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Novel Unified Variants, Properties, and Applications of Ostrowski, Jensen, and Hermite–Hadamard Inequalities for Generalized (η, p, h)-Convex Stochastic Processes

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Abstract. Stochastic processes play a crucial role in functional analysis and the study of inequalities under uncertainty, with Wiener processes providing a fundamental model for random perturbations. In this work, we introduce a new class of generalized (η, p, h)-convex stochastic processes, which unifies and extends several existing notions of convexity in the stochastic setting. We investigate essential properties of this class and derive novel Ostrowski-, Jensen-, and Hermite–Hadamard-type inequalities. The validity and effectiveness of the obtained results are illustrated through non-trivial examples and graphical comparisons highlighting the impact of Wiener processes relative to deterministic components.

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1. Introduction

A stochastic process is a mathematical model that is applied to describe systems or phenomena that change over time as influenced by randomness. In a more formal sense it is a set of random variables governed by time or space, with the help of which one can describe uncertain dynamics in a rigorously probabilistic way. Stochastic processes are critical in most scientific and engineering fields. They are applied in physics to describe Brownian motion, diffusion of particles and thermal noise. In finance and economics, asset pricing, risks, option pricing, and interest rate model are based on stochastic processes. In fields such as biology and medicine, they explain the dynamics of populations, expression of genes, the spread of an epidemic, and physiological cues. They have been used in the application of signal processing, control systems, and communications networks in engineering in order to support noise and uncertainty. Also, in computer science and artificial intelligence, stochastic processes play a central role in machine learning algorithms, Markov decision processes and reinforcement learning. Stochastic processes play a crucial role in the study and forecasting of complex real-world systems because their power to achieve randomness and time evolution is essential. For some applications in different fields, and also in the form of inequalities, we refer [1–4].

The mathematical analysis of stochastic convex inequalities has become a foundation of the modern probability and optimization theory. Its major concern is the behavior of stochastic processes that are subject to convexity conditions, in a sample-path sense or in the mean-square sense. This area was first formalised by the seminal contribution of Nikodem (1980), who coined the term convex stochastic processes and built their fundamental properties into existence [5]. On this basis, Skowroowski (1992) developed the study of Jensen-convex stochastic processes where a more in-depth insight can be developed on how the classical functional analysis can be applied to probabilistic contexts [6].

Another important development in the area was the fact that the classical Hermite-Hadamard inequality was adapted to the stochastic space. This inequality was later established using the mean-square integral by Kotrys (2012) to prove this inequality with the use of convex stochastic processes, which has since become a classic reference point by researchers in the field of research to date [7]. This paper led to several important extensions, including the investigation of strongly convex stochastic processes within convexity theory by Kotrys [8], as well as the development of generalized notions of convexity in random environments, such as h-convexity [9] and η-convexity, studied in [10] and [11], respectively. Such inequalities form one of the pillars of many fields of mathematics, such as optimization theory, numerical analysis, and functional analysis. In particular, Hermite-Hadamard inequality, and Jensen inequality, are related to the geometry of convex functions. Over the past few decades, stochastic processes have come to be as a result of the shift in classical deterministic analysis to the probabilistic one. Preliminary research on these ideas was followed up by examining the Hadamard and Jensen inequalities of s-convex fuzzy processes in a similar way as it is done with s-convex fuzzy processes [12]. This preconditioned the additional improvements, including the construction of Hermite-Hadamard-type inequalities in this case, when stochastic processes are strongly-log convex, in particular, the ones of the type of the Hermite process-Hadamard process [13].

With the maturity of the field, the stochastic environment was generalized to incorporate, and generalize, the use of the fractional calculus, and generalized convexity, to the stochastic environment. Specific convergence to the use of the Hermite-Hadamard and Jensen types of fractional stochastic inequalities was established to apply to the convex processes [14], and specific results were also provided regarding the use of the h-convex stochastic processes [15]. This body of inequalities was extended to other more complicated geometries and dimensions, such as the study of harmonically convex processes [16] of interest to harmonically convex dynamics, convex processes on a n-dimensional manifold [17], and p-convex stochastic processes of interest to p-convex dynamical processes [18].

In recent years, the study of integral inequalities within the framework of stochastic processes has garnered significant attention due to its wide-ranging applications in various fields such as finance, engineering, and risk analysis. Classical inequalities, including those of Hermite–Hadamard, Jensen, and Ostrowski, have been extended to stochastic settings to capture the inherent uncertainty in real-world phenomena. In this context, [19] introduced new stochastic fractional integrals and established related inequalities of Jensen–Mercer and Hermite–Hadamard–Mercer type for convex stochastic processes. Subsequently, [20] explored properties and inequalities for a generalized class of harmonical Godunova–Levin functions using the center radius order relation, while [21] derived generalized fractional inequalities of the Hermite–Hadamard type for convex stochastic processes. Further advancements were made by [22], who investigated weighted Fejér, Hermite–Hadamard, and trapezium-type inequalities for (h_1, h_2)-Godunova–Levin preinvex functions, and by [23], who presented Ostrowski-type inequalities via h-convex stochastic processes. The study of fractional mean-square inequalities for (P, m)-superquadratic stochastic processes was undertaken by [24], with applications to stochastic divergence measures. Meanwhile, [25] provided estimations of various integral inequalities for a generalized class of Godunova–Levin convex and preinvex functions using pseudo and standard order relations. [26] focused on strongly generalized convex stochastic processes, and [27] explored Hermite–Hadamard inequalities using fractional integrals for MT-convex stochastic processes. New structural properties and Hermite–Hadamard inequalities for Godunova–Levin mappings were presented by [28], while [29] offered generalizations of integral inequalities for harmonical cr-(h_1, h_2)-Godunova–Levin functions. [30] studied k-Riemann–Liouville Maclaurin-type inequalities for s-convex stochastic processes, and [31] introduced novel fractional Hermite–Hadamard and product-type inequalities via the Raina function and preinvex mappings with entropy applications. [32] examined Hermite–Hadamard-type inequalities for convex stochastic processes using the Katugampola fractional integral, while [33] provided novel estimates of Hermite–Hadamard and Jensen-type inequalities for (h_1, h_2)-convex functions pertaining to total order relations. Further contributions include [34] on fractional integral inequalities for stochastic processes with quasi-convex derivatives, [35] on inequalities for (h_1, h_2)-convex stochastic processes via interval set inclusion, and [36] on parametric inequalities for s-convex stochastic processes via Caputo fractional derivatives. [37] discussed Ostrowski and Čebyšev-type inequalities for interval-valued functions, while [38] presented Hermite–Hadamard and Jensen-type inequalities via the Riemann integral operator for a generalized class of Godunova–Levin functions. [39] introduced new variations and structural refinements of discrete weighted Jensen and Hermite–Hadamard inequalities using (α, m)-convex mappings. [40] derived new Hermite–Hadamard and Jensen inequalities for log-s-convex fuzzy-interval-valued functions, and finally, [41] investigated Ostrowski-type inequalities via convex, s-convex, and quasi-convex stochastic processes. This work synthesizes these diverse contributions and aims to further advance the theory of stochastic integral inequalities.

The literature in the present decade has been further enriched with the incorporation of sophisticated methods of calculus; that is, quantum calculus [42] and other fractional operators. Recent work has proposed quantum Hermite-Hadamard integral inequalities [43], both with generalized versions of those classical bounds, including with generalized P-convex stochastic processes in mind, have been proposed [44]. The most recent attention that has been drawn by researchers is on interval-valued processes and center- radius order relations. New fractional integrals of Jensen-Mercer and Hermite-Hadamard-Mercer types [45] have been developed, along with new estimates of h-Godunova-Levin processes [46,47] and other types as well as inequalities in fractional Hermite-Hadamard-Mercer types [48]. For a broader discussion of inequalities associated with stochastic processes, the reader is referred to the following articles and the references therein [49–51,53].

