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On Quantum Montgomery and Ostrowski-Type Inequalities for Uniformly Convex Functions

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Abstract. In this paper, we employ the quantum Montgomery identity together with tools from q-calculus to derive new Ostrowski-type inequalities for uniformly convex functions. The resulting estimates involve a modulus function associated with uniform convexity. Several corollaries are obtained by considering particular choices of the parameters. In addition, the relationships between the proposed results and some existing results in the literature are briefly discussed. Numerical examples and graphical illustrations are included to support the theoretical findings.

1. Introduction

The study of integral inequalities has attracted sustained attention for more than a century. Among the classical results in this area, the well-known inequality due to Ostrowski [1] plays a fundamental role, as it provides an explicit bound for the deviation of a function from its integral mean.

Theorem 1.1. *Let $\Upsilon : I \to \mathbb{R}$ be a differentiable function on I°, such that $\Upsilon' \in L[\delta_1, \delta_2]$, where $\delta_1 < \delta_2$. If $|\Upsilon'(\varphi)| \le \eta$ for all $\varphi \in [\delta_1, \delta_2]$, then*

$$\left|\Upsilon(\varphi) - \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \Upsilon(\theta)\, d\theta\right| \le \frac{\eta\left((\varphi - \delta_1)^2 + (\delta_2 - \varphi)^2\right)}{2(\delta_2 - \delta_1)}. \tag{1.1}$$

Over the years, Ostrowski's inequality has been extended in many directions under different assumptions, including convexity, Lipschitz continuity, bounded variation, absolute continuity, and higher-order differentiability. These extensions have proved useful in various fields such as probability theory, numerical integration, and approximation theory (see, for example, [3–13]).

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A refined version of Ostrowski's inequality, obtained by Alomari et al. [2], is recalled below.

Theorem 1.2. *Let* $\Upsilon : I \to \mathbb{R}$ *be differentiable with* $\Upsilon' \in L[\delta_1, \delta_2]$, $\delta_1 < \delta_2$. *If* $|\Upsilon'|^{\frac{r}{r-1}}$, $r > 1$, *is convex on* $[\delta_1, \delta_2]$ *and* $|\Upsilon'| \le \eta$, *then*

$$\left|\Upsilon(\varphi) - \frac{1}{\delta_2 - \delta_1}\int_{\delta_1}^{\delta_2} \Upsilon(\theta)\, d\theta\right| \le \frac{\eta\Big((\varphi - \delta_1)^2 + (\delta_2 - \varphi)^2\Big)}{(\delta_2 - \delta_1)(r+1)^{1/r}}, \tag{1.2}$$

for each $\varphi \in [\delta_1, \delta_2]$.

It is well known that inequality (1.1) can be derived by means of the classical Montgomery identity (see [14]), which is given by

$$\Upsilon(\varphi) = \frac{1}{\delta_2 - \delta_1}\int_{\delta_1}^{\delta_2} \Upsilon(\theta)\, d\theta + \int_{\delta_1}^{\varphi} \frac{\theta - a}{\delta_2 - \delta_1}\Upsilon'(\theta)\, d\theta + \int_{\varphi}^{\delta_2} \frac{\theta - b}{\delta_2 - \delta_1}\Upsilon'(\theta)\, d\theta. \tag{1.3}$$

By applying an appropriate change of variables, the Montgomery identity can equivalently be written as

$$\Upsilon(\varphi) - \frac{1}{\delta_2 - \delta_1}\int_{\delta_1}^{\delta_2} \Upsilon(\theta)\, d\theta = (\delta_2 - \delta_1)\int_0^1 T(\theta)\, \Upsilon'\Big(\theta\delta_2 + (1-\theta)\delta_1\Big)\, d\theta, \tag{1.4}$$

where

$$T(\theta) = \begin{cases} \theta, & \theta \in \left[0, \dfrac{\varphi - \delta_1}{\delta_2 - \delta_1}\right], \\ \theta - 1, & \theta \in \left(\dfrac{\varphi - \delta_1}{\delta_2 - \delta_1}, 1\right]. \end{cases}$$

Motivated by developments in quantum calculus, Kunt et al. [15] introduced a quantum analogue of the Montgomery identity for ${}_aq$-differentiable functions, which can be stated as follows.

Lemma 1.1. *Let* $\Upsilon : [\delta_1, \delta_2] \to \mathbb{R}$ *be a function such that* ${}_{\delta_1}d_q f$ *is quantum integrable on* $[\delta_1, \delta_2]$. *Then*

$$\Upsilon(\varphi) - \frac{1}{\delta_2 - \delta_1}\int_{\delta_1}^{\delta_2} \Upsilon(\theta)\, {}_{\delta_1}d_q\theta = (\delta_2 - \delta_1)\int_0^1 T_q(\theta)\, {}_{\delta_1}d_q\Upsilon\Big(\theta\delta_2 + (1-\theta)\delta_1\Big)\, d_q\theta,$$

where

$$T_q(\theta) = \begin{cases} q\theta, & \theta \in \left[0, \dfrac{\varphi - \delta_1}{\delta_2 - \delta_1}\right], \\ q\theta - 1, & \theta \in \left(\dfrac{\varphi - \delta_1}{\delta_2 - \delta_1}, 1\right]. \end{cases}$$

Strong convexity, introduced by Polyak [16], strengthens the classical notion of convexity by imposing an additional quadratic term, as described by

$$\Upsilon(\alpha c + (1-\alpha)d) \le \alpha\Upsilon(c) + (1-\alpha)\Upsilon(d) - \xi\alpha(1-\alpha)(c-d)^2, \tag{1.5}$$

for all $\alpha \in [0,1]$, $c, d \in [\delta_1, \delta_2]$ and $\xi > 0$.

A more flexible framework is provided by the concept of uniform convexity, which is characterized by a modulus function Φ. This notion allows for finer control of convexity behavior and includes a wide class of functions as special cases [17].

Definition 1.1. *A function* $\Upsilon : [\delta_1, \delta_2] \to \mathbb{R}$ *is* uniformly convex *with modulus* $\Phi : [0, \infty) \to [0, \infty)$ *if* Φ *is increasing, vanishes only at* 0*, and*

$$\Upsilon(\alpha c + (1-\alpha)d) + \alpha(1-\alpha)\,\Phi(|c-d|) \leq \alpha\Upsilon(c) + (1-\alpha)\Upsilon(d),$$

for all $\alpha \in [0,1]$ *and* $c, d \in [\delta_1, \delta_2]$.

Quantum calculus, commonly referred to as q-calculus, is often described as a branch of calculus developed without the use of limits. Its origins can be traced back to the work of Euler (1707–1783), who investigated ideas related to Newton's infinite series. In the early twentieth century, Jackson [18, 19] made significant contributions by introducing the notions of the q-derivative and the q-integral for continuous functions defined on $(0, \infty)$. These concepts were formulated as quantum counterparts of classical calculus, with the classical case recovered as $q \to 1$. In recent years, q-calculus has attracted growing interest due to its wide range of applications in mathematics and physics (see [20–33]). A systematic account of its fundamental theory can be found in the monograph by Kac and Cheung [34].

