

## On Almost Nearly $\tau^*(\sigma_1, \sigma_2)$ -Continuous Functions

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**Abstract.** This paper presents a new class of continuous functions defined between an ideal topological space and a bitopological space, called almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous functions. Moreover, several characterizations and some properties concerning almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous functions are discussed.

### 1. INTRODUCTION

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. Weaker and stronger forms of open sets in topological spaces such as semi-open sets, preopen sets,  $\alpha$ -open sets and  $\beta$ -open sets play an important role in the research of generalizations of continuity. By utilizing these sets several authors introduced and studied various types of generalizations of continuity for functions. Singal and Singal [31] introduced the concept of almost continuous functions as a generalization of continuity. Popa [29] defined almost quasi-continuous functions as a generalization of almost continuity and quasi-continuity [25]. Munshi and Bassan [26] studied the notion of almost semi-continuous functions. Maheshwari et al. [23] introduced the concept of almost feebly continuous functions as a generalization of almost continuity. Malghan and Hanchinamani [24] introduced the notion of N-continuous functions. Noiri and Ergun [28] investigated some characterizations of N-continuous functions. Janković and Hamlett [17] introduced the concept of  $\mathcal{I}$ -open sets in topological spaces via ideals. Abd El-Monsef et al. [1] studied some properties of  $\mathcal{I}$ -open sets and

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$\mathcal{I}$ -continuous functions. As generalizations of open sets in an ideal topological space, many authors introduced the notions of semi- $\mathcal{I}$ -open sets, pre- $\mathcal{I}$ -open sets,  $\alpha$ - $\mathcal{I}$ -open sets and  $\beta$ - $\mathcal{I}$ -open sets. Hatir and Noiri [16] introduced and studied the concepts of semi- $\mathcal{I}$ -continuous functions and  $\alpha$ - $\mathcal{I}$ -continuous functions by utilizing the notions of semi- $\mathcal{I}$ -open sets and  $\alpha$ - $\mathcal{I}$ -open sets, respectively. Moreover, Hatir and Noiri [15] introduced and investigated the notions of  $\beta$ - $\mathcal{I}$ -continuous functions and  $\beta$ - $\mathcal{I}$ -irresolute. Yüksel et al. [39] presented three classes of continuous functions defined from an ideal topological space into an ideal topological space, namely strongly  $\theta$ - $\mathcal{I}$ -continuous functions,  $\delta$ - $\mathcal{I}$ -continuous functions and almost  $\mathcal{I}$ -continuous functions. Al-Omeri and Noiri [2] introduced and studied the notion of almost  $e$ - $\mathcal{I}$ -continuous functions. On the other hand, the present authors introduced and studied the concepts of  $(\tau_1, \tau_2)$ -continuous functions [6], almost  $(\tau_1, \tau_2)$ -continuous functions [4], weakly  $(\tau_1, \tau_2)$ -continuous functions [5], almost quasi  $(\tau_1, \tau_2)$ -continuous functions [21], weakly quasi  $(\tau_1, \tau_2)$ -continuous functions [12],  $\star$ -continuous functions [7], weakly  $\star$ -continuous functions [9],  $\theta(\star)$ -continuous functions [9],  $\theta(\star)$ -precontinuous functions [8], almost  $\star$ -precontinuous functions [8], weakly  $\star$ -precontinuous functions [8] and  $p$  $\mathcal{I}$ -continuous functions [3]. Khampakdee et al. [19] introduced and investigated notion of  $\tau^\star(\sigma_1, \sigma_2)$ -continuous functions. Viriyapong et al. [34] introduced and studied the concepts of almost  $\tau^\star(\sigma_1, \sigma_2)$ -continuous functions and weakly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous functions. Chutiman et al. [13] introduced and investigated the notion of nearly  $(\tau_1, \tau_2)$ -continuous functions. Quite recently, Kong-ied et al. [20] presented a new class of continuous functions defined between bitopological spaces, namely almost nearly  $(\tau_1, \tau_2)$ -continuous functions. In this paper, we introduce the concept of almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous functions. We also investigate several characterizations of almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous functions.

