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Characterizations of Bipolar Fuzzy Bi-Interior Ideals in Gamma-Semirings

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Abstract. This study investigates the structure of bipolar fuzzy bi-interior ideals (BFBII) within the framework of a Γ-semiring. The main objective is to explore and establish essential properties of BFBII, including conditions based on characteristic functions and level-set criteria. Furthermore, the research provides a comprehensive characterization of bipolar fuzzy bi-interior ideals by relating them to the bi-interior ideals of the underlying Γ-semiring, thereby offering deeper insights into their algebraic behavior and logical structure.

1. Introduction

The foundational concept of fuzzy sets, introduced by Zadeh in 1965 [11], initiated a major paradigm shift in the mathematical modeling of uncertainty and imprecision. This idea—characterized by membership values ranging over the unit interval $[0,1]$—paved the way for extensive generalizations across algebraic structures. In 1994, Zhang [12] extended the classical fuzzy framework to bipolar-valued fuzzy sets, where the membership degree is taken from the interval $[-1,1]$, thus allowing for both positive and negative preferences in decision-making contexts.

On the other hand, the algebraic structure known as a Γ-semiring (GSR) was proposed by Murali Krishna Rao in 1995 [8] as a natural generalization of rings, semirings, and ternary semirings. He

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established a rich ideal theory for GSRs, introducing concepts such as left and right bi-quasi-ideals and bi-interior ideals. These structures not only broaden the scope of semiring theory but also enable new methods for characterizing algebraic and fuzzy behaviors.

The intersection of fuzzy theory and Γ-semirings has since attracted considerable attention. Researchers such as Bhargavi and Eswarlal [1] have significantly expanded the theory by incorporating fuzzy ideals into the GSR framework. Madhulatha and Bhargavi further developed the notion of bipolar fuzzy Γ-semirings (BFGSRs) and introduced several classes of bipolar fuzzy ideals—including bi-ideals, interior ideals, and normal ideals—within this extended setting [5,6].

Additional developments in this area include Jagatap's study of interior ideals in Γ-semirings [4], in which key structural properties and characterizations are explored. Mandal [7] examined fuzzy and interior ideals in ordered semirings, providing foundational insights relevant to their behavior under fuzzy extensions. Bhargavi et al. [3] proposed the notion of vague bi-interior ideals, contributing to the generalization of fuzzy concepts in Γ-semirings through vague set theory. Most recently, Parvatham et al. [10] investigated bipolar fuzzy quasi-ideals in Γ-semirings, offering a new perspective on how fuzzy logic intertwines with algebraic structures in more nuanced ways.

These cumulative efforts demonstrate the depth and growing relevance of fuzzy ideal theory in Γ-semirings. They also establish a robust foundation for exploring new forms of fuzzy ideals, such as bipolar fuzzy bi-interior ideals, which are the focus of this study.

In this work, we introduce and examine the notion of bipolar fuzzy bi-interior ideals (BFBII) in Γ-semirings, extending the existing concepts of bipolar fuzzy bi-ideals and bipolar fuzzy interior ideals. We also explore the structural relationship between BFBII and crisp bi-interior ideals, providing an alternative characterization of these fuzzy ideals in terms of their crisp counterparts within the Γ-semiring framework.

2. Preliminaries

In this section, we recall some basic concepts and definitions that we will need in the sequel.

Definition 2.1. *[8] Let $\aleph$ and Γ be two additive commutative semigroups. Then $\aleph$ is called a Γ-semiring if there exists a mapping $\aleph \times \Gamma \times \aleph \to \aleph$, denoted by $\ddot{e}\ddot{\varrho}\ddot{o}$ for $\ddot{e}, \ddot{o} \in \aleph$ and $\ddot{\varrho} \in \Gamma$, satisfying the following conditions for all $\ddot{e}, \ddot{o}, \ddot{v} \in \aleph$ and $\ddot{\varrho}, \ddot{\tau} \in \Gamma$:*
(i) $\ddot{e}\ddot{\varrho}(\ddot{o} + \ddot{v}) = \ddot{e}\ddot{\varrho}\ddot{o} + \ddot{e}\ddot{\varrho}\ddot{v}$
(ii) $(\ddot{e} + \ddot{o})\ddot{\varrho}\ddot{v} = \ddot{e}\ddot{\varrho}\ddot{v} + \ddot{o}\ddot{\varrho}\ddot{v}$
(iii) $\ddot{e}(\ddot{\varrho} + \ddot{\tau})\ddot{v} = \ddot{e}\ddot{\varrho}\ddot{v} + \ddot{e}\ddot{\tau}\ddot{v}$
(iv) $\ddot{e}\ddot{\varrho}(\ddot{o}\ddot{\tau}\ddot{v}) = (\ddot{e}\ddot{\varrho}\ddot{o})\ddot{\tau}\ddot{v}$

Definition 2.2. *[11] Let $\aleph$ be any non-empty set. A mapping $\Im : \aleph \to [0,1]$ is called a fuzzy set in $\aleph$.*

Definition 2.3. *[12] Let $\aleph$ be the universe of discourse. A bipolar-valued fuzzy set $\Im$ in $\aleph$ is an object of the form*

$$\Im = \{(\ddot{v},\ \Im^{-}(\ddot{v}),\ \Im^{+}(\ddot{v})) \mid \ddot{v} \in \aleph\},$$

where $\mathfrak{I}^{-}:\aleph\to[-1,0]$ *and* $\mathfrak{I}^{+}:\aleph\to[0,1]$ *are mappings representing the negative and positive degrees of membership, respectively.*

For simplicity, we shall denote the bipolar-valued fuzzy set as

$$\mathfrak{I}=\{\aleph;\ \mathfrak{I}^{-},\ \mathfrak{I}^{+}\},$$

and henceforth refer to such sets as bipolar fuzzy sets (BFSs).

Definition 2.4. *[1] Let* $\mathfrak{I}=\{\aleph;\ \mathfrak{I}^{-},\ \mathfrak{I}^{+}\}$ *be a BFS, and let* $s\times t\in[-1,0]\times[0,1]$. *Then the sets*

$$\mathfrak{I}_s^N=\{\ddot{v}\in\aleph\mid\mathfrak{I}^{-}(\ddot{v})\le s\},\quad \mathfrak{I}_t^P=\{\ddot{v}\in\aleph\mid\mathfrak{I}^{+}(\ddot{v})\ge t\}$$

are called the negative s-cut and positive t-cut of $\mathfrak{I}$*, respectively. The intersection*

$$\mathfrak{I}_{(s,t)}=\mathfrak{I}_s^N\cap\mathfrak{I}_t^P$$

is called the (s,t)*-cut of the BFS* $\mathfrak{I}=\{\aleph;\ \mathfrak{I}^{-},\ \mathfrak{I}^{+}\}$.