In 2013, Tariboon and Ntouyas [35, 36] extended the framework of q-calculus to continuous functions defined on closed intervals and investigated several of its basic properties. Since then, considerable attention has been devoted to quantum integral inequalities. Numerous results have been established, including quantum analogues of the Hermite–Hadamard, Simpson, Ostrowski, Fejér, Newton, and Hahn-type inequalities, as well as various generalizations of the Hermite–Hadamard inequality (see [37–53]).

Within this context, the use of the quantum Montgomery identity in combination with uniformly convex functions has led to more refined forms of integral inequalities. Several quantum inequalities of Ostrowski type, midpoint type, and related variants have been derived, together with applications to the approximation of different types of mean values.

Motivated by these developments, the present paper aims to further contribute to the theory by establishing new quantum results based on Montgomery-type identities and corresponding Ostrowski-type inequalities under the assumption of uniform convexity. For clarity, the paper is organized as follows. The first part recalls the necessary background and preliminary concepts. The second part is devoted to the main theoretical results. The final part presents numerical examples and graphical illustrations that support and visualize the theoretical findings. The results and techniques developed in this work may be of interest to researchers in quantum calculus and inequality theory.

2. PRELIMINARIES

In this section, we present the fundamental concepts of q-calculus employed in this work. Throughout the paper, we assume that q is a fixed constant satisfying $0 < q < 1$. We introduce the following notation.

$$[\vartheta]_q := \frac{1-q^{\vartheta}}{1-q} = 1+q+q^2+\cdots+q^{\vartheta-1}, \quad \vartheta \in \mathbb{N},$$

which is the q-number of ϑ, see [34] for more details.

Definition 2.1. *Let* $\Upsilon : [\delta_1, \delta_2] \to \mathbb{R}$ *be a continuous function. The* q_{δ_1}*-derivative of* Υ *at a point* $\varphi \in [\delta_1, \delta_2]$ *is defined by*

$$_{\delta_1}D_q\Upsilon(\varphi) = \frac{\Upsilon(\varphi) - \Upsilon(q\varphi + (1-q)\delta_1)}{(1-q)(\varphi - \delta_1)}, \quad \varphi \neq \delta_1. \tag{2.1}$$

Since $\Upsilon : [\delta_1, \delta_2] \to \mathbb{R}$ is continuous, we may further define

$$_{\delta_1}D_q\Upsilon(\delta_1) = \lim_{\varphi \to \delta_1} {}_{\delta_1}D_q\Upsilon(\varphi).$$

The function Υ is said to be q_{δ_1}-differentiable on $[\delta_1, \delta_2]$ if $_{\delta_1}D_q\Upsilon(\varphi)$ exists for all $\varphi \in [\delta_1, \delta_2]$. In particular, by setting $\delta_1 = 0$ in (2.1), we obtain $_0D_q\Upsilon(\varphi) = D_q\Upsilon(\varphi)$, which reduces to

$$D_q\Upsilon(\varphi) = \frac{\Upsilon(\varphi) - \Upsilon(q\varphi)}{(1-q)\varphi}, \quad \varphi \neq 0,$$

known as the classical q-derivative. For further details, see [34,35].

Example 2.1. *Let* $\Upsilon : [\delta_1, \delta_2] \to \mathbb{R}$ *be defined by* $\Upsilon(\varphi) = \mu\varphi^2 + \nu$*, where* μ *and* ν *are constants and* $\varphi \in [\delta_1, \delta_2]$*. By applying Definition 2.1, we obtain*

$$\begin{aligned} {}_{\delta_1}D_q(\mu\varphi^2 + \nu) &= \frac{(\mu\varphi^2 + \nu) - \left[\mu(q\varphi + (1-q)\delta_1)^2 + \nu\right]}{(1-q)(\varphi - \delta_1)} \\ &= \frac{\mu\varphi^2(1+q) - 2\mu q\delta_1\varphi - \mu(\delta_1)^2(1-q)}{(\varphi - \delta_1)} \\ &= \mu(1+q)\varphi + \mu(1-q)\delta_1. \end{aligned}$$

Definition 2.2. *Let* $\Upsilon : [\delta_1, \delta_2] \to \mathbb{R}$ *be a continuous function. The* q_{δ_1}*-integral of* Υ *on* $[\delta_1, \delta_2]$ *is defined by*

$$\int_{\delta_1}^{\delta_2} \Upsilon(\varphi)\ {}_{\delta_1}d_q\varphi = (1-q)(\delta_2 - \delta_1)\sum_{n=0}^{\infty} q^n \Upsilon\left(q^n\delta_2 + (1-q^n)\,\delta_1\right). \tag{2.2}$$

Note that if $\delta_1 = 0$, then (2.2) reduces to

$$\int_0^{\delta_2} \Upsilon(\varphi)\ {}_0d_q\varphi = \int_0^{\delta_2} \Upsilon(\varphi)d_q\varphi = (1-q)\delta_2\sum_{n=0}^{\infty} q^n \Upsilon\left(q^n\delta_2\right),$$

which coincides with the classical q-integral; see [34,35] for further details.

Example 2.2. *Let* $\Upsilon : [\delta_1, \delta_2] \to \mathbb{R}$ *be defined by* $\Upsilon(\varphi) = \kappa\varphi^2$*, where* κ *is a constant and* $\varphi \in [\delta_1, \delta_2]$*. By applying Definition 2.2, we obtain*

$$\int_{\delta_1}^{\delta_2} \Upsilon(\varphi)\ {}_{\delta_1}d_q\varphi = \int_{\delta_1}^{\delta_2} \kappa\varphi^2\ {}_{\delta_1}d_q\varphi$$

$$= \kappa(1-q)(\delta_2-\delta_1)\sum_{n=0}^{\infty} q^n \left(q^n\delta_2 + (1-q^n)\delta_1\right)$$
$$= \kappa(\delta_2-\delta_1)\left(\frac{(\delta_2-\delta_1)^2}{1+q+q^2} + \frac{2\delta_1(\delta_2-\delta_1)}{1+q} + \delta_1^2\right).$$

3. Main Results

In this section, we derive new Ostrowski-type inequalities for uniformly convex functions via the left-sided quantum Montgomery identity, as presented in Theorems 3.1, 3.3, and 3.2.

