Int. J. Anal. Appl. (2026), 24:37

International Journal of Analysis and Applications

PT-Essential and PST-Essential Submodules

Omar Kareem Ali[1,*], Akram Hatem Shadher[2], Mohammed Khalid Shahoodh[3]

[1]*Department of Mathematics, College of Science, University of Anbar, Iraq*
[2]*Department of Computer Engineering, Al-Farabi University college, Iraq*
[3]*Ministry of Education, General Directorate of Education in Ramadi, Iraq*
**Corresponding author:* omar.k.ali@uoanbar.edu.iq

ABSTRACT. Many generalizations have been presented in modules theory in order to present some new results in this direction. In this article, we introduced the concepts of PT-essential submodules and PST-essential submodules as a generalization to the concept of P-essential submodules. In particular, let Ⱥ be an R-module with $\mathrm{T} \leq$ Ⱥ. A submodule $\mathfrak{A}$ of Ⱥ is called PT-essential in Ⱥ ($\mathfrak{A} \leq_{PTE}$ Ⱥ) in case there exist a submodule $\mathfrak{D}$ of prime submodule P of Ⱥ in which $\mathfrak{A} \nleq \mathrm{T}, \mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ implies $\mathfrak{D} \leq \mathrm{T}$. Furthermore, let Ⱥ be an R-module and $\mathrm{T} \leq$ Ⱥ. The submodule $\mathfrak{A}$ of Ⱥ is said to be P-Small T-essential in Ⱥ ($\mathfrak{A} \leq_{PST}$ Ⱥ) in condition that any small submodule $\mathfrak{D}$ of prime submodule P of Ⱥ in which $\mathfrak{A} \nleq \mathrm{T}, \mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ implies $\mathfrak{D} \leq \mathrm{T}$. Besides that, the concept of PT-complement submodule has been introduced with some properties about it. Finally, some basic properties of these concepts have been established. Then, several examples are given to illustrate the mentioned concepts.

1. Introduction

Let Ⱥ be an R-module and $\mathfrak{A}$ be a submodule of Ⱥ. Then, $\mathfrak{A}$ is called small in Ⱥ, if for any submodule $\mathfrak{D}$ of Ⱥ in which $\mathfrak{A} + \mathfrak{D} \neq$ Ⱥ and abbreviated as $\mathfrak{A} \ll$ Ⱥ. This definition was given by T. Y. Lam [1]. Furthermore, F. Kasch [2] given the definition of essential submodule as follows. A submodule $\mathfrak{A}$ of Ⱥ is called essential in Ⱥ, if for any submodule $\mathfrak{D}$ of Ⱥ for which $\mathfrak{A} \cap \mathfrak{D} = 0$ implies $\mathfrak{D} = 0$ and denoted by $\mathfrak{A} \leq_e$ Ⱥ [2]. The concept of T-essential submodule was given by Safaeeyan and Shirazi [3] as a generalization of essential submodules. The authors studied many basic properties of the mentioned concept. Beside that, the concept of T-small submodule has been presented by Beyranvand and Moradi [4] as a generalization to the concept of small submodules.

Received Dec. 3, 2025

2020 Mathematics Subject Classification. 16D10.

Key words and phrases. Small submodule, P-essential submodule, Prime submodule,T-essential submodule.

https://doi.org/10.28924/2291-8639-24-2026-37

ISSN: 2291-8639

Many properties of this concept have been discussed such as the sum, the image and the pre-image of given T-small submodules. By following these two definitions, many new generalizations have been provided. For instance, Elewi [5] generalized the results of [4] by introducing the concept of Semi-T-small submodules with some basic properties about it have been investigated. Mohammad and Yassin generalized the results of [3] by providing the concept of Small T-essential submodules with some properties about it [6]. Recently, Shahad and Al-Mothafar presented the concept of P-essential submodules as another generalization of essential submodules where P is prime submodule [7]. Very recently, Baanoon and Khalid provided another generalization of essential submodule was named as e*-essential submodule [8]. While, in [9] they introduced the concept of e*-essential small submodule as a generalization of small submodule. Finally, Shahoodh and Ali [10] presented some new types of submodules and studied some basic properties about them. Motivated by these works, we introduced in this paper the concepts of PT-essential submodule and PST-essential submodule as a generalization to the concepts of essential and small submodules. The structure of this paper is presented as follows. Section two deals with some important results that are needed in the next section. The main results of this work are given in section three. The conclusions of the current study are given in section four.