Definition 2.5. *[1] Let* $\mathfrak{I}=\{\aleph;\mathfrak{I}^{-},\mathfrak{I}^{+}\}$ *and* $\omega=\{\aleph;\omega^{-},\omega^{+}\}$ *be two BFSs in the universe of discourse* $\aleph$*. The intersection of* $\mathfrak{I}$ *and* ω *is defined as:*

$$(\mathfrak{I}^{-}\cap\omega^{-})(\ddot{v})=\min\{\mathfrak{I}^{-}(\ddot{v}),\ \omega^{-}(\ddot{v})\},\quad(\mathfrak{I}^{+}\cap\omega^{+})(\ddot{v})=\min\{\mathfrak{I}^{+}(\ddot{v}),\ \omega^{+}(\ddot{v})\}$$

The union of $\mathfrak{I}$ *and* ω *is defined as:*

$$(\mathfrak{I}^{-}\cup\omega^{-})(\ddot{v})=\max\{\mathfrak{I}^{-}(\ddot{v}),\ \omega^{-}(\ddot{v})\},\quad(\mathfrak{I}^{+}\cup\omega^{+})(\ddot{v})=\max\{\mathfrak{I}^{+}(\ddot{v}),\ \omega^{+}(\ddot{v})\}$$

A BFS $\mathfrak{I}$ *is said to be contained in another BFS* ω *(denoted* $\mathfrak{I}\subseteq\omega$*) if and only if:*

$$\mathfrak{I}^{-}(\ddot{v})\ge\omega^{-}(\ddot{v})\quad\text{and}\quad\mathfrak{I}^{+}(\ddot{v})\le\omega^{+}(\ddot{v}),\quad\forall\,\ddot{v}\in\aleph.$$

Definition 2.6. *[1] Let I be a subset of a* Γ*-semiring* $\aleph$*. The fuzzy characteristic set of I is the fuzzy set* δ_I *defined by:*

$$\delta_I(\ddot{v})=\begin{cases}1, & \text{if } \ddot{v}\in I\\ 0, & \text{if } \ddot{v}\notin I\end{cases}$$

Then δ_I *is called the fuzzy characteristic function of I in the interval* $[0,1]$.

Definition 2.7. *[5] Let I be a subset of a* Γ*-semiring* $\aleph$*. The bipolar fuzzy characteristic function of I is defined as follows:*

$$\delta_I^{+}(\ddot{v})=\begin{cases}1, & \text{if } \ddot{v}\in I\\ 0, & \text{if } \ddot{v}\notin I\end{cases},\quad \delta_I^{-}(\ddot{v})=\begin{cases}-1, & \text{if } \ddot{v}\in I\\ 0, & \text{if } \ddot{v}\notin I\end{cases}$$

Then, $\delta_I=\{\delta_I^{-},\delta_I^{+}\}$ *is called the bipolar fuzzy characteristic function of I.*

Definition 2.8. *[5] A fuzzy set* $\mathfrak{I}$ *in* $\aleph$ *is called a fuzzy* Γ*-semiring if it satisfies the following properties for all* $\ddot{e},\ddot{o}\in\aleph$ *and* $\varrho\in\Gamma$*:*

(i) $\mathfrak{I}(\ddot{e}+\ddot{o})\ge\min\{\mathfrak{I}(\ddot{e}),\ \mathfrak{I}(\ddot{o})\}$
(ii) $\mathfrak{I}(\ddot{e}\varrho\ddot{o})\ge\min\{\mathfrak{I}(\ddot{e}),\ \mathfrak{I}(\ddot{o})\}$

Definition 2.9. *[5] A BFS $\mathfrak{I} = (\aleph; \mathfrak{I}^-, \mathfrak{I}^+)$ in $\aleph$ is called a bipolar fuzzy Γ-semiring if it satisfies the following properties for all $\ddot{e}, \ddot{o} \in \aleph$ and $\varrho \in \Gamma$:*

(i) $\mathfrak{I}^-(\ddot{e} + \ddot{o}) \leq \max\{\mathfrak{I}^-(\ddot{e}),\ \mathfrak{I}^-(\ddot{o})\}$
(ii) $\mathfrak{I}^-(\ddot{e}\varrho\ddot{o}) \leq \max\{\mathfrak{I}^-(\ddot{e}),\ \mathfrak{I}^-(\ddot{o})\}$
(iii) $\mathfrak{I}^+(\ddot{e} + \ddot{o}) \geq \min\{\mathfrak{I}^+(\ddot{e}),\ \mathfrak{I}^+(\ddot{o})\}$
(iv) $\mathfrak{I}^+(\ddot{e}\varrho\ddot{o}) \geq \min\{\mathfrak{I}^+(\ddot{e}),\ \mathfrak{I}^+(\ddot{o})\}$

Definition 2.10. *[2] Let $\mathfrak{I}$ be a fuzzy set in a semiring $\aleph$. Then $\mathfrak{I}$ is called a fuzzy left (respectively, fuzzy right) ideal of $\aleph$ if for all $\ddot{e}, \ddot{o} \in \aleph$, the following hold:*

(i) $\mathfrak{I}(\ddot{e} + \ddot{o}) \geq \min\{\mathfrak{I}(\ddot{e}),\ \mathfrak{I}(\ddot{o})\}$
(ii) $\mathfrak{I}(\ddot{e}\ddot{o}) \geq \mathfrak{I}(\ddot{o})$ *(respectively, $\mathfrak{I}(\ddot{e}\ddot{o}) \geq \mathfrak{I}(\ddot{e})$)*

Furthermore, $\mathfrak{I}$ is called a fuzzy ideal of $\aleph$ if it is both a fuzzy left ideal and a fuzzy right ideal.

Definition 2.11. *[2] A BFS $\mathfrak{I} = (\aleph; \mathfrak{I}^-, \mathfrak{I}^+)$ is called a BF left (respectively, BF right) ideal of a Γ-semiring $\aleph$ if for all $\ddot{e}, \ddot{o} \in \aleph$ and $\varrho \in \Gamma$, the following conditions hold:*

(i) $\mathfrak{I}^-(\ddot{e} + \ddot{o}) \leq \max\{\mathfrak{I}^-(\ddot{e}),\ \mathfrak{I}^-(\ddot{o})\}$
(ii) $\mathfrak{I}^-(\ddot{e}\varrho\ddot{o}) \leq \mathfrak{I}^-(\ddot{o})$ *(respectively, $\mathfrak{I}^-(\ddot{e}\varrho\ddot{o}) \leq \mathfrak{I}^-(\ddot{e})$)*
(iii) $\mathfrak{I}^+(\ddot{e} + \ddot{o}) \geq \min\{\mathfrak{I}^+(\ddot{e}),\ \mathfrak{I}^+(\ddot{o})\}$
(iv) $\mathfrak{I}^+(\ddot{e}\varrho\ddot{o}) \geq \mathfrak{I}^+(\ddot{o})$ *(respectively, $\mathfrak{I}^+(\ddot{e}\varrho\ddot{o}) \geq \mathfrak{I}^+(\ddot{e})$)*

Moreover, $\mathfrak{I}$ is called a BF ideal of the Γ-semiring $\aleph$ if it is both a BF left ideal and a BF right ideal.

