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Modeling Dual-Uncertainty In Sustainable Supplier Selection Using Bipolar Complex Pythagorean Fuzzy Sets

Murad M. Arar[1], **Hariwan Z. Ibrahim**[2,*]

[1]*Department of Mathematics, College of Sciences and Humanities,Prince Sattam bin Abdulaziz University, Al Kharj, Saudi Arabia*

[2]*Department of Mathematics, College of Education, University of Zakho, Zakho, Kurdistan Region-Iraq*

[*]*Corresponding author:* hariwan_math@yahoo.com

Abstract. Managing sustainability in modern supply chains requires decision-making tools that can accommodate conflicting criteria, uncertain data, and evaluations that involve both positive and negative impacts. To address these challenges, this study develops a new decision-making framework based on bipolar complex Pythagorean fuzzy sets (BCPFSs). The model integrates bipolarity, complex-valued membership degrees, and Pythagorean structures to capture the nuanced interplay of economic, environmental, and social considerations. On this foundation, two aggregation operators—the bipolar complex Pythagorean fuzzy weighted averaging (BCPFWA) and weighted geometric (BCPFWG) operators—are introduced to synthesize multidimensional information while preserving uncertainty and dual evaluations. The applicability of the framework is demonstrated through a case study on green supply chain management (GSCM). Six supplier strategies, ranging from cost-oriented to fully balanced sustainability-focused approaches, are assessed against eight attributes including cost efficiency, product quality, carbon emissions, waste management, technological integration, and social responsibility. The analysis reveals that the balanced sustainability supplier emerges as the most effective choice, consistently ranked highest by both operators. Comparative results with conventional fuzzy aggregation approaches show that the proposed operators provide richer, more stable, and more interpretable rankings, especially when trade-offs between cost and sustainability are present. This research contributes to both theory and practice: it extends the scope of fuzzy decision-making by unifying multiple existing models as special cases, and it offers a practical toolset for organizations seeking resilient and environmentally responsible supply chain solutions. The findings demonstrate that BCPF-based aggregation can enhance strategic decision-making in contexts where sustainability and uncertainty are inseparably linked.

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1. Introduction

Fuzzy set (FS) theory, introduced by Zadeh [1], fundamentally transformed classical set theory by allowing elements to possess degrees of membership ranging from 0 to 1, thereby enabling the representation of uncertainty and imprecision beyond binary inclusion or exclusion. To overcome limitations in expressing hesitation and partial knowledge, Atanassov [2] proposed intuitionistic fuzzy sets (IFSs), which incorporate both membership and non-membership functions while introducing an implicit hesitation degree, improving the modeling of uncertainty in decision-making, expert systems, and computational intelligence. Yager [3] further extended this concept with Pythagorean fuzzy sets (PFSs), relaxing the constraints of IFSs by allowing the squared sum of membership and non-membership degrees to be bounded by one, thereby offering a wider range of admissible membership pairs and enhancing the flexibility in capturing expert judgments under uncertainty. Complementing these developments, Ramot et al. [4] introduced complex fuzzy sets (CFSs), where membership functions are represented as complex numbers, combining magnitude and phase information, which is particularly effective in modeling oscillatory, cyclical, or time-dependent phenomena. This idea was expanded by Alkouri and Salleh [5] in complex intuitionistic fuzzy sets (CIFSs), which incorporate both complex membership and non-membership functions to handle multi-factor, phase-dependent uncertainty, and further generalized by Ullah et al. [6] through complex Pythagorean fuzzy sets (CPFSs), which integrate Pythagorean constraints within the complex domain, allowing a broader and more flexible representation of uncertain information. Alongside these complex extensions, bipolar fuzzy sets (BFSs), initially proposed by Zhang [7,8], introduced dual membership functions—positive and negative—to capture both favorable and unfavorable evaluations simultaneously, which were later extended into bipolar intuitionistic fuzzy sets (BIFSs) [9] and bipolar Pythagorean fuzzy sets (BPFSs) [10], enabling richer and more nuanced representations of dual-sided uncertainty in multi-criteria decision-making contexts. Collectively, these advancements in fuzzy set theory, including intuitionistic, Pythagorean, complex, and bipolar frameworks, have significantly enhanced the mathematical tools available for representing and analyzing imprecise, uncertain, and multi-dimensional information, forming a robust foundation for developing sophisticated aggregation operators and decision-support methodologies in diverse application domains such as artificial intelligence, pattern recognition, and computational intelligence.

Bipolar complex fuzzy sets (BCFSs) represent a significant evolution of both bipolar fuzzy sets and complex fuzzy sets by combining the dual nature of positive and negative membership functions with complex-valued representations. This hybridization allows for a more detailed and nuanced modeling of uncertainty, capturing both the magnitude and phase information of positive and negative evaluations simultaneously. Alkouri et al. [12] investigated the theoretical properties and practical applications of bipolar CFSs, demonstrating their effectiveness in representing uncertainty in dynamic, oscillatory, and time-dependent systems. Similarly, Al-Husban et al. [13] explored foundational properties of these sets, establishing their relevance for decision-making and cognitive modeling under complex uncertainty. Gulistan et al. [14] extended the practical application of bipolar complex fuzzy sets to real-world domains, such as transportation management, highlighting their capacity to capture evolving positive and negative influences over time. Mahmood and Ur Rehman [15] introduced innovative approaches for incorporating bipolar CFSs into similarity measures, broadening their applicability to pattern recognition, system optimization, and multi-criteria decision analysis. Building upon this framework, bipolar complex intuitionistic fuzzy sets (BCIFSs) further enhance modeling flexibility by integrating both membership and non-membership as complex numbers. Al-Husban [16] formally introduced bipolar BCIFSs, describing their properties, while Alkouri and Alshboul [17] proposed alternative formulations, demonstrating their suitability for representing environmental impact data and supporting multi-criteria evaluation in sustainability-focused decision problems. These extensions offer a more versatile tool for managing uncertainties with dual characteristics compared to traditional intuitionistic fuzzy sets. Further innovations by Nandhinii and Amsaveni [18] expanded these concepts into bipolar complex intuitionistic fuzzy graphs, facilitating applications in network analysis, multi-agent systems, and dynamic decision-making contexts. The incorporation of Pythagorean constraints into this structure—forming bipolar complex Pythagorean fuzzy sets (BCPFSs)—introduces an even richer uncertainty representation by enforcing a squared sum relationship between membership and non-membership degrees. Nandhinii and Amsaveni [19] emphasized the potential of bipolar CPF graphs for mathematical analysis and computational intelligence applications. Overall, BCPFSs provide a powerful and flexible framework for modeling complex, dual-phase uncertainties in multi-criteria decision-making, pattern recognition, and other advanced computational intelligence scenarios, enabling more comprehensive and refined evaluation of real-world problems.

Multi-attribute decision-making (MADM) serves as a core methodology for tackling problems where alternatives must be evaluated across multiple, often conflicting criteria. Traditional MADM techniques, while foundational, face limitations when confronted with uncertain, imprecise, or dual-natured information common in real-world scenarios. To address these challenges, researchers have progressively extended classical MADM frameworks through the integration of fuzzy set theory, enabling more nuanced representations of uncertainty and preference. CFSs and BFSs, along with their hybrid formulations such as bipolar CFSs, have emerged as particularly effective tools, as they can simultaneously model positive and negative evaluations while accounting for magnitude and phase variations. Recent innovations have included the development of fuzzy Ostrowski integral inequalities for convex fuzzy-valued mappings [20], and fuzzy N-bipolar soft sets tailored for multi-criteria decision-making applications [21], providing advanced mathematical foundations for uncertainty modeling. On the aggregation side, operators such as complex T-spherical fuzzy aggregation [22] and Aczel-Alsina power-based aggregation [23] have been proposed to handle complex and high-dimensional data, including applications extending to quantum computing. Similarly, specialized operators like bipolar complex Maclaurin symmetric mean and Hamy mean operators [24, 25] have enhanced the precision and robustness of decision support systems by capturing intricate relationships among criteria. Building upon these advances, CIFSs, bipolar IFSs, and bipolar CIFSs allow even richer representations by incorporating both membership and non-membership degrees along with hesitation, supporting methodologies such as Aczel-Alsina aggregation [26], medical diagnostic decision models [27], and TOPSIS-based ranking systems [28]. Furthermore, CPFSs, and bipolar PFSs introduce higher-order flexibility, accommodating more expressive uncertainty in multi-attribute evaluations, with practical implementations demonstrated through Aczel-Alsina-based aggregation methods [29] and supplier selection scenarios [30].

1.1. **Motivation.** In contemporary decision-making scenarios, particularly within sustainability-driven domains, decision-makers often encounter highly complex environments characterized by multidimensional uncertainty, conflicting objectives, and both positive and negative evaluations. Conventional decision-making approaches, based on classical set theory and deterministic models, frequently fall short in capturing the nuanced interplay of hesitation, bipolar preferences, and interdependent or phase-dependent criteria. To address these gaps, extensions of fuzzy set theory—including intuitionistic, Pythagorean, complex, and bipolar fuzzy sets—have been developed to model uncertainty more comprehensively. While Pythagorean fuzzy sets allow for richer representation of hesitation, bipolar fuzzy sets incorporate dual-sided positive and negative assessments, and complex fuzzy sets introduce magnitude-phase components, relying on any single framework often remains insufficient to capture the full spectrum of dynamic, conflicting, and oscillatory information present in real-world decision problems.

In the context of green supply chain management, organizations face the critical task of selecting suppliers that balance economic efficiency, product quality, environmental responsibility, technological integration, and social impact. Classical approaches, including cost-benefit analysis, life-cycle assessment, and traditional multi-criteria decision-making methods such as analytic hierarchy process (AHP), technique for order preference by similarity to ideal solution (TOPSIS), and vlseKriterijumska optimizacija i kompromisno resenje (VIKOR), provide structured evaluation and analytical clarity. However, these techniques generally assume precise and deterministic input data, limiting their ability to capture uncertainty, stakeholder heterogeneity, and the complex trade-offs inherent in modern supply chains. Recent studies in sustainable supply chain and resource management have highlighted that conventional methods often fail to reflect the dynamic, multi-dimensional, and uncertain nature of supplier performance and sustainability criteria [31].

To overcome these limitations, bipolar complex Pythagorean fuzzy sets offer a unified framework that simultaneously integrates dual (positive-negative) information, hesitation, and complex-valued uncertainty. This framework is particularly suitable for MADM problems where supplier alternatives must be evaluated across cost, quality, environmental, technological, and social dimensions, and where assessments may include both supportive and opposing perspectives. The use of BCPFS-based aggregation operators—specifically BCPFWA and BCPFWG—facilitates a systematic evaluation of supplier alternatives, including cost-oriented, quality-driven, socially responsible, environmentally responsible, technologically advanced, and balanced sustainability suppliers. By incorporating uncertainty, bipolar evaluations, and interdependent criteria, the proposed methodology enhances decision reliability, improves ranking stability, and supports the robust selection of suppliers that optimize both sustainability and performance objectives. Consequently, this research bridges theoretical developments in fuzzy-based MADM with practical applications in sustainable supply chain management, offering a versatile and computationally implementable decision-support framework for organizations aiming to achieve long-term environmental, economic, and social goals.

1.2. **Significant Contributions.** This study presents several notable contributions to the advancement of MADM under uncertainty:

(1) A novel fuzzy framework, BCPFSs, is proposed by integrating the strengths of BPFSs and CPFSs. By incorporating real and complex membership and non-membership degrees along with bipolar and Pythagorean structures, BCPFSs provide an enhanced ability to capture both qualitative and quantitative uncertainties. The paper formalizes the foundational mathematical properties of BCPFSs and validates them through illustrative numerical examples, demonstrating robustness and interpretability in practical scenarios.
(2) Two new aggregation operators—the BCPFWA and BCPFWG operators—are developed. These operators effectively synthesize multi-criteria information while preserving the inherent bipolarity and complex characteristics of decision data, supporting nuanced evaluations.
(3) The proposed BCPFS-based MADM framework is applied to GSCM within a circular economy context, evaluating diverse supplier alternatives. Results show enhanced ranking stability, flexible handling of uncertainty, and more reliable decision-making for sustainable supplier selection and resource management.
(4) Comparative analyses with existing fuzzy MADM methods confirm the superiority of the proposed framework, highlighting improved accuracy, robustness, and the capacity to model complex interdependencies among multiple criteria in real-world decision environments.
(5) Graphical representations of the BCPFWA and BCPFWG operators illustrate their computational behavior, providing intuitive insights into their effectiveness in aggregating bipolar complex information for practical decision-making.
(6) By combining bipolarity, Pythagorean fuzziness, and complex membership structures, this study significantly enriches the theoretical foundations of fuzzy MADM and introduces a flexible tool capable of handling complex, uncertain, and conflicting decision problems.

(7) The framework establishes a foundation for future enhancements, including scalability to large-scale decision problems, extension to dynamic and time-varying decision contexts, integration with machine learning and optimization algorithms, and potential applications in areas such as financial risk analysis, healthcare decision-making, and engineering optimization.

In summary, this research positions BCPFSs and the associated aggregation operators as a comprehensive, flexible, and computationally implementable framework, substantially advancing MADM methodologies for complex, uncertain, and sustainability-oriented decision-making challenges.

1.3. **Paper organization.** The structure of this paper is organized as follows:

- Section 1 reviews the pertinent literature, identifying existing research gaps and motivating the development of the proposed framework.
- Section 2 provides the theoretical foundations, including detailed explanations of CIFSs, CPFSs, BIFSs, BPFSs, and the integrated BCPFSs.
- Section 3 formalizes the core definitions and mathematical properties of BCPFSs. It also establishes new operational laws for BCPF numbers (BCPFNs), laying the groundwork for the development of aggregation techniques.
- Section 4 introduces the newly proposed BCPFWA and BCPFWG aggregation operators, highlighting their theoretical properties and demonstrating their suitability for complex multi-criteria evaluations.
- Section 5 presents a structured MADM framework utilizing these operators and illustrates its practical applicability through a real-world GSCM case study.
- Section 6 offers a comparative analysis between the proposed methodology and existing fuzzy MADM approaches, emphasizing improvements in ranking reliability, computational robustness, and decision-making flexibility.
- Section 7 concludes the paper by summarizing the key contributions and suggesting potential directions for future research and applications.

2. Preliminaries

This section presents the essential groundwork for the research.

Definition 2.1. Let $\mathfrak{D}$ be a universal set. We define the set $\mathfrak{P}$ as follows:

$$\mathfrak{P} = \{\langle \varsigma, \mathfrak{m}\mathfrak{R}(\varsigma), \mathfrak{n}\mathfrak{R}(\varsigma)\rangle : \varsigma \in \mathfrak{D}\},$$

where $\mathfrak{m}\mathfrak{R} : \mathfrak{D} \to \mathfrak{Z}_1 : \mathfrak{Z}_1 \in \mathfrak{P}, |\mathfrak{Z}_1| \leq 1$ and
$\mathfrak{n}\mathfrak{R} : \mathfrak{D} \to \mathfrak{Z}_2 : \mathfrak{Z}_2 \in \mathfrak{P}, |\mathfrak{Z}_2| \leq 1$ satisfy the conditions:

$$\mathfrak{m}\mathfrak{R}(\varsigma) = \mathfrak{Z}_1 = h_1 + if_1, \quad \mathfrak{n}\mathfrak{R}(\varsigma) = \mathfrak{Z}_2 = h_2 + if_2,$$

These functions are represented in polar form as follows:

$$\mathfrak{m}\mathfrak{R}(\varsigma) = \mathfrak{m}\mathfrak{R}_{\mathfrak{P}}(\varsigma)e^{i2\pi\mathfrak{m}\mathfrak{I}_{\mathfrak{P}}(\varsigma)}, \text{ and } \mathfrak{n}\mathfrak{R}(\varsigma) = \mathfrak{n}\mathfrak{R}_{\mathfrak{P}}(\varsigma)e^{i2\pi\mathfrak{n}\mathfrak{I}_{\mathfrak{P}}(\varsigma)},$$

Here, $\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}_{\mathfrak{P}}, \mathfrak{m}\mathfrak{I}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{I}_{\mathfrak{P}} \in [0,1]$, with $i = \sqrt{-1}$. Then, $\mathfrak{P}$ is called a:

(1) CIFS [5] if $0 \leq \mathfrak{m}\mathfrak{R}_{\mathfrak{P}}(\varsigma) + \mathfrak{n}\mathfrak{R}_{\mathfrak{P}}(\varsigma) \leq 1$, and $0 \leq \mathfrak{m}\mathfrak{I}_{\mathfrak{P}}(\varsigma) + \mathfrak{n}\mathfrak{I}_{\mathfrak{P}}(\varsigma) \leq 1$.
(2) CPFS [6] if $0 \leq \mathfrak{m}\mathfrak{R}^2_{\mathfrak{P}}(\varsigma) + \mathfrak{n}\mathfrak{R}^2_{\mathfrak{P}}(\varsigma) \leq 1$, and $0 \leq \mathfrak{m}\mathfrak{I}^2_{\mathfrak{P}}(\varsigma) + \mathfrak{n}\mathfrak{I}^2_{\mathfrak{P}}(\varsigma) \leq 1$.

Definition 2.2. Consider a universal set $\mathfrak{D}$. We then define the corresponding set $\mathfrak{P}$ as:

$$\mathfrak{P} = \left\{ \left\langle \varsigma, \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}}(\varsigma), \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}}(\varsigma), \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}(\varsigma), \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}(\varsigma) \right\rangle : \varsigma \in \mathfrak{D} \right\}.$$

Then, $\mathfrak{P}$ is called a:

(1) BIFS [9] if $0 \leq \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}} + \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}} \leq 1$, and $-1 \leq \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}} + \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}} \leq 0$.
(2) BPFS [10,30] if $0 \leq (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}})^2 + (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}})^2 \leq 1$, and $0 \leq (\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}})^2 + (\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}})^2 \leq 1$.

Definition 2.3. Take $\mathfrak{D}$ to be a universal set. Then, $\mathfrak{P} = \left\{ \left\langle \varsigma, \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}}(\varsigma)e^{i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}(\varsigma)}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}}(\varsigma)e^{i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}(\varsigma)}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}(\varsigma)e^{i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}(\varsigma)}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}(\varsigma)e^{i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}(\varsigma)} \right\rangle : \varsigma \in \mathfrak{D} \right\}$ is considered a BCPFS [19] if:

$$0 \leq (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}}(\varsigma))^2 + (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}}(\varsigma))^2 \leq 1, \quad 0 \leq (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}(\varsigma))^2 + (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}(\varsigma))^2 \leq 2\pi,$$

and

$$-1 \leq -[(\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}(\varsigma))^2 + (\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}(\varsigma))^2] \leq 0, \quad 0 \leq (\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}(\varsigma))^2 + (\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}(\varsigma))^2 \leq 2\pi.$$

where

$$\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}}(\varsigma),\ \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}}(\varsigma) \in [0,1], \quad \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}(\varsigma),\ \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}(\varsigma) \in [-1,0],$$

and

$$\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}(\varsigma), \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}(\varsigma), \mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}(\varsigma),\ \mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}(\varsigma) \in [0,2\pi].$$

It is well known that the cartesian form of a complex number can be expressed as $\mathfrak{z}_1 = \mathfrak{m}\mathfrak{R} + i\mathfrak{m}\mathfrak{I}$ and $\mathfrak{z}_2 = \mathfrak{n}\mathfrak{R} + i\mathfrak{n}\mathfrak{I}$, where the real and imaginary parts represent amplitude-related components. Alternatively, in the polar form, the complex number $\mathfrak{z}_1$ is written as $\mathfrak{z}_1 = \alpha_1 e^{i\beta_1}$, where $\alpha_1 = |\mathfrak{z}_1| = \sqrt{\mathfrak{m}\mathfrak{R}^2 + \mathfrak{m}\mathfrak{I}^2}$ denotes the modulus (amplitude) and $\beta_1 = \tan^{-1}\left(\frac{\mathfrak{m}\mathfrak{I}}{\mathfrak{m}\mathfrak{R}}\right)$ is the phase angle. Similarly, for $\mathfrak{z}_2$, the polar form is given by $\mathfrak{z}_2 = \alpha_2 e^{i\beta_2}$, where $\alpha_2 = |\mathfrak{z}_2| = \sqrt{\mathfrak{n}\mathfrak{R}^2 + \mathfrak{n}\mathfrak{I}^2}$ and $\beta_2 = \tan^{-1}\left(\frac{\mathfrak{n}\mathfrak{I}}{\mathfrak{n}\mathfrak{R}}\right)$. From these representations, it is evident that both α_1 and α_2 are non-negative real numbers, that is, $\alpha_1, \alpha_2 \geq 0$, as the modulus of a complex number cannot be negative. However, in the definition of BCPFS introduced in [19], the amplitude terms associated with the negative membership and non-membership degrees, namely $\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}$ and $\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}$, are allowed to take values within the closed interval $[-1,0]$. This creates a contradiction with the standard polar form, where amplitudes must be non-negative. To resolve this inconsistency, we propose a revised definition for BCPFS, as outlined in the following section.

3. Bipolar Complex Pythagorean Fuzzy Sets

This section provides a detailed overview of the fundamental concepts and essential operations associated with bipolar complex PFS.