Theorem 3.1. *Let* $\Upsilon : [\delta_1, \delta_2] \subset \mathbb{R} \to \mathbb{R}$ *be a q-differentiable function with* $|_{\delta_1}d_q\Upsilon|$ *is a uniformly convex function. Then the following inequality holds:*

$$\left|\Upsilon(x) - \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \Upsilon(\theta)\ {}_{\delta_1}d_q\theta\right|$$
$$\leq (\delta_2-\delta_1)\Big\{(T_2(\delta_1,\delta_2,x,q) + T_7(\delta_1,\delta_2,x,q))|_{\delta_1}d_q\Upsilon(\delta_2)| + (T_3(\delta_1,\delta_2,x,q) + T_8(\delta_1,\delta_2,x,q))|_{\delta_1}d_q\Upsilon(\delta_1)|$$
$$- (T_5(\delta_1,\delta_2,x,q) + T_{10}(\delta_1,\delta_2,x,q))\Phi(\delta_2-\delta_1)\Big\} \tag{3.1}$$

for all $x \in [\delta_1, \delta_2]$, *where*

$$T_1(\delta_1,\delta_2,x,q) = \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta\, d_q\theta = \frac{q}{1+q}\left(\frac{x-\delta_1}{\delta_2-\delta_1}\right)^2,$$

$$T_2(\delta_1,\delta_2,x,q) = \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta^2\, d_qt = \frac{q}{1+q+q^2}\left(\frac{x-\delta_1}{\delta_2-\delta_1}\right)^3,$$

$$T_3(\delta_1,\delta_2,x,q) = \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} (q\theta - q\theta^2)\, d_q\theta = \frac{q}{1+q}\left(\frac{x-\delta_1}{\delta_2-\delta_1}\right)^2 - \frac{q}{1+q+q^2}\left(\frac{x-\delta_1}{\delta_2-\delta_1}\right)^3$$
$$= T_1(\delta_1,\delta_2,x,q) - T_2(\delta_1,\delta_2,x,q),$$

$$T_4(\delta_1,\delta_2,x,q) = \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta^3\, d_q\theta = \frac{q}{1+q+q^2+q^3}\left(\frac{x-\delta_1}{\delta_2-\delta_1}\right)^4,$$

$$T_5(\delta_1,\delta_2,x,q) = \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} (q\theta^2 - q\theta^3)\, d_q\theta$$
$$= \frac{q}{1+q+q^2}\left(\frac{x-\delta_1}{\delta_2-\delta_1}\right)^3 - \frac{q}{1+q+q^2+q^3}\left(\frac{x-\delta_1}{\delta_2-\delta_1}\right)^4$$
$$= T_2(\delta_1,\delta_2,x,q) - T_4(\delta_1,\delta_2,x,q),$$

$$T_6(\delta_1,\delta_2,x,q) = \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1 (1-q\theta)\, d_q\theta$$
$$= \int_0^1 (1-q\theta)\, d_q\theta - \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} (1-q\theta)\, d_q\theta$$

$$= \frac{1}{1+q} - \frac{x-\delta_1}{\delta_2-\delta_1} + \frac{q}{1+q}(\frac{x-\delta_1}{\delta_2-\delta_1})^2,$$

$$\begin{aligned}
T_7(\delta_1,\delta_2,x,q) &= \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1} \theta(1-q\theta)\, d_q\theta \\
&= \int_0^1 (\theta - q\theta^2)\, d_q\theta - \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} (\theta - q\theta^2)\, d_q\theta \\
&= \frac{1}{(1+q)(1+q+q^2)} - \frac{1}{(1+q)}(\frac{x-\delta_1}{\delta_2-\delta_1})^2 + \frac{q}{1+q+q^2}(\frac{x-\delta_1}{\delta_2-\delta_1})^3,
\end{aligned}$$

$$\begin{aligned}
T_8(\delta_1,\delta_2,x,q) &= \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1} (1-q\theta)(1-\theta)\, d_q\theta = \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1} (1-q\theta-\theta+q\theta^2)\, d_q\theta \\
&= \frac{1}{1+q} - \frac{x-\delta_1}{\delta_2-\delta_1} + \frac{q}{1+q}(\frac{x-\delta_1}{\delta_2-\delta_1})^2 - \frac{1}{(1+q)(1+q+q^2)} + \frac{1}{(1+q)}(\frac{x-\delta_1}{\delta_2-\delta_1})^2 \\
&- \frac{q}{1+q+q^2}(\frac{x-\delta_1}{\delta_2-\delta_1})^3 \\
&= T_6(\delta_1,\delta_2,x,q) - T_7(\delta_1,\delta_2,x,q).
\end{aligned}$$

$$\begin{aligned}
T_9(\delta_1,\delta_2,x,q) &= \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1} (\theta^2 - q\theta^3)\, d_q\theta \\
&= \int_0^1 (\theta^2 - q\theta^3)\, d_q\theta - \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} (\theta^2 - q\theta^3)\, d_q\theta \\
&= \frac{1}{(1+q+q^2)} - \frac{q}{(1+q+q^2+q^3)} - \frac{1}{(1+q+q^2)}(\frac{x-\delta_1}{\delta_2-\delta_1})^3 \\
&+ \frac{q}{1+q+q^2+q^3}(\frac{x-\delta_1}{\delta_2-\delta_1})^4,
\end{aligned}$$

$$\begin{aligned}
T_{10}(\delta_1,\delta_2,x,q) &= \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1} (1-q\theta)\theta(1-\theta)\, d_q\theta = \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1} (\theta - q\theta^2 - \theta^2 + q\theta^3)\, d_q\theta \\
&= \frac{1}{(1+q)(1+q+q^2)} - \frac{1}{(1+q)}(\frac{x-\delta_1}{\delta_2-\delta_1})^2 + \frac{q}{1+q+q^2}(\frac{x-\delta_1}{\delta_2-\delta_1})^3 \\
&- \frac{1}{(1+q+q^2)} + \frac{q}{(1+q+q^2+q^3)} + \frac{1}{(1+q+q^2)}(\frac{x-\delta_1}{\delta_2-\delta_1})^3 \\
&- \frac{q}{1+q+q^2+q^3}(\frac{x-\delta_1}{\delta_2-\delta_1})^4 \\
&= T_7(\delta_1,\delta_2,x,q) - T_9(\delta_1,\delta_2,x,q).
\end{aligned}$$

Proof. Using Lemma 1.1 and the property of modulus, we have

$$\left|\Upsilon(x) - \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \Upsilon(\theta)\_\delta_1 d_q\theta\right| \leq (\delta_2-\delta_1)\int_0^1 |T_q(\theta)|\, |{}_{\delta_1}d_q\Upsilon(\theta\delta_2 + (1-\theta)\delta_1)|_0 d_q\theta$$

$$\begin{aligned}
&\leq (\delta_2-\delta_1)\bigg\{\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta\,|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|\,{}_0d_q\theta\\
&\quad+\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1}(1-q\theta)\,|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|\,{}_0d_q\theta\bigg\}.
\end{aligned}$$

