1} & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m-1,1} & \vdots & \cdots & f_{m-1,m-1} + c_2 & f_{m-1,m} \\ f_{m,1} & f_{m,2} & \cdots & f_{m,m-1} & c_2 \end{pmatrix},$$

$a_{i,j}, f_{i,j} \in O_1$, $a_{i,i}, f_{i,i} \in S_1$, $i \neq j$, $c_1, c_2 \in Z_{p^r}$. One can write $X = (J + (c_1 I \oplus c_2 I) + G)$, where

$$G = \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & 0 & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & f_{1,2} & \cdots & f_{1,m-1} & f_{1,m} \\ f_{2,1} & 0 & \cdots & f_{2,m-1} & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m,1} & f_{m,2} & \cdots & f_{m,m-1} & 0 \end{pmatrix}$$

is a matrix with all elements in S_1, and the matrix

$$J = \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix} \oplus \begin{pmatrix} f_{1,1} & 0 & \cdots & 0 \\ 0 & f_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{m,m} \end{pmatrix}$$

is a diagonal matrix where the main diagonal consists of elements that belong to O_1. Similarly

$$Y=\begin{pmatrix} b_{1,1}+c_3 & b_{1,2} & \cdots & b_{1,m-1} & b_{1,m}\\ b_{2,1} & b_{2,2}+c_3 & \cdots & b_{2,m-1} & b_{2,m}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ b_{m-1,1} & b_{m-1,2} & \cdots & b_{m-1,m-1}+c_2 & b_{m-1,m}\\ b_{m,1} & b_{m,2} & \cdots & b_{m,m-1} & c_3 \end{pmatrix}\oplus\begin{pmatrix} l_{1,1}+c_4 & l_{1,2} & \cdots & l_{1,m-1} & l_{1,m}\\ l_{2,1} & l_{2,2}+c_4 & \cdots & l_{2,m-1} & l_{2,m}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ l_{m-1,1} & l_{m-1,2} & \cdots & l_{m-1,m-1}+c_4 & l_{m-1,m}\\ l_{m,1} & l_{m,2} & \cdots & l_{m,m-1} & c_4 \end{pmatrix}$$

and $Y=(J^*+(c_3I\oplus c_4I)+G^*)$, where

$$G^*=\begin{pmatrix} 0 & b_{1,2} & \cdots & b_{1,m}\\ b_{2,1} & 0 & \cdots & b_{2,m}\\ \vdots & \vdots & \ddots & \vdots\\ b_{m,1} & b_{m,2} & \cdots & 0 \end{pmatrix}\oplus\begin{pmatrix} 0 & l_{1,2} & \cdots & l_{1,m}\\ l_{2,1} & 0 & \cdots & l_{2,m}\\ \vdots & \vdots & \ddots & \vdots\\ l_{m,1} & l_{m,2} & \cdots & 0 \end{pmatrix}$$

is a matrix with all elements in S_1

$$J^*=\begin{pmatrix} b_{1,1} & 0 & \cdots & 0\\ 0 & b_{2,2} & \ddots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & b_{m,m} \end{pmatrix}\oplus\begin{pmatrix} l_{1,1} & 0 & \cdots & 0\\ 0 & l_{2,2} & \ddots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & l_{m,m} \end{pmatrix}$$

is a diagonal matrix where the main diagonal consists elements belonging to O_1. Then $XY=(J+(c_1I\oplus c_2I)+G)(J^*+(c_3I\oplus c_4I)+G^*)=JJ^*+c_2J+0+c_1J^*+c_1c_2I+c_1G^*+0+c_2G+0=J^*J+c_1J^*+0+c_2J+c_2c_1I+c_2G+0+c_1G^*+0=(J^*+c_2I+G^*)(J+c_1I+G)=YX$. Hence X and Y commute and the induced subgraph on L is complete.

