

## New Results of K-g-Frames for Hilbert $C^*$ -Modules

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**Abstract.** In this paper, we will probe into new constructions of K-g-frames for Hilbert  $C^*$ -module, and we characterize them through some properties of K-g-orthonormal bases. Finally, some results concerning the K-g-dual of a K-g-frame are obtained.

### 1. INTRODUCTION

After it was first launched in 1952 by Duffin and Schaefer [1] in the attempt of studying the nonharmonic Fourier series, frames emerged as a significant tool in signal and image processing, data compression, sampling theory, wavelet analysis, irregular sampling theory, and many other fields. The frames theory has been generalized so fast from Hilbert spaces to Hilbert  $C^*$ -modules, and valuable results that enrich the theory were obtained.

The organization of the paper will be as follows: in section 2, we are going to recall briefly the definitions and basic properties of K-g-frames in Hilbert  $C^*$  modules. In section 3, we will construct new K-g-frames from a g-frame. We will also define K-g-orthonormal bases for Hilbert  $C^*$  modules. Then, we will characterize g-Bessel sequences and K-g-frames by using K-g-orthonormal bases. After that, we will provide sufficient conditions for two g-Bessel sequences to be K g frames. By the end, we will have obtained results about the K-g-dual of a g-frame.

### 2. PRELIMINARIES

Let  $I$  and  $J$  be two finite or countable sets. This section will be a brief reminder of the definitions and basic properties of  $C^*$  algebra, Hilbert  $C^*$ -modules, and K-g-frames in Hilbert  $C^*$  modules.

Received: Dec. 9, 2025.

2020 *Mathematics Subject Classification.* Primary 42C15; Secondary 46L06.

*Key words and phrases.* g-frame; K-g-frame; Hilbert  $C^*$ -modules.

An element  $a$  in a  $C^*$  algebra  $A$ , is said to be positive ( $a \geq 0$ ) if  $a = a^*$  and  $sp(a) \subset \mathbb{R}^+$ .  $A^+$  denotes the set of all positive elements of  $A$ .

**Definition 2.1.** [4]. Let  $A$  be a unital  $C^*$  algebra and  $\mathcal{H}$  be a left  $A$ -module, such that the linear structures of  $A$  and  $\mathcal{H}$  are compatible.  $\mathcal{H}$  is a pre-Hilbert  $A$ -module if  $\mathcal{H}$  is equipped with an  $A$ -valued inner product  $\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \rightarrow A$  sesquilinear, positive definite and respects the module action. In other words,

- (i)  $\langle x, x \rangle_A \geq 0$  for all  $x \in \mathcal{H}$  and  $\langle x, x \rangle_A = 0$  if and only if  $x = 0$ .
- (ii)  $\langle ax + y, z \rangle_A = a\langle x, z \rangle_A + \langle y, z \rangle_A$  for all  $a \in A$  and  $x, y, z \in \mathcal{H}$ .
- (iii)  $\langle x, y \rangle_A = \langle y, x \rangle_A^*$  for all  $x, y \in \mathcal{H}$ .

For  $x \in \mathcal{H}$ , we define  $\|x\| = \|\langle x, x \rangle_A\|^{1/2}$ . If  $\mathcal{H}$  is complete with  $\|\cdot\|$ , it is called a Hilbert  $A$ -module or a Hilbert  $C^*$ -module over  $A$ . For every  $a$  in  $C^*$ -algebra  $A$ , we have  $|a| = (a^*a)^{1/2}$  and the  $A$ -valued norm on  $\mathcal{H}$  is defined by  $|x| = \langle x, x \rangle_A^{1/2}$  for  $x \in \mathcal{H}$ .