The motivation of the present work and its significance are demonstrated by the following major contributions, which extend and improve existing studies.

- Our newly defined notion recovers and generalizes several previously established definitions, including classical convex stochastic processes [7], h-convex stochastic processes [15], Godunova–Levin stochastic processes [46], and p-convex stochastic processes [44]; moreover, the associated results in these settings arise as special cases of our new results.
- Furthermore, to validate our results, we construct non-trivial examples and provide graphical verification by defining stochastic functions that incorporate Wiener processes.
- Moreover, we employ several new assumptions and identities to establish these results. Through appropriate remarks, we demonstrate that a number of earlier results are obtained as special cases of our findings.

The arrangement of this article is as follows. In Section 2, we recall some fundamental notions and auxiliary results related to stochastic processes and convexity concepts. Section 3 is devoted to the main results, where we first establish several new properties of the proposed framework and then derive novel Ostrowski-, Hermite–Hadamard-, and Jensen-type inequalities. Finally, in Section 4, we conclude the paper with a summary of our findings and outline some possible directions for future research.

2. Preliminaries and Background

In this section, we recall some fundamental concepts related to random variables and stochastic processes, together with their regularity properties and several relevant auxiliary results.

2.1. **Preliminaries on Stochastic Processes.** We begin by recalling some standard notions related to stochastic processes that will be used throughout this work.

2.2. **Probability Space and Random Variables.** Let $(\Pi, \mathcal{L}, \mathbb{P})$ be a probability space. A *random variable* is a measurable mapping

$$c : \Pi \to \mathbb{R},$$

that is, c is $\mathcal{L}$-measurable.

2.3. **Stochastic Processes.**

Definition 2.1 ([52]). *A* stochastic process *is a mapping*

$$\psi : J \times \Pi \to \mathbb{R},$$

where $J \subset \mathbb{R}$ is an index set. For each fixed $s \in J$, the function

$$\psi(s, \cdot) : \Pi \to \mathbb{R}$$

is a random variable. For each fixed $\omega \in \Pi$, the mapping $s \mapsto \psi(s, \omega)$ is called a sample path *(or realization) of the process.*

Thus, a stochastic process may be viewed as a family of random variables indexed by the parameter $s \in J$.

2.4. **Regularity Properties.** Let $\psi : J \times \Pi \to \mathbb{R}$ be a stochastic process defined on $(\Pi, \mathcal{L}, \mathbb{P})$.
Continuity in Probability. The process ψ is said to be *continuous in probability* at $s_0 \in J$ if

$$\lim_{s \to s_0} \mathbb{P}\Big(\big|\psi(s, \cdot) - \psi(s_0, \cdot)\big| > \varepsilon\Big) = 0 \quad \text{for every } \varepsilon > 0.$$

Mean Square Continuity. The process ψ is called *mean square continuous* at $s_0 \in J$ if

$$\lim_{s \to s_0} \mathbb{E}\Big[\Big(\psi(s, \cdot) - \psi(s_0, \cdot)\Big)^2\Big] = 0.$$

Mean square continuity implies continuity in probability.
Mean Square Differentiability. The process ψ is said to be *mean square differentiable* at $s_0 \in J$ if there exists a random variable $\psi'(s_0, \cdot)$ such that

$$\lim_{s \to s_0} \mathbb{E}\left[\left(\frac{\psi(s, \cdot) - \psi(s_0, \cdot)}{s - s_0} - \psi'(s_0, \cdot)\right)^2\right] = 0.$$

Mean Square Integrability. Assume that

$$\mathbb{E}\Big[\psi(s,\cdot)^2\Big] < \infty \quad \text{for all } s \in J.$$

A random variable $w : \Pi \to \mathbb{R}$ is called the *mean square integral* of ψ over $[a,b] \subset J$ if

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{k=1}^{n} \psi(\xi_k,\cdot)(s_k - s_{k-1}) - w(\cdot)\right)^2\right] = 0,$$

for every normal sequence of partitions

$$a = s_0 < s_1 < \cdots < s_n = b, \quad \xi_k \in [s_{k-1}, s_k].$$

In this case, we write

$$w(\cdot) = \int_a^b \psi(s,\cdot)\, ds.$$

The mean square integral satisfies the monotonicity property: if

$$\psi_1(s,\cdot) \le \psi_2(s,\cdot) \quad \text{a.e. on } [a,b],$$

then

$$\int_a^b \psi_1(s,\cdot)\, ds \le \int_a^b \psi_2(s,\cdot)\, ds.$$

We now present examples that clarify the above concepts and motivate the convexity-based inequalities studied in subsequent sections.

Example 2.1 (Deterministic Process)**.** *Let*

$$\psi(s,\omega) = s^2, \quad (s,\omega) \in J \times \Pi.$$

Then ψ is a stochastic process with deterministic sample paths. It is continuous, mean square continuous, mean square differentiable, and

$$\int_a^b \psi(s,\cdot)\, ds = \frac{b^3 - a^3}{3}.$$

Example 2.2 (Process Involving a Wiener Process)**.** *Let $\{W(s) : s \ge 0\}$ be a standard Wiener process on $(\Pi, \mathcal{L}, \mathbb{P})$, and define*

$$\psi(s,\omega) = s^2 + W(s,\omega), \quad s \in J \subset (0,\infty).$$

Then ψ is a stochastic process with non-deterministic sample paths. Since

$$\mathbb{E}[W(s)] = 0, \qquad \mathbb{E}[W(s)^2] = s,$$

we obtain

$$\mathbb{E}[\psi(s,\cdot)^2] = s^4 + s < \infty,$$

which shows that ψ is mean square integrable and mean square continuous. However, ψ is not mean square differentiable due to the irregularity of the Wiener process.

Example 2.3 (Convexity-Oriented Stochastic Process). *Let $p \geq 1$ and $\sigma > 0$, and define*

$$\psi(s,\omega) = s^p + \sigma W(s,\omega).$$

Then

$$\mathbb{E}[\psi(s,\cdot)] = s^p,$$

which is convex on $(0,\infty)$. Hence, ψ provides a natural stochastic framework for validating Jensen-, Hermite–Hadamard-, and Ostrowski-type inequalities under random perturbations.

Remark 2.1. *The above examples illustrate that stochastic processes involving Wiener noise naturally satisfy the integrability and continuity assumptions required in convexity-based stochastic inequalities, while reducing to the classical deterministic setting in expectation.*

We now proceed to the definition of an η-convex stochastic process.

2.5. η-Convex Stochastic Processes.

Definition 2.2 ([11]). *Let $(\Pi, \mathcal{A}, \mathbb{P})$ be a probability space and let $J \subset \mathbb{R}$ be an interval. A stochastic process $\psi : J \times \Pi \to \mathbb{R}$ is said to be an η-*convex stochastic process *if*

$$\psi(sm + (1-s)n, \cdot) \leq \psi(n,\cdot) + s\,\eta(\psi(m,\cdot), \psi(n,\cdot))\,, \tag{2.1}$$

for all $m, n \in J$ and $s \in [0,1]$, where $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given bifunction.

Remarks.