2. Preliminaries

This section includes some basics results that are important in completing the results of this work.

Definition 2.1 [3] Let ₳ be an R-module and $\mathrm{T} \leq ₳$. A submodule $\mathfrak{A}$ of ₳ is called T-essential in case that any submodule $\mathfrak{D}$ of ₳ in which $\mathfrak{A} \nleq \mathrm{T}$ and $\mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ implies $\mathfrak{D} \leq \mathrm{T}$.

Definition 2.2 [7] Let ₳ be an R-module and P is a prime submodule of ₳. A submodule $\mathfrak{A}$ of ₳ is called P-essential in case that any submodule $\mathfrak{D}$ of P in which $\mathfrak{A} \nleq \mathrm{T}$ and $\mathfrak{A} \cap \mathfrak{D} = 0$ implies $\mathfrak{D} = 0$.

Definition 2.3 [11] Let ₳ be an R-module and $\mathfrak{A}$ is a submodule of ₳. Then, $\mathfrak{A}$ is called prime submodule if $xr \in \mathfrak{A}$ where $x \in$ ₳ and $r \in \mathrm{R}$, implies $x \in \mathfrak{A}$ or $r \in [\mathfrak{A} : ₳]$.

Proposition 2.1 [7] Let $\mathfrak{E} \leq \mathfrak{F} \leq \mathfrak{G}$ are submodules of ₳. If $\mathfrak{E} \leq_{pe} \mathfrak{F}$ and $\mathfrak{F} \leq_{pe} \mathfrak{G}$ then $\mathfrak{E} \leq_{pe} \mathfrak{G}$.

Proposition 2.2 [12] Let h: ₳ $\longrightarrow$ ₳$'$ be an epimorphism. If P is prime submodule of ₳$'$ then $h^{-1}(P)$ is prime submodule of ₳.

3. Main Results

This section presents the main results of this work. We started as follows.

Definition 3.1 Suppose ₳ be an R-module and $\mathrm{T} \leq$ ₳. The submodule $\mathfrak{A}$ of ₳ is said to be PT-essential in ₳ ($\mathfrak{A} \leq_{PTE}$ ₳) in case there exist proper submodule $\mathfrak{D}$ of prime submodule P of ₳ in which $\mathfrak{A} \nleq \mathrm{T}, \mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ implies $\mathfrak{D} \leq \mathrm{T}$.

Remarks and Examples

1. Consider Z_{24} as Z-module and let $P =< 3 >, \mathfrak{D} =< 12 >, \mathrm{T} =< 6 >$ with $\mathfrak{A} =< 2 >$. Then, $\mathfrak{A} \nleq \mathrm{T}$ and $\mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ which implies $\mathfrak{D} \leq \mathrm{T}$. Thus, $\mathfrak{A} \leq_{PTE} Z_{24}$.
2. Consider Z_{36} as Z-module and let $P =< 2 >, \mathfrak{D} =< 12 >, < 18 >$ and $\mathrm{T} =< 6 >$ with $\mathfrak{A} =< 3 >$. Then, $\mathfrak{A} \leq_{PTE} Z_{36}$. Since $\mathfrak{A} \nleq \mathrm{T}$ and $\mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ implies $\mathfrak{D} \leq \mathrm{T}$.
3. Consider Z_{12} as Z-module and let $P =< 3 >, \mathfrak{D} =< 6 >, \mathrm{T} =< 4 >$ with $\mathfrak{A} =< 2 >$. Then, $\mathfrak{A} \nleq_{PTE} Z_{12}$. Since $\mathfrak{A} \cap \mathfrak{D} \nleq \mathrm{T}$.
4. Any PT-essential submodule is P-essential but not conversely. For example, consider Z_{24} as Z-module and let $P =< 2 >, \mathfrak{D} = \{< 12 >, < 6 >, < 18 >\}, \mathrm{T} =< 4 >$ with $\mathfrak{A} =< 6 >$. Then, $\mathfrak{A}$ is P-essential but not PT-essential because $< 6 > \cap < 6 > =< 6 > \nleq \mathrm{T}$ and $< 6 > \cap < 18 > =< 18 > \nleq \mathrm{T}$.
5. Any ST-essential submodule is PT-essential but not conversely. For example, consider Z_{24} as Z-module and let $\mathfrak{A} =< 2 >, \mathrm{T} =< 6 >$, and $P =< 3 >$ with $\mathfrak{D} = \{< 0 >, < 6 >, < 12 >, < 18 >\}$. Thus, $\mathfrak{A} \nleq \mathrm{T}$ and $\mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T} \Longrightarrow \mathfrak{D} \leq \mathrm{T}$. Thus, $\mathfrak{A}$ is PT-essential but not ST-essential in Z_{24}. Because $< 18 >$ is not small in Z_{24}.
6. Any T-essential submodule is PT-essential but not conversely. For example, consider Z_{24} as Z-module and let $P =< 3 >, \mathrm{T} =< 6 >, \mathfrak{A} =< 2 >$ and $\mathfrak{D} = \{< 6 >, < 12 >, < 18 >\}$. Then, $\mathfrak{A}$ is PT-essential but not T-essential in Z_{24}. Since, if $\mathfrak{D} =< 4 >, < 8 >$, then $< 2 > \cap < 4 > \nleq \mathrm{T}$ and $< 2 > \cap < 8 > \nleq \mathrm{T}$.
7. Any PT-essential submodule is P-essential iff T={0}.