**Example 2.1.** [5] If  $\{\mathcal{H}_k\}_{k \in \mathbb{N}}$  is a set of Hilbert  $A$  modules, then we define their direct sum:  $\bigoplus_{k \in \mathbb{N}} \mathcal{H}_k$ . On the  $A$ -module  $\bigoplus_{k \in \mathbb{N}} \mathcal{H}_k$ , of all sequences  $x = (x_k)_{k \in \mathbb{N}} : x_k \in \mathcal{H}_k$ , such that the series  $\sum_{k \in \mathbb{N}} \langle x_k, x_k \rangle_A$  is convergent at norm in  $A$ , Let us define the inner product:

$$\langle x, y \rangle := \sum_{k \in \mathbb{N}} \langle x_k, y_k \rangle_{\mathcal{H}_k} \quad \text{for } x, y \in \bigoplus_{k \in \mathbb{N}} \mathcal{H}_k.$$

Then  $\bigoplus_{k \in \mathbb{N}} \mathcal{H}_k$  is a Hilbert  $A$ -module.

The notation  $l^2(\mathcal{H})$  refers to the direct sum of a countable copies of the Hilbert  $C^*$  module  $\mathcal{H}$ .

Consider two Hilbert  $A$  modules,  $\mathcal{H}$  and  $\mathcal{K}$ . A map  $T : \mathcal{H} \rightarrow \mathcal{K}$  is called adjointable whenever there exists a map  $T^* : \mathcal{K} \rightarrow \mathcal{H}$  such that  $\langle Tx, y \rangle_A = \langle x, T^*y \rangle_A$  for every  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ .

We also adopt the notation  $End_A^*(\mathcal{H}, \mathcal{K})$  for the set consisting of all adjointable operators from  $\mathcal{H}$  to  $\mathcal{K}$  and  $End_A^*(\mathcal{H}, \mathcal{H})$  is abbreviated to  $End_A^*(\mathcal{H})$ .

**Definition 2.2.** [6] Let  $K \in End_A^*(H)$ , The sequence  $\{\Lambda_i\}_{i \in I}$  is called a  $K$ -g frame for  $H$  relative to  $\{H_i\}_{i \in I}$  if there are positive constants  $A$  and  $B$  satisfying:

$$A\langle K^*f, K^*f \rangle \leq \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle \leq B\langle f, f \rangle, \quad f \in H.$$

The constants  $A$  and  $B$  are called the lower and upper frame bounds, respectively. When only the upper bound exists,  $\{\Lambda_i\}_{i \in I}$  is called a  $g$  Bessel sequence.

**Theorem 2.1.** [3] Consider  $K \in End_A^*(\mathcal{U})$ , a family  $\{\Lambda_i \in End_A^*(\mathcal{U}, \mathcal{V}_i), i \in I\}$  is a  $K$  g frame if and only if, for every  $f \in \mathcal{U}$ :

$$C\| \langle K^*f, K^*f \rangle \| \leq \left\| \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle \right\| \leq D\| \langle f, f \rangle \|^2$$

for some  $0 < C; D < \infty$ .

One advantage of this equivalent definition of  $K$  g frames lies in the fact that comparing the norms of elements is much easier to compare elements in a  $C^*$ -algebras.

**Lemma 2.1.** [2] Let  $U, V \in \text{End}_A^*(\mathcal{H})$ . The following assertions are equivalent :

- 1)  $\mathcal{R}(U) \subset \mathcal{R}(V)$ .
- 2) There exists  $\lambda \geq 0$  such that:  $UU^* \leq \lambda VV^*$ .
- 3)  $U = VC$  for some  $C \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ .

If 1), 2) et 3) hold, then there exists a unique operator  $C$  satisfying:

- 1)  $\|C\|^2 = \text{Inf}\{\alpha : UU^* \leq \alpha VV^*\}$ .
- 2)  $\mathcal{N}(U) = \mathcal{N}(C)$ .
- 3)  $\mathcal{R}(C) \subset \overline{\mathcal{R}(V^*)}$ .

