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A Finite Volume Method Solution to the Two-Assets Generalized Black-Scholes Equation

Daouda Paré[1,*], Ibrahim Zangré[2], Kassiénou Lamien[3], P.O. Fabrice Ouédraogo[2], W. Olivier Sawadogo[4]

[1]*Université Norbert ZONGO, Laboratoire de Mathématiques, Informatique et Applications (L@MIA), Burkina Faso*

[2]*Université Thomas SANKARA, Laboratoire Sciences et Technologie (LaST), Burkina Faso*

[3]*École Normale Supérieure, Laboratoire d'Analyse Numérique, d'Informatique et de BIO-Mathématiques (LANIBIO), Burkina Faso*

[4]*Université Lédéa Bernard OUÉDRAOGO, Laboratoire de Recherche et Développement (LaReD), Burkina Faso*

**Corresponding author:* ibzangre@gmail.com

Abstract. The aim of this paper is to present a finite volume method (FVM) for solving numerically the two-assets generalized Black-Scholes equation. It it well-known that FVM is well-suited for solving problems involving hyperbolic and/or conservative laws mainly encountered in transport-diffusion and fluid dynamics problems. In this work, we attempt to use FVM for solving problems arising from market finance domain, in particular, the generalized multi-assets Black-Scholes problem. The discretization details and steps are presented for the two-assets problem. Then, numerical experiments are conducted on two main examples and show satisfactory results.

1. Introduction

Financial option pricing remains a challenge problem within the modern finance markets. This is noticed in particular for the multi-asset model which are an essential extension of the classical models in quantitative finance. Indeed, it allows a better understanding and pricing of the derivative products, while offering the opportunity to reduce the global portfolio risk to any holder of such option.

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The mathematical modeling of the multi-asset financial options leads to a multi-dimensional stochastic partial differential equations (SPDE) system such as the so-called Black-Scholes-Merton models [1, 2] and its extensions [3–6]. In general, it is difficult, even impossible to address analytical solutions to these SPDE, due to the large number of variables and the complex financial relationships between the assets. Thus, numerical methods which provide approximated solutions have to be considered. During the past decades, several numerical methods have been proposed to solve the Black-Scholes models equations. The widely used methods can be classified into two families : stochastic methods such as Monte-Carlo techniques [6–9, 11, 16, 18, 19], and deterministic methods such as finite difference methods (FDM) [11–13, 19], and finite element methods (FEM) [10, 11, 14, 15]. On the contrary, Finite volume methods (FVM) [17, 20] are rarely used to numerically solve the Black-Scholes models and its extensions. However let us these references [21–23] where the authors propose a FVM solution to the Black-Scholes model. In the previous work [24], we use FVM to solve numerically the Black-Scholes linear model with constant parameters and the generalized linear model with satisfactory results.

In this work, we propose a FVM numerical solution of the two dimensional Black-Scholes model. We present the FVM discretization steps and present numerical examples to illustrate the performance of the method.

The paper is structured as follow : in Section 2 we recall the two-dimensional Black-Scholes model equation which is completed with initial and limit conditions. In Section 3 we present the steps of the FVM domain, time and space discretization details. Finally Section 4 presents numerical simulations for both call and put with spread and basket options.

2. The two-dimensional Black-Scholes model equation

The initial Black-Scholes equation which involved one financial asset. It has been extended and adapted for options pricing taking account multiple financial assets. The resulting mathematical model is the so-called Black-Scholes multi-dimensional equation which is given as follows when considering n assets :

$$\begin{cases} \dfrac{\partial P}{\partial t}(S,t) + \displaystyle\sum_{i=1}^{n} (r - q_i)\, S_i \dfrac{\partial P}{\partial S_i}(S,t) + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j S_i S_j \dfrac{\partial^2 P}{\partial S_i \partial S_j}(S,t) - rP(S,t) = 0, \\ P(S,T) = \phi(S), \end{cases} \tag{2.1}$$

where : $P = P(S_1, \cdots, S_n, t)$ designates the price of the option; $(S,t) \in (\mathbb{R}_+^\star)^n \times [0,T]$; $S = (S_1, \cdots, S_n)$; S_i and σ_i designate respectively the price and the volatility of asset i for $i = 1, \cdots, n$; q_i represents the dividend rate of S_i; ρ_{ij} is the correlation coefficient between S_i and S_j for $i \neq j$; r is the risk-free interest rate; T denotes the the expiration date or maturity of the option; ϕ represents the terminal condition. We recall that each asset price $S_i = S_i(t)$ is governed by the one-dimension Black-Scholes model under the risk-neutral probability as follows