Let $A\oplus B=\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m}\\ x_{2,1} & x_{2,2} & \cdots & x_{2,m}\\ \vdots & \vdots & \ddots & \vdots\\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix}\oplus\begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m}\\ y_{2,1} & y_{2,2} & \cdots & y_{2,m}\\ \vdots & \vdots & \ddots & \vdots\\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}$; $x_{i,j},y_{i,j}\in Z_{p^r}$ be any matrix that commutes with all elements in L. The matrix $A\oplus B$ commute with $p^{\frac{r-1}{2}}E_{1,1}\oplus p^{\frac{r-1}{2}}E_{1,1}$ and hence $(Ap^{\frac{r-1}{2}}E_{1,1})\oplus(Bp^{\frac{r-1}{2}}E_{1,1})=(p^{\frac{r-1}{2}}E_{1,1}A)\oplus(p^{\frac{r-1}{2}}E_{1,1}B)$. From that we get $x_{1,2}p^{\frac{r-1}{2}}=0=\cdots=x_{1,m}p^{\frac{r-1}{2}}=0$, $x_{2,1}p^{\frac{r-1}{2}}=0=\cdots=x_{m,1}p^{\frac{r-1}{2}}=0$ and $y_{1,2}p^{\frac{r-1}{2}}=0=\cdots=y_{1,m}p^{\frac{r-1}{2}}=0$, $y_{2,1}p^{\frac{r-1}{2}}=0=\cdots=y_{m,1}p^{\frac{r-1}{2}}=0$. Similarly $A\oplus B$ commutes with all the matrices $p^{\frac{r-1}{2}}E_{2,2}\oplus p^{\frac{r-1}{2}}E_{2,2}\cdots p^{\frac{r-1}{2}}E_{m,m}\oplus p^{\frac{r-1}{2}}E_{m,m}$. From that we get $x_{1,m}p^{\frac{r-1}{2}}=0=\cdots=x_{m-1,m}p^{\frac{r-1}{2}}$, $x_{m,1}p^{\frac{r-1}{2}}=0=\cdots=x_{m,m-1}p^{\frac{r-1}{2}}$. So, $x_{l,j}=(c_{l,j}+id_{l,j})p^e$, $p^e\in\{0,p^{\frac{r+1}{2}},\ldots,p^{r-1}\}$, $l\neq j$ $x_{1,m}p^{\frac{r-1}{2}}=0=\cdots=x_{m-1,m}p^{\frac{r-1}{2}}$, and $y_{m,1}p^{\frac{r-1}{2}}=0=\cdots=y_{m,m-1}p^{\frac{r-1}{2}}$. So, $y_{l,j}=(c_{l,j}+id_{l,j})p^e$, $p^e\in\{0,p^{\frac{r+1}{2}},\ldots,p^{r-1}\}$, $l\neq j$. So

$$A\oplus B=\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m}\\ x_{2,1} & x_{2,2} & \cdots & x_{2,m}\\ \vdots & \vdots & \ddots & \vdots\\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix}\oplus\begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m}\\ y_{2,1} & y_{2,2} & \cdots & y_{2,m}\\ \vdots & \vdots & \ddots & \vdots\\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix};$$

where $x_{1,1}, x_{1,2}, \dots, x_{m,m}, y_{1,1}, y_{1,2}, \dots, y_{m,m} \in Z_{p^r}$, $x_{i,j}, y_{i,j} \in S_1$ $\forall i \neq j$. Now, we need to show that $A \oplus B \in L$. Since $A \oplus B$ commutes with $p^{\frac{r+1}{2}} E_{1,2} \oplus p^{\frac{r+1}{2}} E_{1,2}, \dots, p^{\frac{r+1}{2}} E_{m-1,m} \oplus p^{\frac{r+1}{2}} E_{m-1,m}$, we have $(Ap^{\frac{r+1}{2}} E_{1,2}) \oplus (Bp^{\frac{r+1}{2}} E_{1,2}) = (p^{\frac{r+1}{2}} E_{1,2}A) \oplus (p^{\frac{r+1}{2}} E_{1,2}B), \dots, (Ap^{\frac{r+1}{2}} E_{m-1,m}) \oplus (Bp^{\frac{r+1}{2}} E_{m-1,m}) = (p^{\frac{r+1}{2}} E_{m-1,m}A) \oplus (p^{\frac{r+1}{2}} E_{m-1,m}B)$. So,

$$(Ap^{\frac{r+1}{2}} E_{1,2}) \oplus (Bp^{\frac{r+1}{2}} E_{1,2}) = \begin{pmatrix} 0 & x_{1,1}p^{\frac{r+1}{2}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & y_{1,1}p^{\frac{r+1}{2}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= (p^{\frac{r+1}{2}} E_{1,2}A) \oplus (p^{\frac{r+1}{2}} E_{1,2}B) = \begin{pmatrix} 0 & x_{2,2}p^{\frac{r+1}{2}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & y_{2,2}p^{\frac{r+1}{2}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