Throughout the paper,  $\mathcal{H}$  denotes a Hilbert  $A$  module, and  $\{\mathcal{H}_i\}_{i \in I}$  a collection of Hilbert  $A$  modules and  $I$  and  $J$  two countable index sets.

### 3. MAIN RESULTS

**Theorem 3.1.** Suppose  $K \in \text{End}_A^*(\mathcal{H})$ . The following statements hold:

- i) If  $\{\Lambda_j \in \text{End}_A^*(\mathcal{H}, \mathcal{H}_j) : j \in J\}$  is a  $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$  with bounds  $A, B$ , then  $\{\Lambda_j K^* : j \in J\}$  is a  $K$ - $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$  and with the same bounds if  $\|K^*\| \leq 1$ .
- ii) If  $\{\Lambda_i\}_{i \in I}$  is a  $K$   $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in I}$  with bounds  $A, B$ , then  $\{\Lambda_i K^*\}_{i \in I}$  is a  $K$   $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in I}$  whenever  $K^*$  is an idempotent operator.

*Proof.* If  $\{\Lambda_j K^*\}_{j \in J}$  a  $K$   $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$ , then there exists two constants  $0 < A \leq B < \infty$  such that, for every  $f \in \mathcal{H}$

$$A \langle K^* f, K^* f \rangle \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq B \langle f, f \rangle,$$

We replace  $f$  by  $K^* f$ ,

$$A \langle K^{*2} f, K^{*2} f \rangle \leq \sum_{j \in J} \langle \Lambda_j K^* f, \Lambda_j K^* f \rangle \leq B \langle K^* f, K^* f \rangle,$$

As  $K^*$  is idempotent, we obtain:

$$A \langle K^* f, K^* f \rangle \leq \sum_{j \in J} \langle \Lambda_j K^* f, \Lambda_j K^* f \rangle \leq B \langle K^* f, K^* f \rangle \leq B \|K^*\|^2 \langle f, f \rangle,$$

This shows that  $\{\Lambda_j K^*\}_{j \in J}$  is a  $K$   $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$ . □

**Theorem 3.2.** If  $\{\Lambda_j\}_{j \in J}$  is a  $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$ , with bounds  $A$  and  $B$ , and  $K \in \text{End}_A^*(\mathcal{H})$  so that  $\|K^*\| \leq 1$ , then  $\{\Lambda_j\}_{j \in J}$  is a  $K$   $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$ , and with the same bounds.

*Proof.* Since  $\{\Lambda_j\}_{j \in J}$  is a  $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$ , then

$$A \langle f, f \rangle \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq B \langle f, f \rangle,$$

Conversely:

$$A \langle K^* f, K^* f \rangle \leq A \|K^*\|^2 \langle f, f \rangle \leq A \langle f, f \rangle$$

So

$$A\langle K^*f, K^*f \rangle \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq B\langle f, f \rangle,$$

□

**Theorem 3.3.** Let  $K_1$  and  $K_2$  in  $\text{End}_A^*(\mathcal{H})$ , if  $\{\Lambda_i\}_{i \in I}$  is a  $K_1$  g frame and a  $K_2$  g frame, and  $\alpha, \beta$  are a scalars. Then  $\{\Lambda_i\}_{i \in I}$  is a  $K_1 K_2$  g frame and a  $(\alpha K_1 + \beta K_2)$  g frame .

*Proof.* As  $\{\Lambda_i\}_{i \in I}$  is a  $K_1$  g frame and a  $K_2$  g frame. Then there exists positives constants  $A_n, B_n > 0$  ( $n = 1, 2$ ) satisfying:

$$A_n \langle K_n^* f, K_n^* f \rangle \leq \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle \leq B_n \langle f, f \rangle, \quad f \in \mathcal{H} \quad (3.1)$$

for  $n \in \{1, 2\}$ .