Definition 3.1. Within a given universe of discourse $\mathfrak{D}$, a BCPFS, denoted by $\mathfrak{P}$, is formally defined as:

$$\mathfrak{P} = \Big\{\Big\langle \varsigma, \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}}(\varsigma) + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}(\varsigma), \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}}(\varsigma) + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}(\varsigma),$$
$$\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}(\varsigma) + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}(\varsigma), \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}(\varsigma) + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}(\varsigma)\Big\rangle : \varsigma \in \mathfrak{D}\Big\}$$

where

$$\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}}(\varsigma),\ \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}}(\varsigma) \in [0,1] \quad \text{(real positive components)},$$
$$\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}(\varsigma),\ \mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}(\varsigma) \in [0,1] \quad \text{(imaginary positive components)},$$

with

$$0 \le (\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}}(\varsigma))^2 + (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}}(\varsigma))^2 \le 1, \quad 0 \le (\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}(\varsigma))^2 + (\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}(\varsigma))^2 \le 1;$$

and

$$\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}(\varsigma),\ \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}(\varsigma) \in [-1,0] \quad \text{(real negative components)},$$
$$\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}(\varsigma),\ \mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}(\varsigma) \in [-1,0] \quad \text{(imaginary negative components)},$$

with

$$0 \le |\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}(\varsigma)|^2 + |\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}(\varsigma)|^2 \le 1, \quad 0 \le |\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}(\varsigma)|^2 + |\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}(\varsigma)|^2 \le 1.$$

The value of ς is evaluated as

$$\mathfrak{P} = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}\rangle$$

which defines the BCPF number.

Figure 1 depicts the graded space of bipolar complex Pythagorean fuzzy values, emphasizing the interplay between real and imaginary components for both positive and negative dimensions. The illustration is structured to reveal the distinct characteristics of these components within the bipolar complex domain, thereby facilitating a clearer understanding of their interrelationships and inherent constraints.

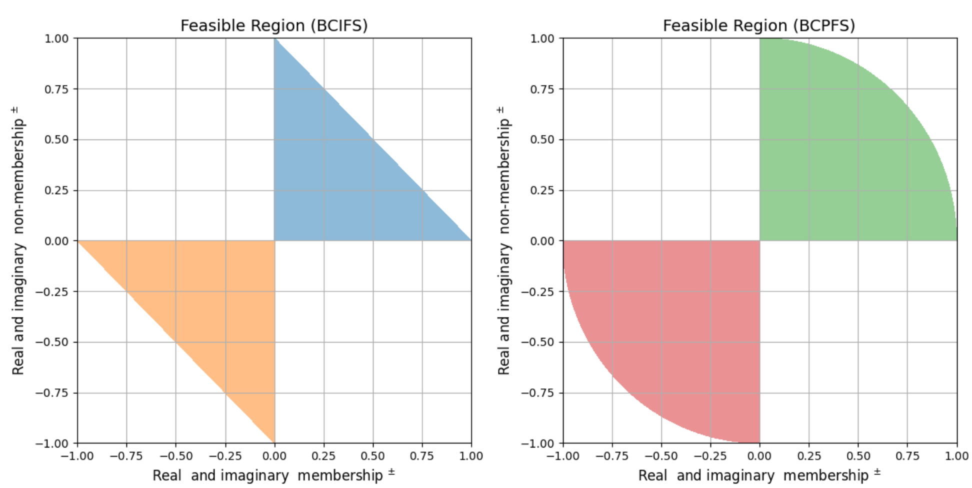

Figure 1. Graded space illustrating the feasible regions of BCIF [17] and BCPF values.

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Definition 3.2. Given two BCPFNs, define them as:
$\mathfrak{P}_1 = \left\langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1} \right\rangle$ and
$\mathfrak{P}_2 = \left\langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2} \right\rangle$. Then:

(1) $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$ if and only if
$\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1} \leq \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}, \quad \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1} \geq \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}, \quad \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1} \geq \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}, \quad \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1} \leq \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}$
for real terms
and $\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1} \leq \mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}, \quad \mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1} \geq \mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}, \quad \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1} \geq \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}, \quad \mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1} \leq \mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}$
for imaginary terms.

(2) $\mathfrak{P}_1 = \mathfrak{P}_2$ if and only if
$\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1} = \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}, \quad \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1} = \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}, \quad \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1} = \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}, \quad \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1} = \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2},$
$\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1} = \mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}, \quad \mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1} = \mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}, \quad \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1} = \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}, \quad \mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1} = \mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}.$

(3) $\mathfrak{P}_1{}^c = \langle \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}, -|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}| + i(-|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|), -|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}| + i(-|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}|) \rangle.$

(4) $$\mathfrak{P}_1 \cap \mathfrak{P}_2 = \begin{array}{c} \langle \min\{\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\} + i\min\{\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}\}, \\ \max\{\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\} + i\max\{\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}\}, \\ \max\{\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\} + i\max\{\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}\}, \\ \min\{\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}\} + i\min\{\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}\}\rangle. \end{array}$$

(5) $$\mathfrak{P}_1 \cup \mathfrak{P}_2 = \begin{array}{c} \langle \max\{\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\} + i\max\{\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}\}, \\ \min\{\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\} + i\min\{\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}\}, \\ \min\{\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\} + i\min\{\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}\}, \\ \max\{\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}\} + i\max\{\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}\}\rangle. \end{array}$$

Example 3.1. Let the set be defined as $\mathfrak{D} = \{\varsigma_1, \varsigma_2, \varsigma_3\}$. For the sake of conciseness and to ensure uniformity in numerical notation, all decimal values within the open interval $(-1, 1)$ are represented without a leading zero. Accordingly, the set $\mathfrak{P}_1$ can be expressed as:
$$\mathfrak{P}_1 = \left\{ \begin{array}{c} \langle \varsigma_1, .7 + (.1)i, .6 + (.1)i, -.1 + (-.1)i, -.6 + (-.1)i \rangle, \\ \langle \varsigma_2, .3 + (.1)i, .5 + (.1)i, -.1 + (-.1)i, -.1 + (-.2)i \rangle, \\ \langle \varsigma_3, .2 + (.2)i, .7 + (.1)i, -.5 + (-.1)i, -.4 + (-.1)i \rangle \end{array} \right\}.$$
Likewise, $\mathfrak{P}_2$ can be expressed as:
$$\mathfrak{P}_2 = \left\{ \begin{array}{c} \langle \varsigma_1, .6 + (.1)i, .7 + (.1)i, -.6 + (-.2)i, -.1 + (-.4)i \rangle, \\ \langle \varsigma_2, .5 + (.1)i, .3 + (.1)i, -.1 + (-.4)i, -.1 + (-.2)i \rangle, \\ \langle \varsigma_3, .5 + (.1)i, .4 + (.1)i, -.7 + (-.2)i, -.2 + (-.4)i \rangle \end{array} \right\}.$$
Since both $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are BCPFNs, it follows that

(1) $$\mathfrak{P}_1^c = \left\{ \begin{array}{c} \langle \varsigma_1, .6 + (.1)i, .7 + (.1)i, -.6 + (-.1)i, -.1 + (-.1)i \rangle, \\ \langle \varsigma_2, .5 + (.1)i, .3 + (.1)i, -.1 + (-.2)i, -.1 + (-.1)i \rangle, \\ \langle \varsigma_3, .7 + (.1)i, .2 + (.2)i, -.4 + (-.1)i, -.5 + (-.1)i \rangle \end{array} \right\}.$$

(2) $$\mathfrak{P}_1 \cap \mathfrak{P}_2 = \left\{ \begin{array}{c} \langle \varsigma_1, .6 + (.1)i, .7 + (.1)i, -.1 + (-.1)i, -.6 + (-.4)i \rangle \\ \langle \varsigma_2, .3 + (.1)i, .5 + (.1)i, -.1 + (-.1)i, -.1 + (-.2)i \rangle \\ \langle \varsigma_3, .2 + (.1)i, .7 + (.1)i, -.5 + (-.1)i, -.4 + (-.4)i \rangle \end{array} \right\}.$$

(3) $\mathfrak{P}_1 \cup \mathfrak{P}_2 = \left\{ \begin{array}{l} \langle \varsigma_1, .7 + (.1)i, .6 + (.1)i, -.6 + (-.2)i, -.1 + (-.1)i \rangle \\ \langle \varsigma_2, .5 + (.1)i, .3 + (.1)i, -.1 + (-.4)i, -.1 + (-.2)i \rangle \\ \langle \varsigma_3, .5 + (.2)i, .4 + (.1)i, -.7 + (-.2)i, -.2 + (-.1)i \rangle \end{array} \right\}.$

Theorem 3.1. Consider a BCPFN
$\mathfrak{P} = \left\langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}} \right\rangle$.
In a similar fashion, define the BCPFNs
$\mathfrak{P}_1 = \left\langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1} \right\rangle$ and
$\mathfrak{P}_2 = \left\langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2} \right\rangle$. Thus,

(1) $\mathfrak{P}^c$ is a BCPFN, and taking the complement twice gives $(\mathfrak{P}^c)^c = \mathfrak{P}$.
(2) Both $\mathfrak{P}_1 \cup \mathfrak{P}_2$ and $\mathfrak{P}_1 \cap \mathfrak{P}_2$ are also BCPFN.

Proof. (1) Since

$$0 \le (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}})^2 + (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}})^2 \le 1,$$
$$0 \le (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}})^2 + (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}})^2 \le 1,$$
$$0 \le \left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}\right|^2 + \left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}\right|^2 \le 1,$$
and
$$0 \le |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}|^2 \le 1$$

then

$$0 \le (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}})^2 + (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}})^2 = (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}})^2 + (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}})^2 \le 1,$$
$$0 \le (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}})^2 + (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}})^2 = (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}})^2 + (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}})^2 \le 1,$$
$$0 \le \left|-\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}\right|\right|^2 + \left|-\left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}\right|\right|^2 = \left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}\right|^2 + \left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}\right|^2 \le 1,$$
and
$$0 \le \left|-|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}|\right|^2 + \left|-|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}|\right|^2 = |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}|^2 \le 1.$$

Hence, $\mathfrak{P}^c$ is a BCPFN, and it is evident that
$$(\mathfrak{P}^c)^c = \langle \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}, -|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}| + i(-|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}|), -|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}| + i(-|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}|)\rangle^c$$
$$= \langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}, -|-|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}|| + i(-|-|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}||), -|-|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}|| + i(-|-|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}||)\rangle$$
$$= \langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}\rangle,$$

where

$$\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}} = -\left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}\right|, \quad \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}} = -\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}\right|,$$
$$\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}} = -\left|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}\right|, \text{and} \quad \mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}} = -\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}\right|.$$

(2) As the following inequalities are valid:

$$0 \le \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}, \mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2} \le 1,$$

and

$$0 \le |\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}|, |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}|, |\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}|, |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}|, |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|, |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|, |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|, |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}| \le 1,$$

it is clear that the following conditions are satisfied:

$$0 \leq \left(\max\left\{\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2}\right\}\right)^{q} + \left(\min\left\{\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2}\right\}\right)^{q} \leq 1,$$

$$0 \leq \left(\max\left\{\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2}\right\}\right)^{q} + \left(\min\left\{\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2}\right\}\right)^{q} \leq 1,$$

$$0 \leq \left(\min\left\{|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1}|, |\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2}|\right\}\right)^{q} + \left(\max\left\{|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}|, |\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}|\right\}\right)^{q} \leq 1,$$

and

$$0 \leq \left(\max\left\{|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|, |\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}|\right\}\right)^{q} + \left(\min\left\{|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}|, |\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}|\right\}\right)^{q} \leq 1.$$

Hence, $\mathfrak{P}_1 \cup \mathfrak{P}_2$ is a BCPFN. A similar argument can be employed to demonstrate that $\mathfrak{P}_1 \cap \mathfrak{P}_2$ is likewise a BCPFN.

□

Theorem 3.2. Let $\mathfrak{P} = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}\rangle$, $\mathfrak{P}_1 = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}\rangle$, and $\mathfrak{P}_2 = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}\rangle$ be BCPFNs. Then:

(1) $\mathfrak{P}_2 \cap \mathfrak{P}_1 = \mathfrak{P}_1 \cap \mathfrak{P}_2$.
(2) $\mathfrak{P}_2 \cup \mathfrak{P}_1 = \mathfrak{P}_1 \cup \mathfrak{P}_2$.
(3) $\mathfrak{P}_2 = (\mathfrak{P}_1 \cup \mathfrak{P}_2) \cap \mathfrak{P}_2$.
(4) $\mathfrak{P}_2 = (\mathfrak{P}_1 \cap \mathfrak{P}_2) \cup \mathfrak{P}_2$.
(5) $(\mathfrak{P} \cap \mathfrak{P}_1) \cap \mathfrak{P}_2 = \mathfrak{P} \cap (\mathfrak{P}_1 \cap \mathfrak{P}_2)$.
(6) $(\mathfrak{P} \cup \mathfrak{P}_1) \cup \mathfrak{P}_2 = \mathfrak{P} \cup (\mathfrak{P}_1 \cup \mathfrak{P}_2)$.
(7) $(\mathfrak{P} \cup \mathfrak{P}_2) \cap (\mathfrak{P}_1 \cup \mathfrak{P}_2) = (\mathfrak{P} \cap \mathfrak{P}_1) \cup \mathfrak{P}_2$.
(8) $(\mathfrak{P} \cap \mathfrak{P}_2) \cup (\mathfrak{P}_1 \cap \mathfrak{P}_2) = (\mathfrak{P} \cup \mathfrak{P}_1) \cap \mathfrak{P}_2$.

Proof. As these statements are clear, the corresponding results are readily observed. □

Theorem 3.3. Let $\mathfrak{P}_1 = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}\rangle$ and $\mathfrak{P}_2 = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}\rangle$ be BCPFNs. Then,

(1) ${\mathfrak{P}_1}^{c} \cup {\mathfrak{P}_2}^{c} = (\mathfrak{P}_1 \cap \mathfrak{P}_2)^{c}$.
(2) ${\mathfrak{P}_1}^{c} \cap {\mathfrak{P}_2}^{c} = (\mathfrak{P}_1 \cup \mathfrak{P}_2)^{c}$.

Proof. (1)
$$(\mathfrak{P}_1 \cap \mathfrak{P}_2)^{c} = \left\{ \begin{array}{c} \langle\{\min\left\{\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2}\right\} + i\min\left\{\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2}\right\}, \\ \max\left\{\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2}\right\} + i\max\left\{\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2}\right\}, \\ \max\left\{\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2}\right\} + i\max\left\{\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}\right\}, \\ \min\left\{\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}\right\} + i\min\left\{\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}\right\}\}\rangle^{c} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \langle\max\{\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2}\} + i\max\{\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2}\}, \\ \min\{\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2}\} + i\min\{\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2}\}, \\ \min\{-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}|, -|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}|\} + i\min\{-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|, -|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}|\}, \\ \max\{-|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1}|, -|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2}|\} + i\max\{-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}|, -|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}|\}\rangle \end{array} \right\}$$

$$= \langle(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1}) + i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}), (\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1}) + i(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1}), -|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}| + i(-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|), -|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1}| + i(-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}|)\rangle$$

$$\cup$$

$$\langle(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2})+i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}),(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})+i(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}),-|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|+i(-|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|),-|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}|+i(-|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}|)\rangle$$

$$=\mathfrak{P}_1{}^c\cup\mathfrak{P}_2{}^c.$$

(2) A similar argument to (1) can be applied to prove this.

□

Definition 3.3. Let $\mathfrak{P}$ be defined as $\langle\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}}+i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}},\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}}+i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}},\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}+i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}},\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}+i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}\rangle$, $\mathfrak{P}_1$ as $\langle\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}+i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1},\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}+i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1},\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}+i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1},\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}+i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}\rangle$, and $\mathfrak{P}_2$ as $\langle\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}+i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2},\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}+i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2},\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}+i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2},\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}+i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}\rangle$, where $\mathfrak{P}$, $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are BCPFNs, and let $\mathscr{I}$ be a positive real number with $\mathscr{I}>0$. Then,

(1)

$$\begin{gathered}\mathfrak{P}_1\oplus\mathfrak{P}_2=\\
\left\langle\left(\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}\right.\\
+i\left(\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}},\\
\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}+i\left(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}\right),-\left(\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\right)+i\left(-\left(\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}\right)\right),\\
-\left(\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}\right|^2+\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}\right|^2-\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}\right|^2\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}\\
\left.+i\left(-\left(\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}\right|^2+\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}\right|^2-\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}\right|^2\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}\right)\right\rangle.\end{gathered}$$

(2)

$$\begin{gathered}\mathfrak{P}_1\otimes\mathfrak{P}_2=\left\langle\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)+i\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}\right),\right.\\
\left(\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}\\
+i\left(\left(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}},\\
-\left(\left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\right|^2+\left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\right|^2-\left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\right|^2\left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}\\
+i\left(-\left(\left|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}\right|^2+\left|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}\right|^2-\left|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}\right|^2\left|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}\right),\\
\left.-\left(\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}\right)+i\left(-\left(\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}\right)\right)\right\rangle.\end{gathered}$$

(3)

$$\begin{gathered}\mathscr{I}\mathfrak{P}=\left\langle\left(1-\left(1-\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}}\right)^2\right)^{\mathscr{I}}\right)^{\frac{1}{2}}+i\left(1-\left(1-\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}\right)^2\right)^{\mathscr{I}}\right)^{\frac{1}{2}},\right.\\
\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}}\right)^{\mathscr{I}}+i\left(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}\right)^{\mathscr{I}},\\
-\left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}\right|^{\mathscr{I}}+i\left(-\left|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}\right|^{\mathscr{I}}\right),\end{gathered}$$

$$-\left(1-\left(1-\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}\right|^2\right)^{\mathscr{I}}\right)^{\frac{1}{2}}+i\left(-\left(1-\left(1-\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}\right|^2\right)^{\mathscr{I}}\right)^{\frac{1}{2}}\right)\Bigg\rangle.$$

(4)

$$\mathfrak{P}^{\mathscr{I}}=\Bigg\langle\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}}\right)^{\mathscr{I}}+i\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}\right)^{\mathscr{I}},$$

$$\left(1-\left(1-\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}}\right)^2\right)^{\mathscr{I}}\right)^{\frac{1}{2}}+i\left(1-\left(1-\left(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}\right)^2\right)^{\mathscr{I}}\right)^{\frac{1}{2}},$$

$$-\left(1-\left(1-\left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}\right|^2\right)^{\mathscr{I}}\right)^{\frac{1}{2}}+i\left(-\left(1-\left(1-\left|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}\right|^2\right)^{\mathscr{I}}\right)^{\frac{1}{2}}\right),$$

$$-\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}\right|^{\mathscr{I}}+i\left(-\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}\right|^{\mathscr{I}}\right)\Bigg\rangle.$$

Example 3.2. Two BCPFNs are defined as follows:

$$\mathfrak{P}_1=\langle .01+i(.03), .03+i(.02), -.04+i(-.01), -.03+i(-.02)\rangle$$

and

$$\mathfrak{P}_2=\langle .06+i(.01), .04+i(.03), -.01+i(-.02), -.02+i(-.01)\rangle.$$

Assuming $\mathscr{I}=3$, the subsequent results follow:

(1) $\mathfrak{P}_1\oplus\mathfrak{P}_2=$

$$\Bigg\langle\left(\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}$$

$$+i\left(\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}},$$

$$\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}+i\left(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}\right),-\left(\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\right)+i\left(-\left(\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}\right)\right),$$

$$-\left(\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}\right|^2+\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}\right|^2-\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}\right|^2\left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}$$

$$+i\left(-\left(\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}\right|^2+\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}\right|^2-\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}\right|^2\left|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}\right)\Bigg\rangle$$

$$=\Big\langle((.01)^6+(.06)^6-(.01)^6(.06)^6)^{\frac{1}{6}}+i((.03)^6+(.01)^6-(.03)^6(.01)^6)^{\frac{1}{6}},$$

$$(.03)(.04)+i((.02)(.03)),-((-.04)(-.01))+i(-((-.01)(-.02))),$$

$$-(|-.03|^6+|-.02|^6-|-.03|^6|-.02|^6)^{\frac{1}{6}}$$

$$+i(-(|-.02|^6+|-.01|^6-|-.02|^6|-.01|^6)^{\frac{1}{6}}\Big\rangle$$

$$\approx\langle .0608+.0316i, .0012+.0006i, -.0004+(-.0002)i, -.0361+(-.0224)i\rangle.$$

(2) $\mathfrak{P}_1 \otimes \mathfrak{P}_2 =$

$$\Big\langle \Big(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\Big) + i\Big(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}\Big),$$
$$\Big(\Big(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\Big)^2 + \Big(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\Big)^2 - \Big(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\Big)^2\Big(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\Big)^2\Big)^{\frac{1}{2}}$$
$$+i\Big(\Big(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\Big)^2 + \Big(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}\Big)^2 - \Big(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\Big)^2\Big(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}\Big)^2\Big)^{\frac{1}{2}},$$
$$-\Big(\Big|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\Big|^2 + \Big|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\Big|^2 - \Big|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\Big|^2\Big|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\Big|^2\Big)^{\frac{1}{2}}$$
$$+i\Big(-\Big(\Big|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}\Big|^2 + \Big|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}\Big|^2 - \Big|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}\Big|^2\Big|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}\Big|^2\Big)^{\frac{1}{2}}\Big),$$
$$-\Big(\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}\Big) + i\Big(-\Big(\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}\Big)\Big)\Big\rangle$$
$$= \langle((.01)(.06)) + i((.03)(.01)),$$
$$((.03)^6 + (.04)^6 - ((.03)^6(.04)^6))^{\frac{1}{6}} + i((.02)^6 + (.03)^6 - ((.02)^6(.03)^6))^{\frac{1}{6}},$$
$$-(|-.04|^6 + |-.01|^6 - |-.04|^6|-.01|^6)^{\frac{1}{6}}$$
$$+i(-(|-.01|^6 + |-.02|^6 - |-.01|^6|-.02|^6)^{\frac{1}{6}}),$$
$$-((-.03)(-.02)) + i(-((-.02)(-.01)))\rangle$$
$$\approx \langle .0006 + .0003i, .0500 + .0361i, -.0412 + (-.0224)i, -.0006 + (-.0002)i\rangle.$$

(3) $3\mathfrak{P}_1 =$

$$\Big\langle(1-(1-(.01)^6)^3)^{\frac{1}{6}} + i(1-(1-(.03)^6)^3)^{\frac{1}{6}},$$
$$(.03)^3 + i(.02)^3,$$
$$-|-.04|^3 + i(-|-.01|^3),$$
$$-(1-(1-|-.03|^6)^3)^{\frac{1}{6}} + i(-(1-(1-|-.02|^6)^3)^{\frac{1}{6}})\Big\rangle$$
$$\approx \langle .0173 + .0519i, .0000 + .0000i, -.0000 + (-.0000)i, -.0519 + (-.0346)i\rangle.$$

(4) $\mathfrak{P}_1^3 =$

$$\Big\langle(.01)^3 + i(.03)^3,$$
$$(1-(1-(.03)^6)^3)^{\frac{1}{6}} + i(1-(1-(.02)^6)^3)^{\frac{1}{6}},$$
$$-(1-(1-|-.04|^6)^3)^{\frac{1}{6}} + i(-(1-(1-|-.01|^6)^3)^{\frac{1}{6}}),$$
$$-|-.03|^3 + i(-|-.02|^3)\Big\rangle$$
$$\approx \langle .0000 + .0000i, .0519 + .0346i, -.0692 + (-.0173)i, -.0000 + (-.0000)i\rangle.$$

Theorem 3.4. Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ denote two BCPFNs, given by $\mathfrak{P}_1 = \Big\langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}\Big\rangle$ and $\mathfrak{P}_2 = \Big\langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}\Big\rangle$. As a result, both $\mathfrak{P}_1 \oplus \mathfrak{P}_2$ and $\mathfrak{P}_1 \otimes \mathfrak{P}_2$ are also BCPFNs.