Definition 2.12. *[2] Let $\mathfrak{I}$ be a fuzzy set in a Γ-semiring $\aleph$. Then $\mathfrak{I}$ is called a fuzzy bi-ideal of $\aleph$ if for all $\ddot{e}, \ddot{o}, \ddot{b} \in \aleph$ and $\alpha, \beta \in \Gamma$, the following conditions hold:*

(i) $\mathfrak{I}(\ddot{e} + \ddot{o}) \geq \min\{\mathfrak{I}(\ddot{e}),\ \mathfrak{I}(\ddot{o})\}$
(ii) $\mathfrak{I}(\ddot{e}\alpha\ddot{o}\beta\ddot{b}) \geq \min\{\mathfrak{I}(\ddot{e}),\ \mathfrak{I}(\ddot{b})\}$

Definition 2.13. *[9] A BFS $\mathfrak{I} = (\aleph; \mathfrak{I}^-, \mathfrak{I}^+)$ on a Γ-semiring $\aleph$ is called a bipolar fuzzy bi-ideal if, for all $\ddot{e}, \ddot{o}, \ddot{b} \in \aleph$ and $\alpha, \beta \in \Gamma$, the following conditions hold:*

(i) $\mathfrak{I}^-(\ddot{e} + \ddot{o}) \leq \max\{\mathfrak{I}^-(\ddot{e}),\ \mathfrak{I}^-(\ddot{o})\}$
(ii) $\mathfrak{I}^-(\ddot{e}\alpha\ddot{o}\beta\ddot{b}) \leq \max\{\mathfrak{I}^-(\ddot{e}),\ \mathfrak{I}^-(\ddot{b})\}$
(iii) $\mathfrak{I}^+(\ddot{e} + \ddot{o}) \geq \min\{\mathfrak{I}^+(\ddot{e}),\ \mathfrak{I}^+(\ddot{o})\}$
(iv) $\mathfrak{I}^+(\ddot{e}\alpha\ddot{o}\beta\ddot{b}) \geq \min\{\mathfrak{I}^+(\ddot{e}),\ \mathfrak{I}^+(\ddot{b})\}$

Definition 2.14. *[2] A non-empty subset I of a Γ-semiring $\aleph$ is called an interior ideal of $\aleph$ if:*

(i) *I is an additive subsemigroup of $\aleph$*
(ii) $\aleph\Gamma I\Gamma\aleph \subseteq I$

Definition 2.15. *[5] A fuzzy set $\mathfrak{I}$ in a Γ-semiring $\aleph$ is called a fuzzy interior ideal of $\aleph$ if, for all $\ddot{e}, \ddot{o}, \ddot{v} \in \aleph$ and $\varrho, \tau \in \Gamma$, the following conditions hold:*

(i) $\mathfrak{I}(\ddot{e} + \ddot{o}) \geq \min\{\mathfrak{I}(\ddot{e}),\ \mathfrak{I}(\ddot{o})\}$
(ii) $\mathfrak{I}(\ddot{e}\varrho\ddot{o}) \geq \min\{\mathfrak{I}(\ddot{e}),\ \mathfrak{I}(\ddot{o})\}$

(iii) $\Im(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) \geq \Im(\ddot{o})$

Definition 2.16. *A BFS $\Im = (\aleph; \Im^+, \Im^-)$ in a Γ-semiring $\aleph$ is called a bipolar fuzzy interior ideal of $\aleph$ if, for all $\ddot{e}, \ddot{o}, \ddot{v} \in \aleph$ and $\varrho, \tau \in \Gamma$, the following conditions hold:*

(i) $\Im^-(\ddot{e} + \ddot{o}) \leq \max\{\Im^-(\ddot{e}),\ \Im^-(\ddot{o})\}$
(ii) $\Im^-(\ddot{e}\varrho\ddot{o}) \leq \max\{\Im^-(\ddot{e}),\ \Im^-(\ddot{o})\}$
(iii) $\Im^+(\ddot{e} + \ddot{o}) \geq \min\{\Im^+(\ddot{e}),\ \Im^+(\ddot{o})\}$
(iv) $\Im^+(\ddot{e}\varrho\ddot{o}) \geq \min\{\Im^+(\ddot{e}),\ \Im^+(\ddot{o})\}$
(v) $\Im^-(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) \leq \Im^-(\ddot{o})$
(vi) $\Im^+(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) \geq \Im^+(\ddot{o})$

3. Notations

In this section, we recall some fundamental definitions and notational conventions that will be used throughout the subsequent discussions. Unless stated otherwise, the symbol $\aleph$ denotes a Γ-semiring.

For brevity, we adopt the following abbreviations:

(1) GSR: Γ-Semiring
(2) BFS: Bipolar Fuzzy Set
(3) BFGS: Bipolar Fuzzy Γ-Semiring
(4) BFI: Bipolar Fuzzy Ideal
(5) BFBI: Bipolar Fuzzy Bi-Ideal
(6) BFBII: Bipolar Fuzzy Bi-Interior Ideal

4. Bipolar Fuzzy Bi-Interior Ideals of Gamma-Semirings

In this section, we introduce and investigate the concept of bipolar fuzzy bi-interior ideals (BFBII) in the setting of Γ-semirings. We establish several key characterizations of BFBII, particularly in terms of bipolar fuzzy bi-ideals (BFBI), bipolar fuzzy interior ideals (BFII), level cuts, and characteristic functions. These results not only unify and extend existing classes of bipolar fuzzy ideals but also provide deeper structural insight into the interplay between fuzziness and algebraic ideal theory in Γ-semirings.

Definition 4.1. *A BFS $\Im = (\aleph; \Im^+, \Im^-)$ on $\aleph$ is called a bipolar fuzzy bi-interior ideal (BFBII) of a GSR if, for all $\ddot{e}, \ddot{o} \in \aleph$, the following conditions hold:*

(i) $\Im^-(\ddot{e} + \ddot{o}) \leq \max\{\Im^-(\ddot{e}),\ \Im^-(\ddot{o})\}$
(ii) $(\delta^-\Gamma\Im^-\Gamma\delta^-) \cup (\Im^-\Gamma\delta^-\Gamma\Im^-) \supseteq \Im^-$
(iii) $\Im^+(\ddot{e} + \ddot{o}) \geq \min\{\Im^+(\ddot{e}),\ \Im^+(\ddot{o})\}$
(iv) $(\delta^+\Gamma\Im^+\Gamma\delta^+) \cap (\Im^+\Gamma\delta^+\Gamma\Im^+) \subseteq \Im^+$

Example 4.1. *Let D be the set of all negative integers and let Γ be the set of all negative even integers. Then both D and Γ form additive commutative semigroups. Define a mapping $D \times \Gamma \times D \to D$ by $\ddot{e}\varrho\ddot{o} = \ddot{e} \cdot \varrho \cdot \ddot{o}$, where $\ddot{e}, \ddot{o} \in D$ and $\varrho \in \Gamma$. Under this operation, D forms a GSR, i.e., D is a GSR.*

Define a BFS $\mathfrak{I} = (D; \mathfrak{I}^+, \mathfrak{I}^-)$ in D by:

$$\mathfrak{I}^-(\psi) = \begin{cases} -0.5 & \text{if } \psi = -1 \\ -0.6 & \text{if } \psi = -2 \\ -0.8 & \text{if } \psi < -2 \end{cases} \quad \text{and} \quad \mathfrak{I}^+(\psi) = \begin{cases} 0.5 & \text{if } \psi = -1 \\ 0.6 & \text{if } \psi = -2 \\ 0.7 & \text{if } \psi < -2 \end{cases}$$

Then $\mathfrak{I}$ is a BFBII of D.