## 2. PRELIMINARIES

Throughout the present paper, spaces  $(X, \tau_1, \tau_2)$  and  $(Y, \sigma_1, \sigma_2)$  (or simply  $X$  and  $Y$ ) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let  $A$  be a subset of a bitopological space  $(X, \tau_1, \tau_2)$ . The closure of  $A$  and the interior of  $A$  with respect to  $\tau_i$  are denoted by  $\tau_i\text{-Cl}(A)$  and  $\tau_i\text{-Int}(A)$ , respectively, for  $i = 1, 2$ . A subset  $A$  of a bitopological space  $(X, \tau_1, \tau_2)$  is called  $\tau_1\tau_2$ -closed [11] if  $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$ . The complement of a  $\tau_1\tau_2$ -closed set is called  $\tau_1\tau_2$ -open. Let  $A$  be a subset of a bitopological space  $(X, \tau_1, \tau_2)$ . The intersection of all  $\tau_1\tau_2$ -closed sets of  $X$  containing  $A$  is called the  $\tau_1\tau_2$ -closure [11] of  $A$  and is denoted by  $\tau_1\tau_2\text{-Cl}(A)$ . The union of all  $\tau_1\tau_2$ -open sets of  $X$  contained in  $A$  is called the  $\tau_1\tau_2$ -interior [11] of  $A$  and is denoted by  $\tau_1\tau_2\text{-Int}(A)$ .

**Lemma 2.1.** [11] *Let  $A$  and  $B$  be subsets of a bitopological space  $(X, \tau_1, \tau_2)$ . For the  $\tau_1\tau_2$ -closure, the following properties hold:*

- (1)  $A \subseteq \tau_1\tau_2\text{-Cl}(A)$  and  $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Cl}(A)) = \tau_1\tau_2\text{-Cl}(A)$ .
- (2) If  $A \subseteq B$ , then  $\tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(B)$ .
- (3)  $\tau_1\tau_2\text{-Cl}(A)$  is  $\tau_1\tau_2$ -closed.

- (4)  $A$  is  $\tau_1\tau_2$ -closed if and only if  $A = \tau_1\tau_2\text{-Cl}(A)$ .  
 (5)  $\tau_1\tau_2\text{-Cl}(X - A) = X - \tau_1\tau_2\text{-Int}(A)$ .

A subset  $A$  of a bitopological space  $(X, \tau_1, \tau_2)$  is said to be  $(\tau_1, \tau_2)$ - $r$ -open [38] (resp.  $(\tau_1, \tau_2)$ - $s$ -open [10],  $(\tau_1, \tau_2)$ - $p$ -open [10],  $(\tau_1, \tau_2)$ - $\beta$ -open [10]) if  $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$  (resp.  $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$ ,  $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ ,  $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$ ). The complement of a  $(\tau_1, \tau_2)$ - $r$ -open (resp.  $(\tau_1, \tau_2)$ - $s$ -open,  $(\tau_1, \tau_2)$ - $p$ -open,  $(\tau_1, \tau_2)$ - $\beta$ -open) set is called  $(\tau_1, \tau_2)$ - $r$ -closed (resp.  $(\tau_1, \tau_2)$ - $s$ -closed,  $(\tau_1, \tau_2)$ - $p$ -closed,  $(\tau_1, \tau_2)$ - $\beta$ -closed). A subset  $A$  of a bitopological space  $(X, \tau_1, \tau_2)$  is said to be  $\alpha(\tau_1, \tau_2)$ -open [37] if  $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$ . The complement of an  $\alpha(\tau_1, \tau_2)$ -open set is said to be  $\alpha(\tau_1, \tau_2)$ -closed. Let  $A$  be a subset of a bitopological space  $(X, \tau_1, \tau_2)$ . The intersection of all  $(\tau_1, \tau_2)$ - $p$ -closed (resp.  $(\tau_1, \tau_2)$ - $s$ -closed,  $\alpha(\tau_1, \tau_2)$ -closed) sets of  $X$  containing  $A$  is called the  $(\tau_1, \tau_2)$ - $p$ -closure [35] (resp.  $(\tau_1, \tau_2)$ - $s$ -closure [38],  $\alpha(\tau_1, \tau_2)$ -closure [36]) of  $A$  and is denoted by  $(\tau_1, \tau_2)$ - $p\text{Cl}(A)$  (resp.  $(\tau_1, \tau_2)$ - $s\text{Cl}(A)$ ,  $\alpha(\tau_1, \tau_2)\text{-Cl}(A)$ ). The union of all  $(\tau_1, \tau_2)$ - $p$ -open (resp.  $(\tau_1, \tau_2)$ - $s$ -open,  $\alpha(\tau_1, \tau_2)$ -open) sets of  $X$  contained in  $A$  is called the  $(\tau_1, \tau_2)$ - $p$ -interior [35] (resp.  $(\tau_1, \tau_2)$ - $s$ -interior [38],  $\alpha(\tau_1, \tau_2)$ -interior [36]) of  $A$  and is denoted by  $(\tau_1, \tau_2)$ - $p\text{Int}(A)$  (resp.  $(\tau_1, \tau_2)$ - $s\text{Int}(A)$ ,  $\alpha(\tau_1, \tau_2)\text{-Int}(A)$ ).