Since

$$\begin{aligned} \|K_1^* f\|^2 &= \frac{1}{|\alpha|^2} \|\alpha K_1^* f\|^2 \\ &= \frac{1}{|\alpha|^2} \|(\alpha K_1^* + \beta K_2^*) f - \beta K_2^* f\|^2 \\ &\geq \frac{1}{|\alpha|^2} \|(\alpha K_1^* + \beta K_2^*) f\|^2 - \frac{1}{|\alpha|^2} \|\beta K_2^* f\|^2. \end{aligned}$$

Then,

$$\begin{aligned} \|(\alpha K_1^* + \beta K_2^*) f\|^2 &\leq |\alpha|^2 \|K_1^* f\|^2 + |\beta|^2 \|K_2^* f\|^2 \\ &\leq \frac{1}{2} (|\alpha|^2 \|K_1^* f\|^2 + |\beta|^2 \|K_2^* f\|^2) + \frac{A_1}{A_2} |\beta|^2 \|K_1^* f\|^2 + \frac{A_2}{A_1} |\alpha|^2 \|K_2^* f\|^2 \\ &= \frac{A_2 |\alpha|^2 + A_1 |\beta|^2}{2A_1 A_2} (A_1 \|K_1^* f\|^2 + A_2 \|K_2^* f\|^2) \end{aligned}$$

Hence,

$$\left\| \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle \right\| \geq \frac{1}{2} (A_1 \|K_1^* f\|^2 + A_2 \|K_2^* f\|^2) \geq \frac{A_1 A_2}{A_2 |\alpha|^2 + A_1 |\beta|^2} (\|(\alpha K_1^* + \beta K_2^*) f\|^2).$$

From inequalities (3.1), we get :

$$\left\| \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle \right\| \leq \frac{B_1 + B_2}{2} \|f\|^2, \quad f \in H.$$

Therefore,  $\{\Lambda_i\}_{i \in I}$  is an  $(\alpha K_1 + \beta K_2)$  g frame.

Now for each  $f \in \mathcal{H}$ :

$$\begin{aligned} \|(K_1 K_2)^* f\|^2 &= \|K_2^* K_1^* f\|^2 \\ &\leq \|K_2^*\|^2 \|K_1^* f\|^2 \end{aligned}$$

Hence  $\{\Lambda_i\}_{i \in I}$  is a  $K_1$  g frame, we have

$$\frac{A_1}{\|K_2^*\|^2} \|(K_1 K_2)^* f\|^2 \leq A_1 \|K_1^* f\|^2 \leq \left\| \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle \right\| \leq B_1 \|f\|^2, \quad f \in H.$$

It follows that  $\{\Lambda_i\}_{i \in I}$  is a  $K_1 K_2$  g frame. □

**Theorem 3.4.** Let  $K \in \text{End}_A^*(\mathcal{H})$ ,  $\{\Lambda_j\}_{j \in J}$  and  $\{\Omega_j\}_{j \in J}$  two g Bessel sequences with bounds  $B_1$  and  $B_2$  and the analysis operators  $T_\Lambda$  and  $T_\Omega$  respectively, suppose that  $T_\Omega T_\Lambda^* = K^*$ .

Then  $\{\Lambda_j\}_{j \in J}$  and  $\{\Omega_j\}_{j \in J}$  are two  $K$  g frames for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$ .

*Proof.*

$$\begin{aligned} \|\langle K^* f, K^* f \rangle\|^2 &= \|\langle T_\Omega T_\Lambda^* f, K^* f \rangle\|^2 \\ &= \|\langle T_\Lambda^* f, T_\Omega^* K^* f \rangle\|^2 \\ &\leq \|\langle T_\Lambda^* f, T_\Lambda^* f \rangle\| \|\langle T_\Omega^* K^* f, T_\Omega^* K^* f \rangle\| \\ &\leq B_2 \|\langle T_\Lambda^* f, T_\Lambda^* f \rangle\| \|\langle K^* f, K^* f \rangle\| \end{aligned}$$

then

$$\frac{2}{B_1} \|\langle K^* f, K^* f \rangle\| \leq \|\langle T_\Lambda^* f, T_\Lambda^* f \rangle\|.$$

