Int. J. Anal. Appl. (2026), 24:30

International Journal of Analysis and Applications

Estimating the Norm of the Error Functional in Sobolev Space of Periodic Functions

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Abstract. This article is devoted to obtaining an upper estimate of the error of an optimal quadrature formula for approximating the integral of periodic functions in the Sobolev space $\widetilde{W_2^{(2,1,0)}}(0,1]$. In the quadrature formulas, a complex exponential weight function of the form $e^{2\pi i\omega x}$ is used. To minimize the norm of the error functional of the quadrature formula, a corresponding extremal function is found, and using it, an expression for the norm of the error functional is derived. The optimal coefficients that give the smallest value to this norm are obtained. Using Fourier analysis and the extremal function method, explicit formulas for the optimal coefficients are derived. These results extend the classical theory of quadrature formulas to the exponential-weight and oscillatory cases, providing efficient schemes for the numerical integration of periodic functions.

1. Introduction

Optimal quadrature formulas are fundamental tools in approximation and numerical integration. Classical quadrature formulas, such as the Gaussian or Newton–Cotes rules, are effective for polynomial approximation in non-weighted spaces. The problems of constructing optimal quadrature formulas in various spaces, including both periodic and non-periodic ones, have been addressed in the following works [1–4, 6, 7]. However, in numerous applications involving oscillatory phenomena or exponential modulation, weighted quadrature formulas offer improved accuracy [5, 8, 13]. The construction of quadrature formulas in Hilbert spaces has also been discussed in the following works [9–12]. In [13], an optimal quadrature formula with weights was

Received: Dec. 1, 2025.

2020 *Mathematics Subject Classification.* 65L05, 65L12, 65L20.

Key words and phrases. optimal quadrature formula; exponential weight; complex exponential function; periodic Sobolev space; Fourier transform coefficients.

https://doi.org/10.28924/2291-8639-24-2026-30
ISSN: 2291-8639

constructed in the Hilbert space. The algorithms for solving the integral equation are given using the constructed optimal quadrature formula.

$\widetilde{W_2^{(2,1,0)}}(0,1] = \{\varphi : (0,1] \to \mathbb{C}$, φ-absolutely continuous and $\varphi'' \in \widetilde{L_2}\}$ and for $\forall\varphi \in \widetilde{W_2^{(2,1,0)}}$ The function satisfies the 1-periodicity condition: $\varphi(x+\beta) = \varphi(x)$, $\forall x \in \mathbb{R}$, $\beta \in \mathbb{Z}$.

The inner product in this space is defined as follows:

$$\langle \psi, \varphi \rangle_{\widetilde{W}_2^{(2,1,0)}(0,1)} = \int_0^1 \left(\bar{\psi}''(x)\varphi''(x) + 2\bar{\psi}'(x)\varphi'(x) + \bar{\psi}(x)\varphi(x)\right) dx.$$

We consider the following quadrature formula

$$\int_0^1 e^{2\pi i\omega x}\varphi(x)dx \cong \sum_{k=1}^{N} C_k\varphi(hk), \tag{1.1}$$

with the error

$$(\ell, \varphi) = \int_0^1 e^{2\pi i\omega x}\varphi(x)dx - \sum_{k=1}^{N} C_k\varphi(hk), \tag{1.2}$$

and the corresponding error functional is:

$$\ell(x) = e^{2\pi i\omega x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x - hk - \beta). \tag{1.3}$$

Here C_k are the coefficients of quadrature formula (1.1), $h = \frac{1}{N}$, $N \in \mathbb{N}$, $\omega \in \mathbb{Z}$.

2. Problem Statement

Our goal is to estimate the error of the considered quadrature formula (1.2) from above, for which it is sufficient to calculate the norm of the error functional (1.3). This leads to the solution of the following two problems; we will first consider them for $m = 2$. Now, we consider the following problem.

Problem 1: Find the analytical representation of the norm of the error functional (1.3) in $\widetilde{W_2^{(2,1,0)}}$ space.