So, we get $x_{1,1}p^{\frac{r+1}{2}} = x_{2,2}p^{\frac{r+1}{2}}$ and hence $(x_{1,1} - x_{2,2})p^{\frac{r+1}{2}} = 0$, $y_{1,1}p^{\frac{r+1}{2}} = y_{2,2}p^{\frac{r+1}{2}}$ and hence $(y_{1,1} - y_{2,2})p^{\frac{r+1}{2}} = 0$. From that we get $y_{1,1} = y_{2,2} + v_1 p^{\frac{r-1}{2}}$; $v_1 \in Z_{p^r}$. Similarly since

$$(Ap^{\frac{r+1}{2}} E_{2,3}) \oplus (Bp^{\frac{r+1}{2}} E_{2,3}) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_{2,2}p^{\frac{r+1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & y_{2,2}p^{\frac{r+1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \end{pmatrix}$$

$$= (p^{\frac{r+1}{2}} E_{2,3}A) \oplus (p^{\frac{r+1}{2}} E_{2,3}B) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_{3,3}p^{\frac{r+1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & y_{3,3}p^{\frac{r+1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \end{pmatrix},$$

we get $x_{2,2} = x_{3,3} + u_2 p^{\frac{r-1}{2}}$; $u_2 \in Z_{p^r}$. So $x_{1,1} = x_{2,2} + u_1 p^{\frac{r-1}{2}}$ then $x_{1,1} = x_{3,3} + u_1^{(1)} p^{\frac{r-1}{2}}$, and $y_{2,2} = y_{3,3} + v_2 p^{\frac{r-1}{2}}$; $v_2 \in Z_{p^r}$. So $y_{1,1} = y_{2,2} + v_1 p^{\frac{r-1}{2}}$ then $y_{1,1} = y_{3,3} + v_1^{(1)} p^{\frac{r-1}{2}}$.

By the same technique we get

$$(Ap^{\frac{r+1}{2}} E_{m-1,m}) \oplus (Bp^{\frac{r+1}{2}} E_{m-1,m}) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & p^{\frac{r+1}{2}} x_{m-1,m-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & p^{\frac{r+1}{2}} y_{m-1,m-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= (p^{\frac{r+1}{2}} E_{m-1,m}A) \oplus (p^{\frac{r+1}{2}} E_{m-1,m}B) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & p^{\frac{r+1}{2}} x_{m,m} \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & p^{\frac{r+1}{2}} y_{m,m} \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

then $x_{m-1,m-1} = x_{m,m} + u_{m-1}p^{\frac{r-1}{2}}$; $u_{m-1} \in Z_{p^r}$. So $x_{1,1} = x_{m,m} + u_{m-1}^{(m)}p^{\frac{r-1}{2}}$; $u_{m-1}^{(m)} \in Z_{p^r}$ and $y_{m-1,m-1} = y_{m,m} + v_{m-1}p^{\frac{r-1}{2}}$; $v_{m-1} \in Z_{p^r}$. So $y_{1,1} = y_{m,m} + v_{m-1}^{(m)}p^{\frac{r-1}{2}}$; $v_{m-1}^{(m)} \in Z_{p^r}$.

Then

$$A \oplus B = \begin{pmatrix} d_{1,1} + x_{m,m} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & d_{2,2} + x_{m,m} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} f_{1,1} + y_{m,m} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & f_{2,2} + y_{m,m} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix};$$

$d_{1,1}, d_{1,2}, \ldots, d_{m-1,m-1}, f_{1,1}, f_{1,2}, \ldots, f_{m-1,m-1} \in O_1$, $x_{m,m} \in Z_{p^r}$ and $x_{i,j}, y_{i,j} \in S_1$ $\forall i \neq j$. Then $A \oplus B$ is a matrix in L. So, the set L is a maximal clique of $\Gamma(M(m \oplus m, Z_{p^r}))$. □

Observe that $|D(m \oplus m, Z_{p^r})| = (p^r)^{2m} - p^{2r}$, since D contains diagonal matrices except scalar matrices. We have

$$L = \left\{ \begin{pmatrix} x_{1,1}+c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2}+c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2}+c_1 & \cdots & x_{m-1,m-1}+c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix} \oplus \begin{pmatrix} y_{1,1}+c_2 & y_{1,2} & \cdots & y_{1,m-1} & y_{1,m} \\ y_{2,1} & y_{2,2}+c_2 & \cdots & y_{2,m-1} & y_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ y_{m-1,1} & y_{m-1,2}+c_2 & \cdots & y_{m-1,m-1}+c_2 & y_{m-1,m} \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m-1} & c_2 \end{pmatrix} : x_{i,i}, y_{i,i} \in O_1, x_{i,j}, y_{i,j} \in S_1; i \neq j, c_1, c_2 \in Z_{p^r} \right\}.$$

Where $S_1 = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r} \text{ and } t \geq \frac{r+1}{2}\}$. And $O_1 = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r} \text{ and } t \geq \frac{r-1}{2}\}$. Observe that $|S_1| = p^{r-1}$, $|O_1| = p^{r+1}$. Hence

$$|L| = (p^{r-1})^{m^2-m}(p^{r+1})^{m-1}p^{2r} - p^{2r}. \tag{3.2}$$

Now, we give an example of a maximal clique of the graph $\Gamma(M(m \oplus m, Z_{p^r}))$ where r is an even integer.

