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Characterizations of Almost $\tau^{\star}(\sigma_1, \sigma_2)$-Continuous Functions

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Abstract. This paper deals with the concept of almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous functions. Moreover, several characterizations and some properties concerning almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous functions are investigated. Furthermore, the relationships between $\tau^{\star}(\sigma_1, \sigma_2)$-continuity and almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuity are considered.

1. Introduction

In 1968, Singal and Singal [31] introduced the concept of almost continuous functions as a generalization of continuity. Popa [30] defined almost quasi-continuous functions as a generalization of almost continuity [31] and quasi-continuity [24]. Munshi and Bassan [26] studied the notion of almost semi-continuous functions. Maheshwari et al. [23] introduced the concept of almost feebly continuous functions as a generalization of almost continuity [31]. Noiri [29] introduced and investigated the concept of almost α-continuous functions. Nasef and Noiri [27] introduced two classes of functions, namely almost precontinuous functions and almost β-continuous functions by utilizing the notions of preopen sets and β-open sets due to Mashhour et al [25] and Abd El-Monsef et al. [2], respectively. The class of almost precontinuity is a generalization of each of almost feeble continuity and almost α-continuity. The class of almost β-continuity is a generalization of almost quasi-continuity and almost semi-continuity. Keskin and Noiri [18] introduced the concept of almost b-continuous functions by utilizing the notion of b-open sets due to Andrijević [4]. The class of almost b-continuity is a generalization of almost precontinuity

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and almost semi-continuity. The class of almost β-continuity is a generalization of almost b-continuity. Janković and Hamlett [17] introduced the concept of $\mathscr{I}$-open sets in topological spaces via ideals. Abd El-Monsef et al. [1] studied some properties of $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. Yüksel et al. [37] presented three classes of continuous functions defined from an ideal topological space into an ideal topological space, namely strongly θ-$\mathscr{I}$-continuous functions, δ-$\mathscr{I}$-continuous functions and almost $\mathscr{I}$-continuous functions. Al-Omeri and Noiri [3] introduced and studied the notion of almost e-$\mathscr{I}$-continuous functions. On the other hand, the present authors introduced and investigated the concepts of (τ_1, τ_2)-continuous functions [9], almost (τ_1, τ_2)-continuous functions [6], weakly (τ_1, τ_2)-continuous functions [7], almost quasi (τ_1, τ_2)-continuous functions [21], weakly quasi (τ_1, τ_2)-continuous functions [15], s-$(\tau_1, \tau_2)p$-continuous functions [34], $\star$-continuous functions [10], weakly $\star$-continuous functions [12], $\theta(\star)$-continuous functions [12], $\theta(\star)$-precontinuous functions [11], almost $\star$-precontinuous functions [11], weakly $\star$-precontinuous functions [11], weakly α-$\star$-continuous functions [8] and $p\iota$-continuous functions [5]. Khampakdee et al. [19] presented a new class of continuous functions defined from an ideal topological space into a bitopological space, namely $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. Kongied et al. [20] introduced and investigated the concepts of almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions and weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions [20]. Quite recently, Viriyapong et al. [33] introduced and studied the notions of almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions and weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. In this paper, we investigate some characterizations of almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions.

2. PRELIMINARIES

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called *$\tau_1\tau_2$-closed* [14] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called *$\tau_1\tau_2$-open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the *$\tau_1\tau_2$-closure* [14] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the *$\tau_1\tau_2$-interior* [14] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [14] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

1. $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ *and* $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.
2. *If* $A \subseteq B$, *then* $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.
3. $\tau_1\tau_2\text{-}Cl(A)$ *is* $\tau_1\tau_2$*-closed.*
4. A *is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.
5. $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$(\tau_1, \tau_2)r$-open* [36] (resp. *$(\tau_1, \tau_2)s$-open* [13], *$(\tau_1, \tau_2)p$-open* [13], *$(\tau_1, \tau_2)\beta$-open* [13]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called *$(\tau_1, \tau_2)r$-closed* (resp. *$(\tau_1, \tau_2)s$-closed*, *$(\tau_1, \tau_2)p$-closed*, *$(\tau_1, \tau_2)\beta$-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$\alpha(\tau_1, \tau_2)$-open* [35] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be *$\alpha(\tau_1, \tau_2)$-closed*.