Proof. For $\mathfrak{P}_1$ and $\mathfrak{P}_2$, the following inequalities are satisfied:

$$0 \le \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 \le 1, \quad 0 \le \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 \le 1, \quad 0 \le \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 + \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 \le 1,$$
$$0 \le \left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\right|^2 \le 1, \quad 0 \le \left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}\right|^2 \le 1, \quad 0 \le \left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\right|^2 + \left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}\right|^2 \le 1,$$
$$0 \le \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \le 1, \quad 0 \le \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \le 1, \quad 0 \le \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 + \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \le 1,$$
$$0 \le \left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\right|^2 \le 1, \quad 0 \le \left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}\right|^2 \le 1, \quad \text{and} \quad 0 \le \left|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\right|^2 + \left|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}\right|^2 \le 1.$$

In addition, given that:

$$0 \le \mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1} \le 1, \quad 0 \le \mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2} \le 1, \quad 0 \le \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1} \le 1, \quad 0 \le \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2} \le 1,$$
$$0 \le (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2 + (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1})^2 \le 1, \quad 0 \le (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2 + (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2})^2 \le 1,$$
$$0 \le |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}| \le 1, \quad 0 \le |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}| \le 1, \quad 0 \le |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}| \le 1, \quad 0 \le |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}| \le 1,$$
$$0 \le |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2 \le 1, \quad \text{and} \quad 0 \le |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2 \le 1.$$

Consequently, we arrive at the following inequalities:

$$\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 \ge \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2, \quad \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \ge \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2,$$
$$1 \ge \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \ge 0, \quad \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 \ge \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2,$$
$$\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \ge \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2, \quad \text{and} \quad 1 \ge \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \ge .$$

It follows from these relationships that:

$$\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 + \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 - \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \ge 0$$

which further leads to:

$$\left(\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 + \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 - \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}} \ge .$$

Similarly, we have:

$$\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 + \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 - \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \ge 0,$$

which implies:

$$\left(\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 + \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 - \left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}} \ge .$$

Given that

$$\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \le 1 \quad \text{and} \quad 0 \le 1 - \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2,$$

we obtain the inequality

$$\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2\left(1 - \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\right) \le 1 - \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2.$$

From this, it follows that

$$\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2 + \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 - \left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}\right)^2 \le 1.$$

Hence

$$\left(\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}\leq 1.$$

ollowing the same approach, we arrive at an equivalent bound:

$$\left(\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}\right)^2\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}\leq 1.$$

Furthermore, it is evident that

$$0\leq\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}\right)^2\leq 1-\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2,\quad 0\leq\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\right)^2\leq 1-\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2.$$

Using these bounds, we establish the inequality

$$\begin{aligned}&\left(\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}+\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\right)^2\\ \leq&\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2+\left(1-\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2\right)\left(1-\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2\right)=1.\end{aligned}$$

Thus, we conclude that

$$0\leq\left(\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}\leq 1,$$

$$0\leq\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\leq 1,$$

and

$$0\leq\left(\left(\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\right)^2\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}\right)^2+\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\right)^2\leq 1.$$

Analogously, we have the following inequalities:

(1)

$$\begin{aligned}&0\leq\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\leq 1,\\ &0\leq\left(\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}\right)^2\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}\leq 1,\\ &0\leq\left(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}\mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\right)^2+\left(\left(\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}\right)^2+\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\right)^2-\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}\right)^2\left(\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}\right)^2\leq 1.\end{aligned}$$

(2)

$$\begin{aligned}&-1\leq-\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}\mathfrak{m}\Re^{-}_{\mathfrak{P}_2}\leq 0,\\ &-1\leq-\left(\left|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}\right|^2+\left|\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}\right|^2-\left|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}\right|^2\left|\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}\leq 0,\\ &0\leq\left|-\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}\mathfrak{m}\Re^{-}_{\mathfrak{P}_2}\right|^2+\left|-\left(\left|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}\right|^2+\left|\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}\right|^2-\left|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}\right|^2\left|\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}\right|^2\leq 1.\end{aligned}$$

(3)

$$-1 \leq -\left(\left|\mathfrak{mR}^{-}_{\mathfrak{P}_1}\right|^2 + \left|\mathfrak{mR}^{-}_{\mathfrak{P}_2}\right|^2 - \left|\mathfrak{mR}^{-}_{\mathfrak{P}_1}\right|^2 \left|\mathfrak{mR}^{-}_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}} \leq 0,$$

$$-1 \leq -\mathfrak{nR}^{-}_{\mathfrak{P}_1}\mathfrak{nR}^{-}_{\mathfrak{P}_2} \leq 0,$$

$$0 \leq \left|-\left(\left|\mathfrak{mR}^{-}_{\mathfrak{P}_1}\right|^2 + \left|\mathfrak{mR}^{-}_{\mathfrak{P}_2}\right|^2 - \left|\mathfrak{mR}^{-}_{\mathfrak{P}_1}\right|^2 \left|\mathfrak{mR}^{-}_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}\right|^2 + \left|-\mathfrak{nR}^{-}_{\mathfrak{P}_1}\mathfrak{nR}^{-}_{\mathfrak{P}_2}\right|^2 \leq 1.$$

By the same method, we derive:

- (i)

$$0 \leq \left(\left(\mathfrak{mI}^{+}_{\mathfrak{P}_1}\right)^2 + \left(\mathfrak{mI}^{+}_{\mathfrak{P}_2}\right)^2 - \left(\mathfrak{mI}^{+}_{\mathfrak{P}_1}\right)^2 \left(\mathfrak{mI}^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}} \leq 1, \quad 0 \leq \left(\mathfrak{nI}^{+}_{\mathfrak{P}_1}\right)\left(\mathfrak{nI}^{+}_{\mathfrak{P}_2}\right) \leq 1,$$

  and

$$0 \leq \left(\left(\left(\mathfrak{mI}^{+}_{\mathfrak{P}_1}\right)^2 + \left(\mathfrak{mI}^{+}_{\mathfrak{P}_2}\right)^2 - \left(\mathfrak{mI}^{+}_{\mathfrak{P}_1}\right)^2 \left(\mathfrak{mI}^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}\right)^2 + \left(\left(\mathfrak{nI}^{+}_{\mathfrak{P}_1}\right)\left(\mathfrak{nI}^{+}_{\mathfrak{P}_2}\right)\right)^2 \leq 1.$$

- (ii)

$$0 \leq \left|-\left(\left|\mathfrak{nI}^{-}_{\mathfrak{P}_1}\right|^2 + \left|\mathfrak{nI}^{-}_{\mathfrak{P}_2}\right|^2 - \left|\mathfrak{nI}^{-}_{\mathfrak{P}_1}\right|^2 \left|\mathfrak{nI}^{-}_{\mathfrak{P}_2}\right|^2\right)\right|^{\frac{1}{2}} \leq 1, \quad 0 \leq \left|\mathfrak{mI}^{-}_{\mathfrak{P}_1}\right|\left|\mathfrak{mI}^{-}_{\mathfrak{P}_2}\right| \leq 1,$$

  and

$$0 \leq \left|-\left(\left|\mathfrak{nI}^{-}_{\mathfrak{P}_1}\right|^2 + \left|\mathfrak{nI}^{-}_{\mathfrak{P}_2}\right|^2 - \left|\mathfrak{nI}^{-}_{\mathfrak{P}_1}\right|^2 \left|\mathfrak{nI}^{-}_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}\right|^2 + \left(\left|\mathfrak{mI}^{-}_{\mathfrak{P}_1}\right|\left|\mathfrak{mI}^{-}_{\mathfrak{P}_2}\right|\right)^2 \leq 1.$$

- (iii)

$$0 \leq \left|-\left(\left|\mathfrak{mI}^{-}_{\mathfrak{P}_1}\right|^2 + \left|\mathfrak{mI}^{-}_{\mathfrak{P}_2}\right|^2 - \left|\mathfrak{mI}^{-}_{\mathfrak{P}_1}\right|^2 \left|\mathfrak{mI}^{-}_{\mathfrak{P}_2}\right|^2\right)\right|^{\frac{1}{2}} \leq 1, \quad 0 \leq \left|\mathfrak{nI}^{-}_{\mathfrak{P}_1}\right|\left|\mathfrak{nI}^{-}_{\mathfrak{P}_2}\right| \leq 1,$$

  and

$$0 \leq \left|-\left(\left|\mathfrak{mI}^{-}_{\mathfrak{P}_1}\right|^2 + \left|\mathfrak{mI}^{-}_{\mathfrak{P}_2}\right|^2 - \left|\mathfrak{mI}^{-}_{\mathfrak{P}_1}\right|^2 \left|\mathfrak{mI}^{-}_{\mathfrak{P}_2}\right|^2\right)^{\frac{1}{2}}\right|^2 + \left(\left|\mathfrak{nI}^{-}_{\mathfrak{P}_1}\right|\left|\mathfrak{nI}^{-}_{\mathfrak{P}_2}\right|\right)^2 \leq 1.$$

- (iv)

$$0 \leq \left(\left(\mathfrak{nI}^{+}_{\mathfrak{P}_1}\right)^2 + \left(\mathfrak{nI}^{+}_{\mathfrak{P}_2}\right)^2 - \left(\mathfrak{nI}^{+}_{\mathfrak{P}_1}\right)^2 \left(\mathfrak{nI}^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}} \leq 1, \quad 0 \leq \left(\mathfrak{mI}^{+}_{\mathfrak{P}_1}\right)\left(\mathfrak{mI}^{+}_{\mathfrak{P}_2}\right) \leq 1,$$

  and

$$0 \leq \left(\left(\left(\mathfrak{nI}^{+}_{\mathfrak{P}_1}\right)^2 + \left(\mathfrak{nI}^{+}_{\mathfrak{P}_2}\right)^2 - \left(\mathfrak{nI}^{+}_{\mathfrak{P}_1}\right)^2 \left(\mathfrak{nI}^{+}_{\mathfrak{P}_2}\right)^2\right)^{\frac{1}{2}}\right)^2 + \left(\left(\mathfrak{mI}^{+}_{\mathfrak{P}_1}\right)\left(\mathfrak{mI}^{+}_{\mathfrak{P}_2}\right)\right)^2 \leq 1.$$

Hence, both $\mathfrak{P}_1 \oplus \mathfrak{P}_2$ and $\mathfrak{P}_1 \otimes \mathfrak{P}_2$ fulfill the conditions required to be BCPFNs.

□

Theorem 3.5. Let $\mathfrak{P} = \left\langle \mathfrak{mR}^{+}_{\mathfrak{P}} + i\,\mathfrak{mI}^{+}_{\mathfrak{P}}, \quad \mathfrak{mR}^{-}_{\mathfrak{P}} + i\,\mathfrak{mI}^{-}_{\mathfrak{P}}, \quad \mathfrak{nR}^{+}_{\mathfrak{P}} + i\,\mathfrak{nI}^{+}_{\mathfrak{P}}, \quad \mathfrak{nR}^{-}_{\mathfrak{P}} + i\,\mathfrak{nI}^{-}_{\mathfrak{P}} \right\rangle$ be a BCPFN, and $\mathscr{I} > 0$. Then, both $\mathscr{I}\mathfrak{P}$ and $\mathfrak{P}^{\mathscr{I}}$ are also BCPFNs.

Proof. Let us first note the following inequalities:

$$0 \leq \left(\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{2} \leq 1, \quad 0 \leq \left(\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{2} \leq 1, \quad 0 \leq \left(\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{2} + \left(\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{2} \leq 1,$$

$$0 \leq \left|\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{2} \leq 1, \quad 0 \leq \left|\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{2} \leq 1, \quad 0 \leq \left|\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{2} + \left|\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{2} \leq 1.$$

Implies

$$0 \leq \left(\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{2} \leq 1 - \left(\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{2}, \quad 0 \leq \left|\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{2} \leq 1 - \left|\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{2}.$$

Hence,

$$0 \leq \left(1 - \left(\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{2}\right)^{\mathscr{I}}, \quad 0 \leq \left(1 - \left|\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{2}\right)^{\mathscr{I}}.$$

Following this, we consider the following relations:

$$1 - \left(1 - \left(\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{2}\right)^{\mathscr{I}} \leq 1, \quad 1 - \left(1 - \left|\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{2}\right)^{\mathscr{I}} \leq 1.$$

This yields the following bounds:

$$0 \leq \left(1 - \left(1 - \left(\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}} \leq 1, \quad 0 \leq \left(1 - \left(1 - \left|\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}} \leq 1.$$

It is also observed that:

$$0 \leq \left(\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{\mathscr{I}} \leq 1, \quad -1 \leq -\left|\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{\mathscr{I}} \leq .$$

From this, it follows that:

$$0 \leq \left(\left(1 - \left(1 - \left(\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}}\right)^{2} + \left(\left(\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{+}\right)^{\mathscr{I}}\right)^{2} \leq 1,$$

and

$$0 \leq \left|-\left|\mathfrak{m}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{\mathscr{I}}\right|^{2} + \left(1 - \left(1 - \left|\mathfrak{n}\mathfrak{R}_{\mathfrak{P}}^{-}\right|^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}} \leq 1.$$

Following this, similar relations are considered for the other terms:

(1)

$$0 \leq \left(1 - \left(1 - \left(\mathfrak{m}\mathfrak{I}_{\mathfrak{P}}^{+}\right)^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}} \leq 1, \quad 0 \leq \left(\mathfrak{n}\mathfrak{I}_{\mathfrak{P}}^{+}\right)^{\mathscr{I}} \leq 1,$$

and

$$0 \leq \left(\left(1 - \left(1 - \left(\mathfrak{m}\mathfrak{I}_{\mathfrak{P}}^{+}\right)^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}}\right)^{2} + \left(\left(\mathfrak{n}\mathfrak{I}_{\mathfrak{P}}^{+}\right)^{\mathscr{I}}\right)^{2} \leq 1.$$

(2)

$$0 \leq \left|\mathfrak{m}\mathfrak{I}_{\mathfrak{P}}^{-}\right|^{\mathscr{I}} \leq 1, \quad -1 \leq -\left(1 - \left(1 - \left|\mathfrak{n}\mathfrak{I}_{\mathfrak{P}}^{-}\right|^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}} \leq 0,$$

and

$$0 \leq \left|-\left|\mathfrak{m}\mathfrak{I}_{\mathfrak{P}}^{-}\right|^{\mathscr{I}}\right|^{2} + \left|-\left(1 - \left(1 - \left|\mathfrak{n}\mathfrak{I}_{\mathfrak{P}}^{-}\right|^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}}\right|^{2} \leq 1.$$

By the same reasoning, we arrive at the following relations:

$$0 \leq \left(\left(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}}\right)^{\mathscr{I}}\right)^{2} + \left(\left(1-\left(1-\left(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}}\right)^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}}\right)^{2} \leq 1,$$

and

$$0 \leq \left|-\left(1-\left(1-\left|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}\right|^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}}\right|^{2} + \left|-\left|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}\right|^{\mathscr{I}}\right|^{2} \leq 1.$$

From the preceding results, it follows that:

(1)

$$0 \leq \left(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}\right)^{\mathscr{I}} \leq 1, \quad 0 \leq \left(1-\left(1-\left(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}\right)^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}} \leq 1,$$

and

$$0 \leq \left(\left(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}\right)^{\mathscr{I}}\right)^{2} + \left(\left(1-\left(1-\left(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}\right)^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}}\right)^{2} \leq 1.$$

(2)

$$-1 \leq -\left(1-\left(1-\left|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}\right|^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}} \leq 0, \quad 0 \leq \left|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}\right|^{\mathscr{I}} \leq 1,$$

and

$$0 \leq \left|-\left(1-\left(1-\left|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}\right|^{2}\right)^{\mathscr{I}}\right)^{\frac{1}{2}}\right|^{2} + \left|-\left|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}\right|^{\mathscr{I}}\right|^{2} \leq 1.$$

Thus, both $\mathscr{I}\mathfrak{P}$ and $\mathfrak{P}^{\mathscr{I}}$ satisfy the conditions for being BCPFNs. □

Theorem 3.6. Let $\mathfrak{P} = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}\rangle$, $\mathfrak{P}_1 = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}\rangle$, and $\mathfrak{P}_2 = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}\rangle$ be BCPFNs. Then:

(1) $\mathfrak{P}_2 \oplus \mathfrak{P}_1 = \mathfrak{P}_1 \oplus \mathfrak{P}_2$.

(2) $\mathfrak{P}_2 \otimes \mathfrak{P}_1 = \mathfrak{P}_1 \otimes \mathfrak{P}_2$.

(3) $\mathfrak{P} \oplus (\mathfrak{P}_1 \oplus \mathfrak{P}_2) = (\mathfrak{P} \oplus \mathfrak{P}_1) \oplus \mathfrak{P}_2$.

(4) $\mathfrak{P} \otimes (\mathfrak{P}_1 \otimes \mathfrak{P}_2) = (\mathfrak{P} \otimes \mathfrak{P}_1) \otimes \mathfrak{P}_2$.

(5) $\mathfrak{P} \oplus (\mathfrak{P}_1 \cup \mathfrak{P}_2) = (\mathfrak{P} \oplus \mathfrak{P}_1) \cup (\mathfrak{P} \oplus \mathfrak{P}_2)$.

(6) $\mathfrak{P} \oplus (\mathfrak{P}_1 \cap \mathfrak{P}_2) = (\mathfrak{P} \oplus \mathfrak{P}_1) \cap (\mathfrak{P} \oplus \mathfrak{P}_2)$.

(7) $\mathfrak{P} \otimes (\mathfrak{P}_1 \cup \mathfrak{P}_2) = (\mathfrak{P} \otimes \mathfrak{P}_1) \cup (\mathfrak{P} \otimes \mathfrak{P}_2)$.

(8) $\mathfrak{P} \otimes (\mathfrak{P}_1 \cap \mathfrak{P}_2) = (\mathfrak{P} \otimes \mathfrak{P}_1) \cap (\mathfrak{P} \otimes \mathfrak{P}_2)$.

(9) $(\mathfrak{P}_1 \cup \mathfrak{P}_2) \oplus (\mathfrak{P}_1 \cap \mathfrak{P}_2) = \mathfrak{P}_1 \oplus \mathfrak{P}_2$.

(10) $(\mathfrak{P}_1 \cup \mathfrak{P}_2) \otimes (\mathfrak{P}_1 \cap \mathfrak{P}_2) = \mathfrak{P}_1 \otimes \mathfrak{P}_2$.

Proof. We show parts (1), (3), and (5) here; the other parts follow similarly.