Proposition 3.1 The intersection of two PT-essential submodules is not necessary PT-essential submodule.

Example 3.1 By remarks and examples (5),(6) we have $< 2 > \leq_{PTE} Z_{24}, < 3 > \leq_{PTE} Z_{24}$. But $< 2 > \cap < 3 > =< 6 > \nleq_{PTE} Z_{24}$.

Proposition 3.2 Suppose Ⱥ be an R-module and $\mathfrak{A}, \mathrm{T}$ are submodules of Ⱥ for which $\mathrm{T} \nleq \mathfrak{A}$. Then, $\mathfrak{A}$ is PT-essential submodule iff there exist a submodule $\mathfrak{D}$ of prime submodule P of Ⱥ s.t $\mathfrak{A} \cap \mathfrak{D} \nleq \mathrm{T}$ implies $\mathfrak{D} \nleq \mathrm{T}$.

Proof: Let $\mathfrak{A} \leq_{PTE}$ Ⱥ and $\mathfrak{A} \cap \mathfrak{D} \nleq \mathrm{T}$ for some submodule $\mathfrak{D}$ of prime submodule P of Ⱥ. Assume $\mathfrak{D} \leq \mathrm{T}$ then since $\mathfrak{A} \leq_{PTE}$ Ⱥ we get $\mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ which is contradiction. Conversely, let $\mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ for some $\mathfrak{D} \leq P$ to prove $\mathfrak{D} \leq \mathrm{T}$. Assume $\mathfrak{D} \nleq \mathrm{T}$ then $\mathfrak{A} \cap \mathfrak{D} \nleq \mathrm{T}$ which is contradiction. Thus, $\mathfrak{A} \leq_{PTE}$ Ⱥ. ■

Proposition 3.3 Let h: Ⱥ $\longrightarrow$ Ⱥ$'$ be a module epimorphism. Moreover, let $\mathfrak{A}, \mathrm{T}$ are submodules of Ⱥ in which $\mathfrak{A} \leq_{PTE}$ Ⱥ. Then, $h(\mathfrak{A}) \leq_{Ph(T)E}$ Ⱥ$'$.

Proof: Let $h(\mathfrak{A}) \cap \mathfrak{D} \leq h(\mathrm{T})$ for some $\mathfrak{D} \leq P$ of Ⱥ$'$. Our claim that $\mathfrak{D} \leq h(\mathrm{T})$. Now, $\mathfrak{A} \cap h^{-1}(\mathfrak{D}) \leq \mathrm{T}$. Since $\mathfrak{A} \leq_{PTE}$ Ⱥ then $h^{-1}(\mathfrak{D}) \leq \mathrm{T}$. By Proposition 2.2, $h^{-1}(P)$ is prime submodule of Ⱥ which gives that $h^{-1}(\mathfrak{D}) \leq h^{-1}(P)$. That is $\mathfrak{D} \leq h(\mathrm{T})$. Therefore, $h(\mathfrak{A}) \leq_{Ph(T)E}$ Ⱥ$'$.