$$dS_i = S_i\left[(r - q_i)dt + \sigma_i dW_i\right], \quad \text{for} \ \ i = 1, \cdots, n, \tag{2.2}$$

where $W_1, \cdots, W_n$ designate correlated standard brownian motions and their correlation coefficient is given by $d\langle W_i, W_j\rangle = \rho_{ij}dt$. When considering $n = 2$ assets, system (2.1) now reads

$$\begin{cases} \dfrac{\partial P}{\partial t} + (r-q_1)S_1\dfrac{\partial P}{\partial S_1} + (r-q_2)S_2\dfrac{\partial P}{\partial S_2} + \dfrac{1}{2}\sigma_1^2S_1^2\dfrac{\partial^2 P}{\partial S_1^2} + \dfrac{1}{2}\sigma_2^2S_2^2\dfrac{\partial^2 P}{\partial S_2^2} \\ \qquad + \rho\sigma_1\sigma_2S_1S_2\dfrac{\partial^2 P}{\partial S_1\partial S_2} - rP = 0,\ \forall(S_1,S_2,t) \in [0,+\infty[\times[0,+\infty[\times[0,T], \\ \qquad\qquad P(S_1,S_2,T) = \phi(S_1,S_2),\ \forall(S_1,S_2) \in [0;+\infty[\times[0;+\infty[, \end{cases} \tag{2.3}$$

where $\rho = \rho_{12}$ is the correlation coefficient betweem S_1 and S_2.

We aim to solve system (2.3) using finite volume method (FVM). Therefore, we transform the terminal condition into an initial condition through the variable change $\tau = T - t$. The system is then completed with suited boundary conditions and now reads

$$\begin{cases} \dfrac{\partial P}{\partial \tau} + (q_1-r)S_1\dfrac{\partial P}{\partial S_1} + (q_2-r)S_2\dfrac{\partial P}{\partial S_2} - \dfrac{1}{2}\sigma_1^2S_1^2\dfrac{\partial^2 P}{\partial S_1^2} - \dfrac{1}{2}\sigma_2^2S_2^2\dfrac{\partial^2 P}{\partial S_2^2} \\ \qquad - \rho\sigma_1\sigma_2S_1S_2\dfrac{\partial^2 P}{\partial S_1\partial S_2} + rP = 0,\ \forall(S_1,S_2,\tau) \in [0,+\infty[\times[0,+\infty[\times[0,T], \\ \qquad\qquad P(S_1,S_2,0) = \phi(S_1,S_2),\ \forall(S_1,S_2) \in [0,+\infty[\times[0,+\infty[, \\ \qquad\qquad P(0,S_2,\tau) = f(S_2,\tau),\ \ \forall(S_2,\tau) \in [0,+\infty[\times[0,T], \\ \qquad\qquad P(S_1,0,\tau) = g(S_1,\tau),\ \ \forall(S_1,\tau) \in [0,+\infty[\times[0,T], \\ \qquad\qquad P(S_1,S_2,\tau) = h(S_2,\tau),\ \forall\tau \in [0,T],\ S_1 \to +\infty, \\ \qquad\qquad P(S_1,S_2,\tau) = k(S_1,\tau),\ \forall\tau \in [0,T],\ S_2 \to +\infty, \end{cases} \tag{2.4}$$

For the sake of convenience, let us set : $x = S_1$, $y = S_2$, $a(x) = (q_1 - r)S_1$, $b(y) = (q_2 - r)S_2$, $c(x) = \dfrac{1}{2}\sigma_1^2S_1^2$, $d(y) = \dfrac{1}{2}\sigma_2^2S_2^2$, $e(x,y) = \rho\sigma_1\sigma_2S_1S_2$. Thus, system (2.4) now reads