(1) The operation $\mathfrak{P}_1 \oplus \mathfrak{P}_2$ is given by:

$$\mathfrak{P}_1 \oplus \mathfrak{P}_2 =$$

$$\left\langle \begin{array}{c} ((\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2)^{\frac{1}{2}} \\ +i((\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2)^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}) + i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}), \\ -(\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}) + i(-(\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2})), \\ -(|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2)^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2)^{\frac{1}{2}}) \end{array} \right\rangle$$

which may also be written as:

$$\left\langle \begin{array}{c} ((\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2 + (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2 - (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2)^{\frac{1}{2}} \\ +i((\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2 + (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2 - (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2)^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}) + i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}), \\ -(\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}) + i(-(\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1})), \\ -(|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2 + |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2 - |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2)^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2)^{\frac{1}{2}}) \end{array} \right\rangle.$$

Thus,

$$\mathfrak{P}_1 \oplus \mathfrak{P}_2 = \mathfrak{P}_2 \oplus \mathfrak{P}_1.$$

(3)

$$\begin{array}{c} \mathfrak{P} \oplus (\mathfrak{P}_1 \oplus \mathfrak{P}_2) = \\ \langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}} \rangle \\ \oplus \\ \left\langle \begin{array}{c} ((\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2)^{\frac{1}{2}} \\ +i((\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2)^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}) + i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}), \\ -(\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}) + i(-(\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2})), \\ -(|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2)^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2)^{\frac{1}{2}}) \end{array} \right\rangle \end{array}$$

$$= \left\langle \begin{array}{c} ((\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}})^2 + ((\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2 \\ -(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2) - (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}})^2((\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2 \\ -(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2))^{\frac{1}{2}} \\ +i((\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}})^2 + ((\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2 \\ -(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2) \\ -(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}})^2((\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2))^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}}\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}) + i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}), \\ -(|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}||\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}||\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}|) \\ +i(-(|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}||\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}||\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}|)), \\ -(|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}|^2 + (|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2) \\ -|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}|^2(|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2))^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}|^2 + (|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2) \\ -|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}|^2(|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2))^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} ((\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2 + (\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2 - (\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2)^{\frac{1}{2}} \\ +i((\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2 + (\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2 - (\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2(\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2)^{\frac{1}{2}}, \\ (\mathfrak{n}\Re^{+}_{\mathfrak{P}}\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}) + i(\mathfrak{n}\Im^{+}_{\mathfrak{P}}\mathfrak{n}\Im^{+}_{\mathfrak{P}_1}), \\ -(\mathfrak{m}\Re^{-}_{\mathfrak{P}}\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}) + i(-(\mathfrak{m}\Im^{-}_{\mathfrak{P}}\mathfrak{m}\Im^{-}_{\mathfrak{P}_1})), \\ -(|\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2 + |\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2 - |\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2)^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2 + |\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2 - |\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2|\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2)^{\frac{1}{2}}) \end{array} \right\rangle$$

$$\oplus$$

$$\langle \mathfrak{m}\Re^{+}_{\mathfrak{P}_2} + i\mathfrak{m}\Im^{+}_{\mathfrak{P}_2}, \mathfrak{m}\Re^{-}_{\mathfrak{P}_2} + i\mathfrak{m}\Im^{-}_{\mathfrak{P}_2}, \mathfrak{n}\Re^{+}_{\mathfrak{P}_2} + i\mathfrak{n}\Im^{+}_{\mathfrak{P}_2}, \mathfrak{n}\Re^{-}_{\mathfrak{P}_2} + i\mathfrak{n}\Im^{-}_{\mathfrak{P}_2} \rangle$$

$$= (\mathfrak{P} \oplus \mathfrak{P}_1) \oplus \mathfrak{P}_2.$$

(5) $\mathfrak{P} \oplus (\mathfrak{P}_1 \cup \mathfrak{P}_2) = \langle \mathfrak{m}\Re^{+}_{\mathfrak{P}} + i\mathfrak{m}\Im^{+}_{\mathfrak{P}}, \mathfrak{m}\Re^{-}_{\mathfrak{P}} + i\mathfrak{m}\Im^{-}_{\mathfrak{P}}, \mathfrak{n}\Re^{+}_{\mathfrak{P}} + i\mathfrak{n}\Im^{+}_{\mathfrak{P}}, \mathfrak{n}\Re^{-}_{\mathfrak{P}} + i\mathfrak{n}\Im^{-}_{\mathfrak{P}} \rangle$

$$\oplus$$

$$\left\langle \begin{array}{c} \max\{\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}, \mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\} + i\max\{\mathfrak{m}\Im^{+}_{\mathfrak{P}_1}, \mathfrak{m}\Im^{+}_{\mathfrak{P}_2}\}, \min\{\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\} + i\min\{\mathfrak{n}\Im^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Im^{+}_{\mathfrak{P}_2}\}, \\ \min\{\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}, \mathfrak{m}\Re^{-}_{\mathfrak{P}_2}\} + i\min\{\mathfrak{m}\Im^{-}_{\mathfrak{P}_1}, \mathfrak{m}\Im^{-}_{\mathfrak{P}_2}\}, \max\{\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}, \mathfrak{n}\Re^{-}_{\mathfrak{P}_2}\} + i\max\{\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}, \mathfrak{n}\Im^{-}_{\mathfrak{P}_2}\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} ((\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2 + \max\{\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}, \mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\}^2 - (\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2\max\{\mathfrak{m}\Re^{+}_{\mathfrak{P}_1}, \mathfrak{m}\Re^{+}_{\mathfrak{P}_2}\}^2)^{\frac{1}{2}} \\ +i((\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2 + \max\{\mathfrak{m}\Im^{+}_{\mathfrak{P}_1}, \mathfrak{m}\Im^{+}_{\mathfrak{P}_2}\}^2 - (\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2\max\{\mathfrak{m}\Im^{+}_{\mathfrak{P}_1}, \mathfrak{m}\Im^{+}_{\mathfrak{P}_2}\}^2)^{\frac{1}{2}}, \\ (\mathfrak{n}\Re^{+}_{\mathfrak{P}}\min\{\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\}) + i(\mathfrak{n}\Im_{\mathfrak{P}}\min\{\mathfrak{n}\Im^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Im^{+}_{\mathfrak{P}_2}\}), \\ -(\mathfrak{m}\Re^{-}_{\mathfrak{P}}\min\{\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}, \mathfrak{m}\Re^{-}_{\mathfrak{P}_2}\}) + i(-(\mathfrak{m}\Im^{-}_{\mathfrak{P}}\min\{\mathfrak{m}\Im^{-}_{\mathfrak{P}_1}, \mathfrak{m}\Im^{-}_{\mathfrak{P}_2}\})), \\ -(|\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2 + |\max\{\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}, \mathfrak{n}\Re^{-}_{\mathfrak{P}_2}\}|^2 - |\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2|\max\{\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}, \mathfrak{n}\Re^{-}_{\mathfrak{P}_2}\}|^2)^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2 + |\max\{\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}, \mathfrak{n}\Im^{-}_{\mathfrak{P}_2}\}|^2 - |\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2|\max\{\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}, \mathfrak{n}\Im^{-}_{\mathfrak{P}_2}\}|^2)^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} ((\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2 + (1 - (\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2)\max\{(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2, (\mathfrak{m}\Re^{+}_{\mathfrak{P}_2})^2\})^{\frac{1}{2}} + \\ i((\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2 + (1 - (\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2)\max\{(\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2, (\mathfrak{m}\Im^{+}_{\mathfrak{P}_2})^2\})^{\frac{1}{2}}, \\ \min\{\mathfrak{n}\Re^{+}_{\mathfrak{P}}\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Re^{+}_{\mathfrak{P}}\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\} + i(\min\{\mathfrak{n}\Im^{+}_{\mathfrak{P}}\mathfrak{n}\Im^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Im^{+}_{\mathfrak{P}}\mathfrak{n}\Im^{+}_{\mathfrak{P}_2}\}), \\ \min\{-(\mathfrak{m}\Re^{-}_{\mathfrak{P}}\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}), -(\mathfrak{m}\Re^{-}_{\mathfrak{P}}\mathfrak{m}\Re^{-}_{\mathfrak{P}_2})\} + \\ i(\min\{-(\mathfrak{m}\Im^{-}_{\mathfrak{P}}\mathfrak{m}\Im^{-}_{\mathfrak{P}_1}), -(\mathfrak{m}\Im^{-}_{\mathfrak{P}}\mathfrak{m}\Im^{-}_{\mathfrak{P}_2})\}), \\ -(|\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2 + (1 - |\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2)\max\{|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2, |\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}|^2\})^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2 + (1 - |\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2)\max\{|\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2, |\mathfrak{n}\Im^{-}_{\mathfrak{P}_2}|^2\})^{\frac{1}{2}}) \end{array} \right\rangle.$$

However,

$$(\mathfrak{P} \oplus \mathfrak{P}_1) \cup (\mathfrak{P} \oplus \mathfrak{P}_2) =$$

$$\left\langle \begin{array}{c} ((\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2 + (\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2 - (\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2)^{\frac{1}{2}} \\ +i((\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2 + (\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2 - (\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2(\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2)^{\frac{1}{2}}, \\ (\mathfrak{n}\Re^{+}_{\mathfrak{P}}\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}) + i(\mathfrak{n}\Im^{+}_{\mathfrak{P}}\mathfrak{n}\Im^{+}_{\mathfrak{P}_1}), \\ -(\mathfrak{m}\Re^{-}_{\mathfrak{P}}\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}) + i(-(\mathfrak{m}\Im^{-}_{\mathfrak{P}}\mathfrak{m}\Im^{-}_{\mathfrak{P}_1})), \\ -(|\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2 + |\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2 - |\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2)^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2 + |\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2 - |\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2|\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2)^{\frac{1}{2}}) \end{array} \right\rangle$$

$$\cup$$

$$\left\langle \begin{array}{c} ((\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2 + (\mathfrak{m}\Re^{+}_{\mathfrak{P}_2})^2 - (\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2})^2)^{\frac{1}{2}} \\ +i((\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2 + (\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2 - (\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2(\mathfrak{m}\Im^{+}_{\mathfrak{P}_2})^2)^{\frac{1}{2}}, \\ (\mathfrak{n}\Re^{+}_{\mathfrak{P}}\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}) + i(\mathfrak{n}\Im^{+}_{\mathfrak{P}}\mathfrak{n}\Im^{+}_{\mathfrak{P}_2}), \\ -(\mathfrak{m}\Re^{-}_{\mathfrak{P}}\mathfrak{m}\Re^{-}_{\mathfrak{P}_2}) + i(-(\mathfrak{m}\Im^{-}_{\mathfrak{P}}\mathfrak{m}\Im^{-}_{\mathfrak{P}_2})), \\ -(|\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2 + |\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2|\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}|^2)^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2 + |\mathfrak{n}\Im^{-}_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2|\mathfrak{n}\Im^{-}_{\mathfrak{P}_2}|^2)^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \max\{((\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2 + (\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2 - (\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2)^{\frac{1}{2}}, \\ ((\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2 + (\mathfrak{m}\Re^{+}_{\mathfrak{P}_2})^2 - (\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2})^2)^{\frac{1}{2}}\} \\ +i(\max\{((\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2 + (\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2 - (\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2(\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2)^{\frac{1}{2}}, \\ ((\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2 + (\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2 - (\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2(\mathfrak{m}\Im^{+}_{\mathfrak{P}_2})^2)^{\frac{1}{2}}\}), \\ \min\{\mathfrak{n}\Re^{+}_{\mathfrak{P}}\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Re^{+}_{\mathfrak{P}}\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\}+ \\ i(\min\{\mathfrak{n}\Im^{+}_{\mathfrak{P}}\mathfrak{n}\Im^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Im^{+}_{\mathfrak{P}}\mathfrak{n}\Im^{+}_{\mathfrak{P}_2}\}), \\ \min\{-(\mathfrak{m}\Re^{-}_{\mathfrak{P}}\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}), -(\mathfrak{m}\Re^{-}_{\mathfrak{P}}\mathfrak{m}\Re^{-}_{\mathfrak{P}_2})\}+ \\ i(\min\{-(\mathfrak{m}\Im^{-}_{\mathfrak{P}}\mathfrak{m}\Im^{-}_{\mathfrak{P}_1}), \\ -(\mathfrak{m}\Im^{-}_{\mathfrak{P}}\mathfrak{m}\Im^{-}_{\mathfrak{P}_2})\}), \\ \max\{-(|\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2 + |\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2 - |\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2)^{\frac{1}{2}}, \\ -(|\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2 + |\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2|\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}|^2)^{\frac{1}{2}}\} \\ +i(\max\{-(|\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2 + |\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2 - |\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2|\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2)^{\frac{1}{2}}, \\ -(|\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2 + |\mathfrak{n}\Im^{-}_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2|\mathfrak{n}\Im^{-}_{\mathfrak{P}_2}|^2)^{\frac{1}{2}}\}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} ((\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2 + (1 - (\mathfrak{m}\Re^{+}_{\mathfrak{P}})^2)\max\{(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2, \\ (\mathfrak{m}\Re^{+}_{\mathfrak{P}_2})^2\})^{\frac{1}{2}} + i((\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2 + (1 - (\mathfrak{m}\Im^{+}_{\mathfrak{P}})^2)\max\{(\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2, (\mathfrak{m}\Im^{+}_{\mathfrak{P}_2})^2\})^{\frac{1}{2}}, \\ \min\{\mathfrak{n}\Re^{+}_{\mathfrak{P}}\mathfrak{n}\Re^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Re^{+}_{\mathfrak{P}}\mathfrak{n}\Re^{+}_{\mathfrak{P}_2}\} \\ +i(\min\{\mathfrak{n}\Im^{+}_{\mathfrak{P}}\mathfrak{n}\Im^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Im^{+}_{\mathfrak{P}}\mathfrak{n}\Im^{+}_{\mathfrak{P}_2}\}), \\ \min\{-(\mathfrak{m}\Re^{-}_{\mathfrak{P}}\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}), -(\mathfrak{m}\Re^{-}_{\mathfrak{P}}\mathfrak{m}\Re^{-}_{\mathfrak{P}_2})\}+ \\ i(\min\{-(\mathfrak{m}\Im^{-}_{\mathfrak{P}}\mathfrak{m}\Im^{-}_{\mathfrak{P}_1}), -(\mathfrak{m}\Im^{-}_{\mathfrak{P}}\mathfrak{m}\Im^{-}_{\mathfrak{P}_2})\}), \\ -(|\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2 + (1 - |\mathfrak{n}\Re^{-}_{\mathfrak{P}}|^2)\max\{|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2, |\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}|^2\})^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2 + (1 - |\mathfrak{n}\Im^{-}_{\mathfrak{P}}|^2)\max\{|\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2, |\mathfrak{n}\Im^{-}_{\mathfrak{P}_2}|^2\})^{\frac{1}{2}}) \end{array} \right\rangle.$$

$\square$

Theorem 3.7. Let $\mathfrak{P} = \langle \mathfrak{m}\Re^{+}_{\mathfrak{P}} + i\mathfrak{m}\Im^{+}_{\mathfrak{P}}, \mathfrak{n}\Re^{+}_{\mathfrak{P}} + i\mathfrak{n}\Im^{+}_{\mathfrak{P}}, \mathfrak{m}\Re^{-}_{\mathfrak{P}} + i\mathfrak{m}\Im^{-}_{\mathfrak{P}}, \mathfrak{n}\Re^{-}_{\mathfrak{P}} + i\mathfrak{n}\Im^{-}_{\mathfrak{P}}\rangle$, $\mathfrak{P}_1 = \langle \mathfrak{m}\Re^{+}_{\mathfrak{P}_1} + i\mathfrak{m}\Im^{+}_{\mathfrak{P}_1}, \mathfrak{n}\Re^{+}_{\mathfrak{P}_1} + i\mathfrak{n}\Im^{+}_{\mathfrak{P}_1}, \mathfrak{m}\Re^{-}_{\mathfrak{P}_1} + i\mathfrak{m}\Im^{-}_{\mathfrak{P}_1}, \mathfrak{n}\Re^{-}_{\mathfrak{P}_1} + i\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}\rangle$, and $\mathfrak{P}_2 = \langle \mathfrak{m}\Re^{+}_{\mathfrak{P}_2} + i\mathfrak{m}\Im^{+}_{\mathfrak{P}_2}, \mathfrak{n}\Re^{+}_{\mathfrak{P}_2} + i\mathfrak{n}\Im^{+}_{\mathfrak{P}_2}, \mathfrak{m}\Re^{-}_{\mathfrak{P}_2} + i\mathfrak{m}\Im^{-}_{\mathfrak{P}_2}, \mathfrak{n}\Re^{-}_{\mathfrak{P}_2} + i\mathfrak{n}\Im^{-}_{\mathfrak{P}_2}\rangle$ be BCPFNs. Let $\mathscr{I} > 0$. Then, the following features are fulfilled:

(1) $(\mathfrak{P}_1 \oplus \mathfrak{P}_2)^c = \mathfrak{P}_1^c \otimes \mathfrak{P}_2^c$.
(2) $(\mathfrak{P}_1 \otimes \mathfrak{P}_2)^c = \mathfrak{P}_1^c \oplus \mathfrak{P}_2^c$.
(3) $(\mathfrak{P}^c)^{\mathscr{I}} = (\mathscr{I}\mathfrak{P})^c$.
(4) $\mathscr{I}(\mathfrak{P})^c = (\mathfrak{P}^{\mathscr{I}})^c$.

Proof. Parts (1) and (3) are established here, and the remaining parts can be shown using similar arguments.

(1)

$$(\mathfrak{P}_1 \oplus \mathfrak{P}_2)^c = \left\langle \begin{array}{c} ((\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2)^{\frac{1}{2}} \\ +i((\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2)^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}) + i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}), \\ -(\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}) + i(-(\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2})), \\ -(|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2)^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2)^{\frac{1}{2}}) \end{array} \right\rangle^c$$

$$= \left\langle \begin{array}{c} (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}) + i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}), \\ ((\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2})^2)^{\frac{1}{2}} \\ +i((\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2})^2)^{\frac{1}{2}}, \\ -|-(|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}|^2)^{\frac{1}{2}}| \\ +i(-|-(|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|^2)^{\frac{1}{2}}|), \\ -|-(\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2})| + i(-|-(\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2})|) \end{array} \right\rangle$$

$$= \langle (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1}) + i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}), (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1}) + i(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}), -|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1}| + i(-|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1}|), -|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1}| + i(-|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}|)\rangle$$

$$\otimes$$

$$\langle (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2}) + i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}), (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2}) + i(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}), -|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2}| + i(-|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2}|), -|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2}| + i(-|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}|)\rangle$$

$$= \mathfrak{P}_1^c \otimes \mathfrak{P}_2^c.$$

(3)

$$(\mathfrak{P}^c)^{\mathscr{I}}$$

$$= \left\langle (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}}) + i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}), (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}}) + i(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}), -|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}| + i(-|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}|), -|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}| + i(-|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}|) \right\rangle^{\mathscr{I}}$$

$$= \left\langle \begin{array}{c} (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}})^{\mathscr{I}} + i(\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}})^{\mathscr{I}}, (1-(1-(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}})^2)^{\mathscr{I}})^{\frac{1}{2}} \\ +i(1-(1-(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}})^2)^{\mathscr{I}})^{\frac{1}{2}}, \\ -(1-(1-|-|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}||^2)^{\mathscr{I}})^{\frac{1}{2}} \\ +i(-(1-(1-|-|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}||^2)^{\mathscr{I}})^{\frac{1}{2}}), \\ -|-|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}||^{\mathscr{I}} + i(-|-|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}||^{\mathscr{I}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} ((\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}})^{\mathscr{I}}) + i((\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}})^{\mathscr{I}}), ((1-(1-(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}})^2)^{\mathscr{I}})^{\frac{1}{2}}) \\ +i((1-(1-(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}})^2)^{\mathscr{I}})^{\frac{1}{2}}), \\ -|-(1-(1-|-|\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}|^2|^{\mathscr{I}})^{\frac{1}{2}}| \\ +i(-|-(1-(1-|\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}|^2)^{\mathscr{I}})^{\frac{1}{2}}|), \\ -|-|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}|^{\mathscr{I}}| + i(-|-|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}|^{\mathscr{I}}|) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}})^{\frac{1}{2}} \\ +i(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}})^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}})^{\mathscr{I}}+i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}})^{\mathscr{I}}, \\ -|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}|^{\mathscr{I}}+i(-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}|^{\mathscr{I}}), \\ -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2|)^{\mathscr{I}})^{\frac{1}{2}} \\ +i(-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2|)^{\mathscr{I}})^{\frac{1}{2}}) \end{array} \right\rangle^{c}$$
$$= (\mathscr{I}\mathfrak{P})^c.$$

□

Theorem 3.8. Let $\mathfrak{P} = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}\rangle$, $\mathfrak{P}_1 = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}\rangle$, and $\mathfrak{P}_2 = \langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}\rangle$ be BCPFNs. For any $\mathscr{I}, \mathscr{I}_1, \mathscr{I}_2 > 0$, the following features are fulfilled:

(1) $\mathscr{I}(\mathfrak{P}_1 \oplus \mathfrak{P}_2) = \mathscr{I}\mathfrak{P}_1 \oplus \mathscr{I}\mathfrak{P}_2$.
(2) $(\mathscr{I}_1 + \mathscr{I}_2)\mathfrak{P} = \mathscr{I}_1\mathfrak{P} \oplus \mathscr{I}_2\mathfrak{P}$.
(3) $(\mathfrak{P}_1 \otimes \mathfrak{P}_2)^{\mathscr{I}} = \mathfrak{P}_1^{\mathscr{I}} \otimes \mathfrak{P}_2^{\mathscr{I}}$.
(4) $\mathfrak{P}^{(\mathscr{I}_1+\mathscr{I}_2)} = \mathfrak{P}^{\mathscr{I}_1} \otimes \mathfrak{P}^{\mathscr{I}_2}$.