Problem 2: Finding the optimal coefficients of $C_k = \overset{0}{C}_k$ that minimize

To solve Problem 1, we use the concept of an extremal function introduced by Sobolev. Using Riesz's theorem for the space $\widetilde{W_2^{(2,1,0)}}(0,1)$, we can write the following

$$(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle_{\widetilde{W}_2^{(2,1,0)}(0,1)} = \int_0^1 \varphi''(x)\bar{\psi}_\ell''(x)dx + 2\int_0^1 \varphi'(x)\bar{\psi}_\ell'(x)dx + \int_0^1 \varphi(x)\bar{\psi}_\ell(x)dx =$$

$$= \int_0^1 (\bar{\psi}_\ell^{(4)}(x) - 2\bar{\psi}_\ell^{(2)}(x) + \bar{\psi}_\ell(x))\varphi(x)dx.$$

From the above equation, we get the following equation,

$$\bar{\psi}_\ell^{(4)}(x) - 2\bar{\psi}_\ell^{(2)}(x) + \bar{\psi}_\ell(x) = \ell(x) \tag{2.1}$$

Theorem 1. In the Sobolev space of $\widetilde{W_2^{(2,1,0)}}(0,1)$ periodic functions, the extremal function of the quadrature formula (1.2) has the following form:

$$\psi_\ell(x) = \frac{e^{-2\pi i\omega x}}{(2\pi\omega)^4 + 2(2\pi\omega)^2 + 1} - \sum_{k=1}^{N} \bar{C}_k \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta(x-hk)}}{(2\pi\beta)^4 + 2(2\pi\beta)^2 + 1}. \tag{2.2}$$

Proof: To find the generalized periodic solution of the differential equation (2.1), we apply the Fourier transform to both sides of the equation and use the following properties of the Fourier transform:

$$F[\varphi] = \int_{-\infty}^{\infty} \varphi(x) e^{2\pi ipx} dx,$$

$$F^{-1}[\varphi] = \int_{-\infty}^{\infty} \varphi(p) e^{-2\pi ipx} dp,$$

$$F\left[\varphi^{(\alpha)}\right] = (-2\pi ip)^{\alpha} F[\varphi],\ (\alpha \in \mathbb{N}),$$

$$F[\phi_0(x)] = \phi_0(p).$$

$$F^{-1}[F[\varphi(x)]] = \varphi(x).$$

Here, $*$ denotes the convolution operation.

We apply the Fourier Transform to both sides of equation (2.1)

$$F\left[\bar{\psi}_\ell^{(4)}(x) - 2\bar{\psi}_\ell^{(2)}(x) + \bar{\psi}_\ell(x)\right] = F[\ell(x)].$$

Since the Fourier Transform is a linear operator, we have

$$\left((2\pi p)^2 + 1\right)^2 F[\bar{\psi}_\ell] = F\left[e^{2\pi i\omega x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x - hk - \beta)\right],$$

where

$$F[\delta(x - hk - \beta)] = \int_{-\infty}^{\infty} \delta(x - hk - \beta) e^{2\pi ipx} dx = e^{2\pi ip(hk+\beta)},$$

$$F\left[e^{2\pi i\omega x}\right] = \int_{-\infty}^{\infty} e^{2\pi i\omega x} e^{2\pi ipx} dx = \delta(p + \omega),$$

$$F[\bar{\psi}_\ell] = \frac{\delta(p+\omega)}{\left((2\pi\omega)^2 + 1\right)^2} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta hk}\delta(p-\beta)}{(2\pi\beta)^4 + 2(2\pi\beta)^2 + 1}.$$

Then, we apply the inverse Fourier transform to the above equality, and we obtain the following

$$\bar{\psi}_\ell(x) = \frac{e^{2\pi i\omega x}}{(2\pi\omega)^4 + 2(2\pi\omega)^2 + 1} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta(x-hk)}}{(2\pi\beta)^4 + 2(2\pi\beta)^2 + 1},$$

$$\psi_\ell(x) = \frac{e^{-2\pi i\omega x}}{(2\pi\omega)^4 + 2(2\pi\omega)^2 + 1} - \sum_{k=1}^{N} \bar{C}_k \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta(x-hk)}}{(2\pi\beta)^4 + 2(2\pi\beta)^2 + 1}.$$

And so, Theorem 1 is proved from the last equality.