Corollary 3.1 If h: Ⱥ $\longrightarrow$ Ⱥ$'$ be a module epimorphism. Moreover, let $\mathfrak{A}$, T are submodules of Ⱥ in which $\mathfrak{A} \leq_{PTE}$ Ⱥ$'$. Then, $h^{-1}(\mathfrak{A}) \leq_{Ph^{-1}(T)E}$ Ⱥ.

Proof: Let $h^{-1}(\mathfrak{A}) \cap \mathfrak{D} \leq h^{-1}(\mathrm{T})$ for some $\mathfrak{D} \leq P$ of Ⱥ. We claim that $\mathfrak{D} \leq h^{-1}(\mathrm{T})$. This gives that $\mathfrak{A} \cap h(\mathfrak{D}) \leq \mathrm{T}$. Since $\mathfrak{A} \leq_{PTE}$ Ⱥ$'$ then $h(\mathfrak{D}) \leq \mathrm{T}$. By Proposition 2.2, $h(P)$ is prime submodule of Ⱥ$'$ which gives that $h(\mathfrak{D}) \subseteq h(P)$. That is $\mathfrak{D} \leq h^{-1}(\mathrm{T})$. Thus, $h^{-1}(\mathfrak{A}) \leq_{Ph^{-1}(T)E}$ Ⱥ.■

Proposition 3.4 The submodule of PT-essential submodule is not necessary PT-essential. For example, in remarks and example point (3) $< 6 > \leq_{PTE} Z_{24}$ and $< 12 > \leq < 6 >$ but $< 12 > \nleq_{PTE} Z_{24}$. Since $< 12 > \cap < 8 > = < 0 > \leq \mathrm{T}$ but $< 8 > \nleq \mathrm{T}$ where $\mathrm{T} = < 3 >$.

Proposition 3.5 Let $\mathfrak{E} \leq \mathfrak{F} \leq \mathfrak{G}$ with T are submodules of Ⱥ. If $\mathfrak{E} \leq_{PTE} \mathfrak{F}$ and $\mathfrak{F} \leq_{PTE} \mathfrak{G}$ then $\mathfrak{E} \leq_{PTE} \mathfrak{G}$.

Proof: Assume $\mathfrak{E} \nleq_{PTE} \mathfrak{F}$ then by Definition 3.1, $\mathfrak{D} \nleq \mathrm{T}$ and $\mathfrak{E} \cap \mathfrak{D} \nleq \mathrm{T}$ for some proper submodule $\mathfrak{D}$ of prime submodule P of Ⱥ. Since any PT-essential submodule is P-essential and $\mathfrak{E} \leq \mathfrak{F}$, then we get $\mathfrak{F} \nleq_{PTE} \mathfrak{G}$ and will be contradiction by Proposition 2.1. ■

Definition 3.2 The R-homomorphism modules h: Ⱥ $\longrightarrow$ Ⱥ$'$ is called PT-essential iff $Im(h)$ is PT-essential.

Proposition 3.6 Let h_1: Ⱥ $\longrightarrow$ Ⱥ$'$, h_2: Ⱥ$'$ $\longrightarrow$ Ⱥ$''$ are PT-essential R-homomorphisms. Then, $(h_2 \circ h_1)$: Ⱥ $\longrightarrow$ Ⱥ$''$ is PT-essential R-homomorphism.

Proof: Assume $(h_2 \circ h_1)$: Ⱥ $\longrightarrow$ Ⱥ$''$ is not PT-essential R-homomorphism. Then, By Definition 3.2, for some submodule $\mathfrak{D}$ of prime submodule P of Ⱥ$''$ we have $Im((h_2 \circ h_1)) \cap \mathfrak{D} \nleq \mathrm{T}$ and $\mathfrak{D} \nleq \mathrm{T}$. Then, since $Im((h_2 \circ h_1)) \subseteq Im(h_2)$ implies $Im(h_2) \cap \mathfrak{D} \nleq \mathrm{T}$ which contradict the assumption. Therefore, as required. ■

Definition 3.3 Suppose Ⱥ be an R-module and $\mathfrak{A}, \mathrm{T} \leq$ Ⱥ. The submodule $\mathfrak{D}$ of a prime submodule P of Ⱥ said to be PT-complement of $\mathfrak{A}$ in Ⱥ ($\mathfrak{D}$ PT-c of $\mathfrak{A}$ in Ⱥ), in case $\mathfrak{D}$ is maximal and $\mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$.