**Lemma 2.2.** [35] For subsets  $A$  and  $B$  of a bitopological space  $(X, \tau_1, \tau_2)$ , the following properties hold:

- (1)  $A \subseteq (\tau_1, \tau_2)\text{-pCl}(A)$  and  $(\tau_1, \tau_2)\text{-pCl}((\tau_1, \tau_2)\text{-pCl}(A)) = (\tau_1, \tau_2)\text{-pCl}(A)$ .
- (2) If  $A \subseteq B$ , then  $(\tau_1, \tau_2)\text{-pCl}(A) \subseteq (\tau_1, \tau_2)\text{-pCl}(B)$ .
- (3)  $(\tau_1, \tau_2)\text{-pCl}(A)$  is  $(\tau_1, \tau_2)$ - $p$ -closed.
- (4)  $A$  is  $(\tau_1, \tau_2)$ - $p$ -closed if and only if  $A = (\tau_1, \tau_2)\text{-pCl}(A)$ .
- (5)  $(\tau_1, \tau_2)\text{-pCl}(X - A) = X - (\tau_1, \tau_2)\text{-pInt}(A)$ .
- (6)  $x \in (\tau_1, \tau_2)\text{-pCl}(A)$  if and only if  $A \cap U \neq \emptyset$  for every  $(\tau_1, \tau_2)$ - $p$ -open set  $U$  of  $X$  containing  $x$ .

**Lemma 2.3.** [36] For subsets  $A$  and  $B$  of a bitopological space  $(X, \tau_1, \tau_2)$ , the following properties hold:

- (1)  $A \subseteq \alpha(\tau_1, \tau_2)\text{-Cl}(A)$  and  $\alpha(\tau_1, \tau_2)\text{-Cl}(\alpha(\tau_1, \tau_2)\text{-Cl}(A)) = \alpha(\tau_1, \tau_2)\text{-Cl}(A)$ .
- (2) If  $A \subseteq B$ , then  $\alpha(\tau_1, \tau_2)\text{-Cl}(A) \subseteq \alpha(\tau_1, \tau_2)\text{-Cl}(B)$ .
- (3)  $\alpha(\tau_1, \tau_2)\text{-Cl}(A)$  is  $\alpha(\tau_1, \tau_2)$ -closed.
- (4)  $A$  is  $\alpha(\tau_1, \tau_2)$ -closed if and only if  $A = \alpha(\tau_1, \tau_2)\text{-Cl}(A)$ .
- (5)  $\alpha(\tau_1, \tau_2)\text{-Cl}(X - A) = X - \alpha(\tau_1, \tau_2)\text{-Int}(A)$ .

**Lemma 2.4.** For a subset  $A$  of a bitopological space  $(X, \tau_1, \tau_2)$ , the following properties hold:

- (1)  $(\tau_1, \tau_2)\text{-sCl}(A) = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)) \cup A$  [10];
- (2)  $(\tau_1, \tau_2)\text{-sInt}(A) = \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)) \cap A$  [30].

A subset  $A$  of a bitopological space  $(X, \tau_1, \tau_2)$  is said to be  $\mathcal{N}(\tau_1, \tau_2)$ -closed [32] if every cover of  $A$  by  $(\tau_1, \tau_2)$ - $r$ -open sets of  $X$  has a finite subcover.

**Lemma 2.5.** [14] Let  $(X, \tau_1, \tau_2)$  be a bitopological space. If  $V$  is a  $\tau_1\tau_2$ -open set of  $X$  having  $\mathcal{N}(\tau_1, \tau_2)$ -closed complement, then  $\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(V))$  is a  $(\tau_1, \tau_2)$ - $r$ -open set having  $\mathcal{N}(\tau_1, \tau_2)$ -closed complement.