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [22], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [17] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$\tau^\star$-semi-open* [28] (resp. *$\tau^\star$-pre-open* [28]) if $A \subseteq \text{Cl}^\star(\text{Int}^\star(A))$ (resp. $A \subseteq \text{Int}^\star(\text{Cl}^\star(A))$). The complement of a $\tau^\star$-semi-open (resp. $\tau^\star$-pre-open) set is called *$\tau^\star$-semi-closed* (resp. $\tau^\star$-pre-closed).

3. CHARACTERIZATIONS OF ALMOST $\tau^\star(\sigma_1, \sigma_2)$-CONTINUOUS FUNCTIONS

In this section, we investigate some characterizations of almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. The relationships between $\tau^\star(\sigma_1, \sigma_2)$-continuous functions and almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions are also discussed.

Definition 3.1. [33] *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost $\tau^\star(\sigma_1, \sigma_2)$-continuous if f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Lemma 3.1. [33] *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;*
(2) *$f^{-1}(V) \subseteq Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;*
(5) *$f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ for every subset B of Y;*
(6) *$f^{-1}(V)$ is $\star$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;*
(7) *$f^{-1}(K)$ is $\star$-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y.*

Recall that a subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-δ-*open* if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$-δ-*closed*. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$-δ-*interior* of A and is denoted by $\tau_1\tau_2$-δ-Int(A). The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$-δ-*closure* of A and is denoted by $\tau_1\tau_2$-δ-Cl(A) [6].

Theorem 3.1. *For a function $(X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

1. *f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;*
2. *$f^{-1}(V)$ is $\star$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;*
3. *$f^{-1}(K)$ is $\star$-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y;*
4. *$f(Cl^\star(A)) \subseteq \sigma_1\sigma_2$-$\delta$-$Cl(f(A))$ for every subset A of X;*
5. *$Cl^\star(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2$-$\delta$-$Cl(B))$ for every subset B of Y;*
6. *$f^{-1}(K)$ is $\star$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;*
7. *$f^{-1}(V)$ is $\star$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y.
Then, we have $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(V))) = V$ and by Lemma 3.1,

$$f^{-1}(V) = f^{-1}(\sigma_1\sigma_2\text{-Int}(V)) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V))))) = \text{Int}^\star(f^{-1}(V)).$$

Thus, $f^{-1}(V)$ is $\star$-open in X.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Let A be any subset of Y and K be any $\sigma_1\sigma_2$-δ-closed set of Y containing $f(A)$. Observe that $K = \sigma_1\sigma_2$-δ-Cl$(K) = \cap\{F \mid K \subseteq F$ and F is $(\sigma_1, \sigma_2)r$-closed$\}$ and so

$$f^{-1}(K) = \cap\{f^{-1}(F) \mid K \subseteq F \text{ and } F \text{ is } (\sigma_1, \sigma_2)r\text{-closed}\}.$$

Now, by (3) we have $f^{-1}(K)$ is $\star$-closed and $A \subseteq f^{-1}(K)$. Thus, Cl$^\star(A) \subseteq f^{-1}(K)$ and so $f(\text{Cl}^\star(A)) \subseteq K$. Since this is true for any $\sigma_1\sigma_2$-δ-closed set K containing $f(A)$, we have $f(\text{Cl}^\star(A)) \subseteq \sigma_1\sigma_2$-$\delta$-Cl$(f(A))$.