Remarks and Examples

1. Consider Z_{12} as Z-module and let $P = < 3 >, \mathfrak{D} = < 6 >, \mathrm{T} = < 6 >$ with $\mathfrak{A} = < 2 >$. Then, $\mathfrak{D}$ is maximal and $\mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$. Thus, $\mathfrak{D}$ is PT-c of $\mathfrak{A}$ in Z_{12}.
2. Consider Z_{24} as Z-module and let $\mathfrak{A} = < 6 >, \mathrm{T} = < 3 >$ and $P = < 2 >$. Then, $\mathfrak{E} = < 12 >$, $\mathfrak{F} = < 4 >, \mathfrak{G} = < 8 >$ are PT-c of $\mathfrak{A}$ in Z_{24}.
3. If $\mathfrak{A}, \mathrm{T} \leq$ Ⱥ and P is prime submodule of Ⱥ.Then, the PT-c of $\mathfrak{A}$ is not unique.
4. If $\mathfrak{E}, \mathfrak{F}$ are PT-comlements of $\mathfrak{A}$. Then, $\mathfrak{E} \cap \mathfrak{F}$ is not necessary too. For example, by point (2), $\mathfrak{E}, \mathfrak{G}$ are PT-c of $\mathfrak{A}$ but $\mathfrak{E} \cap \mathfrak{G} = \{0\}$ which is not PT-c of $\mathfrak{A}$ because it is not maximal.

Theorem 3.1 Suppose Ⱥ be R-module and $\mathfrak{A}, \mathrm{T} \leq$ Ⱥ with P is prime submodule of Ⱥ. Then, $\mathfrak{A}$ has PT-c in Ⱥ.

Proof: Let $\mathfrak{A}, \mathrm{T} \leq$ Ⱥ with P is prime submodule of Ⱥ and $\mathfrak{D} \leq P$. Furthermore, let $\Gamma = \{\mathfrak{D} \leq P: \mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}\}$. Then, when $0 \in \Gamma$ implies $\Gamma \neq \Phi$. Now, let $\{\mathfrak{C}_\gamma\}_{\gamma \in I}$ be a chain in Γ, then $\bigcup_{\gamma \in I} \mathfrak{C}_\gamma \leq$ Ⱥ. Since $\mathfrak{A} \cap \left(\bigcup_{\gamma \in I} \mathfrak{C}_\gamma\right) = \bigcup_{\gamma \in I}(\mathfrak{A} \cap \mathfrak{C}_\gamma) \leq \mathrm{T}$, then $\bigcup_{\gamma \in I} \mathfrak{C}_\gamma \in \Gamma$. According to Zoren's lemma Γ has a maximal element say $\mathfrak{E}$. Thus, $\mathfrak{E}$ must be PT-c of $\mathfrak{A}$ in Ⱥ. If $\mathfrak{F} \leq P$ in which $\mathfrak{E} \not\subseteq \mathfrak{F}$ and $\mathfrak{A} \cap \mathfrak{F} \leq \mathrm{T}$ then $\mathfrak{F} \in \Gamma$ which is contradiction. Therefore, $\mathfrak{E} = \mathfrak{F}$ and $\mathfrak{A}$ has PT-c in Ⱥ. ∎

Definition 3.4 Suppose Ⱥ be an R-module and $\mathrm{T} \leq$ Ⱥ. The submodule $\mathfrak{A}$ of Ⱥ is said to be P-Small T-essential in Ⱥ ($\mathfrak{A} \leq_{PST}$ Ⱥ) in case any small submodule $\mathfrak{D}$ of prime submodule P of Ⱥ in which $\mathfrak{A} \not\leq \mathrm{T}, \mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ implies $\mathfrak{D} \leq \mathrm{T}$.