Since $|{}_{\delta_1}d_q\Upsilon|$ is uniformly convex on $[\delta_1,\delta_2]$, we get

$$\begin{aligned}
&\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta\,|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|\,{}_0d_q\theta\\
&\leq \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta\Big((\theta|{}_{\delta_1}d_q\Upsilon(\delta_2)|+(1-\theta)|{}_{\delta_1}d_q\Upsilon(\delta_1)|-\theta(1-\theta)\Phi(\delta_2-\delta_1)\Big)d_q\theta\\
&=\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}\Big(q\theta^2|{}_{\delta_1}d_q\Upsilon(\delta_2)|+q(\theta-\theta^2)|{}_{\delta_1}d_q\Upsilon(\delta_1)|-q(\theta^2-\theta^3)\Phi(\delta_2-\delta_1)\Big)d_q\theta\\
&=T_2(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_2)|+(T_3(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_1)|-T_5(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1}(1-q\theta)\,|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|\,{}_0d_q\theta\\
&\leq \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}(1-q\theta)\Big((\theta|{}_{\delta_1}d_q\Upsilon(\delta_2)|+(1-\theta)|{}_{\delta_1}d_q\Upsilon(\delta_1)|-\theta(1-\theta)\Phi(\delta_2-\delta_1)\Big)d_q\theta\\
&=\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}\Bigg((\theta-q\theta^2)|{}_{\delta_1}d_q\Upsilon(\delta_2)|+(1-\theta-q\theta+q\theta^2)|{}_{\delta_1}d_q\Upsilon(\delta_1)|\\
&\quad-(\theta-\theta^2-q\theta^2+q\theta^3)\Phi(\delta_2-\delta_1)\Bigg)d_q\theta\\
&=T_7(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_2)|+T_8(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_1)|-T_{10}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1).
\end{aligned}$$

We can easily deduce

$$\begin{aligned}
&\left|\Upsilon(x)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Upsilon(\theta)\ {}_{\delta_1}d_q\theta\right|\\
&\leq(\delta_2-\delta_1)\Bigg\{T_2(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_2)|+(T_3(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_1)|-T_5(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\\
&\quad+T_7(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_2)|+T_8(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_1)|-T_9(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Bigg\}\\
&=(\delta_2-\delta_1)\Bigg\{(T_2(\delta_1,\delta_2,x,q)+T_7(\delta_1,\delta_2,x,q))|{}_{\delta_1}d_q\Upsilon(\delta_2)|+(T_3(\delta_1,\delta_2,x,q)\\
&\quad+T_8(\delta_1,\delta_2,x,q))|{}_{\delta_1}d_q\Upsilon(\delta_1)|-(T_5(\delta_1,\delta_2,x,q)+T_{10}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Bigg\}.
\end{aligned}$$

Which completes the proof. □

Theorem 3.2. *Let* $\Upsilon : [\delta_1, \delta_2] \subset \mathbb{R} \to \mathbb{R}$ *be a q-differentiable function. If* $|_{\delta_1}d_q\Upsilon|^r$, $r \geq 1$ *is uniformly convex function. Then the following inequality holds:*

$$\begin{aligned}
&\left|\Upsilon(x) - \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \Upsilon(\theta)\ {}_{\delta_1}d_q\theta\right| \\
&\leq (\delta_2-\delta_1)\Bigg\{\Big(T_1(\delta_1,\delta_2,x,q)\Big)^{1-\frac{1}{r}}\Big(T_2(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r + (T_3(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r \\
&\quad - T_5(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Big)^{\frac{1}{r}} + \Big(T_6(\delta_1,\delta_2,x,q)\Big)^{1-\frac{1}{r}}\Big(T_7(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r \\
&\quad + (T_8(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r - T_{10}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Big)^{\frac{1}{r}}\Bigg\}. \qquad (3.2)
\end{aligned}$$

for all $x \in [\delta_1, \delta_2], r \geq 1$ *and* $\frac{1}{s} + \frac{1}{r} = 1$.

Proof. Using Lemma 1.1, quantum power mean inequality, we obtain

$$\begin{aligned}
&\left|\Upsilon(x) - \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \Upsilon(\theta)\ {}_{\delta_1}d_q\theta\right| \\
&\leq (\delta_2-\delta_1)\int_0^1 |T_q(\theta)|\,|_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|\,d_q\theta \\
&\leq (\delta_2-\delta_1)\Bigg\{\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta\,|_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|\,d_q\theta \\
&\quad + \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1 (1-q\theta)\,|_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|\,d_q\theta\Bigg\} \\
&\leq (\delta_2-\delta_1)\Bigg\{\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta d_q\theta\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta\,|_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|^r\,d_q\theta\right)^{\frac{1}{r}} \\
&\quad + \left(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1 (1-q\theta)d_q\theta\right)^{1-\frac{1}{r}}\left(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1 (1-q\theta)\,|_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|^r\,d_q\theta\right)^{\frac{1}{r}}\Bigg\}.
\end{aligned}$$

Since $|_{\delta_1}d_q\Upsilon|$ is uniformly convex on $[\delta_1, \delta_2]$, we get

$$\begin{aligned}
&\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta d_q\theta\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta\,|_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|^r\,d_q\theta\right)^{\frac{1}{r}} \\
&\leq \left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta d_q\theta\right)^{1-\frac{1}{r}}\Bigg(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}\Big(q\theta^2|_{\delta_1}d_q\Upsilon(\delta_2)|^r + (q\theta-q\theta^2)|_{\delta_1}d_q\Upsilon(\delta_1)|^r \\
&\quad - (q\theta^2-q\theta^3)\Phi(\delta_2-\delta_1)\Big)d_q\theta\Bigg)^{\frac{1}{r}} \\
&= \Big(T_1(\delta_1,\delta_2,x,q)\Big)^{1-\frac{1}{r}}\Big(T_2(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r + (T_3(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r
\end{aligned}$$

$$- T_5(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Bigg)^{\frac{1}{r}},$$

and

$$\Bigg(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1}(1-q\theta)d_q\theta\Bigg)^{1-\frac{1}{r}}\Bigg(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1}(1-q\theta)\,|_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|^r\,d_q\theta\Bigg)^{\frac{1}{r}}\Bigg\}$$

$$\leq\Bigg(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^{1}(1-q\theta)d_q\theta\Bigg)^{1-\frac{1}{r}}\Bigg(\int_{0}^{\frac{x-\delta_1}{\delta_2-\delta_1}}\Bigg((\theta-q\theta^2)|_{\delta_1}d_q\Upsilon(\delta_2)|^r+(1-\theta-q\theta+q\theta^2)|_{\delta_1}d_q\Upsilon(\delta_1)|^r$$

$$-(\theta-\theta^2-q\theta^2+q\theta^3)\Phi(\delta_2-\delta_1)\Bigg)d_q\theta\Bigg)^{\frac{1}{r}}$$

$$=\Bigg(T_6(\delta_1,\delta_2,x,q)\Bigg)^{1-\frac{1}{r}}\Bigg(T_7(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r+(T_8(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r$$

$$-T_{10}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Bigg)^{\frac{1}{r}}.$$