To establish the relationship between BFBIs and BFBIIs in a GSR, we begin with the following result: every BFBI is necessarily a BFBII.

Theorem 4.1. *Every BFBI of $\aleph$ is a BFBII of $\aleph$.*

Proof. Let $\mathfrak{I} = (\aleph; \mathfrak{I}^-, \mathfrak{I}^+)$ be a BFBI of $\aleph$. Then, for all $\ddot{e}, \ddot{o} \in \aleph$, we have

$$\text{(i)} \quad \mathfrak{I}^-(\ddot{e} + \ddot{o}) \leq \max\{\mathfrak{I}^-(\ddot{e}),\ \mathfrak{I}^-(\ddot{o})\}$$
$$\text{(ii)} \quad \mathfrak{I}^+(\ddot{e} + \ddot{o}) \geq \min\{\mathfrak{I}^+(\ddot{e}),\ \mathfrak{I}^+(\ddot{o})\}$$

Now, let $\ddot{x} = \ddot{a}\gamma\ddot{p}\eta\ddot{b}$ for some $\ddot{p} \in \aleph$, $\ddot{a}, \ddot{b} \in \mathfrak{I}$, and $\gamma, \eta \in \Gamma$. We compute

$$\begin{aligned} (\mathfrak{I}^+\Gamma\delta^+\Gamma\mathfrak{I}^+)(\ddot{x}) &= \sup\left\{\min\left\{(\mathfrak{I}^+\Gamma\delta^+)(\ddot{a}\gamma\ddot{p}),\ \mathfrak{I}^+(\ddot{b})\right\}\right\} \\ &= \sup\left\{\min\left\{\min\left\{\mathfrak{I}^+(\ddot{a}),\ \delta^+(\ddot{p})\right\},\ \mathfrak{I}^+(\ddot{b})\right\}\right\} \\ &= \sup\left\{\min\left\{\mathfrak{I}^+(\ddot{a}),\ \mathfrak{I}^+(\ddot{b})\right\}\right\} \\ &\leq \mathfrak{I}^+(\ddot{a}\gamma\ddot{p}\eta\ddot{b}) \\ &= \mathfrak{I}^+(\ddot{x}). \end{aligned}$$

Hence,

$$\mathfrak{I}^+\Gamma\delta^+\Gamma\mathfrak{I}^+ \subseteq \mathfrak{I}^+ \Rightarrow (\delta^+\Gamma\mathfrak{I}^+\Gamma\delta^+) \cap (\mathfrak{I}^+\Gamma\delta^+\Gamma\mathfrak{I}^+) \subseteq \mathfrak{I}^+.$$

Similarly, we compute

$$\begin{aligned} (\mathfrak{I}^-\Gamma\delta^-\Gamma\mathfrak{I}^-)(\ddot{x}) &= \inf\left\{\max\left\{(\mathfrak{I}^-\Gamma\delta^-)(\ddot{a}\gamma\ddot{p}),\ \mathfrak{I}^-(\ddot{b})\right\}\right\} \\ &= \inf\left\{\max\left\{\max\{\mathfrak{I}^-(\ddot{a}),\ \delta^-(\ddot{p})\},\ \mathfrak{I}^-(\ddot{b})\right\}\right\} \\ &= \inf\left\{\max\left\{\mathfrak{I}^-(\ddot{a}),\ \mathfrak{I}^-(\ddot{b})\right\}\right\} \\ &\geq \mathfrak{I}^-(\ddot{a}\gamma\ddot{p}\eta\ddot{b}) \\ &= \mathfrak{I}^-(\ddot{x}). \end{aligned}$$

Hence,

$$\mathfrak{I}^-\Gamma\delta^-\Gamma\mathfrak{I}^- \supseteq \mathfrak{I}^- \Rightarrow (\delta^-\Gamma\mathfrak{I}^-\Gamma\delta^-) \cup (\mathfrak{I}^-\Gamma\delta^-\Gamma\mathfrak{I}^-) \supseteq \mathfrak{I}^-.$$

Therefore, $\mathfrak{I}$ is a BFBII of $\aleph$. $\square$

Next, we consider the connection between BFIIs and BFBII ideals in a GSR. The following theorem establishes that every BFII also satisfies the conditions of a BFBII.

Theorem 4.2. *Every BFII of $\aleph$ is a BFBII of $\aleph$.*

Proof. Let $\Im = (\aleph; \Im^-, \Im^+)$ be a BFII of $\aleph$. Then, for all $\ddot{e}, \ddot{o} \in \aleph$, we have

$$\begin{aligned} &\text{(i)} \quad \Im^-(\ddot{e}+\ddot{o}) \leq \max\{\Im^-(\ddot{e}),\ \Im^-(\ddot{o})\} \\ &\text{(ii)} \quad \Im^+(\ddot{e}+\ddot{o}) \geq \min\{\Im^+(\ddot{e}),\ \Im^+(\ddot{o})\} \end{aligned}$$

Let $\ddot{x} = \ddot{p}\gamma\ddot{a}\eta\ddot{q}$, where $\ddot{p}, \ddot{q} \in \aleph$, $\ddot{a} \in \Im$, and $\gamma, \eta \in \Gamma$. We compute

$$\begin{aligned} (\delta^+\Gamma\Im^+\Gamma\delta^+)(\ddot{x}) &= \sup\left\{\min\left\{(\delta^+\Gamma\Im^+)(\ddot{p}\gamma\ddot{a}),\ \delta^+(\ddot{q})\right\}\right\} \\ &= \sup\left\{\min\left\{\min\left\{\delta^+(\ddot{p}),\ \Im^+(\ddot{a})\right\},\ \delta^+(\ddot{q})\right\}\right\} \\ &= \sup\left\{\min\left\{\Im^+(\ddot{a}),\ \delta^+(\ddot{q})\right\}\right\} \\ &= \Im^+(\ddot{a}) \\ &\leq \Im^+(\ddot{p}\gamma\ddot{a}\eta\ddot{q}) \\ &= \Im^+(\ddot{x}). \end{aligned}$$

Thus,

$$\delta^+\Gamma\Im^+\Gamma\delta^+ \subseteq \Im^+ \Rightarrow (\delta^+\Gamma\Im^+\Gamma\delta^+) \cap (\Im^+\Gamma\delta^+\Gamma\Im^+) \subseteq \Im^+.$$