Remarks and Examples

1. Consider Z_{24} as Z-module and let $\mathfrak{A} =< 8 >, \mathrm{T} =< 3 >$ and $P =< 2 >$ with $\mathfrak{D} =< 12 >, < 6 >$ are small submodules of P in Ⱥ. Then, $\mathfrak{A}$ is PST-essential in Z_{24}.
2. Consider Z_{12} as Z-module and let $P =< 3 >, \mathrm{T} =< 4 >, \mathfrak{D} =< 0 >< 6 >$ are small submodules in Z_{12}. Let $\mathfrak{A} =< 2 >$. Then, $\mathfrak{A}$ is not PST-essential in Z_{12}. Since $< 2 > \cap < 6 > =< 6 > \not\leq \mathrm{T}$ and $< 6 > \not\leq \mathrm{T}$.
3. Consider Z_{36} as Z-module and let $\mathfrak{A} =< 4 >, \mathrm{T} =< 3 >$ and $P =< 2 >$ with $\mathfrak{D} =< 12 >, < 6 >, < 18 >$ are small submodules of P in Z_{36}. Then, $\mathfrak{A}$ is PST-essential in Z_{24}.
4. The concepts of PST-essential and ST-essential are equivalent.
5. Any PST-essential submodule is PT-essenial submodule but not conversely. This is clear by definitions.

Proposition 3.7 Suppose Ⱥ be an R-module and $\mathfrak{A}, \mathrm{T}$ are submodules of Ⱥ for which $\mathrm{T} \not\leq \mathfrak{A}$. Then, $\mathfrak{A}$ is PST-essential submodule iff for any small submodule $\mathfrak{D}$ of prime submodule P of Ⱥ s.t $\mathfrak{A} \cap \mathfrak{D} \not\leq \mathrm{T}$ implies $\mathfrak{D} \not\leq \mathrm{T}$.

Proof: Let $\mathfrak{A} \leq_{PST}$ Ⱥ then for any small submodule $\mathfrak{D}$ of prime submodule P of Ⱥ we have $\mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ and $\mathfrak{D} \leq \mathrm{T}$. Assume $\mathfrak{A} \cap \mathfrak{D} \not\leq \mathrm{T}$ then we get contradiction. Conversely, to show that $\mathfrak{A} \leq_{PST}$ Ⱥ, let $\mathfrak{A} \cap \mathfrak{D} \leq \mathrm{T}$ for any small submodule $\mathfrak{D}$ of prime submodule P of Ⱥ to prove $\mathfrak{D} \leq \mathrm{T}$. Assume $\mathfrak{D} \not\leq \mathrm{T}$ then we have $\mathfrak{A} \cap \mathfrak{D} \not\leq \mathrm{T}$ which is contradiction. Thus, $\mathfrak{A} \leq_{PST}$ Ⱥ. ∎

Proposition 3.8 Let h: Ⱥ $\longrightarrow$ Ⱥ′ be modules epimorphism. Moreover, let $\mathfrak{A}, \mathrm{T}$ are submodules of Ⱥ′ in which $\mathfrak{A} \leq_{PST}$ Ⱥ′. Then, $h^{-1}(\mathfrak{A}) \leq_{PST}$ Ⱥ.

Proof: Suppose $h^{-1}(\mathfrak{A}) \cap \mathfrak{D} \leq h^{-1}(\mathrm{T})$ for any small submodule $\mathfrak{D}$ of P of Ⱥ. We claim that $\mathfrak{D} \leq h^{-1}(\mathrm{T})$. From this we get $\mathfrak{A} \cap h(\mathfrak{D}) \leq \mathrm{T}$ and $\mathfrak{A} \leq$ Ⱥ. By Proposition 2.2, $h(P)$ is prime submodule of Ⱥ′ which gives that $h(\mathfrak{D}) \subseteq h(P)$. Since $\mathfrak{A} \leq_{PST}$ Ⱥ′ then $h(\mathfrak{D}) \leq \mathrm{T}$ which gives that $\mathfrak{D} \leq h^{-1}(\mathrm{T})$. Thus, $h^{-1}(\mathfrak{A}) \leq_{PSh^{-1}(T)}$ Ⱥ. ∎

Corollary 3.2 Let h: Ⱥ $\longrightarrow$ Ⱥ′ be modules epimorphism. Moreover, let $\mathfrak{A}, \mathrm{T}$ are submodules of Ⱥ in which $\mathfrak{A} \leq_{PST}$ Ⱥ. Then, $h(\mathfrak{A}) \not\leq_{PSh\,(T)}$ Ⱥ.

Example 3.2 [4] Let $h: Z_{10} \longrightarrow Z_{20}$ be a function such that $h(x) = 2x, \forall x \in Z_{10}$. Then, $< 10 >$ is small submodule in Z_{20} but $h^{-1}(< 10 >) =< 5 >$ is not small submodule in Z_{10}.