We can easily deduce

$$\left|\Upsilon(x)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Upsilon(\theta)\ {}_{\delta_1}d_q\theta\right|$$

$$\leq(\delta_2-\delta_1)\Bigg\{\Bigg(T_1(\delta_1,\delta_2,x,q)\Bigg)^{1-\frac{1}{r}}\Bigg(T_2(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r+(T_3(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r$$

$$-T_5(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Bigg)^{\frac{1}{r}}+\Bigg(T_6(\delta_1,\delta_2,x,q)\Bigg)^{1-\frac{1}{r}}\Bigg(T_7(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r$$

$$+(T_8(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r-T_{10}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Bigg)^{\frac{1}{r}}\Bigg\}.$$

Which completes the proof. □

Corollary 3.1. *In Theorem 3.2, the following inequalities are held by the following assumptions:*

1. *If one takes $r=1$ and $x=\frac{q\delta_1+\delta_2}{1+q}$, then one has (a new quantum midpoint type inequality for uniformly convex function)*

$$\left|\Upsilon(x)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Upsilon(\theta)\ {}_{\delta_1}d_q\theta\right|$$

$$\leq(\delta_2-\delta_1)\Bigg\{\Bigg(T_2(\delta_1,\delta_2,\frac{q\delta_1+\delta_2}{1+q},q)|_{\delta_1}d_q\Upsilon(\delta_2)|+(T_3(\delta_1,\delta_2,\frac{q\delta_1+\delta_2}{1+q},q)|_{\delta_1}d_q\Upsilon(\delta_1)|$$

$$-T_5(\delta_1,\delta_2,\frac{q\delta_1+\delta_2}{1+q},q)\Phi(\delta_2-\delta_1)\Bigg)+\Bigg(T_7(\delta_1,\delta_2,\frac{q\delta_1+\delta_2}{1+q},q)|_{\delta_1}d_q\Upsilon(\delta_2)|$$

$$+(T_8(\delta_1,\delta_2,\frac{q\delta_1+\delta_2}{1+q},q)|_{\delta_1}d_q\Upsilon(\delta_1)|-T_{10}(\delta_1,\delta_2,\frac{q\delta_1+\delta_2}{1+q},q)\Phi(\delta_2-\delta_1)\Bigg)\Bigg\}$$

$$\begin{aligned}
&\leq (\delta_2-\delta_1)\Bigg\{\frac{3q}{(1+q)^3(1+q+q^2)}\left|{}_{\delta_1}d_q\Upsilon(\delta_2)\right| + \frac{q(q^2+q-1)}{(1+q)^3(1+q+q^2)}\left|{}_{\delta_1}d_q\Upsilon(\delta_1)\right| \\
&\quad - \frac{q^2}{(1+q)^3(1+q+q^2+q^3)}\Phi(\delta_2-\delta_1)\Bigg\}.
\end{aligned}$$

2. *If one takes* $r = 1$ *and* $x = \frac{\delta_1+\delta_2}{2}$*, then one has (a new quantum midpoint type inequality for uniformly convex function)*

$$\begin{aligned}
&\left|\Upsilon(x) - \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Upsilon(\theta)\,{}_{\delta_1}d_q\theta\right| \\
&\leq (\delta_2-\delta_1)\Bigg\{\Bigg(T_2(\delta_1,\delta_2,\frac{\delta_1+\delta_2}{2},q)|{}_{\delta_1}d_q\Upsilon(\delta_2)| + (T_3(\delta_1,\delta_2,\frac{\delta_1+\delta_2}{2},q)|{}_{\delta_1}d_q\Upsilon(\delta_1)| \\
&\quad - T_5(\delta_1,\delta_2,\frac{\delta_1+\delta_2}{2},q)\Phi(\delta_2-\delta_1)\Bigg) + \Bigg(T_7(\delta_1,\delta_2,\frac{\delta_1+\delta_2}{2},q)|{}_{\delta_1}d_q\Upsilon(\delta_2)| \\
&\quad + (T_8(\delta_1,\delta_2,\frac{\delta_1+\delta_2}{2},q)|{}_{\delta_1}d_q\Upsilon(\delta_1)| - T_{10}(\delta_1,\delta_2,\frac{\delta_1+\delta_2}{2},q)\Phi(\delta_2-\delta_1)\Bigg)\Bigg\} \\
&= (\delta_2-\delta_1)\Bigg\{\frac{3}{4\big(q^3+2q^2+2q+1\big)}\left|{}_{\delta_1}d_q\Upsilon(\delta_2)\right| + \frac{2q^2+2q-1}{4\big(q^3+2q^2+2q+1\big)}\left|{}_{\delta_1}d_q\Upsilon(\delta_1)\right| \\
&\quad - \frac{6q^2-1}{8\big(q^5+2q^4+3q^3+3q^2+2q+1\big)}\Phi(\delta_2-\delta_1)\Bigg\}.
\end{aligned}$$

3. *If one takes* $r = 1$ *and* $x = \frac{\delta_1+q\delta_2}{1+q}$*, then one has (a new quantum midpoint type inequality for uniformly convex functions)*

$$\begin{aligned}
&\left|\Upsilon(x) - \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Upsilon(\theta)\,{}_{\delta_1}d_q\theta\right| \\
&\leq (\delta_2-\delta_1)\Bigg\{\Bigg(T_2\Big(\delta_1,\delta_2,\tfrac{\delta_1+q\delta_2}{1+q},q\Big)\left|{}_{\delta_1}d_q\Upsilon(\delta_2)\right| + T_3\Big(\delta_1,\delta_2,\tfrac{\delta_1+q\delta_2}{1+q},q\Big)\left|{}_{\delta_1}d_q\Upsilon(\delta_1)\right| \\
&\quad - T_5\Big(\delta_1,\delta_2,\tfrac{\delta_1+q\delta_2}{1+q},q\Big)\Phi(\delta_2-\delta_1)\Bigg) + \Bigg(T_7\Big(\delta_1,\delta_2,\tfrac{\delta_1+q\delta_2}{1+q},q\Big)\left|{}_{\delta_1}d_q\Upsilon(\delta_2)\right| \\
&\quad + T_8\Big(\delta_1,\delta_2,\tfrac{\delta_1+q\delta_2}{1+q},q\Big)\left|{}_{\delta_1}d_q\Upsilon(\delta_1)\right| - T_{10}\Big(\delta_1,\delta_2,\tfrac{\delta_1+q\delta_2}{1+q},q\Big)\Phi(\delta_2-\delta_1)\Bigg)\Bigg\} \\
&= (\delta_2-\delta_1)\frac{1}{(1+q)(1+q+q^2)}\Bigg\{\left|{}_{\delta_1}d_q\Upsilon(\delta_2)\right| + \Big(q(1+q+q^2)-1\Big)\left|{}_{\delta_1}d_q\Upsilon(\delta_1)\right| - \Phi(\delta_2-\delta_1)\Bigg\}.
\end{aligned}$$