**Lemma 2.6.** [14] *Let  $(X, \tau_1, \tau_2)$  be a bitopological space. If  $V$  is a  $(\tau_1, \tau_2)$ - $p$ -open set of  $X$  having  $\mathcal{N}(\tau_1, \tau_2)$ -closed complement, then  $\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(V))$  is a  $(\tau_1, \tau_2)$ - $r$ -open set having  $\mathcal{N}(\tau_1, \tau_2)$ -closed complement.*

An ideal  $\mathcal{I}$  on a topological space  $(X, \tau)$  is a nonempty collection of subsets of  $X$  satisfying the following properties: (1)  $A \in \mathcal{I}$  and  $B \subseteq A$  imply  $B \in \mathcal{I}$ ; (2)  $A \in \mathcal{I}$  and  $B \in \mathcal{I}$  imply  $A \cup B \in \mathcal{I}$ . A topological space  $(X, \tau)$  with an ideal  $\mathcal{I}$  on  $X$  is called an ideal topological space and is denoted by  $(X, \tau, \mathcal{I})$ . For an ideal topological space  $(X, \tau, \mathcal{I})$  and a subset  $A$  of  $X$ ,  $A^*(\mathcal{I})$  is defined as follows:

$$A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion,  $A^*(\mathcal{I})$  is simply written as  $A^*$ . In [22],  $A^*$  is called the local function of  $A$  with respect to  $\mathcal{I}$  and  $\tau$  and  $\text{Cl}^*(A) = A^* \cup A$  defines a Kuratowski closure operator for a topology  $\tau^*(\mathcal{I})$  finer than  $\tau$ . A subset  $A$  is said to be  $\star$ -closed [17] if  $A^* \subseteq A$ . The interior of a subset  $A$  in  $(X, \tau^*(\mathcal{I}))$  is denoted by  $\text{Int}^*(A)$ . A subset  $A$  of an ideal topological space  $(X, \tau, \mathcal{I})$  is said to be  $\tau^*$ -semi-open [27] (resp.  $\tau^*$ -pre-open [27]) if  $A \subseteq \text{Cl}^*(\text{Int}^*(A))$  (resp.  $A \subseteq \text{Int}^*(\text{Cl}^*(A))$ ). The complement of a  $\tau^*$ -semi-open (resp.  $\tau^*$ -pre-open) set is called  $\tau^*$ -semi-closed (resp.  $\tau^*$ -pre-closed).

### 3. ALMOST NEARLY $\tau^*(\sigma_1, \sigma_2)$ -CONTINUOUS FUNCTIONS

In this section, we introduce the notion of almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous functions. Furthermore, several characterizations of almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous functions are discussed.

**Definition 3.1.** *A function  $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)$  is said to be almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous at a point  $x \in X$  if for each  $\sigma_1\sigma_2$ -open set  $V$  of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement, there exists a  $\star$ -open set  $U$  of  $X$  containing  $x$  such that  $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ . A function  $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)$  is said to be almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous if  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous at each point  $x$  of  $X$ .*

**Theorem 3.1.** *For a function  $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)$ , the following properties are equivalent:*

- (1)  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous at  $x \in X$ ;
- (2)  $x \in \text{Int}^*(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$  for each  $\sigma_1\sigma_2$ -open set  $V$  of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement;
- (3)  $x \in \text{Int}^*(f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V)))$  for each  $\sigma_1\sigma_2$ -open set  $V$  of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement;
- (4)  $x \in \text{Int}^*(f^{-1}(V))$  for each  $(\sigma_1, \sigma_2)$ - $r$ -open set  $V$  of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement;
- (5) for each  $(\sigma_1, \sigma_2)$ - $r$ -open set  $V$  of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement, there exists a  $\star$ -open set  $U$  of  $X$  containing  $x$  such that  $f(U) \subseteq V$ .

*Proof.* (1)  $\Rightarrow$  (2): Let  $V$  be any  $\sigma_1\sigma_2$ -open set of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. By (1), there exists a  $\star$ -open set  $U$  of  $X$  containing  $x$  such that  $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ . Thus, we have  $x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$  and hence  $x \in \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ .

(2)  $\Rightarrow$  (3): This follows from Lemma 2.4.

(3)  $\Rightarrow$  (4): Let  $V$  be any  $(\sigma_1, \sigma_2)r$ -open set of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. It follows from Lemma 2.4 that  $V = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) = (\sigma_1, \sigma_2)\text{-sCl}(V)$ .