We calculate the analytical form of the error functional norm

$$\|\ell\|^2_{\widetilde{W_2^{(2,1,0)}}(0,1)^*} = \int_0^1 \ell(x)\psi_\ell(x)dx = \int_0^1 \left(e^{2\pi i\omega x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x - hk - \beta) \right) \times$$

$$\times \left(\frac{e^{-2\pi i\omega x}}{(2\pi\omega)^4 + 2(2\pi\omega)^2 + 1} - \sum_{k=1}^{N} \bar{C}_k \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta(x-hk)}}{(2\pi\beta)^4 + 2(2\pi\beta)^2 + 1} \right) dx. \tag{2.3}$$

From the above equality, we obtain the following:

$$\begin{aligned} \|\ell\|^2_{\widetilde{W}_2^{(2,1,0)}(0,1)^*} = &\frac{1}{\left((2\pi\omega)^2 + 1\right)^2} - \sum_{k'=1}^{N} \bar{C}_{k'} \frac{e^{2\pi i\omega hk'}}{\left((2\pi\omega)^2 + 1\right)^2} \\ &- \sum_{k=1}^{N} C_k \frac{e^{-2\pi i\omega hk}}{\left((2\pi\omega)^2 + 1\right)^2} + \sum_{k=1}^{N}\sum_{k'=1}^{N} \bar{C}_{k'} C_k \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta h(k-k')}}{((2\pi\beta)^2 + 1)^2} \end{aligned} \tag{2.4}$$

So Problem 1 is solved.

3. Finding the Coefficient of the Quadrature Formula (1.1)

Theorem 2. The coefficients of the quadrature formula in the form (1.1) that minimizes the norm of the error functional are as follows:

$$\overset{0}{C}_k = C(\omega, h) \cdot e^{2\pi i\omega hk} = \frac{8 \cdot e^{2\pi i\omega hk}}{\left((2\pi\omega)^2 + 1\right)^2} \cdot \left[\frac{2h\lambda e^h + \lambda^2 e^{2h} - 1}{(\lambda e^h - 1)^2} - \frac{\lambda^2 - e^{2h} - 2h\lambda e^h}{(\lambda - e^h)^2} \right]^{-1}. \tag{3.1}$$

Here $\lambda = e^{2\pi i\omega h}$ and k = 1, 2, . . . , N

Proof: Taking the first derivative of the coefficient from equality (2.4) and equating it to zero, we obtain the following equality

$$-\frac{e^{2\pi i\omega hk'}}{(2\pi\omega)^4 + 2(2\pi\omega)^2 + 1} + \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta h(k-k')}}{(2\pi\beta)^4 + 2(2\pi\beta)^2 + 1} = 0. \tag{3.2}$$

Assume the optimal coefficients are as follows: $\overset{0}{C}_k = C(\omega, h) \cdot e^{2\pi i\omega hk}$

$$-\frac{e^{2\pi i\omega hk'}}{(2\pi\omega)^4 + 2(2\pi\omega)^2 + 1} + C(\omega, h) \sum_{k=1}^{N} e^{2\pi i\omega hk} \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta h(k-k')}}{(2\pi\beta)^4 + 2(2\pi\beta)^2 + 1} = 0, \tag{3.3}$$

$$A = C(\omega,h)\sum_{k=1}^{N} e^{2\pi i\omega hk}\sum_{\beta=-\infty}^{\infty}\frac{e^{2\pi i\beta h(k-k')}}{(2\pi\beta)^4+2(2\pi\beta)^2+1}$$

$$= C(\omega,h)\sum_{\beta=-\infty}^{\infty}\frac{e^{-2\pi i\beta hk'}}{\left((2\pi\beta)^2+1\right)^2}\cdot\sum_{k=1}^{N} e^{2\pi i\omega hk}e^{2\pi i\beta hk}$$

$$= C(\omega,h)\sum_{\beta=-\infty}^{\infty}\frac{e^{-2\pi i\beta hk'}}{\left((2\pi\beta)^2+1\right)^2}\cdot\sum_{k=1}^{N} e^{2\pi ihk(\beta+\omega)},$$

$$\sum_{k=1}^{N} e^{2\pi ihk(\beta+\omega)} = \begin{cases}0, & (\beta+\omega)h \notin \mathbb{Z},\\ N, & (\beta+\omega)h \in \mathbb{Z},\end{cases}$$