Remark 3.1. *The obtained inequalities extend several known quantum Ostrowski-type results by incorporating the notion of uniform convexity. In particular, when the modulus function* Φ *vanishes, our estimates reduce to the corresponding inequalities for convex functions available in the literature.*

Theorem 3.3. *Let* $f : [\delta_1, \delta_2] \subset \mathbb{R} \to \mathbb{R}$ *be a q-differentiable function. If* $|_{\delta_1}d_q f|^r$, $r > 1$ *is uniformly convex function. Then the following inequality holds:*

$$\begin{aligned}
&\left| \Upsilon(x) - \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \Upsilon(\theta) \; {}_{\delta_1}d_q\theta \right| \\
&\leq (\delta_2 - \delta_1) \Bigg\{ \Bigg(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} (q\theta)^s d_q\theta \Bigg)^{\frac{1}{s}} \Bigg(T_{11}(\delta_1, \delta_2, x, q) |_{\delta_1}d_q \Upsilon(\delta_2)|^r + (T_{12}(\delta_1, \delta_2, x, q) |_{\delta_1}d_q \Upsilon(\delta_1)|^r \\
&\quad - T_{13}(\delta_1, \delta_2, x, q) \Phi(\delta_2 - \delta_1) \Bigg)^{\frac{1}{r}} + \Bigg(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} (1 - q\theta)^s d_q\theta \Bigg)^{\frac{1}{s}} \Bigg(T_{14}(\delta_1, \delta_2, x, q) |_{\delta_1}d_q \Upsilon(\delta_2)|^r \\
&\quad + (T_{15}(\delta_1, \delta_2, x, q) |_{\delta_1}d_q \Upsilon(\delta_1)|^r - T_{16}(\delta_1, \delta_2, x, q) \Phi(\delta_2 - \delta_1) \Bigg)^{\frac{1}{r}} \Bigg\}
\end{aligned} \tag{3.3}$$

for all $x \in [\delta_1, \delta_2]$, $r > 1$ *and* $\frac{1}{s} + \frac{1}{r} = 1$, *where*

$$T_{11}(\delta_1, \delta_2, x, q) = \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} \theta \, d_q\theta = \frac{1}{1+q} \left(\frac{x - \delta_1}{\delta_2 - \delta_1} \right)^2,$$

$$T_{12}(\delta_1, \delta_2, x, q) = \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} (1 - \theta) \, d_q\theta = \frac{x - \delta_1}{\delta_2 - \delta_1} - \frac{1}{1+q} \left(\frac{x - \delta_1}{\delta_2 - \delta_1} \right)^2,$$

$$\begin{aligned}
T_{13}(\delta_1, \delta_2, x, q) &= \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} \theta(1 - \theta) \, d_q\theta = \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} \theta - \theta^2 \, d_q\theta \\
&= \frac{1}{1+q} \left(\frac{x - \delta_1}{\delta_2 - \delta_1} \right)^2 - \frac{1}{1+q+q^2} \left(\frac{x - \delta_1}{\delta_2 - \delta_1} \right)^3,
\end{aligned}$$

$$\begin{aligned}
T_{14}(\delta_1, \delta_2, x, q) &= \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1 \theta \, d_q\theta = \int_0^1 \theta \, d_q\theta - \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} \theta \, d_q\theta \\
&= \frac{1}{1+q} - \frac{1}{1+q} (\frac{x - \delta_1}{\delta_2 - \delta_1})^2,
\end{aligned}$$

$$\begin{aligned}
T_{15}(\delta_1, \delta_2, x, q) &= \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1 (1 - \theta) \, d_q\theta \\
&= \int_0^1 (1 - \theta) \, d_q\theta - \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} (1 - \theta) \, d_q\theta \\
&= \frac{q}{(1+q)} - (\frac{x - \delta_1}{\delta_2 - \delta_1}) + \frac{1}{1+q} (\frac{x - \delta_1}{\delta_2 - \delta_1})^2,
\end{aligned}$$

and

$$\begin{aligned}
T_{16}(\delta_1, \delta_2, x, q) &= \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1 \theta(1 - \theta) \, d_q\theta \\
&= \int_0^1 (\theta - \theta^2) \, d_q\theta - \int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} (\theta - \theta^2) \, d_q\theta
\end{aligned}$$

$$= \frac{q^2}{(1+q)(1+q+q^2)} - \frac{1}{1+q}(\frac{x-\delta_1}{\delta_2-\delta_1})^2 + \frac{1}{1+q+q^2}(\frac{x-\delta_1}{\delta_2-\delta_1})^3.$$

Proof. Using Lemma 1.1, Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\Upsilon(x) - \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Upsilon(\theta)\ {}_{\delta_1}d_q\theta\right| \\
&\leq (\delta_2-\delta_1)\int_0^1 |T_q(\theta)|\,|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|\,d_q\theta \\
&\leq (\delta_2-\delta_1)\Bigg\{\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}} q\theta\,|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|\,d_q\theta \\
&\quad + \int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1 (1-q\theta)\,|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|\,d_q\theta\Bigg\} \\
&\leq (\delta_2-\delta_1)\Bigg\{\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}(q\theta)^s d_q\theta\right)^{\frac{1}{s}}\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|^r\,d_q\theta\right)^{\frac{1}{r}} \\
&\quad + \left(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1(1-q\theta)^s d_q\theta\right)^{\frac{1}{s}}\left(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|^r\,d_q\theta\right)^{\frac{1}{r}}\Bigg\}.
\end{aligned}$$