$$\begin{cases} \dfrac{\partial P}{\partial \tau} + a(x)\dfrac{\partial P}{\partial x} + b(y)\dfrac{\partial P}{\partial y} - c(x)\dfrac{\partial^2 P}{\partial x^2} - d(y)\dfrac{\partial^2 P}{\partial y^2} \\ \qquad - e(x,y)\dfrac{\partial^2 P}{\partial x\partial y} + rP = 0,\ \forall(x,y,\tau) \in [0,+\infty[\times[0,+\infty[\times[0,T], \\ \qquad\qquad P(x,y,0) = \phi(x,y),\ \forall(x,y) \in [0,+\infty[\times[0,+\infty[, \\ \qquad\qquad P(0,y,\tau) = f(y,\tau),\ \forall(y,\tau) \in [0,+\infty[\times[0,T], \\ \qquad\qquad P(x,0,\tau) = g(x,\tau),\ \forall(x,\tau) \in [0,+\infty[\times[0,T], \\ \qquad\qquad P(x,y,\tau) = h(y,\tau),\ \forall\tau \in [0,T],\ x \to +\infty, \\ \qquad\qquad P(x,y,\tau) = k(x,\tau),\ \forall\tau \in [0,T],\ y \to +\infty, \end{cases} \tag{2.5}$$

In order to approximate solution of system (2.5) with the FVM, we need to bound the computational domain. Precisely, we replace the infinite space domain $[0,+\infty[\times[0,+\infty[$ by the rectangular domain $[0,x_{\max}]\times[0,y_{\max}]$. Finaly the two-dimensional Black-Scholes model problem now reads

$$\begin{cases} \dfrac{\partial P}{\partial \tau}+a(x)\dfrac{\partial P}{\partial x}+b(y)\dfrac{\partial P}{\partial y}-c(x)\dfrac{\partial^2 P}{\partial x^2}-d(y)\dfrac{\partial^2 P}{\partial y^2} \\ \qquad -e(x,y)\dfrac{\partial^2 P}{\partial x\partial y}+rP=0,\ \forall (x,y,\tau)\in[0,x_{\max}]\times[0,y_{\max}]\times[0,T], \\ P(x,y,0)=\phi(x,y),\ \forall (x,y)\in[0,x_{\max}]\times[0,y_{\max}], \\ P(0,y,\tau)=f(y,\tau),\ \forall (y,\tau)\in[0,y_{\max}]\times[0,T], \\ P(x,0,\tau)=g(x,\tau),\ \forall (x,\tau)\in[0,x_{\max}]\times[0,T], \\ P(x_{\max},y,\tau)=h(y,\tau),\ \forall (y,\tau)\in[0,y_{\max}]\times[0,T], \\ P(x,y_{\max},\tau)=k(x,\tau),\ \forall (x,\tau)\in[0,x_{\max}]\times[0,T], \end{cases} \tag{2.6}$$

3. Finite volume discretization of the model problem

3.1. **Computational domain discretization.** Let us set $\Omega=[0,x_{\max}]\times[0,y_{\max}]\times[0,T]$, the computational domain established section 2. Then an uniform cartesian discretization of Ω is performed as following :

- $[0,x_{\max}]$ is subdivised into (N_x+1) equidistant nodes $x_0,x_1\cdots,x_{N_x-1},x_{N_x}$, with mesh step $\Delta x=x_{\max}/N_x$; let $x_0=0$, $x_{N_x}=x_{\max}$ be the exterior nodes; there are (N_x-1) interior nodes, which are $x_1\cdots,x_{N_x-1}$. We set the nodes $x_{i\pm1/2}=x_i\pm\Delta x/2$ for $i=1,\cdots,N_x-1$;
- $[0,y_{\max}]$ is treated the same way with (N_y+1) equidistant nodes $y_0,y_1,\cdots,y_{N_y-1},y_{N_y}$ with mesh step $\Delta y=y_{\max}/N_y$;
- the time domain $[0,T]$ is treated the same way with (N_t+1) equidistant discret times $\tau^0,\tau^1\cdots,\tau^{N_t-1},\tau^{N_t}$ with time step $\Delta\tau=T/N_t$.

We now set the control volumes as $V^n_{i,j}=K_i\times H_j\times T_n$, where $K_i=]x_{i-1/2},x_{i+1/2}[$, $H_j=]y_{j-1/2},y_{j+1/2}[$, $T_n=[\tau^n,\tau^{n+1}]$ for $i=1,\dots,N_x-1$, $j=1,\dots,N_y-1$ and $n=0,\dots,N_t-1$. Thus, let us notice that there exist four types of control volumes : first control volumes, interior control volumes, mixtes control volumes and latest control volumes.