(1) If $\eta(x,y) = x - y$, then η-convexity reduces to the classical convex stochastic process, i.e.,

$$\psi(sm + (1-s)n, \cdot) \leq s\psi(m,\cdot) + (1-s)\psi(n,\cdot).$$

(2) Taking $s = 1$ in the definition yields

$$\psi(m,\cdot) - \psi(n,\cdot) \leq \eta(\psi(m,\cdot), \psi(n,\cdot))\,, \quad \forall\, m, n \in J.$$

(3) Consequently, in order to define an η-convex stochastic process on an interval J, it is natural to assume that

$$\eta(x,y) \geq x - y, \quad \forall\, x, y \in \mathbb{R}. \tag{2.2}$$

Under the above assumption on η, if $\psi : J \times \Pi \to \mathbb{R}$ is a stochastic process and $\eta : \psi(J) \times \psi(J) \to \mathbb{R}$ satisfies $\eta(x,y) \geq x - y$, then for all $m, n \in J$ and $s \in [0,1]$,

$$\psi(sm + (1-s)n, \cdot) \leq \psi(n,\cdot) + s\Big(\psi(m,\cdot) - \psi(n,\cdot)\Big) \leq \psi(n,\cdot) + s\,\eta(\psi(m,\cdot), \psi(n,\cdot)),$$

which shows that ψ is η-convex.

2.5.1. *h-Convex Stochastic Process.*

Definition 2.3 ([15]). *Let $h : (0,1) \to [0,\infty)$ be a given function. A non-negative stochastic process $\psi : J \times \Pi \to \mathbb{R}$ is called an h-*convex stochastic process *if*

$$\psi(sm + (1-s)n, \cdot) \leq h(s)\psi(m, \cdot) + h(1-s)\psi(n, \cdot), \tag{2.3}$$

for all $m, n \in J$ and $s \in (0,1)$.

2.5.2. *p-Convex Stochastic Process.*

Definition 2.4 ([44]). *Let $p > 0$. A non-negative stochastic process $\psi : J \times \Pi \to \mathbb{R}$ is called a p-*convex stochastic process *if*

$$\psi\Big(\big(sm^p + (1-s)n^p\big)^{1/p}, \cdot\Big) \leq s\psi(m, \cdot) + (1-s)\psi(n, \cdot), \tag{2.4}$$

for all $m, n \in J$ and $s \in (0,1)$.

2.5.3. *Generalized p-Convex Stochastic Process.*

Definition 2.5 ([44]). *Let $p > 0$. A non-negative stochastic process $\psi : J \times \Pi \to \mathbb{R}$ is said to be a* generalized p-convex stochastic process *if*

$$\psi\Big(\big(sx^p + (1-s)y^p\big)^{1/p}, \cdot\Big) \leq \psi(y, \cdot) + s\,\eta(\psi(x, \cdot), \psi(y, \cdot)), \tag{2.5}$$

for all $x, y \in J$ and $s \in [0,1]$.

2.5.4. *(p,h)-Convex Stochastic Process.*

Definition 2.6 ([54]). *Let $p > 0$ and let $h : (0,1) \to [0,\infty)$ be a non-zero function. Assume that $J \subset \mathbb{R}$ is a p-convex set. A non-negative stochastic process $\psi : J \times \Pi \to \mathbb{R}$ is called a (p,h)-*convex stochastic process *if*

$$\psi\Big(\big(sm^p + (1-s)n^p\big)^{1/p}, \cdot\Big) \leq h(s)\psi(m, \cdot) + h(1-s)\psi(n, \cdot), \tag{2.6}$$

for all $m, n \in J$ and $s \in (0,1)$.

3. The major results

Definition 3.1 (Generalized Modified (p,h)-Convex Stochastic Process). *Let $p > 0$ and let $h : (0,1) \to [0,\infty)$ be a non-zero function. A non-negative stochastic process $\psi : J \times \Pi \to \mathbb{R}$ is called a* generalized modified (p,h)-convex stochastic process *if*

$$\psi\Big(\big(sm^p + (1-s)n^p\big)^{1/p}, \cdot\Big) \leq \psi(n, \cdot) + \eta(h(s)\psi(m, \cdot),\, h(1-s)\psi(n, \cdot)), \tag{3.1}$$

for all $m, n \in J$ and $s \in (0,1)$.

Remark 3.1. *The class of generalized modified (p,h)-convex stochastic processes introduced above unifies and extends several well-known notions of convexity in the stochastic setting. In particular, under appropriate choices of the functions p, h, and η, one recovers the following special cases:*

- *If $p = 1$, $h(s) = s$, and $\eta(x, y) = x + y$, then the defining inequality reduces to*

$$\psi(sm + (1 - s)n, \cdot) \leq s\psi(m, \cdot) + (1 - s)\psi(n, \cdot),$$

  which corresponds to the classical convex stochastic process.
- *If $p = 1$ and $\eta(x, y) = x + y$, then the above definition reduces to the notion of an h-convex stochastic process.*
- *If $h(s) = s$ and $\eta(x, y) = x + y$, then one obtains the p-convex stochastic process.*
- *If $h(s) = s$ and $\eta(x, y) = x - y$, then the generalized modified (p, h)-convexity reduces to the η-convex stochastic process.*
- *If $\eta(x, y) = x - y$, then the definition coincides with the generalized p-convex stochastic process.*
- *If $\eta(x, y) = x + y$, then the definition reduces to the (p, h)-convex stochastic process.*

Hence, the generalized modified (p, h)-convex stochastic process provides a unified framework that not only recovers classical convex, h-convex, p-convex, η-convex, and (p, h)-convex stochastic processes, but also allows greater flexibility through the interaction of the functions p, h, and η. As a consequence, all results corresponding to these classical notions follow as special cases of the results established in this work.

Proposition 3.1 (Algebraic Properties of Generalized Modified (p, h)-Convex Stochastic Processes)**.** *Let $\psi_1, \psi_2 : I \times \Pi \to \mathbb{R}$ be generalized modified (p, h)-convex stochastic processes, where $I \subset (0, \infty)$, $p \in \mathbb{R} \setminus \{0\}$, and $h : [0, 1] \to [0, \infty)$. Then the following assertions hold:*

(1) ***Sum of Processes:*** *The sum*

$$(\psi_1 + \psi_2)(x, \omega) := \psi_1(x, \omega) + \psi_2(x, \omega)$$

is also a generalized modified (p, h)-convex stochastic process.

(2) ***Scalar Multiplication:*** *For any scalar $\gamma \geq 0$, the function*

$$(\gamma\psi_1)(x, \omega) := \gamma \cdot \psi_1(x, \omega)$$

is a generalized modified (p, h)-convex stochastic process.

Proof. Let $m, n \in I$, $s \in [0, 1]$, and $\omega \in \Pi$ be arbitrary. Define the p-mean point

$$x_{s,m,n} := \left(sm^p + (1 - s)n^p\right)^{1/p}.$$

Sum of Processes: By the definition of generalized modified (p, h)-convexity, for any $\omega \in \Pi$ we have

$$\psi_1(x_{s,m,n}, \omega) \leq h(s)\psi_1(m, \omega) + (1 - h(s))\psi_1(n, \omega),$$
$$\psi_2(x_{s,m,n}, \omega) \leq h(s)\psi_2(m, \omega) + (1 - h(s))\psi_2(n, \omega).$$

Adding these two inequalities, and using the linearity of addition, we get:

$$\begin{aligned}\psi_1(x_{s,m,n}, \omega) + \psi_2(x_{s,m,n}, \omega) \leq &\left[h(s)\psi_1(m, \omega) + (1 - h(s))\psi_1(n, \omega)\right] \\ &+ \left[h(s)\psi_2(m, \omega) + (1 - h(s))\psi_2(n, \omega)\right]\end{aligned}$$

$$= h(s)\Big(\psi_1(m,\omega)+\psi_2(m,\omega)\Big) + (1-h(s))\Big(\psi_1(n,\omega)+\psi_2(n,\omega)\Big)$$
$$= h(s)(\psi_1+\psi_2)(m,\omega) + (1-h(s))(\psi_1+\psi_2)(n,\omega).$$

Since $m, n \in I$ and $\omega \in \Pi$ were arbitrary, this shows that $\psi_1 + \psi_2$ satisfies the generalized modified (p,h)-convexity condition at all points, hence it is a generalized modified (p,h)-convex stochastic process.