Proof. Only (1) and (2) are shown here, while the other expressions follow analogously.
(1)

$$\mathscr{I}(\mathfrak{P}_1 \oplus \mathfrak{P}_2) =$$
$$\mathscr{I} \left\langle \begin{array}{c} ((\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2})^2)^{\frac{1}{2}} \\ +i((\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2})^2)^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2}) + i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2}), \\ -(\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2}) + i(-(\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2})), \\ -(|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}|^2)^{\frac{1}{2}} \\ +i(-(|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}|^2)^{\frac{1}{2}})) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} (1-(1-((\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2})^2))^{\mathscr{I}})^{\frac{1}{2}} \\ +i(1-(1-((\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1})^2 + (\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2})^2 - (\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1})^2(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2})^2))^{\mathscr{I}})^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2})^{\mathscr{I}} + i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2})^{\mathscr{I}}, \\ -(|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1}||\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2}|)^{\mathscr{I}} + i(-(|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}||\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}|)^{\mathscr{I}}), \\ -(1-(1-(-(|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}|^2)))^{\mathscr{I}})^{\frac{1}{2}} \\ +i(-(1-(1-(-(|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|^2 + |\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}|^2 - |\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|^2|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}|^2)))^{\mathscr{I}})^{\frac{1}{2}})\rangle \end{array} \right\rangle.$$

However,

$$\mathscr{I}\mathfrak{P}_1 \oplus \mathscr{I}\mathfrak{P}_2 = \left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1})^2)^{\mathscr{I}})^{\frac{1}{2}} + i(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1})^2)^{\mathscr{I}})^{\frac{1}{2}}, (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1})^{\mathscr{I}} + i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1})^{\mathscr{I}}, \\ -|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1}|^{\mathscr{I}} + i(-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}|^{\mathscr{I}}), -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{I}})^{\frac{1}{2}} + i(-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{I}})^{\frac{1}{2}}) \end{array} \right\rangle$$
$$\oplus$$

$$\left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2})^2)^{\mathscr{I}})^{\frac{1}{2}}+i(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2})^2)^{\mathscr{I}})^{\frac{1}{2}}, (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2})^{\mathscr{I}}+i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2})^{\mathscr{I}}, \\ -|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2}|^{\mathscr{I}}+i(-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}|^{\mathscr{I}}), -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{I}})^{\frac{1}{2}}+i(-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{I}})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} ((1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1})^2)^{\mathscr{I}})+(1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2})^2)^{\mathscr{I}}) \\ -(1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1})^2)^{\mathscr{I}})(1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2})^2)^{\mathscr{I}}))^{\frac{1}{2}} \\ +i((1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1})^2)^{\mathscr{I}})+(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2})^2)^{\mathscr{I}}) \\ -(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1})^2)^{\mathscr{I}})(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2})^2)^{\mathscr{I}}))^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2})^{\mathscr{I}}+i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1}\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2})^{\mathscr{I}}, \\ -(|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1}||\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2}|)^{\mathscr{I}}+i(-(|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}||\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}|)^{\mathscr{I}}), \\ -((1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{I}})+(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{I}}) \\ -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{I}})(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{I}}))^{\frac{1}{2}} \\ +i(-((1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{I}})+(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{I}}) \\ -(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{I}})(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{I}}))^{\frac{1}{2}}) \end{array} \right\rangle$$

$= \mathscr{I}(\mathfrak{P}_1 \oplus \mathfrak{P}_2)$.

(2) $$(\mathscr{I}_1+\mathscr{I}_2)\,\mathfrak{P} = \left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_1+\mathscr{I}_2})^{\frac{1}{2}}+i(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_1+\mathscr{I}_2})^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}})^{\mathscr{I}_1+\mathscr{I}_2}+i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}})^{\mathscr{I}_1+\mathscr{I}_2}, -|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}|^{\mathscr{I}_1+\mathscr{I}_2}+i(-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}|^{\mathscr{I}_1+\mathscr{I}_2}), \\ -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_1+\mathscr{I}_2})^{\frac{1}{2}}+i(-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_1+\mathscr{I}_2})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} ((1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_1})+(1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_2}) \\ -(1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_1})(1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_2}))^{\frac{1}{2}} \\ +i((1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_1})+(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_2}) \\ -(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_1})(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_2}))^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}})^{\mathscr{I}_1}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}})^{\mathscr{I}_2}+i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}})^{\mathscr{I}_1}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}})^{\mathscr{I}_2}, \\ -((-|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}|^{\mathscr{I}_1})(-|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}|^{\mathscr{I}_2})) \\ +i(-((-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}|^{\mathscr{I}_1})(-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}|^{\mathscr{I}_2}))), \\ -(|-(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_1})| \\ +|-(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_2})|-|-(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_1})| \\ |-(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_2})|)^{\frac{1}{2}} \\ +i((-(|-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_1})|+|-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_2})|- \\ |-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_1})||-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_2})|)^{\frac{1}{2}})) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_1})^{\frac{1}{2}} \\ +i(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_1})^{\frac{1}{2}}, (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}})^{\mathscr{I}_1} \\ +i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}})^{\mathscr{I}_1}, \\ -|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}|^{\mathscr{I}_1}+i(-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}|^{\mathscr{I}_1}), \\ -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_1})^{\frac{1}{2}} \\ +i(-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_1})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$\oplus$$

$$\left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_2})^{\frac{1}{2}} \\ +i(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2)^{\mathscr{I}_2})^{\frac{1}{2}}, (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}})^{\mathscr{I}_2} + i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}})^{\mathscr{I}_2}, \\ -|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}|^{\mathscr{I}_2} + i(-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}|^{\mathscr{I}_2}), \\ -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_2})^{\frac{1}{2}} + i(-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2)^{\mathscr{I}_2})^{\frac{1}{2}}) \end{array} \right\rangle$$
$$= \mathscr{I}_1\mathfrak{P} \oplus \mathscr{I}_2\mathfrak{P}.$$

□

4. BCPF Aggregation Operators

This section focuses on the application of BCPF weighted average and geometric aggregation operators in the context of BCPFS. It presents an in-depth examination of the mathematical foundations of these operators and emphasizes their role in facilitating effective data analysis and supporting decision-making tasks.

Definition 4.1. Consider a collection of BCPFNs, denoted as:

$$\mathfrak{P}_i = \left\{\left\langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i} \right\rangle\right\}, \quad \forall i \in \{1,2,\ldots,k\}$$

where each $\mathfrak{P}_i$ represents a BCPFN, and the associated weight vector $\mathscr{X}$ is given by:

$$\mathscr{X} = (\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_k)^T$$

where each weight component fulfills $\mathscr{X}_i > 0$ and together they satisfy the normalization condition $\sum_{i=1}^{k} \mathscr{X}_i = 1$. Accordingly, we present two essential aggregation operators:

(1) The BCPF weighted averaging (BCPFWA) operator is defined as a mapping BCPFWA : $\mathfrak{P}^k \to \mathfrak{P}$, which combines BCPFNs via a weighted average, expressed as:
$\mathrm{BCPFWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k) = \bigoplus_{i=1}^{k} \mathscr{X}_i\mathfrak{P}_i = \mathscr{X}_1\mathfrak{P}_1 \oplus \mathscr{X}_2\mathfrak{P}_2 \oplus \cdots \oplus \mathscr{X}_k\mathfrak{P}_k$.

(2) The BCPF weighted geometric (BCPFWG) operator is an alternative aggregation method based on the weighted geometric approach. It is defined as a mapping BCPFWG : $\mathfrak{P}^k \to \mathfrak{P}$, with its functional form expressed as:
$\mathrm{BCPFWG}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k) = \bigotimes_{i=1}^{k} \mathfrak{P}_i^{\mathscr{X}_i} = \mathfrak{P}_1^{\mathscr{X}_1} \otimes \mathfrak{P}_2^{\mathscr{X}_2} \otimes \cdots \otimes \mathfrak{P}_k^{\mathscr{X}_k}$.

Theorem 4.1. Let $\mathfrak{P}_i$ denote a set of BCPFNs, where each element is specified as

$$\mathfrak{P}_i = \left\langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i} \right\rangle, \quad i = 1,2,\ldots,k.$$

Let $\mathscr{X} = (\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_k)^T$ be the associated weight vector, with $\mathscr{X}_i > 0$ and $\sum_{i=1}^{k} \mathscr{X}_i = 1$. Then, the BCPF weighted averaging and geometric aggregation operators can be alternatively defined as:

(1) The BCPFWA operator combines BCPFNs using a weighted average, with the alternative form:

$$\mathrm{BCPFWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k\right) =$$

$$\left\langle \begin{array}{c} (1-\prod_{i=1}^{k}(1-(\mathfrak{m}\Re^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1-\prod_{i=1}^{k}(1-(\mathfrak{m}\Im^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ \prod_{i=1}^{k}(\mathfrak{n}\Re^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i} + i\prod_{i=1}^{k}(\mathfrak{n}\Im^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}, \\ -(\prod_{i=1}^{k}|\mathfrak{m}\Re^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i}) + i(-(\prod_{i=1}^{k}|\mathfrak{m}\Im^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i})), \\ -(1-\prod_{i=1}^{k}(1-|\mathfrak{n}\Re^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(-(1-\prod_{i=1}^{k}(1-|\mathfrak{n}\Im^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}) \end{array} \right\rangle.$$

(2) The BCPFWG operator aggregates BCPFNs via a weighted geometric approach, given by:

$$\text{BCPFWG}\,(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k) =$$

$$\left\langle \begin{array}{c} \prod_{i=1}^{k}(\mathfrak{m}\Re^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i} + i(\prod_{i=1}^{k}\mathfrak{m}\Im^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}, \\ (1-\prod_{i=1}^{k}(1-(\mathfrak{n}\Re^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1-\prod_{i=1}^{k}(1-(\mathfrak{n}\Im^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ -(1-\prod_{i=1}^{k}(1-|\mathfrak{m}\Re^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(-(1-\prod_{i=1}^{k}(1-|\mathfrak{m}\Im^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}), \\ -(\prod_{i=1}^{k}|\mathfrak{n}\Re^{-}_{\mathfrak{P}_i}|_i^{\mathscr{X}_i}) + i(-(\prod_{i=1}^{k}|\mathfrak{n}\Im^{-}_{\mathfrak{P}_i}|_i^{\mathscr{X}_i})) \end{array} \right\rangle.$$

Proof. (1) To validate the result via mathematical induction, we start with the base case $k = 2$, where the formula simplifies and can be easily verified. This establishes the foundation for extending the proof to larger values of k. The formula then reduces to:

$$\text{BCPFWA}\,(\mathfrak{P}_1, \mathfrak{P}_2) = \mathscr{X}_1\mathfrak{P}_1 \oplus \mathscr{X}_2\mathfrak{P}_2 =$$

$$\left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2)^{\mathscr{X}_1})^{\frac{1}{2}} + i(1-(1-(\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2)^{\mathscr{X}_1})^{\frac{1}{2}}, \\ (\mathfrak{n}\Re^{+}_{\mathfrak{P}_1})^{\mathscr{X}_1} + i(\mathfrak{n}\Im^{+}_{\mathfrak{P}_1})^{\mathscr{X}_1}, -|\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}|^{\mathscr{X}_1} + i(-|\mathfrak{m}\Im^{-}_{\mathfrak{P}_1}|^{\mathscr{X}_1}), \\ -(1-(1-|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{X}_1})^{\frac{1}{2}} + i(-(1-(1-|\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{X}_1})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$\oplus$$

$$\left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2})^2)^{\mathscr{X}_2})^{\frac{1}{2}} + i(1-(1-(\mathfrak{m}\Im^{+}_{\mathfrak{P}_2})^2)^{\mathscr{X}_2})^{\frac{1}{2}}, \\ (\mathfrak{n}\Re^{+}_{\mathfrak{P}_2})^{\mathscr{X}_2} + i(\mathfrak{n}\Im^{+}_{\mathfrak{P}_2})^{\mathscr{X}_2}, -|\mathfrak{m}\Re^{-}_{\mathfrak{P}_2}|^{\mathscr{X}_2} + i(-|\mathfrak{m}\Im^{-}_{\mathfrak{P}_2}|^{\mathscr{X}_2}), \\ -(1-(1-|\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{X}_2})^{\frac{1}{2}} + i(-(1-(1-|\mathfrak{n}\Im^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{X}_2})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} ((1-(1-(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2)^{\mathscr{X}_1}) + (1-(1-(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2})^2)^{\mathscr{X}_2}) \\ -(1-(1-(\mathfrak{m}\Re^{+}_{\mathfrak{P}_1})^2)^{\mathscr{X}_1})(1-(1-(\mathfrak{m}\Re^{+}_{\mathfrak{P}_2})^2)^{\mathscr{X}_2}))^{\frac{1}{2}} \\ +i((1-(1-(\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2)^{\mathscr{X}_1}) + (1-(1-(\mathfrak{m}\Im^{+}_{\mathfrak{P}_2})^2)^{\mathscr{X}_2}) \\ -(1-(1-(\mathfrak{m}\Im^{+}_{\mathfrak{P}_1})^2)^{\mathscr{X}_1})(1-(1-(\mathfrak{m}\Im^{+}_{\mathfrak{P}_2})^2)^{\mathscr{X}_2}))^{\frac{1}{2}}, \\ (\mathfrak{n}\Re^{+}_{\mathfrak{P}_1})^{\mathscr{X}_1}(\mathfrak{n}\Re^{+}_{\mathfrak{P}_2})^{\mathscr{X}_2} \\ +i(\mathfrak{n}\Im^{+}_{\mathfrak{P}_1})^{\mathscr{X}_1}(\mathfrak{n}\Im^{+}_{\mathfrak{P}_2})^{\mathscr{X}_2}, \\ -(|\mathfrak{m}\Re^{-}_{\mathfrak{P}_1}|^{\mathscr{X}_1}|\mathfrak{m}\Re^{-}_{\mathfrak{P}_2}|^{\mathscr{X}_2}) \\ +i(-(|\mathfrak{m}\Im^{-}_{\mathfrak{P}_1}|^{\mathscr{X}_1}|\mathfrak{m}\Im^{-}_{\mathfrak{P}_2}|^{\mathscr{X}_2})), \\ -((1-(1-|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{X}_1}) + (1-(1-|\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{X}_2}) \\ -(1-(1-|\mathfrak{n}\Re^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{X}_1})(1-(1-|\mathfrak{n}\Re^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{X}_2}))^{\frac{1}{2}} \\ +i(-((1-(1-|\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{X}_1}) + (1-(1-|\mathfrak{n}\Im^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{X}_2}) \\ -(1-(1-|\mathfrak{n}\Im^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{X}_1})(1-(1-|\mathfrak{n}\Im^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{X}_2}))^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_1})^2)^{\mathscr{X}_1}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_2})^2)^{\mathscr{X}_2})^{\frac{1}{2}} \\ +i(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_1})^2)^{\mathscr{X}_1}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_2})^2)^{\mathscr{X}_2})^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_1})^{\mathscr{X}_1}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_2})^{\mathscr{X}_2} \\ +i((\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_1})^{\mathscr{X}_1}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_2})^{\mathscr{X}_2}), -((|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_1}|^{\mathscr{X}_1})(|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_2}|^{\mathscr{X}_2})) \\ +i(-((|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_1}|^{\mathscr{X}_1})(|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_2}|^{\mathscr{X}_2}))), \\ -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{X}_1}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{X}_2})^{\frac{1}{2}} \\ +i(-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_1}|^2)^{\mathscr{X}_1}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_2}|^2)^{\mathscr{X}_2})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} (1-\prod_{i=1}^{2}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1-\prod_{i=1}^{2}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ \prod_{i=1}^{2}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i} + i\prod_{i=1}^{2}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}, \\ -(\prod_{i=1}^{2}|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i}) + i(-(\prod_{i=1}^{2}|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i})), \\ -(1-\prod_{i=1}^{2}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} \\ +i(-(1-\prod_{i=1}^{2}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}) \end{array} \right\rangle.$$

Assuming the theorem holds for $k = l$, the statement is valid for l elements. Specifically, we have:

$$\mathrm{BCPFWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_l) = \mathscr{X}_1\mathfrak{P}_1 \oplus \mathscr{X}_2\mathfrak{P}_2 \oplus \cdots \oplus \mathscr{X}_l\mathfrak{P}_l$$

$$= \left\langle \begin{array}{c} (1-\prod_{i=1}^{l}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1-\prod_{i=1}^{l}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ \prod_{i=1}^{l}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i} + i\prod_{i=1}^{l}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}, -(\prod_{i=1}^{l}|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i}) + i(-(\prod_{i=1}^{l}|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i})), \\ -(1-\prod_{i=1}^{l}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(-(1-\prod_{i=1}^{l}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}) \end{array} \right\rangle.$$

To prove the statement for $k = l+1$, we apply the inductive hypothesis. Then, for $k = l+1$, the expression becomes:

$$\mathrm{BCPFWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{l+1}) = \mathscr{X}_1\mathfrak{P}_1 \oplus \mathscr{X}_2\mathfrak{P}_2 \oplus \cdots \oplus \mathscr{X}_{l+1}\mathfrak{P}_{l+1}$$

$$= \left\langle \begin{array}{c} (1-\prod_{i=1}^{l}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1-\prod_{i=1}^{l}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ \prod_{i=1}^{l}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i} + i\prod_{i=1}^{l}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}, -(\prod_{i=1}^{l}|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i}) + i(-(\prod_{i=1}^{l}|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i})), \\ -(1-\prod_{i=1}^{l}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(-(1-\prod_{i=1}^{l}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$\oplus$$

$$\left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_{l+1}})^2)^{\mathscr{X}_{l+1}})^{\frac{1}{2}} \\ +i(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_{l+1}})^2)^{\mathscr{X}_{l+1}})^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_{l+1}})^{\mathscr{X}_{l+1}} + i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_{l+1}})^{\mathscr{X}_{l+1}}, \\ -|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_{l+1}}|^{\mathscr{X}_{l+1}} + i(-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_{l+1}}|^{\mathscr{X}_{l+1}}), \\ -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_{l+1}}|^2)^{\mathscr{X}_{l+1}})^{\frac{1}{2}} \\ +i(-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_{l+1}}|^2)^{\mathscr{X}_{l+1}})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} ((1-\prod_{i=1}^{l}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i}) + (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_{l+1}})^2)^{\mathscr{X}_{l+1}}) - \\ (1-\prod_{i=1}^{l}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})(1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_{l+1}})^2)^{\mathscr{X}_{l+1}}))^{\frac{1}{2}} + \\ i((1-\prod_{i=1}^{l}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i}) + (1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_{l+1}})^2)^{\mathscr{X}_{l+1}}) - \\ (1-\prod_{i=1}^{l}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_{l+1}})^2)^{\mathscr{X}_{l+1}}))^{\frac{1}{2}}, \\ (\prod_{i=1}^{l}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_{l+1}})^{\mathscr{X}_{l+1}}) \\ +i(\prod_{i=1}^{l}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_{l+1}})^{\mathscr{X}_{l+1}}), \\ -((-(\prod_{i=1}^{l}|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i}))(-|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_{l+1}}|^{\mathscr{X}_{l+1}})) \\ +i(-((-(\prod_{i=1}^{l}|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i}))(-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_{l+1}}|^{\mathscr{X}_{l+1}}))), \\ -(|-(1-\prod_{i=1}^{l}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})| + (|-(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_{l+1}}|^2)^{\mathscr{X}_{l+1}})|) - \\ (|-(1-\prod_{i=1}^{l}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})|)(|-(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_{l+1}}|^2)^{\mathscr{X}_{l+1}})|))^{\frac{1}{2}} + \\ i(-(|-(1-\prod_{i=1}^{l}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})| + (|-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_{l+1}}|^2)^{\mathscr{X}_{l+1}})|) - \\ (|-(1-\prod_{i=1}^{l}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})|)(|-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_{l+1}}|^2)^{\mathscr{X}_{l+1}})|))^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} (1-\prod_{i=1}^{l+1}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} \\ +i(1-\prod_{i=1}^{l+1}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ \prod_{i=1}^{l+1}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i} + i\prod_{i=1}^{l+1}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}, \\ -(\prod_{i=1}^{l+1}|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i}) + i(-(\prod_{i=1}^{l+1}|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i})), \\ -(1-\prod_{i=1}^{l+1}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} \\ +i(-(1-\prod_{i=1}^{l+1}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}) \end{array} \right\rangle.$$

This matches the formula for $k = l+1$, thereby completing the proof by induction.

(2) The proof is carried out using the same approach as in the proof of (1).

□

Example 4.1. Consider the following BCPFNs:

$$\mathfrak{P}_1 = \langle .11 + i(.79), .78 + i(.15), -.89 + i(-.63), -.12 + i(-.37) \rangle$$

$$\mathfrak{P}_2 = \langle .77 + i(.78), .19 + i(.12), -.44 + i(-.59), -.30 + i(-.33) \rangle$$

and

$$\mathfrak{P}_3 = \langle .63 + i(.28), .29 + i(.55), -.77 + i(-.20), -.19 + i(-.41) \rangle,$$

with the associated weight vector $\mathscr{X} = (.294, .384, .322)^T$ respectively. Then, we proceed as follows: 1- BCPFWA$(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)$

$$= \left\langle \begin{array}{c} (1-\prod_{i=1}^{3}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1-\prod_{i=1}^{3}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ \prod_{i=1}^{3}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i} + i\prod_{i=1}^{3}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}, \\ -(\prod_{i=1}^{3}|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i}) + i(-(\prod_{i=1}^{3}|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i})), \\ -(1-\prod_{i=1}^{3}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(-(1-\prod_{i=1}^{3}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$\approx \langle .6328 + .7002i, .3298 + .2092i, -.6481 + (-.4246)i, -.2258 + (-.3695)i \rangle.$$

2- BCPFWG $(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)$

$$= \left\langle \begin{array}{c} \prod_{i=1}^{3}(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i} + i(\prod_{i=1}^{3}\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}, \\ (1-\prod_{i=1}^{3}(1-(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1-\prod_{i=1}^{3}(1-(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ -(1-\prod_{i=1}^{3}(1-|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(-(1-\prod_{i=1}^{3}(1-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}), \\ -(\prod_{i=1}^{3}|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i}|_i^{\mathscr{X}_i}) + i(-(\prod_{i=1}^{3}|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}|_i^{\mathscr{X}_i})) \end{array} \right\rangle$$

$$\approx \langle .4074 + .5629i, .5219 + .3470i, -.7521 + (-.5275)i, -.1978 + (-.3660)i\rangle.$$

Theorem 4.2. The results obtained by applying the operators BCPFWA $(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k)$ and BCPFWG $(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k)$ are both classified as BCPFNs.