(4) ⇒ (5): Let B be any subset of Y. By (4), $f(\text{Cl}^\star(f^{-1}(B))) \subseteq \sigma_1\sigma_2$-$\delta$-Cl$(f(f^{-1}(B))) \subseteq \sigma_1\sigma_2$-$\delta$-Cl$(B)$. Thus, Cl$^\star(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2$-$\delta$-Cl$(B))$.

(5) ⇒ (6): Let K be any $\sigma_1\sigma_2$-δ-closed set of Y. Using (5), we have

$$\text{Cl}^\star(f^{-1}(K)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(K)) = f^{-1}(K)$$

and hence $f^{-1}(K)$ is $\star$-closed in X.

(6) ⇒ (7): This is obvious.

(7) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ is $(\sigma_1, \sigma_2)r$-open in Y, we have $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ is $\sigma_1\sigma_2$-δ-open and by (7), $f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ is $\star$-open in X. Put $U = f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$. Then, U is a $\star$-open set of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. Thus, f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at x and so f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous. □

Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)s$-closed sets of X containing A is called the (τ_1, τ_2)*s-closure* [13] of A and is denoted by (τ_1, τ_2)-sCl(A). The union of all $(\tau_1, \tau_2)s$-open sets of X contained in A is called the (τ_1, τ_2)*s-interior* [13] of A and is denoted by (τ_1, τ_2)-sInt(A).

Lemma 3.2. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties of hold:*

1. (τ_1, τ_2)*-sCl*$(A) = \tau_1\tau_2$*-Int*$(\tau_1\tau_2$*-Cl*$(A)) \cup A$ [13].
2. *If A is $\tau_1\tau_2$-open in X, then* (τ_1, τ_2)*-sCl*$(A) = \tau_1\tau_2$*-Int*$(\tau_1\tau_2$*-Cl*$(A))$ [13].
3. *If A is $(\tau_1, \tau_2)p$-open in X, then* (τ_1, τ_2)*-sCl*$(A) = \tau_1\tau_2$*-Int*$(\tau_1\tau_2$*-Cl*$(A))$.

Theorem 3.2. *For a function $(X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

1. *f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;*
2. *for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$;*
3. *for each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq V$;*
4. *for each $x \in X$ and each $\sigma_1\sigma_2$-δ-open set V of Y containing $f(x)$, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq V$.*

Proof. (1) ⇒ (2): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (1), there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. Since V is $(\sigma_1, \sigma_2)p$-open, by Lemma 3.2, $f(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$.

(2) ⇒ (3): Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. Then, V is $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), there exists a $\star$-open set U of X containing x such that $f(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. Since V is $\sigma_1\sigma_2$-open, by Lemma 3.2 we have $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)) = V$.

(3) ⇒ (4): Let $x \in X$ and V be any $\sigma_1\sigma_2$-δ-open set of Y containing $f(x)$. Then, there exists a $\sigma_1\sigma_2$-open set W of Y containing $f(x)$ such that $W \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(W)) \subseteq V$. Since $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(W))$ is a $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$, by (3) there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(W)) \subseteq V$.

(4) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ is a $\sigma_1\sigma_2$-δ-open set V of Y containing $f(x)$. By (4), there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. Thus, f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at x. This shows that f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous. □

Theorem 3.3. *For a function $(X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

1. *f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;*
2. *$f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ is $\star$-open in X for every $\sigma_1\sigma_2$-open set V of Y;*
3. *$f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(K)))$ is $\star$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$. Then, we have $f(x) \in \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ and $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ is $(\sigma_1, \sigma_2)r$-open in

Y. Since f is almost $\tau^\star(\sigma_1,\sigma_2)$-continuous, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $x \in \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$. This shows that $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$. Therefore, $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ is $\star$-open in X.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let K be any $(\sigma_1,\sigma_2)r$-closed set of Y. Then, K is a $\sigma_1\sigma_2$-closed set of Y. By (3), $f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))$ is $\star$-closed in X. Since K is $(\sigma_1,\sigma_2)r$-closed, we have

$$f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = f^{-1}(K)$$

and hence $f^{-1}(K)$ is $\star$-closed in X. Thus by Theorem 3.1, f is almost $\tau^\star(\sigma_1,\sigma_2)$-continuous. □