$t=(\beta+\omega)h, \qquad \beta = tN-\omega.$

By substituting A from the previous equation into equation (3.3), we derive the following:

$$\frac{1}{16\pi^4 h}C(\omega,h)e^{2\pi i\omega hk'}\sum_{t=-\infty}^{\infty}\frac{1}{\left((tN-\omega)^2-\left(\frac{i}{2\pi}\right)^2\right)^2}-\frac{e^{2\pi i\omega hk'}}{(2\pi\omega)^4+2(2\pi\omega)^2+1}=0,$$

$$C(\omega,h)=\frac{16\pi^4 h}{\left((2\pi\omega)^2+1\right)^2}\left(\sum_{t=-\infty}^{\infty}\frac{h^4}{\left((t-\omega h)^2-\left(\frac{hi}{2\pi}\right)^2\right)^2}\right)^{-1}.$$

In order to compute $C(\omega,h)$, it suffices to evaluate the infinite series

$$\sum_{t=-\infty}^{\infty}\frac{h^4}{\left((t-\omega h)^2-\left(\frac{hi}{2\pi}\right)^2\right)^2}$$

This infinite series is computed by means of the residue theorem [5]

$$\sum_{t=-\infty}^{\infty}\frac{h^4}{\left((t-\omega h)^2-\left(\frac{hi}{2\pi}\right)^2\right)^2}=2\pi^4 h\left[\frac{2h\lambda e^h+\lambda^2e^{2h}-1}{(\lambda e^h-1)^2}-\frac{\lambda^2-e^{2h}-2h\lambda e^h}{(\lambda-e^h)^2}\right] \tag{3.4}$$

$$C(\omega,h)=\frac{16\pi^4 h}{\left((2\pi\omega)^2+1\right)^2}\left(\sum_{t=-\infty}^{\infty}\frac{h^4}{\left((t-\omega h)^2-\left(\frac{hi}{2\pi}\right)^2\right)^2}\right)^{-1}=\frac{16\pi^4 h}{\left((2\pi\omega)^2+1\right)^2}\times$$

$$\times\frac{1}{2\pi^4 h}\left[\frac{2h\lambda e^h+\lambda^2e^{2h}-1}{(\lambda e^h-1)^2}-\frac{\lambda^2-e^{2h}-2h\lambda e^h}{(\lambda-e^h)^2}\right]^{-1}=$$

$$=\frac{8}{\left((2\pi\omega)^2+1\right)^2}\cdot\left[\frac{2h\lambda e^h+\lambda^2e^{2h}-1}{(\lambda e^h-1)^2}-\frac{\lambda^2-e^{2h}-2h\lambda e^h}{(\lambda-e^h)^2}\right]^{-1}$$

$$\overset{0}{C}_k = C(\omega,h)\cdot e^{2\pi i\omega hk} = \frac{8\cdot e^{2\pi i\omega hk}}{\left((2\pi\omega)^2+1\right)^2}\cdot\left[\frac{2h\lambda e^h+\lambda^2e^{2h}-1}{\left(\lambda e^h-1\right)^2}-\frac{\lambda^2-e^{2h}-2h\lambda e^h}{\left(\lambda-e^h\right)^2}\right]^{-1}.$$

Here $\lambda = e^{2\pi i\omega h}$. The theorem has been proven.

Next, we proceed to compute the norm of the error functional.

Theorem 3. The square of the norm of the optimal quadratura formula in the space $\widetilde{W}_2^{(2,1,0)}(0,1)$ is expressed as

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1,0)}(0,1)^*} = \frac{1}{((2\pi\omega)^2+1)^2} - \frac{8}{((2\pi\omega)^2+1)^4\cdot h}\times$$
$$\times\left[\frac{2h\lambda e^h+\lambda^2e^{2h}-1}{(\lambda e^h-1)^2}-\frac{\lambda^2-e^{2h}-2h\lambda e^h}{(\lambda-e^h)^2}\right]^{-1}.$$

Proof: By simplifying the square of the norm of the error functional, we obtain the following results.