Proof. Theorems 3.4 and 3.5 directly confirm that the BCPFWA and BCPFWG operators produce BCPFNs.

□

Theorem 4.3. (Idempotency) A set of BCPFNs is defined as:
$\mathfrak{P}_i = \left\{\left\langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}\right\rangle\right\}$. Let $\mathscr{X} = (\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_k)^T$ represent the weight vector associated with $\mathfrak{P}_i$, satisfying the conditions where $\mathscr{X}_i > 0$ and $\sum_{i=1}^{k}\mathscr{X}_i = 1$. If all elements $\mathfrak{P}_i$ are identical to a common BCPFN $\mathfrak{P}$, expressed as:
$\mathfrak{P} = \left\langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}\right\rangle$, then

(1) BCPFWA$(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k) = \mathfrak{P}$.
(2) BCPFWG$(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k) = \mathfrak{P}$.

Proof. It suffices to verify the first case, as the remaining cases follow analogously. Assuming $\mathfrak{P}_i = \mathfrak{P}$ for all $i = 1, 2, \ldots, k$, we obtain:

$$\text{BCPFWA}\ (\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k)$$

$$= \left\langle \begin{array}{c} (1-\prod_{i=1}^{k}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1-\prod_{i=1}^{k}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ \prod_{i=1}^{k}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i} + i\prod_{i=1}^{k}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}, -(\prod_{i=1}^{k}|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i}) + i(-(\prod_{i=1}^{k}|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i})), \\ -(1-\prod_{i=1}^{k}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(-(1-\prod_{i=1}^{k}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} (1-\prod_{i=1}^{k}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1-\prod_{i=1}^{k}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ \prod_{i=1}^{k}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}})^{\mathscr{X}_i} + i\prod_{i=1}^{k}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}})^{\mathscr{X}_i}, -(\prod_{i=1}^{k}|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}|^{\mathscr{X}_i}) + i(-(\prod_{i=1}^{k}|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}|^{\mathscr{X}_i})), \\ -(1-\prod_{i=1}^{k}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(-(1-\prod_{i=1}^{k}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2)^{\sum_{i=1}^{k}\mathscr{X}_i})^{\frac{1}{2}} + i(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2)^{\sum_{i=1}^{k}\mathscr{X}_i})^{\frac{1}{2}}, \\ (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}})^{\sum_{i=1}^{k}\mathscr{X}_i} + i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}})^{\sum_{i=1}^{k}\mathscr{X}_i}, -(|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}|^{\sum_{i=1}^{k}\mathscr{X}_i}) + i(-(|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}|^{\sum_{i=1}^{k}\mathscr{X}_i})), \\ -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2)^{\sum_{i=1}^{k}\mathscr{X}_i})^{\frac{1}{2}} + i(-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2)^{\sum_{i=1}^{k}\mathscr{X}_i})^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} (1-(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}})^2))^{\frac{1}{2}} + i(1-(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}})^2))^{\frac{1}{2}}, (\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}}) + i(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}), \\ -(|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}|) + i(-(|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}|)), -(1-(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}|^2))^{\frac{1}{2}} + i(-(1-(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}|^2))^{\frac{1}{2}}) \end{array} \right\rangle$$

$$= \left\langle \ \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}} \ \right\rangle,$$

where $-|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}| = \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}}$, $-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}| = \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}}$, $-|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}| = \mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}}$ and $-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}| = \mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}}$. □

Theorem 4.4. (Boundedness) Consider a set of BCPFNs defined as:
$\mathfrak{P}_i = \left\{\left\langle \mathfrak{mR}^+_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_i}, \mathfrak{nR}^+_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_i}, \mathfrak{mR}^-_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_i}, \mathfrak{nR}^-_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_i}\right\rangle\right\}, \forall i = 1, 2, \ldots, k$. Let $\mathscr{X} = (\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_k)^T$ be the vector of weights associated with each $\mathfrak{P}_i$, where each weight satisfies $\mathscr{X}_i > 0$ and the sum of all weights is normalized as $\sum_{i=1}^k \mathscr{X}_i = 1$. Now, suppose two BCPFNs, denoted as $\overline{\mathfrak{P}}$ and $\underline{\mathfrak{P}}$, are defined such that:
$\overline{\mathfrak{P}} = \left\langle \mathfrak{mR}^{+^*}_{\overline{\mathfrak{P}}} + i\mathfrak{m}\mathfrak{I}^{+^*}_{\overline{\mathfrak{P}}}, \mathfrak{nR}^{+^\bullet}_{\overline{\mathfrak{P}}} + i\mathfrak{n}\mathfrak{I}^{+^\bullet}_{\overline{\mathfrak{P}}}, \mathfrak{mR}^{-^\bullet}_{\overline{\mathfrak{P}}} + i\mathfrak{m}\mathfrak{I}^{-^\bullet}_{\overline{\mathfrak{P}}}, \mathfrak{nR}^{-^*}_{\overline{\mathfrak{P}}} + i\mathfrak{n}\mathfrak{I}^{-^*}_{\overline{\mathfrak{P}}}\right\rangle$,
which may further be rewritten in terms of the maximum and minimum values of the given BCPFNs as:
$\overline{\mathfrak{P}} = \left\langle \max(\mathfrak{mR}^+_{\overline{\mathfrak{P}}_i}) + i\max(\mathfrak{m}\mathfrak{I}^+_{\overline{\mathfrak{P}}_i}), \min(\mathfrak{nR}^+_{\overline{\mathfrak{P}}_i}) + i\min(\mathfrak{n}\mathfrak{I}^+_{\overline{\mathfrak{P}}_i}), \min(\mathfrak{mR}^-_{\overline{\mathfrak{P}}_i}) + i\min(\mathfrak{m}\mathfrak{I}^-_{\overline{\mathfrak{P}}_i}), \max(\mathfrak{nR}^-_{\overline{\mathfrak{P}}_i}) + i\max(\mathfrak{n}\mathfrak{I}^-_{\overline{\mathfrak{P}}_i})\right\rangle$,
where $1 \le i \le k$.
Similarly, another BCPFN $\underline{\mathfrak{P}}$ is defined as:
$\underline{\mathfrak{P}} = \left\langle \mathfrak{mR}^{+^\bullet}_{\underline{\mathfrak{P}}} + i\mathfrak{m}\mathfrak{I}^{+^\bullet}_{\underline{\mathfrak{P}}}, \mathfrak{nR}^{+^*}_{\underline{\mathfrak{P}}} + i\mathfrak{n}\mathfrak{I}^{+^*}_{\underline{\mathfrak{P}}}, \mathfrak{mR}^{-^*}_{\underline{\mathfrak{P}}} + i\mathfrak{m}\mathfrak{I}^{-^*}_{\underline{\mathfrak{P}}}, \mathfrak{nR}^{-^\bullet}_{\underline{\mathfrak{P}}} + i\mathfrak{n}\mathfrak{I}^{-^\bullet}_{\underline{\mathfrak{P}}}\right\rangle$,
which can also be rewritten in terms of the minimum and maximum values as:
$\underline{\mathfrak{P}} = \left\langle \min(\mathfrak{mR}^+_{\underline{\mathfrak{P}}_i}) + i\min(\mathfrak{m}\mathfrak{I}^+_{\underline{\mathfrak{P}}_i}), \max(\mathfrak{nR}^+_{\underline{\mathfrak{P}}_i}) + i\max(\mathfrak{n}\mathfrak{I}^+_{\underline{\mathfrak{P}}_i}), \max(\mathfrak{mR}^-_{\underline{\mathfrak{P}}_i}) + i\max(\mathfrak{m}\mathfrak{I}^-_{\underline{\mathfrak{P}}_i}), \min(\mathfrak{nR}^-_{\underline{\mathfrak{P}}_i}) + i\min(\mathfrak{n}\mathfrak{I}^-_{\underline{\mathfrak{P}}_i})\right\rangle$,
where $1 \le i \le k$. Then,

(1) $\underline{\mathfrak{P}} \le \text{BCPFWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k) \le \overline{\mathfrak{P}}$.
(2) $\underline{\mathfrak{P}} \le \text{BCPFWG}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k) \le \overline{\mathfrak{P}}$.

Proof. We prove the first result, noting that the others follow similarly. To establish it, we need to verify that:

$$\mathfrak{mR}^{+^\bullet}_{\underline{\mathfrak{P}}} \le (1 - \textstyle\prod_{i=1}^k (1 - (\mathfrak{mR}^+_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} \le \mathfrak{mR}^{+^*}_{\underline{\mathfrak{P}}},$$

$$\mathfrak{m}\mathfrak{I}^{+^\bullet}_{\underline{\mathfrak{P}}} \le (1 - \textstyle\prod_{i=1}^k (1 - (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} \le \mathfrak{m}\mathfrak{I}^{+^*}_{\underline{\mathfrak{P}}},$$

$$\mathfrak{nR}^{+^*}_{\underline{\mathfrak{P}}} \ge \textstyle\prod_{i=1}^k (\mathfrak{nR}^+_{\mathfrak{P}_i})^{\mathscr{X}_i} \ge \mathfrak{nR}^{+^\bullet}_{\underline{\mathfrak{P}}},$$

$$\mathfrak{n}\mathfrak{I}^{+^*}_{\underline{\mathfrak{P}}} \ge \textstyle\prod_{i=1}^k (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_i})^{\mathscr{X}_i} \ge \mathfrak{n}\mathfrak{I}^{+^\bullet}_{\underline{\mathfrak{P}}},$$

$$\mathfrak{mR}^{-^*}_{\underline{\mathfrak{P}}} \ge -(\textstyle\prod_{i=1}^k |\mathfrak{mR}^-_{\mathfrak{P}_i}|^{\mathscr{X}_i}) \ge \mathfrak{mR}^{-^\bullet}_{\underline{\mathfrak{P}}},$$

$$\mathfrak{m}\mathfrak{I}^{-^*}_{\underline{\mathfrak{P}}} \ge -(\textstyle\prod_{i=1}^k |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_i}|^{\mathscr{X}_i}) \ge \mathfrak{m}\mathfrak{I}^{-^\bullet}_{\underline{\mathfrak{P}}},$$

$$\mathfrak{nR}^{-^\bullet}_{\underline{\mathfrak{P}}} \le -(1 - \textstyle\prod_{i=1}^k (1 - |\mathfrak{nR}^-_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} \le \mathfrak{nR}^{-^*}_{\underline{\mathfrak{P}}},$$

and

$$\mathfrak{n}\mathfrak{I}^{-^\bullet}_{\underline{\mathfrak{P}}} \le -(1 - \textstyle\prod_{i=1}^k (1 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} \le \mathfrak{n}\mathfrak{I}^{-^*}_{\underline{\mathfrak{P}}}.$$

Given the following inequalities:

$$\mathfrak{mR}^{+^\bullet}_{\underline{\mathfrak{P}}} \le \mathfrak{mR}^+_{\mathfrak{P}_i} \le \mathfrak{mR}^{+^*}_{\overline{\mathfrak{P}}}, \quad \mathfrak{mR}^{-^\bullet}_{\underline{\mathfrak{P}}} \le \mathfrak{mR}^-_{\mathfrak{P}_i} \le \mathfrak{mR}^{-^*}_{\overline{\mathfrak{P}}},$$

$$\mathfrak{nR}^{+^\bullet}_{\underline{\mathfrak{P}}} \le \mathfrak{nR}^+_{\mathfrak{P}_i} \le \mathfrak{nR}^{+^*}_{\overline{\mathfrak{P}}}, \quad \mathfrak{nR}^{-^\bullet}_{\underline{\mathfrak{P}}} \le \mathfrak{nR}^-_{\mathfrak{P}_i} \le \mathfrak{nR}^{-^*}_{\overline{\mathfrak{P}}},$$

analogously, the following relations apply to the additional terms:

$$\mathfrak{m}\mathfrak{I}^{+\bullet}_{\underline{\mathfrak{P}}} \leq \mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i} \leq \mathfrak{m}\mathfrak{I}^{+*}_{\overline{\mathfrak{P}}}, \quad \mathfrak{m}\mathfrak{I}^{-\bullet}_{\underline{\mathfrak{P}}} \leq \mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i} \leq \mathfrak{m}\mathfrak{I}^{-*}_{\overline{\mathfrak{P}}},$$

$$\mathfrak{n}\mathfrak{I}^{+\bullet}_{\underline{\mathfrak{P}}} \leq \mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i} \leq \mathfrak{n}\mathfrak{I}^{+*}_{\overline{\mathfrak{P}}}, \quad \mathfrak{n}\mathfrak{I}^{-\bullet}_{\underline{\mathfrak{P}}} \leq \mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i} \leq \mathfrak{n}\mathfrak{I}^{-*}_{\overline{\mathfrak{P}}}.$$

From these inequalities, we can conclude that:

$$\begin{aligned}
&\mathfrak{m}\mathfrak{R}^{+\bullet}_{\mathfrak{P}} = (1-(1-(\mathfrak{m}\mathfrak{R}^{+\bullet}_{\mathfrak{P}})^2)^{\sum_{i=1}^{k}\mathscr{X}_i})^{\frac{1}{2}}\\
&= (1-\prod_{i=1}^{k}(1-(\mathfrak{m}\mathfrak{R}^{+\bullet}_{\mathfrak{P}})^2)^{\mathscr{X}_i})^{\frac{1}{2}}\\
&\leq (1-\prod_{i=1}^{k}(1-(\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}\\
&\leq (1-\prod_{i=1}^{k}(1-(\mathfrak{m}\mathfrak{R}^{+^*}_{\mathfrak{P}})^2)^{\mathscr{X}_i})^{\frac{1}{2}}\\
&= (1-(1-(\mathfrak{m}\mathfrak{R}^{+^*}_{\mathfrak{P}})^2)^{\sum_{i=1}^{k}\mathscr{X}_i})^{\frac{1}{2}} = \mathfrak{m}\mathfrak{R}^{+^*}_{\mathfrak{P}},
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{n}\mathfrak{R}^{+\bullet}_{\mathfrak{P}} = (\mathfrak{n}\mathfrak{R}^{+\bullet}_{\mathfrak{P}})^{\sum_{i=1}^{k}\mathscr{X}_i}\\
&= \prod_{i=1}^{k}(\mathfrak{n}\mathfrak{R}^{+\bullet}_{\mathfrak{P}})^{\mathscr{X}_i}\\
&\leq \prod_{i=1}^{k}(\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i}\\
&\leq \prod_{i=1}^{k}(\mathfrak{n}\mathfrak{R}^{+^*}_{\mathfrak{P}})^{\mathscr{X}_i} = (\mathfrak{n}\mathfrak{R}^{+^*}_{\mathfrak{P}})^{\sum_{i=1}^{k}\mathscr{X}_i} = \mathfrak{n}\mathfrak{R}^{+^*}_{\mathfrak{P}},
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{m}\mathfrak{R}^{-\bullet}_{\mathfrak{P}} = -(|\mathfrak{m}\mathfrak{R}^{-\bullet}_{\mathfrak{P}}|^{\sum_{i=1}^{k}\mathscr{X}_i})\\
&= -(\prod_{i=1}^{k}(|\mathfrak{m}\mathfrak{R}^{-\bullet}_{\mathfrak{P}}|)^{\mathscr{X}_i})\\
&\leq -(\prod_{i=1}^{k}(|\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i}|)^{\mathscr{X}_i})\\
&\leq -(\prod_{i=1}^{k}(|\mathfrak{m}\mathfrak{R}^{-^*}_{\mathfrak{P}}|)^{\mathscr{X}_i}) = -(|\mathfrak{m}\mathfrak{R}^{-^*}_{\mathfrak{P}}|^{\sum_{i=1}^{k}\mathscr{X}_i}) = -|\mathfrak{m}\mathfrak{R}^{-^*}_{\mathfrak{P}}| = \mathfrak{m}\mathfrak{R}^{-^*}_{\mathfrak{P}},
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{n}\mathfrak{R}^{-\bullet}_{\mathfrak{P}} = -(1-(1-|\mathfrak{n}\mathfrak{R}^{-\bullet}_{\mathfrak{P}}|^2)^{\sum_{i=1}^{k}\mathscr{X}_i})^{\frac{1}{2}}\\
&= -(1-\prod_{i=1}^{k}(1-|\mathfrak{n}\mathfrak{R}^{-\bullet}_{\mathfrak{P}}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}\\
&\leq -(1-\prod_{i=1}^{k}(1-|\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}\\
&\leq -(1-\prod_{i=1}^{k}(1-|\mathfrak{n}\mathfrak{R}^{-^*}_{\mathfrak{P}}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} = -(1-(1-(\mathfrak{n}\mathfrak{R}^{-^*}_{\mathfrak{P}})^2)^{\sum_{i=1}^{k}\mathscr{X}_i})^{\frac{1}{2}} = -|\mathfrak{n}\mathfrak{R}^{-^*}_{\mathfrak{P}}| = \mathfrak{n}\mathfrak{R}^{-^*}_{\mathfrak{P}}.
\end{aligned}$$

By applying the same method, we further obtain:

$$\mathfrak{m}\mathfrak{I}^{+\bullet}_{\underline{\mathfrak{P}}} \leq (1-\prod_{i=1}^{k}(1-(\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} \leq \mathfrak{m}\mathfrak{I}^{+*}_{\underline{\mathfrak{P}}},$$

$$\mathfrak{n}\mathfrak{I}^{+*}_{\underline{\mathfrak{P}}} \geq \prod_{i=1}^{k}(\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i})^{\mathscr{X}_i} \geq \mathfrak{n}\mathfrak{I}^{+\bullet}_{\underline{\mathfrak{P}}},$$

$$\mathfrak{m}\mathfrak{I}^{-*}_{\underline{\mathfrak{P}}} \geq -(\prod_{i=1}^{k}|\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^{\mathscr{X}_i}) \geq \mathfrak{m}\mathfrak{I}^{-\bullet}_{\underline{\mathfrak{P}}},$$

and

$$\mathfrak{n}\mathfrak{I}^{-\bullet}_{\underline{\mathfrak{P}}} \leq -(1-\prod_{i=1}^{k}(1-|\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} \leq \mathfrak{n}\mathfrak{I}^{-*}_{\underline{\mathfrak{P}}}.$$

□

Theorem 4.5. (Monotonicity) Let
$\mathfrak{P}_i = \left\{\langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_i}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_i}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_i}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_i}\rangle\right\}$ and

$\tilde{\mathfrak{P}}_i = \left\{ \langle \mathfrak{m}\mathfrak{R}^+_{\tilde{\mathfrak{P}}_i} + i\mathfrak{m}\mathfrak{I}^+_{\tilde{\mathfrak{P}}_i}, \mathfrak{n}\mathfrak{R}^+_{\tilde{\mathfrak{P}}_i} + i\mathfrak{n}\mathfrak{I}^+_{\tilde{\mathfrak{P}}_i}, \mathfrak{m}\mathfrak{R}^-_{\tilde{\mathfrak{P}}_i} + i\mathfrak{m}\mathfrak{I}^-_{\tilde{\mathfrak{P}}_i}, \mathfrak{n}\mathfrak{R}^-_{\tilde{\mathfrak{P}}_i} + i\mathfrak{n}\mathfrak{I}^-_{\tilde{\mathfrak{P}}_i} \rangle \right\}$ represent two sets of BCPFNs, indexed by $i = 1, 2, \ldots, k$. If $\mathfrak{P}_i \subset \tilde{\mathfrak{P}}_i$ for every i, then:

(1) $\mathrm{BCPFWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k) \leq \mathrm{BCPFWA}(\tilde{\mathfrak{P}}_1, \tilde{\mathfrak{P}}_2, \ldots, \tilde{\mathfrak{P}}_k)$.
(2) $\mathrm{BCPFWG}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k) \leq \mathrm{BCPFWG}(\tilde{\mathfrak{P}}_1, \tilde{\mathfrak{P}}_2, \ldots, \tilde{\mathfrak{P}}_k)$.