Theorem 3.4. *For a function $(X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

1. *f is almost $\tau^\star(\sigma_1,\sigma_2)$-continuous;*
2. *$Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1,\sigma_2)\beta$-open set V of Y;*
3. *$f^{-1}(\sigma_1\sigma_2\text{-}Int(K)) \subseteq Int^\star(f^{-1}(K))$ for every $(\sigma_1,\sigma_2)\beta$-closed set K of Y;*
4. *$f^{-1}(\sigma_1\sigma_2\text{-}Int(K)) \subseteq Int^\star(f^{-1}(K))$ for every $(\sigma_1,\sigma_2)s$-closed set K of Y;*
5. *$Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1,\sigma_2)s$-open set V of Y;*
6. *$f^{-1}(V) \subseteq Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $(\sigma_1,\sigma_2)p$-open set V of Y.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1,\sigma_2)\beta$-open set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1,\sigma_2)r$-closed, by Theorem 3.1 (3) we have $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) = f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Thus,

$$\text{Cl}^\star(f^{-1}(V)) \subseteq \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) = f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) ⇒ (3): Let K be any $(\sigma_1,\sigma_2)\beta$-closed set of Y. Then $Y-K$ is $(\sigma_1,\sigma_2)\beta$-open in Y. By (2), $\text{Cl}^\star(f^{-1}(Y-K)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y-K))$ and $\text{Cl}^\star(X-f^{-1}(K)) \subseteq f^{-1}(Y-\sigma_1\sigma_2\text{-Cl}(K))$. Thus,

$$X-\text{Int}^\star(f^{-1}(K)) \subseteq X-f^{-1}(\sigma_1\sigma_2\text{-Int}(K))$$

and so $f^{-1}(\sigma_1\sigma_2\text{-Int}(K)) \subseteq \text{Int}^\star(f^{-1}(K))$.

(3) ⇒ (4): It is obvious since every $(\sigma_1,\sigma_2)s$-open set is $(\sigma_1,\sigma_2)\beta$-open.

(4) ⇒ (5): Let Let V be any $(\sigma_1,\sigma_2)s$-open set of Y. Then, $Y-V$ is $(\sigma_1,\sigma_2)s$-closed in Y. By (4), $f^{-1}(\sigma_1\sigma_2\text{-Int}(Y-V)) \subseteq \text{Int}^\star(f^{-1}(Y-V))$ and $f^{-1}(Y-\sigma_1\sigma_2\text{-Cl}(V)) \subseteq \text{Int}^\star(X-f^{-1}(V))$. Thus, $X-f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)) \subseteq X-\text{Cl}^\star(f^{-1}(V))$ and so $\text{Cl}^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(5) ⇒ (1): Let K be any $(\sigma_1,\sigma_2)r$-closed set of Y. Then, K is $(\sigma_1,\sigma_2)s$-open in Y and by (5),

$$\text{Cl}^\star(f^{-1}(K)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(K)) = f^{-1}(K).$$

Thus, $f^{-1}(K)$ is $\star$-closed in X and hence f is almost $\tau^\star(\sigma_1,\sigma_2)$-continuous by Theorem 3.1.