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1,0)}(0,1)^*} = \frac{1}{\left((2\pi\omega)^2+1\right)^2} - \sum_{k'=1}^{N}\bar{C}_{k'}\frac{e^{2\pi i\omega hk'}}{\left((2\pi\omega)^2+1\right)^2} - \sum_{k=1}^{N}C_k\frac{e^{-2\pi i\omega hk}}{\left((2\pi\omega)^2+1\right)^2}+$$
$$+\sum_{k=1}^{N}\sum_{k'=1}^{N}\bar{C}_{k'}C_k\sum_{\beta=-\infty}^{\infty}\frac{e^{-2\pi i\beta h(k-k')}}{\left((2\pi\beta)^2+1\right)^2} = \frac{1}{\left((2\pi\omega)^2+1\right)^2}+$$
$$+\sum_{k'=1}^{N}\bar{C}_{k'}\left(-\frac{e^{2\pi i\omega hk'}}{(2\pi\omega)^4+2(2\pi\omega)^2+1}+\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\frac{e^{2\pi i\beta h(k-k')}}{(2\pi\beta)^4+2(2\pi\beta)^2+1}\right)$$
$$-\sum_{k=1}^{N}C_k\frac{e^{-2\pi i\omega hk}}{\left((2\pi\omega)^2+1\right)^2}.$$

Taking equality (3.2) into account, we obtain the following relation.

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1,0)}(0,1)^*} = \frac{1}{((2\pi\omega)^2+1)^2} - \sum_{k=1}^{N}C_k\frac{e^{-2\pi i\omega hk}}{((2\pi\omega)^2+1)^2} =$$
$$= \frac{1}{((2\pi\omega)^2+1)^2}\left(1-\sum_{k=1}^{N}C_ke^{-2\pi i\omega hk}\right).$$

Substituting the obtained coefficient into the above equality, we derive the following relation.

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1,0)}(0,1)^*} = \frac{1}{\left((2\pi\omega)^2+1\right)^2}\left(1-\sum_{k=1}^{N}C_ke^{-2\pi i\omega hk}\right) = \frac{1}{\left((2\pi\omega)^2+1\right)^2}\times$$
$$\times\left(1-\frac{8\cdot\sum\limits_{k=1}^{N}e^{2\pi i\omega hk}\cdot e^{-2\pi i\omega hk}}{\left((2\pi\omega)^2+1\right)^2}\cdot\left[\frac{2h\lambda e^h+\lambda^2e^{2h}-1}{\left(\lambda e^h-1\right)^2}-\frac{\lambda^2-e^{2h}-2h\lambda e^h}{\left(\lambda-e^h\right)^2}\right]^{-1}\right)$$

$$= \frac{1}{\left((2\pi\omega)^2+1\right)^2} - \frac{8\cdot N}{\left((2\pi\omega)^2+1\right)^4}\cdot\left[\frac{2h\lambda e^h+\lambda^2e^{2h}-1}{(\lambda e^h-1)^2} - \frac{\lambda^2-e^{2h}-2h\lambda e^h}{(\lambda-e^h)^2}\right]^{-1}$$

$$= \frac{1}{\left((2\pi\omega)^2+1\right)^2} - \frac{8}{\left((2\pi\omega)^2+1\right)^4\cdot h}\cdot\left[\frac{2h\lambda e^h+\lambda^2e^{2h}-1}{(\lambda e^h-1)^2} - \frac{\lambda^2-e^{2h}-2h\lambda e^h}{(\lambda-e^h)^2}\right]^{-1}.$$

The theorem has been proven. For the norm of the error functional (1.3) of optimal quadrature formulas of the form (1.1) in the space$\widetilde{W_2^{(2,1,0)}}(0,1]$ for h $\to 0$, the following equality holds

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1,0)}(0,1)^*} = \frac{h^4}{720} - \frac{(1-20\pi^2\omega^2)h^6}{15120} + O(h^8).$$

The table below illustrates a comparison of the error functional norms between the $W_2^{(2,1)}$ and $W_2^{(2,1,0)}$ spaces. The norms are evaluated for different error functions, providing a clear understanding of how the error behaves in each space.