Since $|{}_{\delta_1}d_qf|$ is uniformly convex on $[\delta_1,\delta_2]$, we get

$$\begin{aligned}
&\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}(q\theta)^s d_q\theta\right)^{\frac{1}{s}}\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|^r\,d_q\theta\right)^{\frac{1}{r}} \\
&\leq\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}(q\theta)^s d_q\theta\right)^{\frac{1}{s}}\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}\Big(\theta|{}_{\delta_1}d_q\Upsilon(\delta_2)|^r+(1-\theta)|{}_{\delta_1}d_q\Upsilon(\delta_1)|^r-\theta(1-\theta)\Phi(\delta_2-\delta_1)\Big)\,d_q\theta\right)^{\frac{1}{r}} \\
&=\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}(q\theta)^s d_q\theta\right)^{\frac{1}{s}}\Bigg(T_{11}(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_2)|^r+(T_{12}(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_1)|^r \\
&\quad -T_{13}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Bigg)^{\frac{1}{r}},
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1(1-q\theta)^s d_q\theta\right)^{\frac{1}{s}}\left(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1|{}_{\delta_1}d_q\Upsilon(\theta\delta_2+(1-\theta)\delta_1)|^r\,d_q\theta\right)^{\frac{1}{r}} \\
&\leq\left(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1(1-q\theta)^s d_q\theta\right)^{\frac{1}{s}}\left(\int_{\frac{x-\delta_1}{\delta_2-\delta_1}}^1\Big(\theta|{}_{\delta_1}d_q\Upsilon(\delta_2)|^r+(1-\theta)|{}_{\delta_1}d_q\Upsilon(\delta_1)|^r-\theta(1-\theta)\Phi(\delta_2-\delta_1)\Big)\,d_q\theta\right)^{\frac{1}{r}} \\
&=\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}(1-q\theta)^s d_q\theta\right)^{\frac{1}{s}}\Bigg(T_{14}(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_2)|^r+(T_{15}(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_1)|^r \\
&\quad -T_{16}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Bigg)^{\frac{1}{r}}.
\end{aligned}$$

We can easily deduce

$$\begin{aligned}
&\left|\Upsilon(x)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Upsilon(\theta)\;{}_{\delta_1}d_q\theta\right|\\
&\leq(\delta_2-\delta_1)\Bigg\{\bigg(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}(q\theta)^s d_q\theta\bigg)^{\frac{1}{s}}\bigg(T_{11}(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r+(T_{12}(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r\\
&\quad-T_{13}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\bigg)^{\frac{1}{r}}+\bigg(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}(1-q\theta)^s d_q\theta\bigg)^{\frac{1}{s}}\bigg(T_{14}(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r\\
&\quad+(T_{15}(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r-T_{16}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\bigg)^{\frac{1}{r}}\Bigg\}.
\end{aligned}$$

Which completes the proof. □

Remark 3.2. *Theorem 3.3 allows the derivation of many other inequalities following the same approach as in Corollary 3.1.*

4. Numerical Examples

In this section, we present numerical examples to illustrate the validity of the obtained inequalities. For selected values of $q \in (0,1)$, the left-hand side and right-hand side of the inequalities in Theorem 3.1, 3.2, and 3.3 are computed explicitly.

Example 4.1. *Let* $\Upsilon:[0,1]\to\mathbb{R}$ *be defined by* $\Upsilon(x)=(1+x)^3$. *A direct computation yields*

$$|_{\delta_1}D_q\Upsilon(x)|=3(1+q)x+(1+q+q^2)x^2+3,$$

which is a uniformly convex function with modulus $\Phi(x)=x^2$. *Moreover, straightforward calculations give*

$$|_0D_q\Upsilon(1)|=q^2+4q+7\quad\text{and}\quad|_0D_q\Upsilon(0)|=3.$$

Applying Theorem 3.1 at $x=1$*, the left-hand side of the inequality is expressed as*

$$\begin{aligned}
LHS1:&=\left|\Upsilon(x)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Upsilon(\theta)\;{}_{\delta_1}d_q\theta\right|\\
&=\left|7-\frac{3}{1+q}-\frac{3}{1+q+q^2}-\frac{1}{1+q+q^2+q^3}\right|,
\end{aligned}\tag{4.1}$$

and the right-hand side of the inequality becomes

$$\begin{aligned}
RHS1:=&\,(\delta_2-\delta_1)\Big\{(T_2(\delta_1,\delta_2,x,q)+T_7(\delta_1,\delta_2,x,q))|_{\delta_1}d_q\Upsilon(\delta_2)|+(T_3(\delta_1,\delta_2,x,q)\\
&\quad+T_8(\delta_1,\delta_2,x,q))|_{\delta_1}d_q\Upsilon(\delta_1)|-(T_5(\delta_1,\delta_2,x,q)+T_{10}(\delta_1,\delta_2,x,q))\Phi(\delta_2-\delta_1)\Big\}\\
=&\left(\frac{q}{1+q+q^2}+\frac{1}{(1+q)(1+q+q^2)}-\frac{1}{(1+q)}+\frac{q}{1+q+q^2}\right)(q^2+4q+7)\\
&+\bigg(\frac{q}{1+q}-\frac{q}{1+q+q^2}+\frac{1}{1+q}-1+\frac{q}{1+q}-\frac{1}{(1+q)(1+q+q^2)}
\end{aligned}$$

$$+\frac{1}{(1+q)}-\frac{q}{1+q+q^2}\Bigg)(3)-\Bigg(\frac{q}{1+q+q^2}-\frac{q}{1+q+q^2+q^3}$$
$$+\frac{1}{(1+q)(1+q+q^2)}-\frac{1}{(1+q)}+\frac{q}{1+q+q^2}-\frac{1}{(1+q+q^2)}$$
$$+\frac{q}{(1+q+q^2+q^3)}+\frac{1}{(1+q+q^2)}-\frac{q}{1+q+q^2+q^3}\Bigg). \tag{4.2}$$

By Theorem 3.1, (4.1) *and* (4.2)*, obtain*

$$LHS1 \leq RHS1. \tag{4.3}$$

The graphical illustration of inequality (4.3) *is shown in Figure 1.*

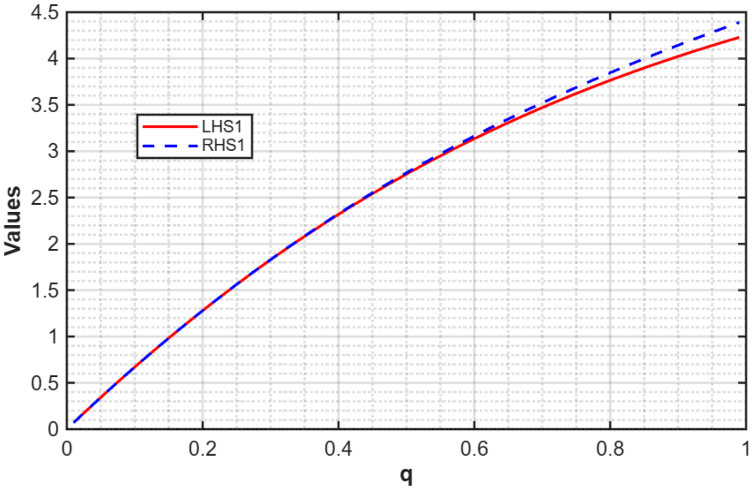

Figure 1. Graphical illustration of the inequalities in (4.3).