(1) ⇒ (6): Let V be any $(\sigma_1,\sigma_2)p$-open set of Y. Then, we have $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1,\sigma_2)r$-open. By Theorem 3.1, $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ is $\star$-open in X. Thus, $f^{-1}(V) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) = \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(6) ⇒ (1): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open and by (6),

$$f^{-1}(V) \subseteq \mathrm{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))) = \mathrm{Int}^\star(f^{-1}(V)).$$

Therefore, $f^{-1}(V)$ is $\star$-open in X. It follows from Theorem 3.1 that f is almost $\tau^\star(\sigma_1,\sigma_2)$-continuous. □

Corollary 3.1. *For a function $(X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

1. *f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;*
2. *$f^{-1}(V) \subseteq Int^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;*
3. *$Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Cl(K)))) \subseteq f^{-1}(K)$ for every $(\sigma_1, \sigma_2)p$-closed set K of Y;*
4. *$Cl^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sInt(K))) \subseteq f^{-1}(K)$ for every $(\sigma_1, \sigma_2)p$-closed set K of Y.*

Definition 3.2. [19] *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be $\tau^\star(\sigma_1, \sigma_2)$-continuous if f is $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Remark 3.1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:*

$\tau^\star(\sigma_1, \sigma_2)$-continuity ⇒ almost $\tau^\star(\sigma_1, \sigma_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 3.1. *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{1\}\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{a\}, \{a, b\}, Y\}$ and $\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is defined as follows: $f(1) = c$, $f(2) = a$ and $f(3) = b$. Then, f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous but f is not $\tau^\star(\sigma_1, \sigma_2)$-continuous.*

Lemma 3.3. [32] *A bitopological space (X, τ_1, τ_2) is $(\tau_1, \tau_2)s$-regular if and only if for each $x \in X$ and each $(\tau_1, \tau_2)s$-open set U containing x, there exists a $(\tau_1, \tau_2)s$-open set V such that $x \in V \subseteq (\tau_1, \tau_2)\text{-}sCl(V) \subseteq U$.*

Lemma 3.4. [32] *Let (X, τ_1, τ_2) be a $(\tau_1, \tau_2)s$-regular space. Then, the following properties hold:*

1. *$\tau_1\tau_2\text{-}Cl(A) = \tau_1\tau_2\text{-}\delta\text{-}Cl(A)$ for every subset A of X.*
2. *Every $\tau_1\tau_2$-open set is $\tau_1\tau_2\text{-}\delta$-open.*

Lemma 3.5. [19] *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is $(\sigma_1, \sigma_2)s$-regular, the following properties are equivalent:*

1. *f is $\tau^\star(\sigma_1, \sigma_2)$-continuous;*
2. *$f^{-1}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ is $\star$-closed in X for every subset B of Y;*
3. *$f^{-1}(K)$ is $\star$-closed in X for every $\sigma_1\sigma_2\text{-}\delta$-closed set K of Y;*
4. *$f^{-1}(V)$ is $\star$-open in X for every $\sigma_1\sigma_2\text{-}\delta$-open set V of Y.*

Theorem 3.5. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is $(\sigma_1, \sigma_2)s$-regular, the following properties are equivalent:*

(1) *f is $\tau^\star(\sigma_1, \sigma_2)$-continuous;*
(2) *$f^{-1}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ is $\star$-closed in X for every subset B of Y;*
(3) *$f^{-1}(K)$ is $\star$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;*
(4) *$f^{-1}(V)$ is $\star$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y;*
(5) *f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous.*

Proof. The proofs of the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are similar as in Lemma 3.5.

$(4) \Rightarrow (5)$: Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $\sigma_1\sigma_2$-open in Y and by Lemma 3.4, V is $\sigma_1\sigma_2$-δ-open in Y. By (4), we have $f^{-1}(V)$ is $\star$-open in X. Thus by Theorem 3.1, f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous.

$(5) \Rightarrow (1)$: Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since (Y, σ_1, σ_2) is $(\sigma_1, \sigma_2)s$-regular, there exists a $(\sigma_1, \sigma_2)r$-open set W containing $f(x)$ and $W \subseteq V$. Since f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq W$; hence $f(U) \subseteq V$. This shows that f is $\tau^\star(\sigma_1, \sigma_2)$-continuous. □

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