Scalar Multiplication: Let $\gamma \geq 0$. By definition,

$$(\gamma\psi_1)(x_{s,m,n},\omega) = \gamma \cdot \psi_1(x_{s,m,n},\omega).$$

Using the (p,h)-convexity of ψ_1, we have

$$\gamma \cdot \psi_1(x_{s,m,n},\omega) \leq \gamma\Big[h(s)\psi_1(m,\omega) + (1-h(s))\psi_1(n,\omega)\Big].$$

Since $\gamma \geq 0$, multiplying by γ preserves the inequality, so we can distribute γ:

$$(\gamma\psi_1)(x_{s,m,n},\omega) \leq h(s)(\gamma\psi_1)(m,\omega) + (1-h(s))(\gamma\psi_1)(n,\omega).$$

This confirms that $\gamma\psi_1$ also satisfies the generalized modified (p,h)-convexity inequality, and hence is a generalized modified (p,h)-convex stochastic process.

□

Proposition 3.2 (Maximum Bound for Generalized Modified (p,h)-Convex Stochastic Processes)**.** *Let $\psi : [b_1, b_2] \times \Pi \to \mathbb{R}$ be a generalized modified (p,h)-convex stochastic process. Then, for any $x \in [b_1, b_2]$ and $\omega \in \Pi$, we have*

$$\psi(x,\omega) \leq \max\Big\{\psi(b_2,\omega),\ \psi(b_2,\omega) + \eta\, h(s)(\psi(b_1,\omega), \psi(b_2,\omega))\Big\}, \tag{3.2}$$

where $\eta(\alpha,\beta) = \alpha - \beta$.

Proof. Let $x \in [b_1, b_2]$ be arbitrary. Since x^p lies between b_1^p and b_2^p, there exists $s \in [0,1]$ such that

$$x^p = sb_1^p + (1-s)b_2^p.$$

Hence, we can write

$$x = \Big(sb_1^p + (1-s)b_2^p\Big)^{1/p},$$

and therefore

$$\psi(x,\omega) = \psi\Big((sb_1^p + (1-s)b_2^p)^{1/p},\omega\Big).$$

By the generalized modified (p,h)-convexity of ψ, we have

$$\psi\Big((sb_1^p + (1-s)b_2^p)^{1/p},\omega\Big) \leq h(s)\psi(b_1,\omega) + (1-h(s))\psi(b_2,\omega).$$

Rewriting the right-hand side, we factor terms to make the η-function explicit:

$$h(s)\psi(b_1,\omega) + (1-h(s))\psi(b_2,\omega) = \psi(b_2,\omega) + h(s)\Big(\psi(b_1,\omega) - \psi(b_2,\omega)\Big)$$
$$= \psi(b_2,\omega) + \eta\, h(s)(\psi(b_1,\omega), \psi(b_2,\omega)).$$

Since $x \in [b_1, b_2]$ was arbitrary, we can bound $\psi(x,\omega)$ by the larger of the two quantities:

$$\psi(x,\omega) \le \max\Big\{\psi(b_2,\omega),\ \psi(b_2,\omega) + \eta\, h(s)(\psi(b_1,\omega), \psi(b_2,\omega))\Big\}.$$

This completes the proof. □

Proposition 3.3 (Maximum of Generalized Modified (p,h)-Convex Processes)**.** *Let $I \subset (0,\infty)$ be an interval, $p \in \mathbb{R} \setminus \{0\}$, and $h : [0,1] \to [0,1]$. Let $\{f_j : I \times \Pi \to \mathbb{R} \mid j \in J\}$ be a non-empty collection of generalized modified (p,h)-convex stochastic processes. Assume that for each $u \in I$ and $\omega \in \Pi$, the maximum*

$$\max_{j\in J} f_j(u,\omega)$$

exists.

Define

$$f(u,\omega) := \max_{j\in J} f_j(u,\omega).$$

Then f is a generalized modified (p,h)-convex stochastic process, and consequently a generalized modified (p,h,η)-convex stochastic function with $\eta(\alpha,\beta) = \alpha - \beta$.

Proof. Let $u, v \in I$, $s \in [0,1]$, and $\omega \in \Pi$, and define the p-mean point

$$x_{s,u,v} := \Big(su^p + (1-s)v^p\Big)^{1/p}.$$

By definition,

$$f(x_{s,u,v},\omega) = \max_{j\in J} f_j(x_{s,u,v},\omega),$$

and since each f_j is generalized modified (p,h)-convex,

$$f_j(x_{s,u,v},\omega) \le h(s) f_j(u,\omega) + (1-h(s)) f_j(v,\omega), \quad \forall j \in J.$$

Taking the maximum over $j \in J$ and using the property $\max_j(a_j + b_j) \le \max_j a_j + \max_j b_j$ gives

$$\begin{aligned} f(x_{s,u,v},\omega) &\le \max_{j\in J}\Big(h(s) f_j(u,\omega) + (1-h(s)) f_j(v,\omega)\Big) \\ &\le h(s)\max_{j\in J} f_j(u,\omega) + (1-h(s))\max_{j\in J} f_j(v,\omega) \\ &= h(s) f(u,\omega) + (1-h(s)) f(v,\omega). \end{aligned}$$

Expanding $1 - h(s)$ yields

$$f(x_{s,u,v},\omega) \le f(v,\omega) + h(s)\Big(f(u,\omega) - f(v,\omega)\Big) = f(v,\omega) + h(s)\,\eta(f(u,\omega), f(v,\omega)),$$

with $\eta(\alpha,\beta) = \alpha - \beta$.

Hence, f is a generalized modified (p,h)-convex stochastic process. □

Lemma 3.1. *Let $(\Pi, \mathcal{L}, \mathbb{P})$ be a probability space and $J \subset \mathbb{R}$ be an interval. Let $\psi : J \times \Pi \to \mathbb{R}$ be a stochastic process of the form*

$$\psi(s,\cdot) = A_1(\cdot)s + A_2(\cdot), \tag{3.3}$$

where $A_1, A_2 : \Pi \to \mathbb{R}$ are random variables such that $\mathbb{E}[A_1^2] < \infty$, $\mathbb{E}[A_2^2] < \infty$ and $[c_1, c_2] \subset J$. Then the mean square integral of ψ exists and satisfies:

$$\int_{c_1}^{c_2} \psi(s,\cdot)\, ds = A_1(\cdot)\frac{c_2^2 - c_1^2}{2} + A_2(\cdot)(c_2 - c_1) \quad (a.e.). \tag{3.4}$$

Proof. Let $P_n = \{c_1 = v_0, v_1, \ldots, v_n = c_2\}$ be a sequence of partitions of $[c_1, c_2]$ such that $\lim_{n\to\infty} \Delta v_k = 0$. Consider the Riemann sum S_n:

$$S_n(\cdot) = \sum_{k=1}^{n} \psi(M_k, \cdot)\Delta v_k, \quad M_k \in [v_{k-1}, v_k].$$