Table 1. Numerical error table

N	$\omega = 1$		$\omega = 11$	
	$W_2^{(2,1,0)}$	$W_2^{(2,1)}$	$W_2^{(2,1,0)}$	$W_2^{(2,1)}$
1	2.46969e-2	2.5015e-2	2.09297e-4	2.09319e-4
5	1.77764e-3	1.77896e-3	2.09290e-4	2.09312e-4
10	3.9022e-4	3.9029e-4	2.09289e-4	2.09312e-4

4. Conclusion

In this work, we derived an upper estimate for the error of an optimal quadrature formula for integrating periodic functions in the Sobolev space $\widetilde{W_2^{(2,1,0)}}(0,1]$, using a complex exponential weight $e^{2\pi i\omega x}$. By finding the corresponding extremal function, we obtained an explicit expression for the error norm and determined the optimal coefficients that minimize this error. The results extend classical quadrature theory to handle exponential weights and oscillatory integrals, providing efficient schemes for numerical integration of periodic functions. These findings offer a foundation for further research in improving quadrature methods for weighted and oscillatory cases.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] K.M. Shadimetov, A.R. Hayotov, Construction of Interpolation Splines Minimizing Semi-Norm in $w_2^{(m,m-1)}(0,1)$ Space, BIT Numer. Math. 53 (2013), 545–563. https://doi.org/10.1007/s10543-012-0407-z.

[2] A.R. Hayotov, G.V. Milovanović, K.M. Shadimetov, Optimal Quadratures in the Sense of Sard in a Hilbert Space, Appl. Math. Comput. 259 (2015), 637–653. https://doi.org/10.1016/j.amc.2015.02.093.

[3] A. Hayotov, S. Jeon, C. Lee, K. Shadimetov, Optimal Quadrature Formulas for Non-Periodic Functions in Sobolev Space and Its Application to CT Image Reconstruction, Filomat 35 (2021), 4177–4195. https://doi.org/10.2298/fil2112177h.

[4] K. Shadimetov, S. Azamov, On the Construction of Optimal Quadrature Formulas with Equally Spaced Nodes, Filomat 38 (2024), 10279–10295. https://doi.org/10.2298/fil2429279s.

[5] K. Shadimetov, A. Hayotov, U. Khayriev, Optimal Quadrature Formulas for Approximating Strongly Oscillating Integrals in the Hilbert Space $W_2^{(m,m-1)}$ of Periodic Functions, J. Comput. Appl. Math. 453 (2025), 116133. https://doi.org/10.1016/j.cam.2024.116133.

[6] Shadimetov, Kh.M., Azamov, K.M. Shadimetov, S.S. Azamov, Discrete Analogues of High Order Differential Operators, J. Math. Sci. 284 (2024), 253–265. https://doi.org/10.1007/s10958-024-07347-y.

[7] S.S. Azamov, On the Construction of Optimal Quadrature Formulas With Equally Spaced Nodes, Uzbek Math. J. 67 (2023), 41–48.

[8] K. Shadimetov, A. Adilkhodjaev, O. Gulomov, Optimal Quadrature Formulas for Approximate Calculation of Rapidly Oscillating Integrals, Results Appl. Math. 27 (2025), 100627. https://doi.org/10.1016/j.rinam.2025.100627.

[9] A.R. Hayotov, U.N. Khayriev, Construction of an Optimal Quadrature Formula in the Hilbert Space of Periodic Functions, Lobachevskii J. Math. 43 (2022), 3151–3160. https://doi.org/10.1134/s199508022214013x.

[10] K. Shadimetov, F. Nuraliev, S. Kuziev, Optimal Quadrature Formula of Hermite Type in the Space of Differentiable Functions, Int. J. Anal. Appl. 22 (2024), 25. https://doi.org/10.28924/2291-8639-22-2024-25.

[11] S.L. Sobolev, Introduction to the Theory of Cubature Formulas, Nauka, (1974).

[12] M.A. Chakhkiev, Linear Differential Operators With Real Spectrum, and Optimal Quadrature Formulas, Math. USSR-Izv. 25 (1985), 391–417. https://doi.org/10.1070/im1985v025n02abeh001289.

[13] M. Shadimetov, S. Azamov, H. Qobilov Optimality of Approximate Integration Formulas for Classes of Periodic Functions, Bull. Inst. Math. 8 (2025), 218–225.