Definition 3.5 The R-homomorphism modules $h: Ꜳ \longrightarrow Ꜳ'$ is called PST-essential iff $Im(h)$ is PST-essential.

Proposition 3.9 Let $h_1: Ꜳ \longrightarrow Ꜳ'$, $h_2: Ꜳ' \longrightarrow Ꜳ''$ are PST-essential R-homomorphisms. Then, $(h_2 \circ h_1): Ꜳ \longrightarrow Ꜳ''$ is not PST-essential R-homomorphism.

Proof: Assume $(h_2 \circ h_1): Ꜳ \longrightarrow Ꜳ''$ is a PST-essential R-homomorphism. Then, $Im((h_2 \circ h_1)) \leq_{PST} Ꜳ''$. By Definition 3.3 we have $\mathrm{T} \leq Ꜳ''$ with prime submodule $P \leq Ꜳ''$ contains a small submodule $\mathfrak{D}$ for which $\left(Im(h_2 \circ h_1)\right) \cap \mathfrak{D} \leq \mathrm{T}$ and $\mathfrak{D} \leq \mathrm{T}$. Since $Im(h_2 \circ h_1) \subseteq Im(h_2)$, then $Im(h_2) \cap \mathfrak{D} \leq \mathrm{T}$. Since $h_2: Ꜳ' \longrightarrow Ꜳ''$ is R-homomorphism and $\mathfrak{D} \ll Ꜳ''$, then ${h_2}^{-1}(\mathfrak{D})$ is small in $Ꜳ'$. But Corollary 3.2. make this way is not true which gives a contradiction with the assumption that $h_1: Ꜳ \longrightarrow Ꜳ'$ is PST-essential. Therefore, as desired. ■

4. Conclusion

This work presented the concepts of PT-essential and PST-essential submodules. The obtained results showed that the image (pre-image) of PT-essential submodule is also PT-essential submodule. Also, the image of PST-essential submodule is PST-essential. While, the inverse image is not PST-essential because the inverse image of small submodule is not necessarily small as shown in example 3.1. Based on this work, many results can be extended.

Acknowledgments: The authors thank the reviewers for their valuable comments that will improve the quality of our paper.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] T.Y. Lam, A First Course in Noncommutative Rings, Springer New York, 2001. https://doi.org/10.1007/978-1-4419-8616-0.

[2] Kasch, F. Modules and Ring. 35, Inc-London: Academic Press, 1982.

[3] S. Safaeeyan, N.S. Shirazi, Essential Submodules with Respect to an Arbitrary Submodule, J. Math. Ext. 7 (2013), 15-27.

[4] R. Beyranvand, F. Moradi, Small Submodules with Respect to An Arbitrary Submodule, J. Algebra Related Top. 3 (2015), 43-51.

[5] A.A. Elewi, Semi-T-Small Submodules, Iraqi J. Sci. 62 (2021), 1956-1960. https://doi.org/10.24996/ijs.2021.62.6.20.

[6] Z.R. Mohammad, S.M. Yassin, On ST-Essential (Complement) Submodules, Iraqi J. Sci. 61 (2020), 838-844. https://doi.org/10.24996/ijs.2020.61.4.17.

[7] H.A. Shahad, N.S. Al-Mothafar, On P-Essential Submodules, Iraqi J. Sci. 62 (2021), 4916-4922. https://doi.org/10.24996/ijs.2021.62.12.29.

[8] H. Baanoon, W. Khalid, E*-Essential Submodule, Eur. J. Pure Appl. Math. 15 (2022), 224-228. https://doi.org/10.29020/nybg.ejpam.v15i1.4215.

[9] H. Baanoon, W. Khalid, E*-Essential Small Submodules and E*-Hollow Modules, Eur. J. Pure Appl. Math. 15 (2022), 478-485. https://doi.org/10.29020/nybg.ejpam.v15i2.4301.

[10] M.K. Shahoodh, O.K. Ali, T-Closed-Coessential and T-Closed-Coclosed Submodules, Int. J. Anal. Appl. 23 (2025), 299. https://doi.org/10.28924/2291-8639-23-2025-299.

[11] C.P. Lu, Prime Submodules of Modules, Comment. Math. Univ. Sancti Pauli 33 (1984), 61-69.

[12] E.A. Athab, Prime and Semiprime Submodules, Msc Thesis, Universty of Baghdad, 1996.