Now compute

$$\begin{aligned} (\delta^-\Gamma\Im^-\Gamma\delta^-)(\ddot{x}) &= \inf\{\max\{(\delta^-\Gamma\Im^-)(\ddot{p}\gamma\ddot{a}),\ \delta^-(\ddot{q})\}\} \\ &= \inf\{\max\{\max\{\delta^-(\ddot{p}),\ \Im^-(\ddot{a})\},\ \delta^-(\ddot{q})\}\} \\ &= \inf\{\max\{\Im^-(\ddot{a}),\ \delta^-(\ddot{q})\}\} \\ &= \Im^-(\ddot{a}) \\ &\geq \Im^-(\ddot{p}\gamma\ddot{a}\eta\ddot{q}) \\ &= \Im^-(\ddot{x}). \end{aligned}$$

Hence,

$$\delta^-\Gamma\Im^-\Gamma\delta^- \supseteq \Im^- \Rightarrow (\delta^-\Gamma\Im^-\Gamma\delta^-) \cup (\Im^-\Gamma\delta^-\Gamma\Im^-) \supseteq \Im^-.$$

Therefore, $\Im$ is a BFBII of $\aleph$. □

We now investigate the role of BFLIs in BFBII structures. The following result shows that every BFLI naturally satisfies the properties of a BFBII in a GSR.

Theorem 4.3. *Every BFLI of $\aleph$ is a BFBII of $\aleph$.*

Proof. Let $\mathfrak{I} = (\aleph; \mathfrak{I}^-, \mathfrak{I}^+)$ be a BFLI of $\aleph$. Then, for all $\ddot{e}, \ddot{o} \in \aleph$, we have

$$\text{(i)} \quad \mathfrak{I}^-(\ddot{e} + \ddot{o}) \leq \max\{\mathfrak{I}^-(\ddot{e}),\ \mathfrak{I}^-(\ddot{o})\}$$
$$\text{(ii)} \quad \mathfrak{I}^+(\ddot{e} + \ddot{o}) \geq \min\{\mathfrak{I}^+(\ddot{e}),\ \mathfrak{I}^+(\ddot{o})\}$$

Let $\ddot{x} = \ddot{p}\gamma\ddot{a}\eta\ddot{q}$, where $\ddot{p}, \ddot{q} \in \aleph$, $\ddot{a} \in \mathfrak{I}$, and $\gamma, \eta \in \Gamma$. Consider

$$\begin{aligned}(\delta^+\Gamma\mathfrak{I}^+\Gamma\delta^+)(\ddot{x}) &= \sup\left\{\min\left\{(\delta^+\Gamma\mathfrak{I}^+)(\ddot{p}\gamma\ddot{a}),\ \delta^+(\ddot{q})\right\}\right\} \\ &= \sup\left\{\min\left\{\min\left\{\delta^+(\ddot{p}),\ \mathfrak{I}^+(\ddot{a})\right\},\ \delta^+(\ddot{q})\right\}\right\} \\ &= \sup\left\{\min\left\{\mathfrak{I}^+(\ddot{a}),\ \delta^+(\ddot{q})\right\}\right\} \\ &= \mathfrak{I}^+(\ddot{a}) \\ &\leq \mathfrak{I}^+(\ddot{p}\gamma\ddot{a}) \\ &\leq \mathfrak{I}^+(\ddot{x}).\end{aligned}$$

Thus,

$$\delta^+\Gamma\mathfrak{I}^+\Gamma\delta^+ \subseteq \mathfrak{I}^+ \Rightarrow (\delta^+\Gamma\mathfrak{I}^+\Gamma\delta^+) \cap (\mathfrak{I}^+\Gamma\delta^+\Gamma\mathfrak{I}^+) \subseteq \mathfrak{I}^+.$$

Next, consider

$$\begin{aligned}(\delta^-\Gamma\mathfrak{I}^-\Gamma\delta^-)(\ddot{x}) &= \inf\{\max\{(\delta^-\Gamma\mathfrak{I}^-)(\ddot{p}\gamma\ddot{a}),\ \delta^-(\ddot{q})\}\} \\ &= \inf\{\max\{\max\{\delta^-(\ddot{p}),\ \mathfrak{I}^-(\ddot{a})\},\ \delta^-(\ddot{q})\}\} \\ &= \inf\{\max\{\mathfrak{I}^-(\ddot{a}),\ \delta^-(\ddot{q})\}\} \\ &= \mathfrak{I}^-(\ddot{a}) \\ &\geq \mathfrak{I}^-(\ddot{p}\gamma\ddot{a}) \\ &\geq \mathfrak{I}^-(\ddot{x}).\end{aligned}$$

Hence,

$$\delta^-\Gamma\mathfrak{I}^-\Gamma\delta^- \supseteq \mathfrak{I}^- \Rightarrow (\delta^-\Gamma\mathfrak{I}^-\Gamma\delta^-) \cup (\mathfrak{I}^-\Gamma\delta^-\Gamma\mathfrak{I}^-) \supseteq \mathfrak{I}^-.$$

Therefore, $\mathfrak{I}$ is a BFBII of $\aleph$. □

Similarly, we examine the structure of BFRIs within the BFBII framework. The following theorem establishes that each BFRI in a GSR inherits the characteristics of a BFBII.

Theorem 4.4. *Every BFRI of $\aleph$ is a BFBII of $\aleph$.*

Proof. Let $\mathfrak{I} = (\aleph; \mathfrak{I}^-, \mathfrak{I}^+)$ be a BFRI of $\aleph$. Then, by definition, $\mathfrak{I}$ satisfies the following conditions for all $\ddot{e}, \ddot{o} \in \aleph$:

$$\text{(i)} \quad \mathfrak{I}^-(\ddot{e} + \ddot{o}) \leq \max\{\mathfrak{I}^-(\ddot{e}),\ \mathfrak{I}^-(\ddot{o})\}$$
$$\text{(ii)} \quad \mathfrak{I}^+(\ddot{e} + \ddot{o}) \geq \min\{\mathfrak{I}^+(\ddot{e}),\ \mathfrak{I}^+(\ddot{o})\}$$

The remaining argument follows analogously to the proof of the previous theorem (Theorem 4.3), by considering the element $\ddot{x} = \ddot{p}\gamma\ddot{a}\eta\ddot{q}$, where $\ddot{p}, \ddot{q} \in \aleph$, $\ddot{a} \in \Im$, and $\gamma, \eta \in \Gamma$. Applying similar reasoning, we obtain

$$(\delta^+\Gamma\Im^+\Gamma\delta^+) \cap (\Im^+\Gamma\delta^+\Gamma\Im^+) \subseteq \Im^+ \quad \text{and} \quad (\delta^-\Gamma\Im^-\Gamma\delta^-) \cup (\Im^-\Gamma\delta^-\Gamma\Im^-) \supseteq \Im^-$$

Hence, $\Im$ is a BFBII of $\aleph$. □

Corollary 4.1. *Every BFI of $\aleph$ is a BFBII of $\aleph$.*

Proof. The result follows directly from Theorems 4.3 and 4.4, since a BFI is both a BFLI and a BFRI. □

To further understand the structure of BFBII, we explore its relationship with the level cuts of a BFS. The following theorem provides a necessary and sufficient condition for a BFS to be a BFBII in terms of its level cuts.