Example 4.2. *Under the same setting as in Example 4.1, a direct application of Theorem 3.2 with $x = 0$ and $r = 2$ yields the left-hand side of the inequality in the form*

$$LHS2 := \left|\Upsilon(x)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Upsilon(\theta)\_\delta_1 d_q\theta\right|$$
$$= \left|-\frac{3}{1+q}-\frac{3}{1+q+q^2}-\frac{1}{1+q+q^2+q^3}\right|, \tag{4.4}$$

and the right-hand side becomes:

$$RHS2 := (\delta_2-\delta_1)\Bigg\{\Big(T_1(\delta_1,\delta_2,x,q)\Big)^{1-\frac{1}{r}}\Big(T_2(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r$$
$$+(T_3(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r-T_5(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\Big)^{\frac{1}{r}}$$
$$+\Big(T_6(\delta_1,\delta_2,x,q)\Big)^{1-\frac{1}{r}}\Big(T_7(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r$$

$$+\left(T_8(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r - T_{10}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\right)^{\frac{1}{r}}\Bigg\}$$

$$=\left(\frac{1}{1+q}\right)^{\frac{1}{2}}\left\{\frac{(q^2+4q+7)^2}{(1+q)(1+q+q^2)}+\frac{9(q+q^2)}{(1+q)(1+q+q^2)}\right. \tag{4.5}$$

$$\left.-\left(\frac{1}{(1+q)(1+q+q^2)}-\frac{1}{1+q+q^2}+\frac{q}{1+q+q^2+q^3}\right)\right\}^{\frac{1}{2}} \tag{4.6}$$

By Theorem 3.2, (4.4) *and* (4.5)*, obtain*

$$LHS2 \leq RHS2. \tag{4.7}$$

The graphical illustration of inequality (4.7) *is presented in Figure 2.*

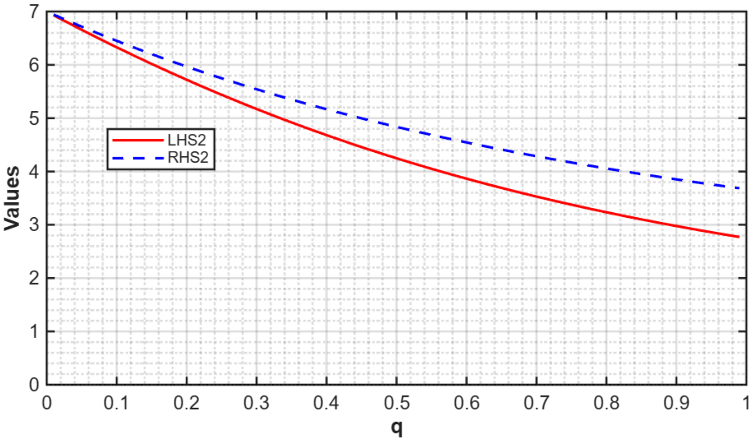

Figure 2. Graphical illustration of the inequalities in (4.7).

Example 4.3. *Under the same setting as in Example 4.1, by applying Theorem 3.3 with $x = 1$ and $r = 2$, the left-hand side of the inequality is given by*

$$\begin{aligned} LHS3 :&= \left|\Upsilon(x)-\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\Upsilon(\theta)\ {}_{\delta_1}d_q\theta\right| \\ &= \left|-\frac{3}{1+q}-\frac{3}{1+q+q^2}-\frac{1}{1+q+q^2+q^3}\right|, \end{aligned} \tag{4.8}$$

and the right-hand side becomes:

$$RHS3 := (\delta_2-\delta_1)\Bigg\{\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}(q\theta)^s d_q\theta\right)^{\frac{1}{s}}\Big(T_{11}(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r$$

$$+\left(T_{12}(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_1)|^r - T_{13}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\right)^{\frac{1}{r}}$$

$$+\left(\int_0^{\frac{x-\delta_1}{\delta_2-\delta_1}}(1-q\theta)^s d_q\theta\right)^{\frac{1}{s}}\Big(T_{14}(\delta_1,\delta_2,x,q)|_{\delta_1}d_q\Upsilon(\delta_2)|^r$$

$$
\begin{aligned}
&+ \left(T_{15}(\delta_1,\delta_2,x,q)|{}_{\delta_1}d_q\Upsilon(\delta_1)|^r - T_{16}(\delta_1,\delta_2,x,q)\Phi(\delta_2-\delta_1)\right)^{\frac{1}{r}}\Bigg\}\\
&= \left\{\left(\frac{q^2}{1+q+q^2}\right)^{\frac{1}{2}}\left(\frac{1}{1+q}(q^2+4q+7)^2+1-\frac{9}{1+q}-\frac{1}{1+q}+\frac{1}{1+q+q^2}\right)\right.\\
&\quad \left. -\left(\frac{q^2}{(1+q)(1+q+q^2)}-\frac{1}{1+q}+\frac{1}{1+q+q^2}\right)\right\}^{\frac{1}{2}}. \qquad (4.9)
\end{aligned}
$$

By Theorem 3.3, (4.8) *and* (4.9)*, obtain*

$$LHS3 \leq RHS3. \qquad (4.10)$$

The graphical illustration of inequality (4.10) *is presented in Figure 3.*

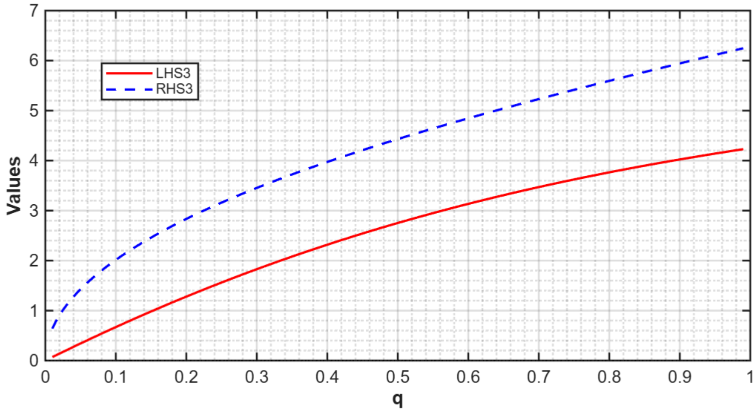

FIGURE 3. Graphical illustration of the inequalities in (4.10).

Remark 4.1. *Examples 4.1–4.3 validate the proposed inequalities for explicit functions and demonstrate their effectiveness through numerical computations and graphical illustrations. Moreover, these examples highlight the improvement obtained under the assumption of uniform convexity in the quantum setting, thereby supporting the theoretical results.*

5. CONCLUSIONS

In this paper, we have derived new quantum Ostrowski-type inequalities for uniformly convex functions within the framework of q-calculus. The obtained bounds involve a modulus function associated with uniform convexity, which yields sharper estimates than those corresponding to the convex case. Several corollaries have been presented to recover known results as special cases, while numerical examples and graphical illustrations have been included to demonstrate the validity of the theoretical findings and to clarify the effect of uniform convexity on the proposed inequalities.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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