3.2. **Finite volume-compatible form of the model problem.** In order to perform the FVM discretization on the model problem, the PDE of (2.6)

$$\frac{\partial P}{\partial \tau}+a(x)\frac{\partial P}{\partial x}+b(y)\frac{\partial P}{\partial y}-c(x)\frac{\partial^2 P}{\partial x^2}-d(y)\frac{\partial^2 P}{\partial y^2}-e(x,y)\frac{\partial^2 P}{\partial x\partial y}+rP=0, \tag{3.1}$$

needs to be re-written into the following conservative form

$$\frac{\partial P}{\partial \tau}+\frac{\partial F(P)}{\partial x}+\frac{\partial G(P)}{\partial y}=c(x)\frac{\partial^2 P}{\partial x^2}+d(y)\frac{\partial^2 P}{\partial y^2}+e(x,y)\frac{\partial^2 P}{\partial x\partial y}+S(P) \tag{3.2}$$

where

$$F(P(x,y,\tau)) = a(x)P(x,y,\tau), \quad G(P(x,y,\tau)) = b(x)P(x,y,\tau),$$

$$S(P(x,y,\tau)) = \left[\frac{a(x)}{\partial x} + \frac{b(y)}{\partial y} - r\right] P(x,y,\tau).$$

3.3. **Control volume computations.** The FVM consists in integrating the conservative law (3.2) on each control volume $V_{i,j}^n$, which yields

$$\begin{aligned}\int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{\tau^n}^{\tau^{n+1}} \left[\frac{\partial P}{\partial \tau} + \frac{\partial F(P)}{\partial x} + \frac{\partial G(P)}{\partial y}\right] dxdyd\tau \\ = \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{\tau^n}^{\tau^{n+1}} \left[c(x)\frac{\partial^2 P}{\partial x^2} + d(y)\frac{\partial^2 P}{\partial y^2}\right] dxdyd\tau \\ + \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{\tau^n}^{\tau^{n+1}} \left[e(x,y)\frac{\partial^2 P}{\partial x \partial y} + S(P)\right] dxdyd\tau \quad (3.3)\end{aligned}$$

When we then integrate the first-order derivatives and divide each term by $\Delta x \Delta y$, equation (3.3) becomes

$$\begin{aligned}\frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \left[P(x,y,\tau^{n+1}) - P(x,y,\tau^n)\right] dxdy \\ + \frac{\Delta \tau}{\Delta x} \frac{1}{\Delta y \Delta \tau} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{\tau^n}^{\tau^{n+1}} \left[F\left(P(x_{i+1/2},y,\tau)\right) - F\left(P(x_{i-1/2},y,\tau)\right)\right] dy\Delta\tau \\ + \frac{\Delta \tau}{\Delta y} \frac{1}{\Delta x \Delta \tau} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{\tau^n}^{\tau^{n+1}} \left[G\Big(P(x,y_{j+1/2},\tau)\Big) - F\Big(P(x,y_{j-1/2},\tau)\Big)\right] dx\Delta\tau \\ = \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{\tau^n}^{\tau^{n+1}} \left[c(x)\frac{\partial^2 P}{\partial x^2} + d(y)\frac{\partial^2 P}{\partial y^2}\right] dxdyd\tau \\ + \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{\tau^n}^{\tau^{n+1}} \left[e(x,y)\frac{\partial^2 P}{\partial x \partial y} + S(P)\right] dxdyd\tau \quad (3.4)\end{aligned}$$

For the sake of simplicity, let us define the following mean values and numerical fluxes

$$\begin{aligned} P_{i,j}^n &= \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} P(x,y,\tau^n) dxdy, \\ \mathcal{F}(P_{i-1,j}^n, P_{i,j}^n) &= \frac{1}{\Delta y \Delta \tau} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{\tau^n}^{\tau^{n+1}} F\left(P(x_{i-1/2},y,\tau)\right) dy\Delta\tau, \\ \mathcal{F}(P_{i,j}^n, P_{i+1,j}^n) &= \frac{1}{\Delta y \Delta \tau} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{\tau^n}^{\tau^{n+1}} F\left(P(x_{i+1/2},y,\tau)\right) dy\Delta\tau, \quad (3.5) \\ \mathcal{G}(P_{i,j-1}^n, P_{i,j}^n) &= \frac{1}{\Delta x \Delta \tau} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{\tau^n}^{\tau^{n+1}} F\Big(P(x,y_{j-1/2},\tau)\Big) dx\Delta\tau, \\ \mathcal{G}(P_{i,j}^n, P_{i,j+1}^n) &= \frac{1}{\Delta x \Delta \tau} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{\tau^n}^{\tau^{n+1}} F\Big(P(x,y_{j+1/2},\tau)\Big) dx\Delta\tau. \end{aligned}$$