(4)  $\Rightarrow$  (5): Let  $V$  be any  $(\sigma_1, \sigma_2)r$ -open set of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. By (4),  $x \in \text{Cl}^\star(f^{-1}(V))$  and therefore there exists a  $\star$ -open set  $U$  of  $X$  such that  $x \in U \subseteq f^{-1}(V)$ ; hence  $f(U) \subseteq V$ .

(5)  $\Rightarrow$  (1): Let  $x \in X$  and let  $V$  be any  $\sigma_1\sigma_2$ -open set of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. By Lemma 2.5,  $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$  is a  $(\sigma_1, \sigma_2)r$ -open set of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. Thus by (5), there exists a  $\star$ -open set  $U$  of  $X$  containing  $x$  such that  $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ . This shows that  $f$  is almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous at  $x$ .  $\square$

**Theorem 3.2.** For a function  $f : (X, \tau, \mathcal{S}) \rightarrow (Y, \sigma_1, \sigma_2)$ , the following properties are equivalent:

- (1)  $f$  is almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous;
- (2)  $f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$  for each  $\sigma_1\sigma_2$ -open set  $V$  of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement;
- (3)  $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq f^{-1}(K)$  for every  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed and  $\sigma_1\sigma_2$ -closed set  $K$  of  $Y$ ;
- (4)  $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$  for every every subset  $B$  of  $Y$  having the  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed  $\sigma_1\sigma_2$ -closure;
- (5)  $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$  for every every subset  $B$  of  $Y$  such that  $Y - \sigma_1\sigma_2\text{-Int}(B)$  is  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed;
- (6)  $f^{-1}(V)$  is  $\star$ -open in  $X$  for each  $(\sigma_1, \sigma_2)r$ -open set  $V$  of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement;
- (7)  $f^{-1}(K)$  is  $\star$ -closed in  $X$  for every  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed and  $(\sigma_1, \sigma_2)r$ -closed set  $K$  of  $Y$ .

*Proof.* (1)  $\Rightarrow$  (2): Let  $V$  be any  $\sigma_1\sigma_2$ -open set of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement and  $x \in f^{-1}(V)$ . Then,  $f(x) \in V$ . By Theorem 3.1, we have  $x \in \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$  and so  $f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ .

(2)  $\Rightarrow$  (3): Let  $K$  be any  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed and  $\sigma_1\sigma_2$ -closed set  $K$  of  $Y$ . Then,  $Y - K$  is a  $\sigma_1\sigma_2$ -open set of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. By (2), we have

$$\begin{aligned} X - f^{-1}(K) &= f^{-1}(Y - K) \\ &\subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K)))) \\ &= \text{Int}^\star(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \\ &= X - \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \end{aligned}$$

Thus,  $\text{Cl}^*(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq f^{-1}(K)$ .

(3)  $\Rightarrow$  (4): Let  $B$  be any subset of  $Y$  having the  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed  $\sigma_1\sigma_2$ -closure. Then,  $\sigma_1\sigma_2\text{-Cl}(B)$  is a  $\sigma_1\sigma_2$ -closed  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed set of  $Y$  and by (3),

$$\text{Cl}^*(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)).$$

(4)  $\Rightarrow$  (5): Let  $B$  be any subset of  $Y$  such that  $Y - \sigma_1\sigma_2\text{-Int}(B)$  is  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed. Since  $Y - \sigma_1\sigma_2\text{-Int}(B)$  is  $\sigma_1\sigma_2$ -closed and  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed. Then by (4), we have

$$\begin{aligned} f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) &= X - f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \text{Cl}^*(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &= X - \text{Cl}^*(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= \text{Int}^*(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))). \end{aligned}$$

(5)  $\Rightarrow$  (6): Let  $V$  be any  $(\sigma_1, \sigma_2)r$ -open set  $V$  of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. Then, we have  $Y - \sigma_1\sigma_2\text{-Int}(V)$  is  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed and by (5),  $f^{-1}(V) \subseteq \text{Int}^*(f^{-1}(V))$ . Thus,  $f^{-1}(V)$  is  $\star$ -open in  $X$ .