Theorem 4.5. *A BFS $\Im = (\aleph; \Im^-, \Im^+)$ is a BFBII of $\aleph$ if and only if its level cut $\Im_{(s,t)}$ is a BII of $\aleph$ for all $s \times t \in [-1, 0] \times [0, 1]$.*

Proof. Suppose $\Im = (\aleph; \Im^-, \Im^+)$ is a BFBII of $\aleph$. Then $\Im$ is a BFGSR of $\aleph$. By Theorem 3.4 of [5], its level cuts are sub-GSRs of $\aleph$. Let $\ddot{x} \in (\aleph\Gamma\Im_t^P\Gamma\aleph) \cap (\Im_t^P\Gamma\aleph\Gamma\Im_t^P)$. Then

$$\ddot{x} = \ddot{p}\gamma\ddot{a}\eta\ddot{q} = \ddot{b}\zeta\ddot{r}\epsilon\ddot{c}$$

for $\ddot{p}, \ddot{q}, \ddot{r} \in \aleph$, $\ddot{a}, \ddot{b}, \ddot{c} \in \Im_t^P$, and $\gamma, \eta, \zeta, \epsilon \in \Gamma$. Now consider

$$\begin{aligned}
(\delta^+\Gamma\Im_t^P\Gamma\delta^+)(\ddot{x}) &= \sup\left\{\min\left\{(\delta^+\Gamma\Im_t^P)(\ddot{p}\gamma\ddot{a}), \delta^+(\ddot{q})\right\}\right\} \\
&= \sup\left\{\min\left\{\min\left\{\delta^+(\ddot{p}), \Im_t^P(\ddot{a})\right\}, \delta^+(\ddot{q})\right\}\right\} \\
&= \sup\left\{\min\left\{\Im_t^P(\ddot{a}), \delta^+(\ddot{q})\right\}\right\} \\
&= \Im_t^P(\ddot{a}) \\
&\geq t.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\Im_t^P\Gamma\delta^+\Gamma\Im_t^P)(\ddot{x}) &= \sup\left\{\min\left\{(\Im_t^P\Gamma\delta^+)(\ddot{b}\eta\ddot{r}), \Im_t^P(\ddot{c})\right\}\right\} \\
&= \sup\left\{\min\left\{\min\left\{\Im_t^P(\ddot{b}), \delta^+(\ddot{r})\right\}, \Im_t^P(\ddot{c})\right\}\right\} \\
&= \sup\left\{\min\left\{\Im_t^P(\ddot{b}), \Im_t^P(\ddot{c})\right\}\right\} \\
&\geq t.
\end{aligned}$$

Since $\Im$ is a BFBII, we have

$$(\delta^+\Gamma\Im^+\Gamma\delta^+) \cap (\Im^+\Gamma\delta^+\Gamma\Im^+) \subseteq \Im^+.$$

Thus,

$$\mathfrak{I}^+(\ddot{x}) \geq \min\left\{(\delta^+\Gamma\mathfrak{I}^+\Gamma\delta^+)(\ddot{x}), (\mathfrak{I}^+\Gamma\delta^+\Gamma\mathfrak{I}^+)(\ddot{x})\right\} \geq t.$$

Hence, $\ddot{x} \in \mathfrak{I}_t^P$, and so

$$(\aleph\Gamma\mathfrak{I}_t^P\Gamma\aleph) \cap (\mathfrak{I}_t^P\Gamma\aleph\Gamma\mathfrak{I}_t^P) \subseteq \mathfrak{I}_t^P.$$

Therefore, $\mathfrak{I}_t^P$ is a BII.

Conversely, suppose that the level cuts of $\mathfrak{I} = (\aleph; \mathfrak{I}^-, \mathfrak{I}^+)$ are BII's of $\aleph$. Then, $\mathfrak{I}_s^N$ and $\mathfrak{I}_t^P$ are sub-GSRs of $\aleph$. Also, by Theorem 3.4 of [5], $\mathfrak{I}$ is a BFGSR of $\aleph$. Let $\ddot{p}, \ddot{q}, \ddot{r} \in \aleph$ and $\ddot{r} \in \Gamma$. Define $s = \max\{\mathfrak{I}^-(\ddot{p}), \mathfrak{I}^-(\ddot{q}), \mathfrak{I}^-(\ddot{r})\}$ and $t = \min\{\mathfrak{I}^+(\ddot{p}), \mathfrak{I}^+(\ddot{q}), \mathfrak{I}^+(\ddot{r})\}$. Then

$$\mathfrak{I}^-(\ddot{p}), \mathfrak{I}^-(\ddot{q}), \mathfrak{I}^-(\ddot{r}) \leq s \quad \text{and} \quad \mathfrak{I}^+(\ddot{p}), \mathfrak{I}^+(\ddot{q}), \mathfrak{I}^+(\ddot{r}) \geq t,$$

so $\ddot{p}, \ddot{q}, \ddot{r} \in \mathfrak{I}_s^N \cap \mathfrak{I}_t^P$. Suppose $\ddot{a}, \ddot{b}, \ddot{c} \in \aleph$ and $\gamma, \eta, \zeta, \epsilon \in \Gamma$. Let $\ddot{x} = \ddot{a}\gamma\ddot{p}\eta\ddot{b} = \ddot{q}\zeta\ddot{c}\epsilon\ddot{r}$. Consider

$$\begin{aligned}
&((\delta^+\Gamma\mathfrak{I}^+\Gamma\delta^+) \cap (\mathfrak{I}^+\Gamma\delta^+\Gamma\mathfrak{I}^+))(\ddot{x}) \\
&= \min\{(\delta^+\Gamma\mathfrak{I}^+\Gamma\delta^+)(\ddot{x}), (\mathfrak{I}^+\Gamma\delta^+\Gamma\mathfrak{I}^+)(\ddot{x})\} \\
&= \min\{\sup\{\min\{(\delta^+\Gamma\mathfrak{I}^+)(\ddot{a}\gamma\ddot{p}), \delta^+(\ddot{b})\}\}, \sup\{\min\{(\mathfrak{I}^+\Gamma\delta^+)(\ddot{q}\zeta\ddot{c}), \mathfrak{I}^+(\ddot{r})\}\}\} \\
&= \min\{\sup\{\min\{\sup\{\min\{\delta^+(\ddot{a}), \mathfrak{I}^+(\ddot{p})\}\}, \delta^+(\ddot{b})\}\}, \sup\{\min\{\sup\{\min\{\mathfrak{I}^+(\ddot{q}), \delta^+(\ddot{c})\}\}, \mathfrak{I}^+(\ddot{r})\}\}\} \\
&= \min\{\mathfrak{I}^+(\ddot{p}), \mathfrak{I}^+(\ddot{q}), \mathfrak{I}^+(\ddot{r})\} \\
&\geq t.
\end{aligned}$$