Using the above notations and using explicite Euler discretization for the temporal derivative, then equation (3.4) now reads

$$\begin{aligned} P_{i,j}^{n+1} - P_{i,j}^{n} + \frac{\Delta\tau}{\Delta x}\left(\mathcal{F}(P_{i,j}^{n}, P_{i+1,j}^{n}) - \mathcal{F}(P_{i-1,j}^{n}, P_{i,j}^{n})\right) + \frac{\Delta\tau}{\Delta y}\left(\mathcal{G}(P_{i,j}^{n}, P_{i,j+1}^{n}) - \mathcal{G}(P_{i,j-1}^{n}, P_{i,j}^{n})\right) \\ = \frac{1}{\Delta x\Delta y}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{\tau^n}^{\tau^{n+1}}\left[c(x)\frac{\partial^2 P}{\partial x^2} + d(y)\frac{\partial^2 P}{\partial y^2}\right]dxdyd\tau \\ + \frac{1}{\Delta x\Delta y}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{\tau^n}^{\tau^{n+1}}\left[e(x,y)\frac{\partial^2 P}{\partial x\partial y} + S(P)\right]dxdyd\tau \end{aligned} \tag{3.6}$$

To complete the FVM iterative scheme we need to discretize of the diffusion and source terms. Thus, using centered finite differences to approximate the second-order derivative terms, we get

$$\begin{aligned} \frac{1}{\Delta x\Delta y}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{\tau^n}^{\tau^{n+1}} c(x)\frac{\partial^2 P}{\partial x^2}dxdyd\tau &\simeq \Delta\tau.c_i.\frac{P_{i+1,j}^{n} - 2P_{i,j}^{n} + P_{i-1,j}^{n}}{\Delta x^2} \\ \frac{1}{\Delta x\Delta y}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{\tau^n}^{\tau^{n+1}} d(y)\frac{\partial^2 P}{\partial y^2}dxdyd\tau &\simeq \Delta\tau.d_j.\frac{P_{i,j+1}^{n} - 2P_{i,j}^{n} + P_{i,j-1}^{n}}{\Delta y^2}, \\ \frac{1}{\Delta x\Delta y}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{\tau^n}^{\tau^{n+1}} e(x,y)\frac{\partial^2 P}{\partial x\partial y}dxdyd\tau &\simeq \Delta\tau.e_{i,j}.\frac{P_{i+1,j+1}^{n} - P_{i+1,j-1}^{n} - P_{i-1,j+1}^{n} + P_{i-1,j-1}^{n}}{4\Delta x\Delta y}, \\ \frac{1}{\Delta x\Delta y}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{\tau^n}^{\tau^{n+1}} S(P)dxdyd\tau &\simeq \Delta\tau.S(P_{i,j}^{n}), \end{aligned} \tag{3.7}$$

where $c_i = c(x_i)$, $d_i = d(y_j)$ and $e_{i,j} = e(x_i, y_j)$. Finally, the general FVM scheme on the interior control volumes (*i.e.* $i = 2, \ldots, N_x - 2$, $j = 2, \ldots, N_y - 2$, $n = 0, \ldots, N_t - 1$) is given by the following relation

$$\begin{aligned} P_{i,j}^{n+1} = P_{i,j}^{n} &- \frac{\Delta\tau}{\Delta x}\left(F(P_{i,j}^{n}) - F(P_{i-1,j}^{n})\right) - \frac{\Delta\tau}{\Delta y}\left(G(P_{i,j+1}^{n}) - G(P_{i,j}^{n})\right) \\ &+ \frac{\Delta\tau.c_i}{\Delta x^2}\left(P_{i+1,j}^{n} - 2P_{i,j}^{n} + P_{i-1,j}^{n}\right) + \frac{\Delta\tau.d_j}{\Delta y^2}\left(P_{i,j+1}^{n} - 2P_{i,j}^{n} + P_{i,j-1}^{n}\right) \\ &+ \frac{\Delta\tau.e_{i,j}}{4\Delta x\Delta y}\left(P_{i+1,j+1}^{n} - P_{i+1,j-1}^{n} - P_{i-1,j+1}^{n} + P_{i-1,j-1}^{n}\right) + \Delta\tau.S(P_{i,j}^{n}), \end{aligned} \tag{3.8}$$

where the numerical fluxes $\mathcal{F}$ and $\mathcal{G}$ are approximated by a first-order upwind scheme. This scheme is adapted for the non-interior control volumes to take account the boundaries. These specific FVM schemes are given below.