(6)  $\Rightarrow$  (7): Let  $K$  be any  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed and  $(\sigma_1, \sigma_2)r$ -closed set of  $Y$ . Then,  $Y - K$  is a  $(\sigma_1, \sigma_2)r$ -open set of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. By (6), we have  $f^{-1}(Y - K) = X - f^{-1}(K)$  is  $\star$ -open in  $X$  and so  $f^{-1}(K)$  is  $\star$ -closed in  $X$ .

(7)  $\Rightarrow$  (1): Let  $x \in X$  and let  $V$  be any  $(\sigma_1, \sigma_2)r$ -open set of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. Then,  $Y - V$  is  $(\sigma_1, \sigma_2)r$ -closed and  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed. By (7),  $f^{-1}(Y - V) = X - f^{-1}(V)$  is  $\star$ -closed in  $X$ . Thus,  $f^{-1}(V)$  is  $\star$ -open in  $X$ . Then, there exists a  $\star$ -open set  $U$  of  $X$  containing  $x$  such that  $f(U) \subseteq V$ . It follows from Theorem 3.1 that  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous at  $x$ . This shows that  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous.  $\square$

**Corollary 3.1.** *A function  $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma_1, \sigma_2)$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous if  $f^{-1}(K)$  is  $\star$ -closed in  $X$  for every  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed set  $K$  of  $Y$ .*

*Proof.* Let  $V$  be any  $(\sigma_1, \sigma_2)r$ -open set of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. Then,  $Y - V$  is  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed and  $(\sigma_1, \sigma_2)r$ -closed. By the hypothesis,  $X - f^{-1}(V) = f^{-1}(Y - V)$  is  $\star$ -closed in  $X$  and hence  $f^{-1}(V)$  is  $\star$ -open in  $X$ . It follows from Theorem 3.2 that  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous.  $\square$

**Theorem 3.3.** *For a function  $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma_1, \sigma_2)$ , the following properties are equivalent:*

- (1)  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous;
- (2)  $\text{Cl}^*(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$  for every every  $(\sigma_1, \sigma_2)\beta$ -open set  $V$  of  $Y$  having the  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed  $\sigma_1\sigma_2$ -closure;
- (3)  $\text{Cl}^*(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$  for every every  $(\sigma_1, \sigma_2)s$ -open set  $V$  of  $Y$  having the  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed  $\sigma_1\sigma_2$ -closure;

- (4)  $f^{-1}(V) \subseteq \text{Int}^*(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$  for every every  $(\sigma_1, \sigma_2)p$ -open set  $V$  of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement.

*Proof.* (1)  $\Rightarrow$  (2): Let  $V$  be any  $(\sigma_1, \sigma_2)\beta$ -open set of  $Y$  having the  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed  $\sigma_1\sigma_2$ -closure. Then,  $\sigma_1\sigma_2\text{-Cl}(V)$  is  $(\sigma_1, \sigma_2)r$ -closed in  $Y$ . Since  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous, by Theorem 3.2 we have  $f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$  is  $\star$ -closed in  $X$ . Thus,  $\text{Cl}^*(f^{-1}(V)) \subseteq \text{Cl}^*(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) = f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ .

(2)  $\Rightarrow$  (3): The proof is obvious since every  $(\sigma_1, \sigma_2)s$ -open set is  $(\sigma_1, \sigma_2)\beta$ -open.

(3)  $\Rightarrow$  (4): Let  $V$  be any  $(\sigma_1, \sigma_2)p$ -open set of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. Then by Lemma 2.6,  $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$  is a  $(\sigma_1, \sigma_2)r$ -open set having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. Then,  $Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$  is a  $(\sigma_1, \sigma_2)r$ -closed and  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed set. Therefore,  $Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$  is a  $(\sigma_1, \sigma_2)s$ -open set having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed  $\sigma_1\sigma_2$ -closure. By (3), we have

$$\begin{aligned} X - \text{Int}^*(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) &= \text{Cl}^*(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - f^{-1}(V) \end{aligned}$$

and hence  $f^{-1}(V) \subseteq \text{Int}^*(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ .

(4)  $\Rightarrow$  (1): Let  $V$  be any  $(\sigma_1, \sigma_2)r$ -open set of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. Then,  $V$  is a  $(\sigma_1, \sigma_2)p$ -open set having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. By (4), we have

$$f^{-1}(V) \subseteq \text{Int}^*(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) = \tau_1\tau_2\text{-Int}(f^{-1}(V))$$

and so  $f^{-1}(V)$  is  $\star$ -open in  $X$ . Thus by Theorem 3.2,  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous.  $\square$

**Lemma 3.1.** [14] For a bitopological space  $(X, \tau_1, \tau_2)$ , the following properties hold:

- (1)  $\alpha(\tau_1, \tau_2)\text{-Cl}(U) = \tau_1\tau_2\text{-Cl}(U)$  for every  $(\tau_1, \tau_2)\beta$ -open set  $U$  of  $X$ ;
- (2)  $(\tau_1, \tau_2)\text{-pCl}(U) = \tau_1\tau_2\text{-Cl}(U)$  for every  $(\tau_1, \tau_2)s$ -open set  $U$  of  $X$ .

**Corollary 3.2.** For a function  $f : (X, \tau, \mathcal{S}) \rightarrow (Y, \sigma_1, \sigma_2)$ , the following properties are equivalent:

- (1)  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous;
- (2)  $\text{Cl}^*(f^{-1}(V)) \subseteq f^{-1}(\alpha(\sigma_1, \sigma_2)\text{-Cl}(V))$  for every every  $(\sigma_1, \sigma_2)\beta$ -open set  $V$  of  $Y$  having the  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed  $\sigma_1\sigma_2$ -closure;
- (3)  $\text{Cl}^*(f^{-1}(V)) \subseteq f^{-1}((\sigma_1, \sigma_2)\text{-pCl}(V))$  for every every  $(\sigma_1, \sigma_2)s$ -open set  $V$  of  $Y$  having the  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed  $\sigma_1\sigma_2$ -closure.

**Theorem 3.4.** For a function  $f : (X, \tau, \mathcal{S}) \rightarrow (Y, \sigma_1, \sigma_2)$ , the following properties are equivalent:

- (1)  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous;
- (2) for each  $x \in X$  and for every  $\sigma_1\sigma_2$ -closed and  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed sets  $K$  of  $Y$  such that  $x \in f^{-1}(Y - K)$ , there exists a  $\star$ -closed set  $H$  of  $X$  such that  $x \in X - H$  and  $f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq H$ ;

- (3)  $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$  is  $\star$ -open in  $X$  for every  $\sigma_1\sigma_2$ -open sets  $V$  of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement;
- (4)  $f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))$  is  $\star$ -closed in  $X$  for every  $\sigma_1\sigma_2$ -closed  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed sets  $K$  of  $Y$ .

*Proof.* (1)  $\Rightarrow$  (2): Let  $x \in X$  and  $K$  be any  $\sigma_1\sigma_2$ -closed and  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed set of  $Y$  such that  $x \in f^{-1}(Y - K)$ . Then,  $Y - K$  is a  $\sigma_1\sigma_2$ -open set having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. Since  $f$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous, there exists a  $\star$ -open set  $U$  of  $X$  containing  $x$  such that

$$U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K))) = X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))).$$

It is clear that  $H = X - U$  is  $\star$ -closed in  $X$  and  $f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq H$ .

(2)  $\Rightarrow$  (1): The proof is similar to the proof (1)  $\Rightarrow$  (2).

(1)  $\Rightarrow$  (3): Let  $V$  be any  $\sigma_1\sigma_2$ -open set of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement and

$$x \in f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Then, we have  $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$  is a  $\sigma_1\sigma_2$ -open set of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. Thus by (1), there exists a  $\star$ -open set  $U$  of  $X$  containing  $x$  such that  $U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ . Since  $U$  is  $\star$ -open, we have  $x \in \text{Int}^*(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$  and hence

$$f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq \text{Int}^*(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))).$$

Thus,  $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$  is  $\star$ -open in  $X$ .

(3)  $\Rightarrow$  (1): The proof is clear.

(3)  $\Rightarrow$  (4): Let  $K$  be any  $\sigma_1\sigma_2$ -closed  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed set of  $Y$ . Then,  $Y - K$  is a  $\sigma_1\sigma_2$ -open set having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement. By (3),  $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K)))$  is  $\star$ -open in  $X$ . Since  $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K)) = Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ , it follows that

$$\begin{aligned} f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K))) &= f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))). \end{aligned}$$

Thus,  $f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))$  is  $\star$ -closed in  $X$ .