$\Rightarrow \ddot{x} \in \mathfrak{I}_t^P$, so $(\delta^+\Gamma\mathfrak{I}^+\Gamma\delta^+) \cap (\mathfrak{I}^+\Gamma\delta^+\Gamma\mathfrak{I}^+) \subseteq \mathfrak{I}^+$. Now consider

$$\begin{aligned}
&((\delta^-\Gamma\mathfrak{I}^-\Gamma\delta^-) \cup (\mathfrak{I}^-\Gamma\delta^-\Gamma\mathfrak{I}^-))(\ddot{x}) \\
&= \max\{(\delta^-\Gamma\mathfrak{I}^-\Gamma\delta^-)(\ddot{x}), (\mathfrak{I}^-\Gamma\delta^-\Gamma\mathfrak{I}^-)(\ddot{x})\} \\
&= \max\{\inf\{\max\{(\delta^-\Gamma\mathfrak{I}^-)(\ddot{a}\gamma\ddot{p}), \delta^-(\ddot{b})\}\}, \inf\{\max\{(\mathfrak{I}^-\Gamma\delta^-)(\ddot{q}\zeta\ddot{c}), \mathfrak{I}^-(\ddot{r})\}\}\} \\
&= \max\{\inf\{\max\{\inf\{\max\{\delta^-(\ddot{a}), \mathfrak{I}^-(\ddot{p})\}\}, \delta^-(\ddot{b})\}\}, \inf\{\max\{\inf\{\max\{\mathfrak{I}^-(\ddot{q}), \delta^-(\ddot{c})\}\}, \mathfrak{I}^-(\ddot{r})\}\}\} \\
&= \max\{\mathfrak{I}^-(\ddot{p}), \mathfrak{I}^-(\ddot{q}), \mathfrak{I}^-(\ddot{r})\} \\
&\leq s.
\end{aligned}$$

$\Rightarrow \ddot{x} \in \mathfrak{I}_s^N$, so $(\delta^-\Gamma\mathfrak{I}^-\Gamma\delta^-) \cup (\mathfrak{I}^-\Gamma\delta^-\Gamma\mathfrak{I}^-) \supseteq \mathfrak{I}^-$. Therefore, $\mathfrak{I}$ is a BFBII of $\aleph$. □

Note 4.1. *Let κ be a non-empty subset of a GSR $\aleph$. Then the characteristic function δ_κ is a GSR of $\aleph$ if and only if κ is a sub-GSR of $\aleph$.*

To connect crisp BIIs with their bipolar fuzzy counterparts, we consider the behavior of characteristic functions. The following theorem establishes an equivalence between a BII and its induced BFBII through the characteristic function.

Theorem 4.6. *Let κ be a non-empty subset of $\aleph$. Then the characteristic function δ_κ is a BFBII of $\aleph$ if and only if κ is a BII of $\aleph$.*

Proof. Suppose δ_κ is a BFBII of $\aleph$. Then

$$(\delta^+\Gamma\delta^+_\kappa\Gamma\delta^+) \cap (\delta^+_\kappa\Gamma\delta^+\Gamma\delta^+_\kappa) \subseteq \delta^+_\kappa.$$

This implies

$$\delta^+_{\aleph\Gamma\kappa\Gamma\aleph} \cap \delta^+_{\kappa\Gamma\aleph\Gamma\kappa} \subseteq \delta^+_\kappa,$$

which leads to

$$\delta^+_{(\aleph\Gamma\kappa\Gamma\aleph)\cap(\kappa\Gamma\aleph\Gamma\kappa)} \subseteq \delta^+_\kappa,$$

and hence

$$(\aleph\Gamma\kappa\Gamma\aleph) \cap (\kappa\Gamma\aleph\Gamma\kappa) \subseteq \kappa.$$

Therefore, κ is a BII of $\aleph$.

Conversely, suppose κ is a BII of $\aleph$. Then, by the previous note, the characteristic function δ_κ is a BFGSR of $\aleph$. Since

$$(\aleph\Gamma\kappa\Gamma\aleph) \cap (\kappa\Gamma\aleph\Gamma\kappa) \subseteq \kappa,$$

we obtain

$$(\delta^+\Gamma\delta^+_\kappa\Gamma\delta^+) \cap (\delta^+_\kappa\Gamma\delta^+\Gamma\delta^+_\kappa) = \delta^+_{\aleph\Gamma\kappa\Gamma\aleph} \cap \delta^+_{\kappa\Gamma\aleph\Gamma\kappa} = \delta^+_{(\aleph\Gamma\kappa\Gamma\aleph)\cap(\kappa\Gamma\aleph\Gamma\kappa)} \subseteq \delta^+_\kappa,$$

and similarly,

$$(\delta^-\Gamma\delta^-_\kappa\Gamma\delta^-) \cup (\delta^-_\kappa\Gamma\delta^-\Gamma\delta^-_\kappa) = \delta^-_{\aleph\Gamma\kappa\Gamma\aleph} \cup \delta^-_{\kappa\Gamma\aleph\Gamma\kappa} = \delta^-_{(\aleph\Gamma\kappa\Gamma\aleph)\cup(\kappa\Gamma\aleph\Gamma\kappa)} \supseteq \delta^-_\kappa.$$

Hence, δ_κ is a BFBII of $\aleph$. □

The family of BFBII sets exhibits closure properties under set-theoretic operations. The following theorem confirms that the intersection of two BFBII sets remains within the same class.

Theorem 4.7. *Let $\mathfrak{I} = (\aleph; \mathfrak{I}^-, \mathfrak{I}^+)$ and $\varkappa = (\aleph; \varkappa^-, \varkappa^+)$ be two BFBIIs of $\aleph$. Then their intersection $\mathfrak{I} \cap \varkappa$ is also a BFBII of $\aleph$.*

Proof. Suppose $\mathfrak{I}$ and $\varkappa$ are BFBIIs of $\aleph$.

From Theorem 3.6 of [5], the intersection of an arbitrary family of BFGSRs is a BFGSR.