Proof. It suffices to prove the first part, as the second can be shown analogously. The following relations hold for each i:

$$\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_i} \leq \mathfrak{m}\mathfrak{R}^+_{\tilde{\mathfrak{P}}_i}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_i} \geq \mathfrak{m}\mathfrak{R}^-_{\tilde{\mathfrak{P}}_i}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_i} \geq \mathfrak{n}\mathfrak{R}^+_{\tilde{\mathfrak{P}}_i}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_i} \leq \mathfrak{n}\mathfrak{R}^-_{\tilde{\mathfrak{P}}_i},$$

$$\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_i} \leq \mathfrak{m}\mathfrak{I}^+_{\tilde{\mathfrak{P}}_i}, \mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_i} \geq \mathfrak{m}\mathfrak{I}^-_{\tilde{\mathfrak{P}}_i}, \mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_i} \geq \mathfrak{n}\mathfrak{I}^+_{\tilde{\mathfrak{P}}_i}, \text{ and } \mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_i} \leq \mathfrak{n}\mathfrak{I}^-_{\tilde{\mathfrak{P}}_i}.$$

This leads to the following set of inequalities:

$$(1 - \prod_{i=1}^k (1 - (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} \leq (1 - \prod_{i=1}^k (1 - (\mathfrak{m}\mathfrak{R}^+_{\tilde{\mathfrak{P}}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}},$$
$$(1 - \prod_{i=1}^k (1 - (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} \leq (1 - \prod_{i=1}^k (1 - (\mathfrak{m}\mathfrak{I}^+_{\tilde{\mathfrak{P}}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}},$$
$$\prod_{i=1}^k (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_i})^{\mathscr{X}_i} \geq \prod_{i=1}^k (\mathfrak{n}\mathfrak{R}^+_{\tilde{\mathfrak{P}}_i})^{\mathscr{X}_i},$$
$$\prod_{i=1}^k (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_i})^{\mathscr{X}_i} \geq \prod_{i=1}^k (\mathfrak{n}\mathfrak{I}^+_{\tilde{\mathfrak{P}}_i})^{\mathscr{X}_i},$$
$$-(\prod_{i=1}^k |\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_i}|^{\mathscr{X}_i}) \geq -(\prod_{i=1}^k |\mathfrak{m}\mathfrak{R}^-_{\tilde{\mathfrak{P}}_i}|^{\mathscr{X}_i}),$$
$$-(\prod_{i=1}^k |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_i}|^{\mathscr{X}_i}) \geq -(\prod_{i=1}^k |\mathfrak{m}\mathfrak{I}^-_{\tilde{\mathfrak{P}}_i}|^{\mathscr{X}_i}),$$
$$-(1 - \prod_{i=1}^k (1 - |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} \leq -(1 - \prod_{i=1}^k (1 - |\mathfrak{n}\mathfrak{R}^-_{\tilde{\mathfrak{P}}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}},$$
and
$$-(1 - \prod_{i=1}^k (1 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} \leq -(1 - \prod_{i=1}^k (1 - |\mathfrak{n}\mathfrak{I}^-_{\tilde{\mathfrak{P}}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}.$$

Therefore, it follows that:

$$\begin{gathered}
\mathrm{BCPFWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k) \\
= \left\langle \begin{matrix} (1 - \prod_{i=1}^k (1 - (\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1 - \prod_{i=1}^k (1 - (\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ \prod_{i=1}^k (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_i})^{\mathscr{X}_i} + i\prod_{i=1}^k (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_i})^{\mathscr{X}_i}, -(\prod_{i=1}^k |\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_i}|^{\mathscr{X}_i}) + i(-(\prod_{i=1}^k |\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_i}|^{\mathscr{X}_i})), \\ -(1 - \prod_{i=1}^k (1 - |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(-(1 - \prod_{i=1}^k (1 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}) \end{matrix} \right\rangle \\
\leq \left\langle \begin{matrix} (1 - \prod_{i=1}^k (1 - (\mathfrak{m}\mathfrak{R}^+_{\tilde{\mathfrak{P}}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(1 - \prod_{i=1}^k (1 - (\mathfrak{m}\mathfrak{I}^+_{\tilde{\mathfrak{P}}_i})^2)^{\mathscr{X}_i})^{\frac{1}{2}}, \\ \prod_{i=1}^k (\mathfrak{n}\mathfrak{R}^+_{\tilde{\mathfrak{P}}_i})^{\mathscr{X}_i} + i\prod_{i=1}^k (\mathfrak{n}\mathfrak{I}^+_{\tilde{\mathfrak{P}}_i})^{\mathscr{X}_i}, -(\prod_{i=1}^k |\mathfrak{m}\mathfrak{R}^-_{\tilde{\mathfrak{P}}_i}|^{\mathscr{X}_i}) + i(-(\prod_{i=1}^k |\mathfrak{m}\mathfrak{I}^-_{\tilde{\mathfrak{P}}_i}|^{\mathscr{X}_i})), \\ -(1 - \prod_{i=1}^k (1 - |\mathfrak{n}\mathfrak{R}^-_{\tilde{\mathfrak{P}}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}} + i(-(1 - \prod_{i=1}^k (1 - |\mathfrak{n}\mathfrak{I}^-_{\tilde{\mathfrak{P}}_i}|^2)^{\mathscr{X}_i})^{\frac{1}{2}}) \end{matrix} \right\rangle \\
= \mathrm{BCPFWA}(\tilde{\mathfrak{P}}_1, \tilde{\mathfrak{P}}_2, \ldots, \tilde{\mathfrak{P}}_k).
\end{gathered}$$

$\square$

Theorem 4.6. Let $\mathfrak{P}_i = \left\{\left\langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_i}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_i}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_i} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_i}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_i} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_i}\right\rangle\right\}$, for $i = 1, 2, \ldots, k$, represent a collection of BCPFNs. Additionally, consider $\mathscr{X} = (\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_k)^T$ is a weight vector associated with $\mathfrak{P}_i$, where each weight satisfies $\mathscr{X}_i > 0$ and the sum of all weights is normalized as $\sum_{i=1}^k \mathscr{X}_i = 1$. Then,

(1) $\mathrm{BCPFWA}(\mathfrak{P}_1^c, \mathfrak{P}_2^c, \ldots, \mathfrak{P}_k^c) = (\mathrm{BCPFWG}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k))^c$.

(2) $\mathrm{BCPFWG}(\mathfrak{P}_1^c, \mathfrak{P}_2^c, \ldots, \mathfrak{P}_k^c) = (\mathrm{BCPFWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k))^c$.

Proof. By Theorem 3.7, we derive the following results:

(1)
$$\begin{aligned}\mathrm{BCPFWA}(\mathfrak{P}_1^c, \mathfrak{P}_2^c, \ldots, \mathfrak{P}_k^c) &= \mathscr{X}_1\mathfrak{P}_1^c \oplus \mathscr{X}_2\mathfrak{P}_2^c \oplus \ldots \oplus \mathscr{X}_k\mathfrak{P}_k^c \\ &= (\mathfrak{P}_1^{\mathscr{X}_1})^c \oplus (\mathfrak{P}_2^{\mathscr{X}_2})^c \oplus \ldots \oplus (\mathfrak{P}_k^{\mathscr{X}_k})^c \\ &= ((\mathfrak{P}_1^{\mathscr{X}_1}) \otimes (\mathfrak{P}_2^{\mathscr{X}_2}) \otimes \ldots \otimes (\mathfrak{P}_k^{\mathscr{X}_k}))^c \\ &= (\mathrm{BCPFWG}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k))^c.\end{aligned}$$

(2)
$$\begin{aligned}\mathrm{BCPFWG}(\mathfrak{P}_1^c, \mathfrak{P}_2^c, \ldots, \mathfrak{P}_k^c) &= (\mathfrak{P}_1^c)^{\mathscr{X}_1} \otimes (\mathfrak{P}_2^c)^{\mathscr{X}_2} \otimes \ldots \otimes (\mathfrak{P}_k^c)^{\mathscr{X}_k} \\ &= (\mathscr{X}_1\mathfrak{P}_1)^c \otimes (\mathscr{X}_2\mathfrak{P}_2)^c \otimes \ldots \otimes (\mathscr{X}_k\mathfrak{P}_k)^c \\ &= (\mathscr{X}_1\mathfrak{P}_1 \oplus \mathscr{X}_2\mathfrak{P}_2 \oplus \ldots \oplus \mathscr{X}_k\mathfrak{P}_k)^c \\ &= (\mathrm{BCPFWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k))^c.\end{aligned}$$

□

The following functions play a central role in the ranking process of BCPFNs:

Definition 4.2. Let $\mathfrak{P} = \langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}\rangle$ be an arbitrary BCPFN. Then, the following functions are defined:

(1) The score function of $\mathfrak{P}$ is expressed as:

$$\begin{aligned}\omega(\mathfrak{P}) = \frac{1}{4}\Big(&\Big[(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}})^2 - (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}})^2\Big] + \Big[(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}})^2 - (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}})^2\Big] \\ &+ \Big[|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}|^2 - |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}|^2\Big] + \Big[|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}|^2 - |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}|^2\Big]\Big).\end{aligned}$$

(2) The accuracy function of $\mathfrak{P}$ is expressed as:

$$\begin{aligned}\varpi(\mathfrak{P}) = \frac{1}{4}\Big(&\Big[(\mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}})^2 + (\mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}})^2\Big] + \Big[(\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}})^2 + (\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}})^2\Big] \\ &+ \Big[|\mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}}|^2 + |\mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}}|^2\Big] + \Big[|\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}|^2 + |\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}}|^2\Big]\Big).\end{aligned}$$

Example 4.2. Consider Example 4.1. The following results are observed:

(1) $\omega\Big(\mathrm{BCPFWA}(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)\Big) \approx .2877$, and
$\omega\Big(\mathrm{BCPFWG}(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)\Big) \approx .1902$.

(2) $\varpi\Big(\mathrm{BCPFWA}(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)\Big) \approx .4578$, and
$\varpi\Big(\mathrm{BCPFWG}(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)\Big) \approx .4731$.

Remark 4.1. For any BCPFN
$\mathfrak{P} = \langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}} \rangle$, The following properties hold:

(1) The score function satisfies $\omega(\mathfrak{P}) \in [-1, 1]$.
(2) The accuracy function satisfies $\varpi(\mathfrak{P}) \in [0, 1]$.

Definition 4.3. Consider two BCPFNs,

$$\mathfrak{P}_1 = \left\langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_1}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_1} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_1}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_1} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_1} \right\rangle$$

and

$$\mathfrak{P}_2 = \left\langle \mathfrak{m}\mathfrak{R}^+_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^+_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^+_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^+_{\mathfrak{P}_2}, \mathfrak{m}\mathfrak{R}^-_{\mathfrak{P}_2} + i\mathfrak{m}\mathfrak{I}^-_{\mathfrak{P}_2}, \mathfrak{n}\mathfrak{R}^-_{\mathfrak{P}_2} + i\mathfrak{n}\mathfrak{I}^-_{\mathfrak{P}_2} \right\rangle.$$

The comparison procedure is defined as follows:

(1) If $\omega(\mathfrak{P}_1) < \omega(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \prec \mathfrak{P}_2$.
(2) If $\omega(\mathfrak{P}_1) > \omega(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \succ \mathfrak{P}_2$.
(3) When $\omega(\mathfrak{P}_1) = \omega(\mathfrak{P}_2)$, the comparison relies on ϖ values:
   (a) If $\varpi(\mathfrak{P}_1) < \varpi(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \prec \mathfrak{P}_2$.
   (b) If $\varpi(\mathfrak{P}_1) > \varpi(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \succ \mathfrak{P}_2$.
   (c) If $\varpi(\mathfrak{P}_1) = \varpi(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \approx \mathfrak{P}_2$.

5. Evaluation of the Suggested MADM Strategies in the BCPFNs Environment

In this section, we introduce a systematic MADM framework constructed under the paradigm of BCPFNs. The essence of this approach lies in its ability to capture both bipolarity and the inherent uncertainty of human reasoning, while simultaneously accommodating the geometric complexity of complex-valued information. The suggested methodology not only provides greater flexibility in representing decision-makers' assessments but also enhances the robustness of the final outcomes when compared to traditional crisp or fuzzy environments. To demonstrate its practicality and reliability, the procedure is later applied to real-life scenarios, showcasing its performance in diverse decision-making contexts.

In many decision-making situations, experts are typically required to evaluate a finite collection of alternatives:

$$\{\mathfrak{P}\mathfrak{L}_1, \mathfrak{P}\mathfrak{L}_2, \ldots, \mathfrak{P}\mathfrak{L}_p\},$$

with respect to a predefined set of evaluation attributes:

$$\{\mathfrak{P}\mathfrak{T}_1, \mathfrak{P}\mathfrak{T}_2, \ldots, \mathfrak{P}\mathfrak{T}_k\}.$$

Since not all attributes carry equal relevance, a normalized weight vector is assigned to capture their relative importance:

$$\mathscr{X} = (\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_k)^T, \quad \mathscr{X}_i > 0, \ \sum_{i=1}^{k} \mathscr{X}_i = 1.$$

Each alternative is then assessed under the BCPFN setting, resulting in the following bipolar complex fuzzy decision matrix:

$$\mathfrak{DM} = \left[\mathfrak{P}_{ji}\right]_{p\times k} = \left[\langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_{ji}} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_{ji}},\ \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_{ji}} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_{ji}},\ \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_{ji}} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_{ji}},\ \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_{ji}} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_{ji}}\rangle\right]_{p\times k}.$$

In expanded form, it can be expressed as:

$$\mathfrak{DM} = \begin{bmatrix} \mathfrak{P}_{11} & \mathfrak{P}_{12} & \cdots & \mathfrak{P}_{1k} \\ \mathfrak{P}_{21} & \mathfrak{P}_{22} & \cdots & \mathfrak{P}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{P}_{p1} & \mathfrak{P}_{p2} & \cdots & \mathfrak{P}_{pk} \end{bmatrix},$$

where $\mathfrak{P}_{ji}$ denotes the evaluation of alternative $\mathfrak{PL}_j$ $(j = 1, 2, \ldots, p)$ with respect to attribute $\mathfrak{PT}_i$ $(i = 1, 2, \ldots, k)$.

The structural components of $\mathfrak{P}_{ji}$ are defined as follows:

- $\mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_{ji}}$: real term of positive membership.
- $\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_{ji}}$: imaginary term of positive membership.
- $\mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_{ji}}$: real term of positive non-membership.
- $\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_{ji}}$: imaginary term of positive non-membership.
- $\mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_{ji}}$: real term of negative membership.
- $\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_{ji}}$: imaginary term of negative membership.
- $\mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_{ji}}$: real term of negative non-membership.
- $\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_{ji}}$: imaginary term of negative non-membership.

Algorithm: MADM Procedure in the BCPFN Framework. The stepwise procedure for carrying out decision analysis within the BCPFN environment is outlined below:

Step 1: Problem Specification and Attribute Identification

- Clearly state the decision problem and objectives.
- Establish the set of feasible alternatives and relevant attributes.
- Determine attribute weights $\mathscr{X}$ using expert knowledge or preference elicitation.

Step 2: Construction of the BCPF Decision Matrix

- Collect evaluations of each alternative with respect to each attribute, represented as BCPFN values, to populate $\mathfrak{DM}$.

Step 3: Normalization of the Decision Matrix

- Standardize or normalize attribute data, ensuring comparability across heterogeneous criteria.

Step 4: Aggregation of Evaluations via BCPF Operators

- Apply the weighted average and weighted geometric aggregation operators defined in the BCPFN setting. Specifically:

(1) Weighted average operator:

$$\mathrm{BCPFWA_j} = \mathrm{BCPFWA}(\mathfrak{P}_{j1}, \ldots, \mathfrak{P}_{jk}),$$

which produces a BCPFN that summarizes the collective evaluation of $\mathfrak{P}\mathfrak{L}_j$.

(2) Weighted geometric operator:

$$\mathrm{BCPFWG_j} = \mathrm{BCPFWG}(\mathfrak{P}_{j1}, \ldots, \mathfrak{P}_{jk}),$$

providing an alternative aggregation form that captures multiplicative interactions.

Step 5: Score Determination

- Transform the aggregated BCPFN evaluations into comparable scalar values (scores). This step synthesizes the outputs of both $\mathrm{BCPFWA_j}$ and $\mathrm{BCPFWG_j}$ for each alternative.

Step 6: Ranking of Alternatives

- Arrange alternatives in descending order of their final scores. The highest-ranked alternative represents the most preferred choice given the decision context.

The above process constitutes a comprehensive methodology for decision-making within the BCPFN setting. By combining the expressive capacity of bipolar complex representations with the rigor of fuzzy aggregation operators, the suggested framework ensures consistent, precise, and rational decision outcomes across a wide variety of MADM applications.

5.1. **Case study: Enhancing green supply chain management using BCPFWA and BCPFWG operators.** Green supply chain management (GSCM) has become a critical priority for industries worldwide, as companies seek to balance cost efficiency with environmental sustainability. In today's competitive market, firms are under increasing pressure to select suppliers that not only meet economic requirements but also comply with environmental regulations, reduce carbon footprints, and contribute positively to corporate social responsibility goals. The challenge lies in integrating sustainability considerations into the supplier selection process, which is inherently multi-dimensional and uncertain.

In this case study, different supplier alternatives are evaluated using MADM techniques with the aid of the BCPFWA and BCPFWG operators. The objective is to identify the most sustainable and economically viable suppliers, thereby ensuring that supply chain strategies contribute effectively to long-term environmental and social goals. The evaluation framework considers both traditional supply chain performance measures and sustainability-oriented factors.

The following supplier alternatives are considered for sustainable GSCM practices:

$\mathscr{S}_1$**: Cost-Oriented Supplier.** A cost-oriented supplier primarily focuses on minimizing production and delivery costs. While this strategy may improve short-term profitability, it often overlooks sustainability aspects such as environmental responsibility and waste management. In highly competitive markets, this approach can be beneficial in the short term but may expose firms to reputational risks and regulatory penalties.

$\mathscr{S}_2$**: Quality-Driven Supplier.** This supplier emphasizes superior product quality, reliability, and long-term durability of goods. Quality-driven suppliers reduce risks related to defective products and customer dissatisfaction. However, quality improvements may come with higher costs and sometimes limited emphasis on environmental performance.

$\mathscr{S}_3$**: Socially Responsible Supplier.** These suppliers prioritize fair labor practices, community development, and ethical business standards. While such initiatives strengthen a company's social responsibility and corporate reputation, the supplier may not always be the most cost-efficient option compared to conventional suppliers.

$\mathscr{S}_4$**: Environmentally Responsible Supplier.** Environmentally responsible suppliers adopt eco-friendly practices such as reducing CO_2 emissions, using renewable energy sources, and ensuring sustainable waste disposal. This approach enhances the green image of companies but may require higher operational costs and investments in green technologies.

$\mathscr{S}_5$**: Technologically Advanced Supplier.** Suppliers that integrate technologies such as IoT, blockchain, and AI into supply chain processes can provide transparency, traceability, and efficiency. These advanced systems improve supply chain resilience and reduce long-term costs. However, initial adoption costs and technological complexities may be a barrier for some firms.

$\mathscr{S}_6$**: Balanced Sustainability Supplier.** Balanced sustainability suppliers aim to optimize cost, quality, environmental, and social performance simultaneously. While this alternative may not lead in any single category, it often provides the most stable and well-rounded option for companies seeking long-term strategic partnerships.

The supplier alternatives are evaluated against eight critical attributes:

$\mathscr{C}_1$: **Cost Efficiency**: The ability of the supplier to minimize operational and procurement costs.

$\mathscr{C}_2$: **Product Quality and Reliability**: The level of consistency and durability of the supplied goods.

$\mathscr{C}_3$: **Carbon Footprint (CO_2 Emissions)**: The extent to which the supplier reduces greenhouse gas emissions during production and transportation.

$\mathscr{C}_4$: **Waste Management and Resource Efficiency**: The supplier's ability to minimize waste generation and optimize resource utilization.

$\mathscr{C}_5$: **Technological Integration**: The extent of adopting modern technologies such as IoT, blockchain, or AI for transparency and efficiency.

$\mathscr{C}_6$: **Social Responsibility**: The supplier's contribution to fair labor practices, ethical sourcing, and community development.

$\mathscr{C}_7$: **Scalability and Adaptability**: The supplier's capability to adjust production levels and adapt to market or regulatory changes.

$\mathscr{C}_8$: **Long-Term Partnership Potential**: The supplier's ability to maintain stable, collaborative, and strategic relationships with firms.

By applying the BCPFWA and BCPFWG operators, the suppliers are aggregated and ranked according to their performance across these attributes. This structured decision-making ensures that trade-offs between cost, quality, environmental performance, and social impact are properly evaluated, supporting both economic and sustainability goals.

The decision matrix is structured using bipolar complex PFNs. Each entry in this matrix signifies the evaluation of a specific supplier alternative, denoted as $\mathscr{S}_j$ $(j = 1, 2, \ldots, p)$, with respect to a given attribute, represented by $\mathscr{C}_i$ $(i = 1, 2, \ldots, k)$. Mathematically, the decision matrix is formulated as follows:

$$\mathfrak{DM} = \left[\langle \mathfrak{m}\mathfrak{R}^{+}_{\mathfrak{P}_{ji}} + i\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_{ji}}, \mathfrak{n}\mathfrak{R}^{+}_{\mathfrak{P}_{ji}} + i\mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_{ji}}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_{ji}} + i\mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_{ji}}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_{ji}} + i\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_{ji}} \rangle \right]_{p \times k}.$$

The attribute weights are given as:

$$\mathscr{X} = (.13, .09, .18, .15, .14, .12, .08, .11)^{T},$$

where the highest weight is assigned to CO_2 emissions (.18), reflecting the growing importance of environmental sustainability in supplier selection, followed by cost efficiency and product quality.

The evaluation procedure follows the systematic steps of the MADM framework:

- Constructing and normalizing the decision matrix (Step 2 and Step 3).
- Applying the aggregation operators:

  $$\mathrm{BCPFWA_j} = \mathrm{BCPFWA}\left(\mathfrak{P}_{j1}, \mathfrak{P}_{j2}, \ldots, \mathfrak{P}_{j8}\right),$$

  and

  $$\mathrm{BCPFWG_j} = \mathrm{BCPFWG}\left(\mathfrak{P}_{j1}, \mathfrak{P}_{j2}, \ldots, \mathfrak{P}_{j8}\right),$$

  to compute the aggregated scores for each supplier.
- Determining the final evaluation scores (Step 5).
- Ranking suppliers in descending order of their scores to identify the most sustainable choice (Step 6).

The ranking obtained using the BCPFWA operator is:

$$\mathscr{S}_6 \succ \mathscr{S}_4 \succ \mathscr{S}_5 \succ \mathscr{S}_3 \succ \mathscr{S}_2 \succ \mathscr{S}_1.$$

Similarly, the ranking produced by the BCPFWG operator is:

$$\mathscr{S}_6 \succ \mathscr{S}_4 \succ \mathscr{S}_5 \succ \mathscr{S}_3 \succ \mathscr{S}_2 \succ \mathscr{S}_1.$$

Both operators consistently indicate that the balanced sustainability supplier ($\mathscr{S}_6$) is the most appropriate choice, as it effectively combines cost efficiency, product quality, environmental responsibility, and social impact. The environmentally responsible supplier ($\mathscr{S}_4$). A graphical illustration of these ranking outcomes is presented in Figure 2.