(4)  $\Rightarrow$  (3): It can be obtained similarly as (3)  $\Rightarrow$  (4). □

A net  $(x_\gamma)$  in an ideal topological space  $(X, \tau, \mathcal{I})$  is said to be *eventually* [33] in the set  $U \subseteq X$  if there exists an index  $\gamma_0 \in \nabla$  such that  $x_\gamma \in U$  for each  $\gamma \geq \gamma_0$ . A net  $(x_\gamma)$  is called  $\star$ -converge [33] to a point  $x \in X$  if for every  $\star$ -open set  $U$  of  $X$  containing  $x$ , there exists an index  $\gamma_0 \in \nabla$  such that  $x_\gamma \in U$  for each  $\gamma \geq \gamma_0$ .

**Theorem 3.5.** A function  $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)$  is almost nearly  $\tau^*(\sigma_1, \sigma_2)$ -continuous if and only if for each  $x \in X$  and for each net  $(x_\gamma)$  which  $\star$ -converges to  $x$  in  $X$  and for each  $\sigma_1\sigma_2$ -open set  $V$  of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement such that  $x \in f^{-1}(V)$ , the net  $(x_\gamma)$  is eventually in  $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ .

*Proof.* Let  $(x_\gamma)$  be a net which  $\star$ -converges to  $x$  in  $X$  and let  $V$  be any  $\sigma_1\sigma_2$ -open set of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement such that  $x \in f^{-1}(V)$ . Since  $f$  is almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous, there exists a  $\star$ -open set  $U$  of  $X$  containing  $x$  such that  $U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ . Since  $(x_\gamma)$   $\star$ -converges to  $x$ , it follows that there exists an index  $\gamma_0 \in \nabla$  such that  $x_\gamma \in U$  for each  $\gamma \geq \gamma_0$ . Thus,

$$x_\gamma \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

for each  $\gamma \geq \gamma_0$ . Thus, the net  $(x_\gamma)$  is eventually in  $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ .

Conversely, suppose that  $f$  is not almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous. Then, there exists a point  $x$  of  $X$  and a  $\sigma_1\sigma_2$ -open set  $V$  of  $Y$  having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement with  $x \in f^{-1}(V)$  such that

$$U \not\subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

for every  $\star$ -open set  $U$  of  $X$  containing  $x$ . Let  $x_U \in U$  and  $x_U \notin f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$  for every  $\star$ -open set  $U$  of  $X$  containing  $x$ . Then, for each  $\star$ -neighbourhood net  $(x_U)$ ,  $(x_U)$   $\star$ -converges to  $x$ , but  $(x_U)$  is not eventually in  $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ . This is a contradiction. Thus,  $f$  is almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous.  $\square$

The  $\star$ -frontier [18] of a subset  $A$  of an ideal topological space  $(X, \tau, \mathcal{I})$ , denoted by  $\text{fr}^\star(A)$ , is defined by

$$\text{fr}^\star(A) = \text{Cl}^\star(A) \cap \text{Cl}^\star(X - A) = \text{Cl}^\star(A) - \text{Int}^\star(A).$$

**Theorem 3.6.** *The set of all points  $x$  of  $X$  at which a function  $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)$  is not almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous is identical with the union of the  $\star$ -frontier of the inverse images of  $(\sigma_1, \sigma_2)r$ -open sets containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement.*

*Proof.* Let  $x$  be a point of  $X$  at which  $f$  is not almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous. Then, by Theorem 3.1 there exists a  $(\sigma_1, \sigma_2)r$ -open set  $V$  of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement such that  $U \cap (X - f^{-1}(V)) \neq \emptyset$  for every  $\star$ -open set  $U$  of  $X$  containing  $x$ . Thus,  $x \in \text{Cl}^\star(X - f^{-1}(V))$ . On the other hand, we have  $x \in f^{-1}(V) \subseteq \text{Cl}^\star(f^{-1}(V))$  and hence  $x \in \text{fr}^\star(f^{-1}(V))$ .

Conversely, suppose that  $V$  is a  $(\sigma_1, \sigma_2)r$ -open set of  $Y$  containing  $f(x)$  and having  $\mathcal{N}(\sigma_1, \sigma_2)$ -closed complement such that  $x \in \text{fr}^\star(f^{-1}(V))$ . If  $f$  is almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous at  $x \in X$ . Then by Theorem 3.1, we have  $x \in \text{Int}^\star(f^{-1}(V))$ . This is a contradiction and so  $f$  is not almost nearly  $\tau^\star(\sigma_1, \sigma_2)$ -continuous at  $x$ .  $\square$

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