Let $\ddot{x} \in \aleph$. Then,

$$\begin{aligned}\delta^+\Gamma(\mathfrak{I}^+ \cap \varkappa^+)(\ddot{x}) &= \sup\{\min\{\delta^+(\ddot{y}), (\mathfrak{I}^+ \cap \varkappa^+)(\ddot{z})\} : \ddot{x} = \ddot{y}\gamma\ddot{z}, \ddot{y}, \ddot{z} \in \aleph, \gamma \in \Gamma\}\\ &= \min\{\delta^+\Gamma\mathfrak{I}^+(\ddot{x}), \delta^+\Gamma\varkappa^+(\ddot{x})\}\\ &= (\delta^+\Gamma\mathfrak{I}^+) \cap (\delta^+\Gamma\varkappa^+)(\ddot{x}).\end{aligned}$$

Therefore,

$$\delta^+\Gamma(\mathfrak{I}^+ \cap \varkappa^+) = (\delta^+\Gamma\mathfrak{I}^+) \cap (\delta^+\Gamma\varkappa^+).$$

Similarly,

$$(\mathfrak{I}^+ \cap \varkappa^+)\Gamma\delta^+\Gamma(\mathfrak{I}^+ \cap \varkappa^+) = (\mathfrak{I}^+\Gamma\delta^+\Gamma\mathfrak{I}^+) \cap (\varkappa^+\Gamma\delta^+\Gamma\varkappa^+).$$

Also,

$$\delta^+\Gamma(\mathfrak{I}^+ \cap \varkappa^+)\Gamma\delta^+ = (\delta^+\Gamma\mathfrak{I}^+\Gamma\delta^+) \cap (\delta^+\Gamma\varkappa^+\Gamma\delta^+).$$

Hence,

$$\begin{aligned}&\{\delta^+\Gamma(\mathfrak{I}^+ \cap \varkappa^+)\Gamma\delta^+\} \cap \{(\mathfrak{I}^+ \cap \varkappa^+)\Gamma\delta^+\Gamma(\mathfrak{I}^+ \cap \varkappa^+)\}\\ &= (\delta^+\Gamma\mathfrak{I}^+\Gamma\delta^+) \cap (\delta^+\Gamma\varkappa^+\Gamma\delta^+) \cap (\mathfrak{I}^+\Gamma\delta^+\Gamma\mathfrak{I}^+) \cap (\varkappa^+\Gamma\delta^+\Gamma\varkappa^+)\\ &\subseteq \mathfrak{I}^+ \cap \varkappa^+.\end{aligned}$$

Therefore, $\mathfrak{I}^+ \cap \varkappa^+$ satisfies the upper membership condition of a BFBII.

Now consider the lower membership functions:

$$\begin{aligned}\delta^-\Gamma(\mathfrak{I}^- \cap \varkappa^-)(\ddot{x}) &= \inf\{\max\{\delta^-(\ddot{y}), (\mathfrak{I}^- \cap \varkappa^-)(\ddot{z})\} : \ddot{x} = \ddot{y}\gamma\ddot{z}, \ddot{y}, \ddot{z} \in \aleph, \gamma \in \Gamma\}\\ &= \min\{\delta^-\Gamma\mathfrak{I}^-(\ddot{x}), \delta^-\Gamma\varkappa^-(\ddot{x})\}\\ &= (\delta^-\Gamma\mathfrak{I}^-) \cap (\delta^-\Gamma\varkappa^-)(\ddot{x}).\end{aligned}$$

Therefore,

$$\delta^-\Gamma(\mathfrak{I}^- \cap \varkappa^-) = (\delta^-\Gamma\mathfrak{I}^-) \cap (\delta^-\Gamma\varkappa^-).$$

Also,

$$(\mathfrak{I}^- \cap \varkappa^-)\Gamma\delta^-\Gamma(\mathfrak{I}^- \cap \varkappa^-) = (\mathfrak{I}^-\Gamma\delta^-\Gamma\mathfrak{I}^-) \cap (\varkappa^-\Gamma\delta^-\Gamma\varkappa^-)$$

and

$$\delta^-\Gamma(\mathfrak{I}^- \cap \varkappa^-)\Gamma\delta^- = (\delta^-\Gamma\mathfrak{I}^-\Gamma\delta^-) \cap (\delta^-\Gamma\varkappa^-\Gamma\delta^-).$$

Hence,

$$\begin{aligned}&\{\delta^-\Gamma(\mathfrak{I}^- \cap \varkappa^-)\Gamma\delta^-\} \cup \{(\mathfrak{I}^- \cap \varkappa^-)\Gamma\delta^-\Gamma(\mathfrak{I}^- \cap \varkappa^-)\}\\ &= (\delta^-\Gamma\mathfrak{I}^-\Gamma\delta^-) \cap (\delta^-\Gamma\varkappa^-\Gamma\delta^-) \cup (\mathfrak{I}^-\Gamma\delta^-\Gamma\mathfrak{I}^-) \cap (\varkappa^-\Gamma\delta^-\Gamma\varkappa^-)\\ &\supseteq \mathfrak{I}^- \cap \varkappa^-.\end{aligned}$$

Thus, $\mathfrak{I}^- \cap \varkappa^-$ satisfies the lower membership condition of a BFBII.

Therefore, $\mathfrak{I} \cap \varkappa$ is a BFBII of $\aleph$. □

The intersection of different types of BFIs can also result in a BFBII under suitable conditions. The following theorems demonstrate that the intersection of a BFBI and a BFII, as well as that of a BFRI and a BFLI, each yields a BFBII in a GSR.

Theorem 4.8. *Let $\mathfrak{I} = (\aleph; \mathfrak{I}^-, \mathfrak{I}^+)$ be a BFBI, and let $\varkappa = (\aleph; \varkappa^-, \varkappa^+)$ be a BFII. Then, $\mathfrak{I} \cap \varkappa$ is a BFBII.*

Proof. Suppose $\mathfrak{I}$ is a BFBI and $\varkappa$ is a BFII of $\aleph$. By Theorems 4.1 and 4.2, both $\mathfrak{I}$ and $\varkappa$ are BFBIIs. Then, by Theorem 4.7, it follows that $\mathfrak{I} \cap \varkappa$ is a BFBII. □

Theorem 4.9. *Let $\mathfrak{I} = (\aleph; \mathfrak{I}^-, \mathfrak{I}^+)$ be a BFRI, and let $\varkappa = (\aleph; \varkappa^-, \varkappa^+)$ be a BFLI. Then, $\mathfrak{I} \cap \varkappa$ is a BFBII.*

Proof. Suppose $\mathfrak{I}$ is a BFRI and $\varkappa$ is a BFLI of $\aleph$. From Theorems 4.3 and 4.4, it follows that both $\mathfrak{I}$ and $\varkappa$ are BFBIIs. Then, by Theorem 4.7, we conclude that $\mathfrak{I} \cap \varkappa$ is a BFBII. □

5. Conclusion

In this paper, we have introduced and developed the concept of bipolar fuzzy bi-interior ideals (BFBII) in the framework of Γ-semirings, extending the notions of bipolar fuzzy bi-ideals (BFBI) and bipolar fuzzy interior ideals (BFII). We established fundamental properties of BFBII, including characterizations via level cuts and characteristic functions. In particular, we demonstrated that a bipolar fuzzy set is a BFBII if and only if all of its level cuts form crisp bi-interior ideals in the underlying Γ-semiring. Furthermore, we showed that the intersection of various bipolar fuzzy ideals—such as BFBI with BFII, and BFRI with BFLI—results in a BFBII. These findings not only deepen the structural understanding of fuzzy ideals in Γ-semirings but also provide a strong foundation for future theoretical investigations and applications in fuzzy algebraic systems.

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