It follows that  $\{\Lambda_j\}_{j \in J}$  is a  $K$  g frames for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$ . In the same way we show that  $\{\Omega_j\}_{j \in J}$  is a  $K$  g frames for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$  with lower bound  $\frac{1}{B_1}$  □

**Definition 3.1.**  $\{\Lambda_j \in \text{End}_A^*(\mathcal{H}, \mathcal{H}_j) : j \in J\}$  is said to be a  $K$  g orthonormal basis for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$  if it fulfills the following:

- 1)  $\langle \Lambda_i^* f_i, \Lambda_j^* f_j \rangle = \delta_{i,j} \langle f_i, f_j \rangle$ , for all  $i, j \in J$ ,  $f_i \in \mathcal{H}_i$  and  $f_j \in \mathcal{H}_j$ .
- 2)  $\sum_{i \in I} \langle \Lambda_i f_i, \Lambda_i f_j \rangle = \langle K^* f, K^* f \rangle$ , for all  $f \in \mathcal{H}$ .

In the following results, we will characterize the g Bessel sequences and  $K$  g frames for  $\mathcal{H}$  via  $K$  g orthonormal bases in  $\mathcal{H}$ .

**Theorem 3.5.** Let  $K \in \text{End}_A^*(\mathcal{H})$ . If  $\{\Lambda_j\}_{j \in J}$  is a  $K$  g orthonormal basis for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$ . ,then  $\{\Omega_j\}_{j \in J}$  is a g-Bessel sequence for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$  ,if and only if there exists an operator  $\Psi \in \text{End}_A^*(\mathcal{H})$  satisfying:  $\Omega_j = \Lambda_j \Psi^*$ ,  $\forall j \in J$ .

*Proof.* Let  $\{\Lambda_j\}_{j \in J}$  be a  $K$  g orthonormal basis for  $\mathcal{H}$  and  $\{\Omega_j\}_{j \in J}$  a g-Bessel sequence for  $\mathcal{H}$  relative to  $\{\mathcal{H}_j\}_{j \in J}$ , let  $\Psi : \mathcal{H} \rightarrow \mathcal{H}$  the operator defined by:

$$\Psi(f) = \sum_{j \in J} \Omega_j^* \Lambda_j \quad , \forall f \in \mathcal{H}.$$

then

$$\begin{aligned} \Psi \Lambda_j^*(f) &= \sum_{i \in J} \Omega_i^* \Lambda_i \Lambda_j^*(f) \\ &= \sum_{i \in J} \Omega_i^* \delta_{ij}(f) \\ &= \Omega_j^*(f). \end{aligned}$$

So  $\Omega_j^* = \Psi \Lambda_j^*$ ; that is  $\Omega_j = \Lambda_j \Psi^*$ .

For the converse, if there exists an operator  $\Psi \in \text{End}_A^*(\mathcal{H})$  such that  $\Omega_j = \Lambda_j \Psi^*$ , then:

$$\begin{aligned} \left\| \sum_{j \in J} \langle \Gamma_j f, \Gamma_j f \rangle \right\| &= \left\| \sum_{j \in J} \langle \Lambda_j \Psi^* f, \Lambda_j \Psi^* f \rangle \right\| \\ &= \left\| \sum_{j \in J} \langle K^* \Psi^* f, K^* \Psi^* f \rangle \right\| \\ &\leq \|K^* \psi^*\|^2 \|\langle f, f \rangle\|. \end{aligned}$$