Substituting the definition of $\psi(s,\cdot)$:

$$\begin{aligned} S_n(\cdot) &= \sum_{k=1}^{n} \left(A_1(\cdot)M_k + A_2(\cdot)\right) \Delta v_k \\ &= A_1(\cdot)\sum_{k=1}^{n} M_k\Delta v_k + A_2(\cdot)\sum_{k=1}^{n} \Delta v_k. \end{aligned}$$

We define the target random variable $w(\cdot) = A_1(\cdot)\frac{c_2^2-c_1^2}{2} + A_2(\cdot)(c_2 - c_1)$. To show $S_n \to w$ in mean square, we examine:

$$\mathbb{E}\left[(S_n - w)^2\right] = \mathbb{E}\left[\left(A_1\left(\sum_{k=1}^{n} M_k\Delta v_k - \frac{c_2^2 - c_1^2}{2}\right) + A_2\left(\sum_{k=1}^{n} \Delta v_k - (c_2 - c_1)\right)\right)^2\right].$$

Let $E_1^{(n)} = \left(\sum M_k\Delta v_k - \int_{c_1}^{c_2} s\, ds\right)$ and $E_2^{(n)} = \left(\sum \Delta v_k - \int_{c_1}^{c_2} 1\, ds\right)$. Both $E_1^{(n)}, E_2^{(n)} \to 0$ as $n \to \infty$ by classical Riemann integration. Using the Cauchy-Schwarz inequality $\mathbb{E}[(X + Y)^2] \le 2\mathbb{E}[X^2] + 2\mathbb{E}[Y^2]$:

$$\mathbb{E}\left[(S_n - w)^2\right] \le 2(E_1^{(n)})^2\mathbb{E}[A_1^2] + 2(E_2^{(n)})^2\mathbb{E}[A_2^2].$$

Since $\mathbb{E}[A_1^2]$ and $\mathbb{E}[A_2^2]$ are finite and $E_i^{(n)} \to 0$, it follows that:

$$\lim_{n\to\infty} \mathbb{E}\left[(S_n - w)^2\right] = 0.$$

Thus, the mean square integral is established. □

Theorem 3.1. *Let $\psi : J \times \Pi \to \mathbb{R}$ be a stochastic process and $h : J \subset \mathbb{R} \to [0, 1]$. Then ψ is a generalized modified (p, h)-convex stochastic process if and only if, for any $m_1, m_2, m_3 \in J$ with $m_1 \le m_2 \le m_3$, the following determinant inequality holds:*

$$\det\begin{pmatrix} h(m_3^p - m_2^p) & \psi(m_2,\cdot) - \psi(m_3,\cdot) \\ h(m_3^p - m_1^p) & \eta\Big(\psi(m_1,\cdot), \psi(m_3,\cdot)\Big) \end{pmatrix} \le 0.$$

Proof. Assume ψ is a generalized modified (p, h)-convex stochastic process. Take any $m_1, m_2, m_3 \in J$ with $m_1 \le m_2 \le m_3$. Then there exists $\beta \in (0, 1)$ such that

$$m_2^p = \beta m_1^p + (1 - \beta)m_3^p, \quad \text{so that} \quad \beta = \frac{m_3^p - m_2^p}{m_3^p - m_1^p}.$$

By the (p,h)-convexity of ψ, we have

$$\psi(m_2,\cdot) = \psi\Big([\beta m_1^p + (1-\beta)m_3^p]^{1/p},\cdot\Big)$$
$$\le \psi(m_3,\cdot) + h(\beta)\,\eta\Big(\psi(m_1,\cdot),\psi(m_3,\cdot)\Big),$$

where $\eta(\alpha,\beta) = \alpha - \beta$.

Rewriting, we get

$$\psi(m_2,\cdot) - \psi(m_3,\cdot) \le h(\beta)\,\eta\Big(\psi(m_1,\cdot),\psi(m_3,\cdot)\Big).$$

Multiplying both sides by $h(m_3^p - m_1^p) > 0$ and noting

$$h(\beta) = \frac{h(m_3^p - m_2^p)}{h(m_3^p - m_1^p)},$$

we obtain

$$h(m_3^p - m_1^p)\Big(\psi(m_2,\cdot) - \psi(m_3,\cdot)\Big) \le h(m_3^p - m_2^p)\,\eta\Big(\psi(m_1,\cdot),\psi(m_3,\cdot)\Big),$$

which is exactly the determinant inequality.

Conversely, assume the determinant inequality holds for all $m_1 \le m_2 \le m_3 \in J$. Fix $m_1, m_3 \in J$ and choose any $\beta \in (0,1)$. Let

$$m_2^p = \beta m_1^p + (1-\beta)m_3^p.$$

By the determinant condition, we have

$$\psi(m_2,\cdot) - \psi(m_3,\cdot) \le h(\beta)\,\eta\Big(\psi(m_1,\cdot),\psi(m_3,\cdot)\Big),$$

or equivalently,

$$\psi\Big([\beta m_1^p + (1-\beta)m_3^p]^{1/p},\cdot\Big) \le \psi(m_3,\cdot) + h(\beta)\,\eta\Big(\psi(m_1,\cdot),\psi(m_3,\cdot)\Big).$$

Since $m_1, m_3 \in J$ and $\beta \in (0,1)$ were arbitrary, this verifies the definition of generalized modified (p,h)-convexity for ψ.

Hence, the necessity and sufficiency are both proved. □

3.1. Novel Recursive Jensen-Type Inequalities for Generalized (η, p, h)-Convex Stochastic Processes.

Theorem 3.2. *Consider $\psi : J \times \Pi \to \mathbb{R}$ as a modified (p,h)-convex stochastic process, and let $\eta : X \times Y \to \mathbb{R}$ be non-decreasing and non-negatively sub-linear in the first variable.*

Define

$$M_r = \sum_{s=1}^{r} \beta_s, \quad r = 1,2,\ldots,k, \quad \text{with } M_k = 1.$$

Then

$$\psi\left[\Big(\sum_{r=1}^{k}(\beta_1 l_1^p + \beta_2 l_2^p)\Big)^{1/p},\cdot\right] \le \psi_k(\cdot) + \sum_{r=1}^{k-1} h(M_r)\,\eta_\psi(l_r, l_{r+1},\ldots,l_k,\cdot), \tag{3.5}$$

where

$$\eta_\psi(l_r, l_{r+1}, \ldots, l_k, \cdot) := \eta\Big(\eta_\psi(l_r, \ldots, l_{k-1}, \cdot), \psi(l_k, \cdot)\Big), \quad \eta_\psi(l, \cdot) := \psi(l, \cdot), \quad \forall l \in J.$$

Proof. By the Jensen-type inequality, we start with

$$\psi\Big[\Big(\sum_{r=1}^{k} \beta_r l_r^p\Big)^{1/p}, \cdot\Big] \leq h(M_{k-1})\, \psi\Big[\Big(\sum_{r=1}^{k-1} \frac{\beta_r l_r^p}{M_{k-1}} + l_k^p\Big)^{1/p}, \cdot\Big] + (1 - h(M_{k-1}))\psi_k(\cdot).$$