Let $S = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r} \text{ and } t \geq \frac{r}{2}\}$. Let L be the set of all matrices $A \oplus B$ of the form

$$A \oplus B = \begin{pmatrix} x_{1,1}+c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2}+c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2}+c_1 & \cdots & x_{m-1,m-1}+c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix} \oplus \begin{pmatrix} y_{1,1}+c_2 & y_{1,2} & \cdots & y_{1,m-1} & y_{1,m} \\ y_{2,1} & y_{2,2}+c_2 & \cdots & y_{2,m-1} & y_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ y_{m-1,1} & y_{m-1,2}+c_2 & \cdots & y_{m-1,m-1}+c_2 & y_{m-1,m} \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m-1} & c_2 \end{pmatrix},$$

such that $x_{i,j}, y_{i,j} \in S$, for all $i, j, c_1, c_2 \in Z_{p^r}$.

Lemma 3.3. *Let r be an even number, L and S are defined as above. Then L is a maximal clique in $\Gamma(M(m \oplus m, Z_{p^r}))$.*

Proof. The proof is similar to that one of Lemma 3.2. □

Now, we look at the size of the set L.

We have

$$L = \left\{ \begin{pmatrix} x_{1,1}+c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2}+c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2}+c_1 & \cdots & x_{m-1,m-1}+c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix} \oplus \begin{pmatrix} y_{1,1}+c_2 & y_{1,2} & \cdots & y_{1,m-1} & y_{1,m} \\ y_{2,1} & y_{2,2}+c_2 & \cdots & y_{2,m-1} & y_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ y_{m-1,1} & y_{m-1,2}+c_2 & \cdots & y_{m-1,m-1}+c_2 & y_{m-1,m} \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m-1} & c_2 \end{pmatrix} : x_{i,i}, y_{i,i} \in O_1, x_{i,j}, y_{i,j} \in S_1; i \neq j, c_1, c_2 \in Z_{p^r} \right\}.$$

Where $S = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r}$ and $t \geq \frac{r}{2}\}$. Observe that $|S| = p^r$. Hence

$$|L| = (p^r)^{m^2-1} p^{2r} - p^{2r} \tag{3.3}$$

.

Corollary 3.1. *For any finite commutative ring R isomorphic to Z_{p^r} with $r \geq 3$ is an odd number,*

$$\omega(\Gamma(M(m,R))) \geq max\{(p^r)^{2m} - p^{2r}, (p^{r-1})^{m^2-m}(p^{r+1})^{m-1}p^{2r} - p^{2r}\}. \tag{3.4}$$

Proof. Using inequality (3.1) and equation (3.2), then we conclude the result. □

Corollary 3.2. *For any finite commutative ring R isomorphic to Z_{p^r} with $r \geq 2$ is an even number,*

$$\omega(\Gamma(M(m,R))) \geq max\{(p^r)^{2m} - p^{2r}, (p^r)^{m^2-1}p^{2r} - p^{2r}\}. \tag{3.5}$$

Proof. Using inequality (3.1) and equation (3.3), then we conclude the result. □

Corollary 3.3. *For any finite commutative ring R isomorphic to Z_{p^r} with $r \geq 3$ is an odd number,*

$$\chi(\Gamma(M(m,R))) \geq max\{(p^r)^{2m} - p^{2r}, (p^{r-1})^{m^2-m}(p^{r+1})^{m-1}p^{2r} - p^{2r}\}. \tag{3.6}$$

Proof. Using inequality (3.4), then we conclude the result. □

Corollary 3.4. *For any finite commutative ring R isomorphic to Z_{p^r} with $r \geq 2$ is an even number,*

$$\chi(\Gamma(M(m,R))) \geq max\{(p^r)^{2m} - p^{2r}, (p^r)^{m^2-1}p^{2r} - p^{2r}\}. \tag{3.7}$$

Proof. Using inequality (3.5), then we conclude the result. □

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