□

**Theorem 3.6.** Let the operator  $\Psi$  be as defined in Thm(3.1), suppose that  $\{\Lambda_i\}_{i \in I}$  is a  $K$   $g$  orthonormal basis for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in I}$ , and  $\{\Gamma_i\}_{i \in I}$  is a  $g$  Bessel sequence for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in I}$  and  $S$  is the  $K$   $g$  frame operator related to  $\{\Gamma_i\}_{i \in I}$  which has closed image. If  $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$  is an onto operator, then the following statements are satisfied:

- i- If  $\{\Gamma_i\}_{i \in I}$  is a  $K$   $g$  frame for  $\mathcal{H}$  with respect to  $\{\mathcal{H}_i\}_{i \in I}$ , then  $\Psi$  is onto.
- ii- If  $\Psi$  is onto and  $K\Psi = K\Psi$ , then  $\{\Gamma_i\}_{i \in I}$  is a  $K$ - $g$ -frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in I}$
- iii-  $\{\Gamma_i\}_{i \in I}$  is a normalized tight  $K$   $g$  frame if and only if  $\Psi$  is an isometry.

*Proof.* i- Suppose that  $\{\Gamma_i\}_{i \in I}$  is a  $K$   $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in I}$ , by definition of the frame operator:

$$Sf = \sum_{i \in I} \Gamma_i^* \Gamma_i f = \sum_{i \in I} \Gamma_i^* \Lambda_i \Psi^* f = \Psi \Psi^* f.$$

since  $K$  is onto, i.e  $\text{Range}(K) = \mathcal{H}$ , then  $S$  is surjective. It follows that  $\psi$  is surjective.

- ii- If  $\{\Gamma_i\}_{i \in I}$  is a  $g$ -Bessel sequence for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in I}$ , and  $\psi$  commute with  $K$  and onto (has closed image) then it admits a Moore–Penrose inverse  $\psi^\dagger$  such that  $\psi^\dagger \psi^* = I$ , then

$$\begin{aligned} \|\langle K^* f, K^* f \rangle\| &= \|\langle \psi^\dagger \psi^* K^* f, \psi^\dagger \psi^* K^* f \rangle\| \\ &= \|\langle \psi^\dagger K^* \psi^* f, \psi^\dagger K^* \psi^* f \rangle\| \\ &\leq \|\psi^\dagger\|^2 \|\langle K^* \psi^* f, K^* \psi^* f \rangle\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| \sum_{i \in I} \langle \Gamma_i f, \Gamma_i f \rangle \right\| &= \left\| \sum_{i \in I} \langle \Lambda_i \Psi^* f, \Lambda_i \Psi^* f \rangle \right\| \\ &= \left\| \sum_{i \in I} \langle K^* \Psi^* f, K^* \Psi^* f \rangle \right\| \\ &\geq \frac{1}{\|\psi^\dagger\|^2} \|\langle K^* f, K^* f \rangle\|. \end{aligned}$$

This means that  $\{\Gamma_i\}_{i \in I}$  has a lower frame bound

- iii-  $\{\Gamma_i\}_{i \in I}$  is a normalized tight  $K$   $g$  frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in I}$  if and only if its  $K$   $g$  frame operator  $S = \Psi \Psi^* = I_{\mathcal{H}}$  if and only if  $\psi^*$  is an isometry.

□

Before stating the next result, we require a the following lemma.

**Lemma 3.1.** Let  $K \in \text{End}_A^*(\mathcal{H})$  be an onto operator. Suppose that  $\{\Lambda_i\}_{i \in \mathbb{I}}$  is a  $K$ -g-frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$  with bounds  $A$  and  $B$ . For any  $i \in \mathbb{I}$ ,  $K_i \in \text{End}_A^*(\mathcal{H}_i)$  an invertible operator such that:

$$0 < m < \inf_{i \in \mathbb{I}} \frac{1}{\|K_i^{-1}\|} \leq \sup_{i \in \mathbb{I}} \|K_i\| = M < \infty$$

If  $\Gamma_i = K_i \Lambda_i K$ , for any  $i \in \mathbb{I}$ , then  $\{\Gamma_i\}_{i \in \mathbb{I}}$  is a g frame for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ .

*Proof.*

$$\begin{aligned} \left\| \sum_{i \in \mathbb{I}} \langle \Lambda_i f, \Lambda_i f \rangle \right\| &= \left\| \sum_{i \in \mathbb{I}} \langle K_i \Gamma_i K^* f, K_i \Gamma_i K^* f \rangle \right\| \\ &\geq \inf_{i \in \mathbb{I}} \|K_i^{-1}\|^{-2} \left\| \sum_{i \in \mathbb{I}} \langle \Gamma_i K f, \Gamma_i K f \rangle \right\| \\ &\geq m^2 A \left\| \sum_{i \in \mathbb{I}} \langle K K^* f, K K^* f \rangle \right\|. \\ &\geq m^2 A \left\| [(K K^*)^* (K K^*)]^{-1} \right\|^{-1} \left\| \sum_{i \in \mathbb{I}} \langle f, f \rangle \right\|. \end{aligned}$$

For the other side

$$\begin{aligned} \left\| \sum_{i \in \mathbb{I}} \langle \Lambda_i f, \Lambda_i f \rangle \right\| &= \left\| \sum_{i \in \mathbb{I}} \langle K_i \Gamma_i K f, K_i \Gamma_i K f \rangle \right\| \\ &\leq M \left\| \sum_{i \in \mathbb{I}} \langle \Gamma_i K f, \Gamma_i K f \rangle \right\| \\ &\leq M B \|K\|^2 \left\| \sum_{i \in \mathbb{I}} \langle f, f \rangle \right\| \end{aligned}$$

□

**Theorem 3.7.** With the data of the previous lemma and if  $K$  is moreover self-adjoint, then  $\Omega := \{K_i^* K_i \Lambda_i K S_\Gamma^{-1} K^2\}_{i \in \mathbb{I}}$  is a  $K$  g dual of  $\{\Lambda_i\}_{i \in \mathbb{I}}$

*Proof.*

$$\begin{aligned} \left\| \sum_{i \in \mathbb{I}} \langle K_i^* K_i \Lambda_i K S_\Gamma^{-1} K^2 f, K_i^* K_i \Lambda_i K S_\Gamma^{-1} K^2 f \rangle \right\| &\leq M^4 \left\| \sum_{i \in \mathbb{I}} \langle \Lambda_i K S_\Gamma^{-1} K^2 f, \Lambda_i K S_\Gamma^{-1} K^2 f \rangle \right\| \\ &\leq M^6 B \|K\|^2 \left\| \sum_{i \in \mathbb{I}} \langle S_\Gamma^{-1} K^2 f, S_\Gamma^{-1} K^2 f \rangle \right\| \\ &\leq \frac{M^6 B \|K\|^8}{A^2} \left\| \sum_{i \in \mathbb{I}} \langle f, f \rangle \right\|. \end{aligned}$$

That is  $\{\Omega_i\}_{i \in \mathbb{I}}$  is a g Bessel sequence for  $\mathcal{H}$  relative to  $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$

$$\begin{aligned} \sum_{i \in \mathbb{I}} \Omega_i^* \Lambda_i f &= \sum_{i \in \mathbb{I}} K^2 S_\Gamma^{-1} K \Lambda_i^* K_i^* K_i \Lambda_i f \\ &= K^2 S_\Gamma^{-1} \sum_{i \in \mathbb{I}} (K_i \Lambda_i K_i)^* (K_i \Lambda_i K) K^{-1} f \\ &= K^2 S_\Gamma^{-1} S_\Gamma K^{-1} f \\ &= K f. \end{aligned}$$

That is  $\{\Omega_i\}_{i \in \mathbb{I}}$  is a  $K$  g dual of  $\{\Lambda_i\}_{i \in \mathbb{I}}$ .

□

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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