Step by step, we have:

$$\leq h(M_{k-1})\, \psi\Big[\Big(\sum_{r=1}^{k-1} \frac{\beta_r l_r^p}{M_{k-1}}\Big)^{1/p}, \cdot\Big] + (1 - h(M_{k-1}))\psi_k(\cdot)$$

$$\Longrightarrow \leq \psi_k(\cdot) + h(M_{k-1})\Big[\psi\Big(\Big(\sum_{r=1}^{k-1} \frac{\beta_r l_r^p}{M_{k-1}}\Big)^{1/p}, \cdot\Big) - \psi_k(\cdot)\Big]$$

$$\leq \psi_k(\cdot) + h(M_{k-1})\, \eta\Big[\psi\Big(\frac{M_{k-2}}{M_{k-1}}\Big(\sum_{r=1}^{k-2} \frac{\beta_r l_r^p}{M_{k-2}}\Big)^{1/p} + \frac{\beta_{k-1} l_{k-1}}{M_{k-2}}, \cdot\Big), \psi_k(\cdot)\Big]$$

$$\leq \psi_k(\cdot) + h(M_{k-1})\, \eta\Big[h\Big(\frac{M_{k-2}}{M_{k-1}}\Big)\psi\Big(\sum_{r=1}^{k-2} \frac{\beta_r l_r^p}{M_{k-2}}^{1/p}, \cdot\Big) + \Big(1 - h\Big(\frac{M_{k-2}}{M_{k-1}}\Big)\Big)\psi(l_{k-1}, \cdot), \psi(l_k, \cdot)\Big]$$

$$\leq \psi_k(\cdot) + h(M_{k-1}) h\Big(\frac{M_{k-2}}{M_{k-1}}\Big)\eta\Big[\psi\Big(\sum_{r=1}^{k-2} \frac{\beta_r l_r^p}{M_{k-2}}^{1/p}, \cdot\Big) + \psi(l_{k-1}, \cdot) - h\Big(\frac{M_{k-2}}{M_{k-1}}\Big)\psi(l_{k-1}, \cdot), \psi(l_k, \cdot)\Big]$$

$$\leq \psi_k(\cdot) + h(M_{k-1})\eta\Big[\psi(l_{k-1}, \cdot) + h\Big(\frac{M_{k-2}}{M_{k-1}}\Big)\eta\Big[\psi\Big(\sum_{r=1}^{k-2} \frac{\beta_r l_r^p}{M_{k-2}}^{1/p}, \cdot\Big), \psi(l_{k-1}, \cdot)\Big], \psi(l_k, \cdot)\Big].$$

Continuing recursively in the same manner, we finally obtain

$$\psi\Big[\Big(\sum_{r=1}^{k} (\beta_1 l_1^p + \beta_2 l_2^p)\Big)^{1/p}, \cdot\Big] \leq \psi_k(\cdot) + \sum_{r=1}^{k-1} h(M_r)\, \eta_\psi(l_r, l_{r+1}, \ldots, l_k, \cdot),$$

which proves the statement. □

Example 3.1. *Under the assumptions of Theorem 3.2, consider $J = [0, 1]$ and let*

$$\psi(x, \omega) = x + 0.1\,\omega,$$

where ω is a fixed sample of a Wiener process at each x (for simplicity, take $\omega = 0.5$).

Take $p = 1$, $k = 3$, $\beta_1 = \beta_2 = \beta_3 = \frac{1}{3}$, and $h(x) = x$, $\eta(a, b) = |a - b|$. Let

$$l_1 = 0.2, \quad l_2 = 0.5, \quad l_3 = 0.8.$$

Compute

$$M_1 = \beta_1 = \frac{1}{3}, \quad M_2 = \beta_1 + \beta_2 = \frac{2}{3}.$$

The left-hand side (LHS) of the inequality is

$$\psi\left(\sum_{r=1}^{3}\beta_r l_r,\omega\right)=\psi\left(\frac{1}{3}(0.2+0.5+0.8),0.5\right)=\psi(0.5,0.5)=0.5+0.1\cdot 0.5=0.55.$$

Now compute the right-hand side (RHS):

$$\psi_3(\omega)=\psi(l_3,0.5)=0.8+0.1\cdot 0.5=0.85,$$

$$\eta_\psi(l_2,l_3,\omega)=|\psi(l_2,0.5)-\psi(l_3,0.5)|=|0.55-0.85|=0.3,$$

$$\eta_\psi(l_1,l_2,l_3,\omega)=|\psi(l_1,0.5)-(\psi(l_2,0.5)+\eta_\psi(l_2,l_3,\omega))|=|0.25-(0.55+0.3)|=0.$$

Then

$$RHS=\psi_3(\omega)+h(M_2)\eta_\psi(l_2,l_3,\omega)+h(M_1)\eta_\psi(l_1,l_2,l_3,\omega)=0.85+\frac{2}{3}\cdot 0.3+\frac{1}{3}\cdot 0=0.85+0.2=1.05.$$

Thus, we verify numerically that

$$\psi\left(\sum_{r=1}^{3}\beta_r l_r,\omega\right)=0.55\le 1.05=\psi_3(\omega)+\sum_{r=1}^{k-1}h(M_r)\,\eta_\psi(l_r,\ldots,l_k,\omega),$$

and the recursive Jensen-type inequality holds for this specific example.

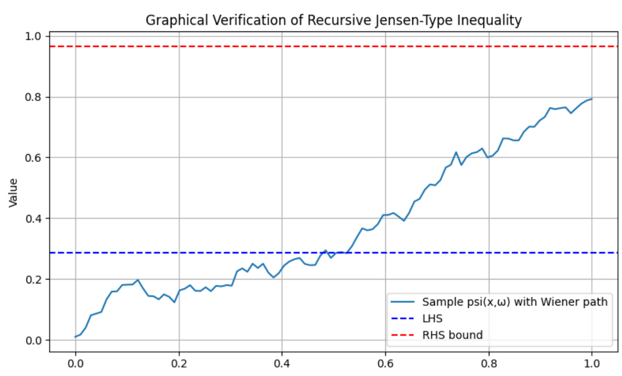

FIGURE 1. Graphical Verification of Jensen-type Inequality

3.2. **Novel Hermite-Hadamard-Type Inequality for Generalized (η,p,h)-Convex Stochastic Processes.** Now, we establish the Hermite-Hadamard type inequality for a generalized modified (p,h)-convex stochastic process.

Theorem 3.3. *Let $J\subset(0,1)$, $p>0$, and let $\psi:[l,m]\times\Pi\to\mathbb{R}$ be a measurable generalized modified (p,h)-convex stochastic process which is integrable. Then, for any $l,m\in J$ with $l<m$, we have*

$$\psi\left(\left[\frac{l^p+m^p}{2}\right]^{1/p},\cdot\right)\le\frac{p}{m^p-l^p}\int_l^m z^{p-1}\psi(z,\cdot)\,dz$$

$$\le \frac{\psi(l,\cdot)+\psi(m,\cdot)}{2} + \frac{1}{2}\int_0^1 \Big[\eta(\psi(l,\cdot),\psi(m,\cdot)) + \eta(\psi(m,\cdot),\psi(l,\cdot))\Big] h(q)\,dq.$$

Proof. Let

$$\delta_1^p = ql^p + (1-q)m^p, \quad \delta_2^p = (1-q)l^p + qm^p.$$

Then

$$\psi\Big(\Big[\frac{l^p+m^p}{2}\Big]^{1/p},\cdot\Big) = \psi\Big(\Big[\frac{\delta_1^p+\delta_2^p}{2}\Big]^{1/p},\cdot\Big).$$

Using the generalized modified (p,h)-convexity property:

$$\psi\Big(\Big[\frac{l^p+m^p}{2}\Big]^{1/p},\cdot\Big) \le \psi\Big([(1-q)l^p+qm^p]^{1/p},\cdot\Big) + h\Big(\frac{1}{2}\Big)\eta\Big(\psi([ql^p+(1-q)m^p]^{1/p},\cdot),\psi([(1-q)l^p+qm^p]^{1/p},\cdot)\Big).$$