TABLE 1. BCPFS values

GSCM	$\mathscr{C}_1$	$\mathscr{C}_2$
$\mathscr{S}_1$	$\langle .35+.45i, .48+.40i, -.40-.42i, -.45-.38i\rangle$	$\langle .28+.40i, .45+.38i, -.36-.35i, -.42-.36i\rangle$
$\mathscr{S}_2$	$\langle .40+.40i, .42+.35i, -.38-.32i, -.42-.34i\rangle$	$\langle .32+.35i, .38+.32i, -.34-.30i, -.36-.31i\rangle$
$\mathscr{S}_3$	$\langle .50+.32i, .35+.28i, -.32-.30i, -.38-.33i\rangle$	$\langle .40+.28i, .30+.25i, -.28-.26i, -.36-.27i\rangle$
$\mathscr{S}_4$	$\langle .45+.38i, .38+.32i, -.35-.33i, -.40-.30i\rangle$	$\langle .38+.32i, .34+.28i, -.30-.30i, -.36-.29i\rangle$
$\mathscr{S}_5$	$\langle .48+.34i, .40+.30i, -.34-.31i, -.39-.28i\rangle$	$\langle .42+.30i, .36+.26i, -.30-.28i, -.36-.27i\rangle$
$\mathscr{S}_6$	$\langle .55+.28i, .28+.25i, -.28-.30i, -.28-.22i\rangle$	$\langle .48+.25i, .25+.22i, -.22-.24i, -.24-.21i\rangle$
GSCM	$\mathscr{C}_3$	$\mathscr{C}_4$
$\mathscr{S}_1$	$\langle .30+.42i, .35+.40i, -.32-.36i, -.38-.45i\rangle$	$\langle .25+.38i, .28+.36i, -.27-.38i, -.30-.42i\rangle$
$\mathscr{S}_2$	$\langle .35+.40i, .30+.36i, -.31-.34i, -.32-.38i\rangle$	$\langle .28+.32i, .25+.30i, -.26-.32i, -.28-.36i\rangle$
$\mathscr{S}_3$	$\langle .45+.32i, .38+.28i, -.32-.35i, -.38-.32i\rangle$	$\langle .35+.28i, .32+.25i, -.28-.29i, -.36-.27i\rangle$
$\mathscr{S}_4$	$\langle .40+.37i, .35+.32i, -.35-.37i, -.36-.30i\rangle$	$\langle .38+.30i, .34+.28i, -.30-.33i, -.36-.27i\rangle$
$\mathscr{S}_5$	$\langle .44+.36i, .40+.30i, -.34-.36i, -.38-.28i\rangle$	$\langle .40+.30i, .36+.28i, -.34-.30i, -.34-.27i\rangle$
$\mathscr{S}_6$	$\langle .50+.28i, .26+.25i, -.28-.30i, -.45-.21i\rangle$	$\langle .26+.22i, .24+.20i, -.22-.24i, -.22-.21i\rangle$
GSCM	$\mathscr{C}_5$	$\mathscr{C}_6$
$\mathscr{S}_1$	$\langle .27+.40i, .36+.38i, -.36-.42i, -.42-.39i\rangle$	$\langle .20+.38i, .27+.36i, -.32-.35i, -.38-.36i\rangle$
$\mathscr{S}_2$	$\langle .33+.36i, .38+.32i, -.34-.40i, -.36-.34i\rangle$	$\langle .25+.32i, .30+.30i, -.36-.34i, -.34-.31i\rangle$
$\mathscr{S}_3$	$\langle .42+.32i, .34+.30i, -.32-.36i, -.38-.31i\rangle$	$\langle .38+.30i, .32+.28i, -.28-.34i, -.36-.33i\rangle$
$\mathscr{S}_4$	$\langle .37+.35i, .33+.30i, -.35-.31i, -.36-.30i\rangle$	$\langle .36+.32i, .34+.30i, -.34-.33i, -.34-.33i\rangle$
$\mathscr{S}_5$	$\langle .40+.34i, .36+.31i, -.30-.32i, -.38-.30i\rangle$	$\langle .32+.30i, .30+.28i, -.34-.32i, -.34-.29i\rangle$
$\mathscr{S}_6$	$\langle .46+.28i, .38+.25i, -.28-.24i, -.45-.22i\rangle$	$\langle .38+.25i, .34+.22i, -.22-.20i, -.42-.21i\rangle$
GSCM	$\mathscr{C}_7$	$\mathscr{C}_8$
$\mathscr{S}_1$	$\langle .22+.35i, .44+.37i, -.34-.36i, -.40-.38i\rangle$	$\langle .24+.36i, .42+.36i, -.35-.36i, -.39-.31i\rangle$
$\mathscr{S}_2$	$\langle .28+.36i, .40+.32i, -.32-.34i, -.36-.35i\rangle$	$\langle .29+.31i, .36+.32i, -.34-.34i, -.36-.29i\rangle$
$\mathscr{S}_3$	$\langle .40+.32i, .28+.30i, -.27-.29i, -.32-.31i\rangle$	$\langle .38+.28i, .36+.28i, -.26-.28i, -.32-.25i\rangle$
$\mathscr{S}_4$	$\langle .36+.30i, .30+.28i, -.30-.31i, -.34-.33i\rangle$	$\langle .34+.28i, .32+.30i, -.28-.28i, -.33-.27i\rangle$
$\mathscr{S}_5$	$\langle .38+.32i, .34+.30i, -.28-.30i, -.36-.27i\rangle$	$\langle .36+.28i, .34+.30i, -.28-.28i, -.33-.27i\rangle$
$\mathscr{S}_6$	$\langle .45+.38i, .38+.36i, -.22-.24i, -.42-.21i\rangle$	$\langle .42+.36i, .40+.33i, -.20-.22i, -.39-.19i\rangle$

TABLE 2. Aggregated BCPFWA information matrix

GSCM	BCPFWA
$\mathscr{S}_1$	$\langle .2722+.3978i, .3648+.3774i, -.3348+(-.3761)i, -.3923+(-.3910)i\rangle$
$\mathscr{S}_2$	$\langle .3203+.3574i, .3362+.3250i, -.3265+(-.3381)i, -.3491+(-.3407)i\rangle$
$\mathscr{S}_3$	$\langle .4164+.3043i, .3356+.2767i, -.2941+(-.3123)i, -.3623+(-.3022)i\rangle$
$\mathscr{S}_4$	$\langle .3846+.3342i, .3396+.2994i, -.3239+(-.3235)i, -.3586+(-.2981)i\rangle$
$\mathscr{S}_5$	$\langle .4074+.3224i, .3600+.2920i, -.3184+(-.3123)i, -.3624+(-.2799)i\rangle$
$\mathscr{S}_6$	$\langle .4481+.2862i, .3042+.2498i, -.2427+(-.2492)i, -.3764+(-.2107)i\rangle$

TABLE 3. Aggregated BCPFWG information matrix

GSCM	BCPFWG
$\mathscr{S}_1$	$\langle .2644 + .3952i, .3815 + .3783i, -.3396 + (-.3785)i, -.3863 + (-.3859)i\rangle$
$\mathscr{S}_2$	$\langle .3131 + .3535i, .3470 + .3266i, -.3310 + (-.3406)i, -.3439 + (-.3380)i\rangle$
$\mathscr{S}_3$	$\langle .4102 + .3030i, .3385 + .2778i, -.2961 + (-.3163)i, -.3607 + (-.2992)i\rangle$
$\mathscr{S}_4$	$\langle .3817 + .3302i, .3410 + .3003i, -.3265 + (-.3259)i, -.3574 + (-.2966)i\rangle$
$\mathscr{S}_5$	$\langle .4011 + .3199i, .3632 + .2928i, -.3206 + (-.3146)i, -.3610 + (-.2795)i\rangle$
$\mathscr{S}_6$	$\langle .4254 + .2778i, .3173 + .2589i, -.2472 + (-.2544)i, -.3487 + (-.2103)i\rangle$

TABLE 4. Score values

GSCM	ω(BCPFWA)	ω(BCPFWG)
$\mathscr{S}_1$	-. 0241	-. 0255
$\mathscr{S}_2$	-. 0013	-. 0027
$\mathscr{S}_3$	.0096	.0091
$\mathscr{S}_4$	.0117	.0114
$\mathscr{S}_5$	.0111	.0097
$\mathscr{S}_6$	.0157	.0126

TABLE 5. Ranking results of alternatives using BCPFWA and BCPFWG operators

Aggregation operator	**Ranking of alternatives**	**Best alternative**
BCPFWA	$\mathscr{S}_6 \succ \mathscr{S}_4 \succ \mathscr{S}_5 \succ \mathscr{S}_3 \succ \mathscr{S}_2 \succ \mathscr{S}_1$	$\mathscr{S}_6$
BCPFWG	$\mathscr{S}_6 \succ \mathscr{S}_4 \succ \mathscr{S}_5 \succ \mathscr{S}_3 \succ \mathscr{S}_2 \succ \mathscr{S}_1$	$\mathscr{S}_6$

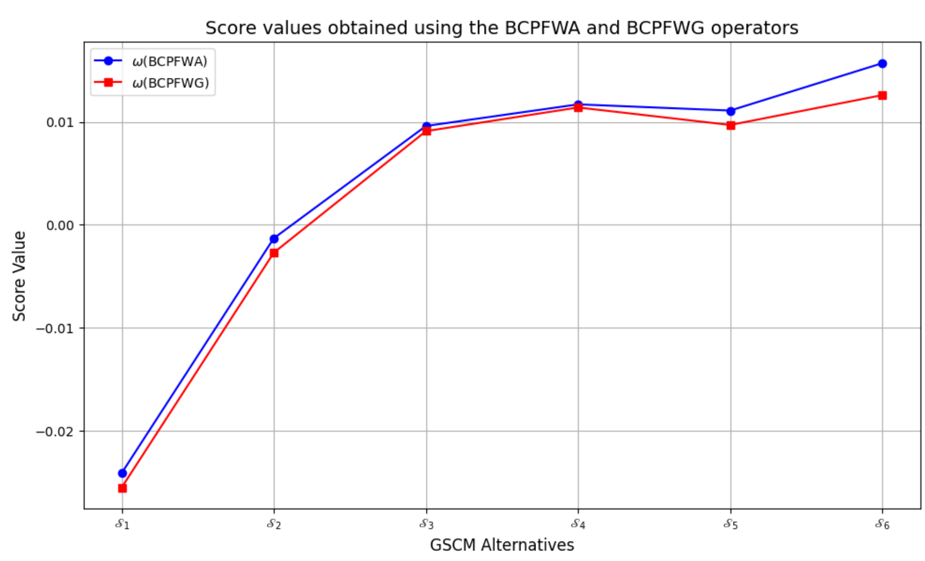

FIGURE 2. Score values obtained using the BCPFWA and BCPFWG operators.

6. Comparison between BCPF Models and Conventional Operators

In this section, we present a thorough comparative analysis between the newly formulated bipolar complex Pythagorean fuzzy weighted averaging (BCPFWA) and weighted geometric (BCPFWG) operators and a variety of aggregation mechanisms already established in the literature. The objective of this comparison is to highlight the broader applicability and generality of the proposed models across multiple fuzzy set extensions.

Our investigation considers several particular scenarios by constraining certain parameters of the general model. Specifically, cases are analyzed where $\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_{ji}}, \mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_{ji}}, \mathfrak{m}\mathfrak{R}^{-}_{\mathfrak{P}_{ji}}, \mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_{ji}}, \mathfrak{n}\mathfrak{R}^{-}_{\mathfrak{P}_{ji}}$, and $\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_{ji}}$ take zero values. Furthermore, bipolar extensions are considered when $\mathfrak{m}\mathfrak{I}^{+}_{\mathfrak{P}_{ji}}, \mathfrak{n}\mathfrak{I}^{+}_{\mathfrak{P}_{ji}}, \mathfrak{m}\mathfrak{I}^{-}_{\mathfrak{P}_{ji}}$, and $\mathfrak{n}\mathfrak{I}^{-}_{\mathfrak{P}_{ji}}$ are restricted to zero. These reductions confirm that the proposed BCPFWA and BCPFWG operators naturally embed and unify existing operators, thus providing an umbrella model that encompasses IFSs, PFSs, BIFSs, BPFSs, and BCFSs. This unified framework ensures a seamless transition between classical and more sophisticated fuzzy environments.

Main findings and benefits

The central novelty of this work is the design and application of BCPFWA and BCPFWG operators, which substantially extend the expressive power of aggregation techniques in the fuzzy decision-making literature. The performance of the suggested operators has been validated through comparative results: the calculated scores of alternatives are presented in Table 6, while Figure 3 provides a visual illustration of the ranking differences and similarities. The integration of complex-valued and bipolar structures equips the proposed model with the capacity to handle multidimensional uncertainty, offering richer insights in MADM contexts.

The primary advantages of this study can be outlined as follows:

(1) **Generalization of existing aggregation operators.** The proposed BCPFWA and BCPFWG models act as a superset of several existing fuzzy aggregation frameworks, each emerging as a special case under certain constraints:
   - *Classical intuitionistic and Pythagorean fuzzy cases:* When imaginary and negative parts are absent, the model simplifies to IFS and PFS operators. This reduction encompasses the IF weighted averaging (IFWA) [32] and IF weighted geometric (IFWG) [33] operators, as well as the Pythagorean fuzzy weighted averaging (PFWA) [34] and weighted geometric (PFWG) [35] operators.
   - *Bipolar intuitionistic and Pythagorean fuzzy cases:* When imaginary parts alone are omitted, the framework reduces to BIFSs and BPFSs, thereby encompassing BIFWA and BIFWG operators [11], as well as bipolar Pythagorean fuzzy weighted averaging (BPFWA) and weighted geometric (BPFWG) operators [30].
   - *Bipolar complex fuzzy cases:* When the real and imaginary components of nonmembership functions vanish simultaneously in both positive and negative evaluations, the model specializes to the bipolar complex fuzzy weighted averaging (BCFWAA) and geometric (BCFWGA) operators introduced in [36].

This chain of reductions verifies that the proposed model is highly flexible and general, capturing a wide spectrum of fuzzy set environments.

(2) **Comprehensive representation of uncertainty.** Unlike traditional models, the BCPFWA and BCPFWG operators incorporate multiple dimensions of uncertainty—positive and negative assessments, hesitation degrees, and complex-valued membership structures—within a unified decision-making framework. This allows for a more realistic and nuanced modeling of uncertain information, especially in contexts involving conflicting, bipolar, or imprecise evaluations.

(3) **Enhanced comparative insights.** The experimental results (Table 6) and their visual comparison (Figure 3) demonstrate that the suggested operators not only align with the outcomes of established methods but also provide distinct variations in rankings. This suggests that BCPFWA and BCPFWG introduce new interpretive dimensions for decision-makers while ensuring backward compatibility with classical intuitionistic and bipolar fuzzy operators.

In summary, the BCPFWA and BCPFWG operators significantly advance the state-of-the-art by generalizing existing models, offering a richer structure for uncertainty representation, and delivering enhanced decision-making capabilities in complex MADM scenarios.

Table 6. Benchmarking existing operators against our approach

Operators	Score values of						Option
	$\mathscr{S}_1$	$\mathscr{S}_2$	$\mathscr{S}_3$	$\mathscr{S}_4$	$\mathscr{S}_5$	$\mathscr{S}_6$	
IFWA [32]	-.0954	-.0183	.0790	.0441	.0456	.1381	$\mathscr{S}_6$
IFWG [33]	-.1120	-.0304	.0727	.0412	.0389	.1126	$\mathscr{S}_6$
BIFWA [11]	-.0199	.0013	.0733	.0392	.0446	.1318	$\mathscr{S}_6$
BIFWG [11]	-.0318	-.0080	.0690	.0365	.0400	.1079	$\mathscr{S}_6$
PFWA [34]	-.0590	-.0104	.0608	.0326	.0364	.1082	$\mathscr{S}_6$
PFWG [35]	-.0756	-.0224	.0537	.0294	.0290	.0803	$\mathscr{S}_6$
BPFWA [30]	-.0086	.0024	.0528	.0281	.0332	.0955	$\mathscr{S}_6$
BPFWG [30]	-.0209	-.0068	.0481	.0253	.0283	.0704	$\mathscr{S}_6$
BCFWAA [36]	.4889	.5023	.5280	.5173	.5241	.5584	$\mathscr{S}_6$
BCFWGA [36]	.4860	.4993	.5257	.5153	.5218	.5513	$\mathscr{S}_6$
Proposed BCPFWA	-.0241	-.0013	.0096	.0117	.0111	.0157	$\mathscr{S}_6$
Proposed BCPFWG	-.0255	-.0027	.0091	.0114	.0097	.0126	$\mathscr{S}_6$

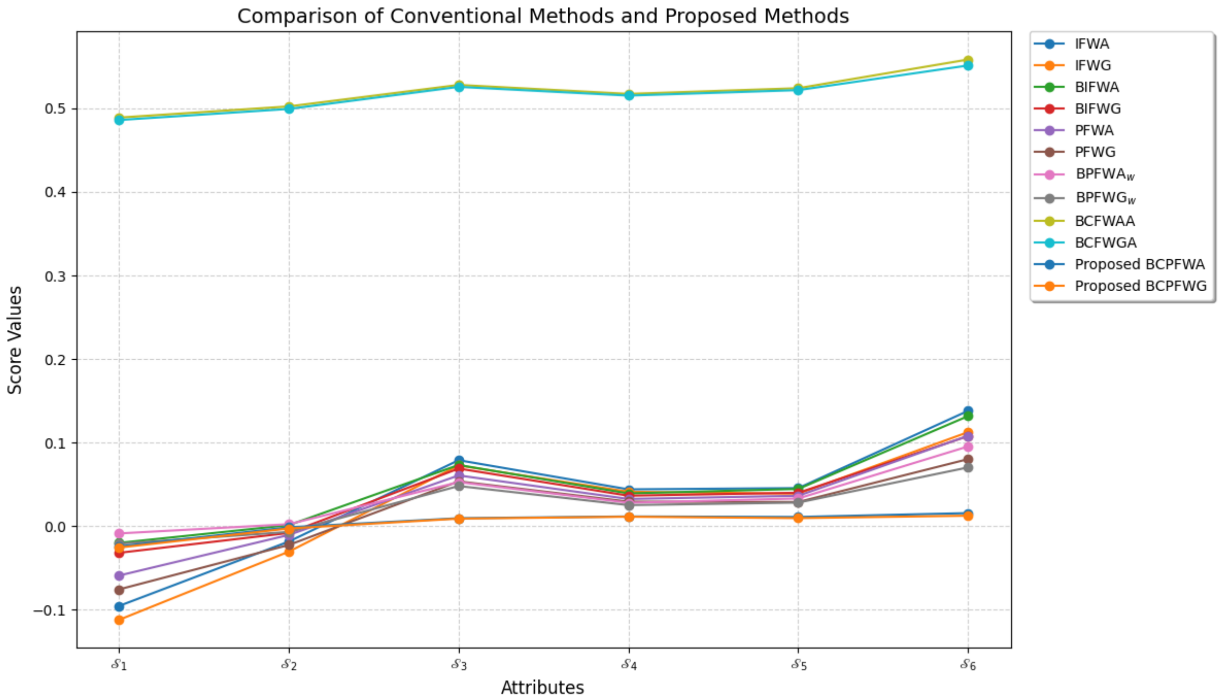

FIGURE 3. Visual representation of evaluation findings.

7. Conclusions

This study has presented a robust MADM framework grounded in BCPFSs. By introducing two novel aggregation operators—the BCPFWA and BCPFWG operators—the framework provides a mathematically consistent and practically adaptable approach for handling multidimensional uncertainty, dual (positive–negative) evaluations, and oscillatory information in complex decision-making contexts. The framework was applied to a green supply chain management case study, where six supplier strategies were assessed across multiple sustainability-oriented attributes. Results confirmed that the balanced sustainability-oriented supplier consistently ranked as the optimal choice, illustrating the effectiveness of the suggested operators in guiding real-world strategic decisions. A comparative analysis with existing fuzzy and bipolar fuzzy aggregation techniques demonstrated that the BCPFWA and BCPFWG operators deliver enhanced ranking stability, improved discrimination among alternatives, and richer interpretability. The generalized nature of the framework allows it to reduce to classical Pythagorean, bipolar, or complex fuzzy models under specific conditions, confirming its versatility.

Despite these contributions, the study has certain limitations. It is based on a single domain case study with expert-driven weight assignments, which may reflect subjective biases. Additionally, dynamic changes in technology, market trends, or policy shifts were not fully incorporated. Future research can address these gaps by extending the framework to other application areas such as renewable energy site selection, smart water management, healthcare prioritization, and climate-resilient infrastructure. Incorporating hybrid approaches that combine BCPFWA/BCPFWG with optimization algorithms, machine learning, and scenario-based simulations could support adaptive and real-time decision-making. Further exploration of alternative fuzzy paradigms, such as spherical or q-rung orthopair fuzzy sets, and flexible parameterized operators (e.g., Dombi-based extensions), may expand modeling precision. Finally, integration with digital technologies—including IoT-driven monitoring systems and AI-enabled predictive analytics—could make the framework more dynamic, scalable, and directly applicable to data-rich decision-making environments.

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