Integrating with respect to q over $[0,1]$:

$$\psi\Big(\Big[\frac{l^p+m^p}{2}\Big]^{1/p},\cdot\Big) \le \int_0^1 \psi\Big([(1-q)l^p+qm^p]^{1/p},\cdot\Big)dq + h\Big(\frac{1}{2}\Big)\int_0^1 \eta\Big(\psi([ql^p+(1-q)m^p]^{1/p},\cdot),\psi([(1-q)l^p+qm^p]^{1/p},\cdot)\Big)dq.$$

Set $z^p = ql^p + (1-q)m^p$, then $pz^{p-1}dz = (m^p - l^p)dq$, so $dq = \frac{p}{m^p-l^p}z^{p-1}dz$. Substituting gives:

$$\int_0^1 \psi([(1-q)l^p+qm^p]^{1/p},\cdot)dq = \frac{p}{m^p-l^p}\int_l^m z^{p-1}\psi(z,\cdot)dz.$$

By convexity and linearity of integrals:

$$\begin{aligned}\frac{p}{m^p-l^p}\int_l^m z^{p-1}\psi(z,\cdot)dz &\le \int_0^1 \Big[h(q)\psi(l,\cdot)+(1-h(q))\psi(m,\cdot)\Big]dq \\ &= \psi(m,\cdot) + \int_0^1 h(q)\Big(\psi(l,\cdot)-\psi(m,\cdot)\Big)dq \\ &= \psi(m,\cdot) + \int_0^1 \eta\Big(h(q)(\psi(l,\cdot),\psi(m,\cdot))\Big)dq.\end{aligned}$$

Similarly, integrating in reversed order gives:

$$\frac{p}{m^p-l^p}\int_l^m z^{p-1}\psi(z,\cdot)dz \le \psi(l,\cdot) + \int_0^1 \eta\Big(h(q)(\psi(m,\cdot),\psi(l,\cdot))\Big)dq.$$

Adding the two bounds:

$$\begin{aligned}\frac{2p}{m^p-l^p}\int_l^m z^{p-1}\psi(z,\cdot)dz \le \psi(l,\cdot)+\psi(m,\cdot) &+ \int_0^1 \Big[\eta(h(q)(\psi(m,\cdot),\psi(l,\cdot))) \\ &+ \eta(h(q)(\psi(l,\cdot),\psi(m,\cdot)))\Big]dq.\end{aligned}$$

Dividing both sides by 2 gives the Hermite-Hadamard type inequality:

$$\frac{p}{m^p - l^p} \int_l^m z^{p-1} \psi(z, \cdot) dz \leq \frac{\psi(l, \cdot) + \psi(m, \cdot)}{2} + \frac{1}{2} \int_0^1 \Big[\eta(\psi(l, \cdot), \psi(m, \cdot)) + \eta(\psi(m, \cdot), \psi(l, \cdot)) \Big] h(q) dq.$$

□

Under the assumptions of Theorem 2, consider the stochastic process:

$$\psi(x, \omega) = x^2 + W_x(\omega), \quad x \in [0.1, 0.5],$$

where W_x is a standard Wiener process. Let $p = 1$, $l = 0.1$, $m = 0.5$.

Example 3.2. *Under the assumptions of Theorem 3.3, consider the stochastic process:*

$$\psi(x, \omega) = x^2 + W_x(\omega), \quad x \in [0.1, 0.5],$$

where W_x is a standard Wiener process. Let $p = 1$, $l = 0.1$, $m = 0.5$. We numerically verify the Hermite-Hadamard inequality:

$$\psi\Big(\frac{l+m}{2}, \omega\Big) \leq \frac{p}{m-l} \int_l^m \psi(z, \omega)\, dz \leq \frac{\psi(l, \omega) + \psi(m, \omega)}{2} + \frac{1}{2} \int_0^1 \Big[\eta(\psi(l, \omega), \psi(m, \omega)) + \eta(\psi(m, \omega), \psi(l, \omega)) \Big] h(q)\, dq.$$

For simplicity, take $h(q) = q$ and $\eta(a, b) = |a - b|$. Consider a sample realization of the Wiener process:

$$W_l = 0.0, \quad W_m = 0.3, \quad W_{\frac{l+m}{2}} = 0.15.$$

Then the values are

$$\psi\Big(\frac{l+m}{2}, \omega\Big) = 0.21, \quad \frac{1}{m-l} \int_l^m \psi(z, \omega)\, dz \approx 0.3617,$$

and the right-hand side bound is

$$\frac{\psi(l, \omega) + \psi(m, \omega)}{2} + \frac{1}{2} \int_0^1 |\psi(l, \omega) - \psi(m, \omega)| h(q)\, dq \approx 0.4375.$$

Hence, the Hermite-Hadamard inequality holds for this sample path:

$$0.21 \leq 0.3617 \leq 0.4375.$$

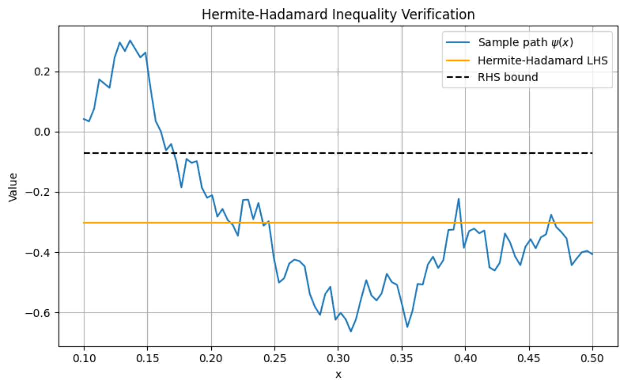

Figure 2. Graphical Verification of Hermite-Hadamard-type Inequality

3.3. **Novel Ostrowski-Type Inequality for Generalized (η, p, h)-Convex Stochastic Processes.** The following lemma is necessary to prove this inequality according to the generalized modified (p, h)-convex stochastic process.

Lemma 3.2. *Let $\psi : J \times \Pi \to \mathbb{R}$ be a generalized modified (p, h)-convex stochastic process that is mean square differentiable in J°. If ψ' is mean square integrable in the interval $[b_1, b_2]$, with $b_1 < b_2 \in J$ and $p \in \mathbb{R}$, then*

$$\begin{aligned}
&\psi(y,\cdot) - \frac{p}{b_2^p - b_1^p}\int_{b_1}^{b_2}\frac{\psi(v,\cdot)}{v^{1-p}}dv \\
&\leq \frac{1}{p(b_2^p - b_1^p)}\left[(y^p - b_1^p)^2\int_0^1 \frac{s}{(sy^p + (1-s)b_1^p)^{1-1/p}}\psi'\Big((sy^p + (1-s)b_1^p)^{1/p},\cdot\Big)ds\right] \\
&- \frac{1}{p(b_2^p - b_1^p)}\left[(b_2^p - y^p)^2\int_0^1 \frac{s}{(sy^p + (1-s)b_2^p)^{1-1/p}}\psi'\Big((sy^p + (1-s)b_2^p)^{1/p},\cdot\Big)ds\right].
\end{aligned}$$

Theorem 3.4. *Let $\psi : J \subseteq (0, \infty) \to \mathbb{R}$ be differentiable on J°, with $b_1, b_2 \in J^\circ$, $b_1 < b_2$, $p \neq 0$, and $\psi' \in L[b_1, b_2]$. If $|\psi'|^k$ is a generalized modified (p, h)-convex stochastic process in $[b_1, b_2]$ for $k \geq 1$, then for all $y \in [b_1, b_2]$:*

$$\begin{aligned}
&\left|\psi(y,\cdot) - \frac{p}{b_2^p - b_1^p}\int_{b_1}^{b_2}\frac{\psi(v,\cdot)}{v^{1-p}}dv\right| \\
&\leq \frac{(y^p - b_1^p)^2}{p(b_2^p - b_1^p)}\left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-1/p}}ds\right)^{1-1/k} \\
&\times\left(\int_0^1 \frac{s^k|\psi'(b_1,\cdot)|^k + h(s)\eta(|\psi'(y,\cdot)|^k, |\psi'(b_1,\cdot)|^k)}{(sy^p + (1-s)b_1^p)^{1-1/p}}ds\right)^{1/k} \\
&- \frac{(b_2^p - y^p)^2}{p(b_2^p - b_1^p)}\left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-1/p}}ds\right)^{1-1/k}
\end{aligned}$$

$$\times\left(\int_0^1 \frac{s^k|\psi'(b_2,\cdot)|^k + h(s)\eta(|\psi'(y,\cdot)|^k, |\psi'(b_2,\cdot)|^k)}{(sy^p + (1-s)b_2^p)^{1-1/p}} ds\right)^{1/k}.$$

Proof. From the lemma, we have:

$$\begin{aligned}
&\left|\psi(y,\cdot) - \frac{p}{b_2^p - b_1^p}\int_{b_1}^{b_2} \frac{\psi(v,\cdot)}{v^{1-p}} dv\right| \\
&\le \frac{(y^p - b_1^p)^2}{p(b_2^p - b_1^p)} \int_0^1 \frac{s\psi'((sy^p + (1-s)b_1^p)^{1/p},\cdot)}{(sy^p + (1-s)b_1^p)^{1-1/p}} ds \\
&- \frac{(b_2^p - y^p)^2}{p(b_2^p - b_1^p)} \int_0^1 \frac{s\psi'((sy^p + (1-s)b_2^p)^{1/p},\cdot)}{(sy^p + (1-s)b_2^p)^{1-1/p}} ds.
\end{aligned}$$

Applying Hölder's inequality and the generalized modified (p,h)-convexity of $|\psi'|^k$, we get:

$$\begin{aligned}
&\left|\psi(y,\cdot) - \frac{p}{b_2^p - b_1^p}\int_{b_1}^{b_2} \frac{\psi(v,\cdot)}{v^{1-p}} dv\right| \\
&\le \frac{(y^p - b_1^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_1^p)^{1-1/p}} ds\right)^{1-1/k} \\
&\quad \times\left(\int_0^1 \frac{s^k|\psi'(b_1,\cdot)|^k + h(s)\eta(|\psi'(y,\cdot)|^k, |\psi'(b_1,\cdot)|^k)}{(sy^p + (1-s)b_1^p)^{1-1/p}} ds\right)^{1/k} \\
&- \frac{(b_2^p - y^p)^2}{p(b_2^p - b_1^p)} \left(\int_0^1 \frac{1}{(sy^p + (1-s)b_2^p)^{1-1/p}} ds\right)^{1-1/k} \\
&\quad \times\left(\int_0^1 \frac{s^k|\psi'(b_2,\cdot)|^k + h(s)\eta(|\psi'(y,\cdot)|^k, |\psi'(b_2,\cdot)|^k)}{(sy^p + (1-s)b_2^p)^{1-1/p}} ds\right)^{1/k}.
\end{aligned}$$

This completes the proof. □

Example 3.3. *Under the assumptions of Theorem 3.4, let*

$$\psi(x,\omega) = x^2 + W(\omega), \quad x \in [1,2],$$

where $W(\omega)$ is a standard Wiener process. Let

$$p = 2, \quad k = 2, \quad h(s) = s, \quad \eta(a,b) = |a-b|.$$

With $y = 1.5$, $b_1 = 1$, and $b_2 = 2$, we evaluate the Ostrowski-type inequality.

The derivative of ψ is

$$\psi'(x,\omega) = 2x.$$

The integral average in the inequality is

$$I = \frac{p}{b_2^p - b_1^p}\int_{b_1}^{b_2} \frac{\psi(v,\omega)}{v^{1-p}} dv = \frac{2}{3}\int_1^2 v(v^2 + W(\omega)) dv = 2.5 + 1.0W(\omega).$$

For the weighting factors,

$$(y^p - b_1^p)^2 = 1.25^2 = 1.5625, \quad (b_2^p - y^p)^2 = 1.75^2 = 3.0625,$$

and the integrals

$$\int_0^1 (sy^p + (1-s)b_1^p)^{-0.5} ds \approx 0.8, \quad \int_0^1 (sy^p + (1-s)b_2^p)^{-0.5} ds \approx 0.789.$$

The weighted derivative integrals are approximated by

$$\int_0^1 s^k \frac{h(s)|\psi'(y)|^k + (1-h(s))|\psi'(b_1)|^k}{(sy^p + (1-s)b_1^p)^{0.5}} ds \approx 1.274,$$

$$\int_0^1 s^k \frac{h(s)|\psi'(y)|^k + (1-h(s))|\psi'(b_2)|^k}{(sy^p + (1-s)b_2^p)^{0.5}} ds \approx 2.448.$$

Hence, the Ostrowski bound becomes

$$\frac{(y^p - b_1^p)^2}{p(b_2^p - b_1^p)} \sqrt{1.274} - \frac{(b_2^p - y^p)^2}{p(b_2^p - b_1^p)} \sqrt{2.448} \approx 1.092.$$

The actual difference is

$$|\psi(y) - I| = |2.25 + W(\omega) - (2.5 + W(\omega))| = 0.25 \leq 1.092,$$

which verifies the Ostrowski-type inequality numerically for this example.

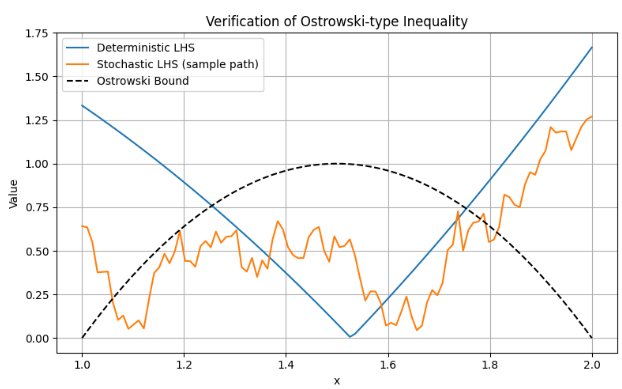

Figure 3. Graphical Verification of Ostrowski-type Inequality

4. Conclusion and Future Directions

In this paper we presented a new generalized version of the generalized by two parameters, (η, p, h)-convex stochastic process, which generalizes and brings together that of a number of existing concepts of convexity in the stochastic context. Basic properties of this class were studied and new Ostrowski-, Jensen-, and Hermite-type inequalities were obtained. The theoretical conclusions were demonstrated by non-trivial cases and graphs, which demonstrated the role of stochastic elements, especially, the Wiener processes, as compared to deterministic elements.

We establish that in addition to a number of classical stochastic convexity theorems being recovered as special cases of the proposed framework, the proposed framework permits a degree of modeling and analysis flexibility in the analysis and description of stochastic systems under uncertainty.

To conduct future research, a number of directions are prospective:

- Generalizations to multiple dimensions Multidimensional stochastic processes and generalized to multidimensional Wiener processes.
- Developing uses in stochastic optimization, stochastic differential equations and financial mathematics, where the convexity of (η,p,h) can provide a better bound and error estimate.
- The further validation of the theoretical constraints in high-dimensional stochastic models through development of numerical methods and simulation-based research.
- The investigation of relationships with other generalized convexity notions and stochastic integral inequalities, which may generate new classes of stochastic inequalities.

On the whole, the suggested framework not only opens up the possibilities of the theoretical aspects but also has numerous practical applications to the stochastic analysis and other areas.

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