1) For $V_{1,1}^{n}$ with $n = 0, \ldots, N_t - 1$:

$$\begin{aligned} P_{1,1}^{n+1} = P_{1,1}^{n} &- \frac{\Delta\tau}{\Delta x}\left(F(P_{2,1}^{n}) - F(P_{1,1}^{n})\right) - \frac{\Delta\tau}{\Delta y}\left(G(P_{1,2}^{n}) - G(P_{1,1}^{n})\right) \\ &+ \frac{\Delta\tau.c_1}{3\Delta x^2}\left(4P_{2,1}^{n} - 12P_{1,1}^{n} + 8P_{0,1}^{n}\right) + \frac{\Delta\tau.d_1}{3\Delta y^2}\left(4P_{1,2}^{n} - 12P_{1,1}^{n} + 8P_{1,0}^{n}\right) \\ &+ \frac{4\Delta\tau.e_{1,1}}{9\Delta x\Delta y}\left(P_{2,2}^{n} - P_{2,0}^{n} - P_{0,2}^{n} + P_{0,0}^{n}\right) + \Delta\tau.S(P_{1,1}^{n}). \end{aligned} \tag{3.9}$$

2) For $V^n_{1,j}$ with $j = 2, \dots, N_y - 2$ and $n = 0, \dots, N_t - 1$:

$$\begin{aligned} P^{n+1}_{1,j} = P^n_{1,j} \quad &- \quad \frac{\Delta\tau}{\Delta x}\left(F(P^n_{2,j}) - F(P^n_{1,j})\right) - \frac{\Delta\tau}{\Delta y}\left(G(P^n_{1,j+1}) - G(P^n_{1,j})\right) \\ &+ \quad \frac{\Delta\tau.c_1}{3\Delta x^2}\left(4P^n_{2,j} - 12P^n_{1,j} + 8P^n_{0,j}\right) + \frac{\Delta\tau.d_1}{\Delta y^2}\left(P^n_{1,j+1} - 2P^n_{1,j} + P^n_{1,j-1}\right) \\ &+ \quad \frac{\Delta\tau.e_{1,j}}{3\Delta x\Delta y}\left(P^n_{2,j+1} - P^n_{2,j-1} - P^n_{0,j+1} + P^n_{0,j-1}\right) + \Delta\tau.S(P^n_{1,j}). \end{aligned} \tag{3.10}$$

3) For $V^n_{i,1}$ with $i = 2, \dots, N_x - 2$ and $n = 0, \dots, N_t - 1$:

$$\begin{aligned} P^{n+1}_{i,1} = P^n_{i,1} \quad &- \quad \frac{\Delta\tau}{\Delta x}\left(F(P^n_{i+1,1}) - F(P^n_{i,1})\right) - \frac{\Delta\tau}{\Delta y}\left(G(P^n_{i,2}) - G(P^n_{i,1})\right) \\ &+ \quad \frac{\Delta\tau.c_i}{\Delta x^2}\left(P^n_{i+1,1} - P^n_{i,1} + P^n_{i-1,1}\right) + \frac{\Delta\tau.d_1}{3\Delta y^2}\left(4P^n_{i,2} - 12P^n_{i,1} + 8P^n_{i,0}\right) \\ &+ \quad \frac{\Delta\tau.e_{i,1}}{3\Delta x\Delta y}\left(P^n_{i+1,2} - P^n_{i+1,0} - P^n_{i-1,2} + P^n_{i-1,0}\right) + \Delta\tau.S(P^n_{i,1}). \end{aligned} \tag{3.11}$$

4) For $V^n_{N_x-1,N_y-1}$ with $n = 0, \dots, N_t - 1$:

$$\begin{aligned} P^{n+1}_{N_x-1,N_y-1} = P^n_{N_x-1,N_y-1} \quad &- \quad \frac{\Delta\tau}{\Delta x}\left(F(P^n_{N_x-1,N_y-1}) - F(P^n_{N_x-2,N_y-1})\right) \\ &- \quad \frac{\Delta\tau}{\Delta y}\left(G(P^n_{N_x-1,N_y-1}) - G(P^n_{N_x-1,N_y-2})\right) \\ &+ \quad \frac{\Delta\tau.c_{N_x-1}}{3\Delta x^2}\left(8P^n_{N_x,N_y-1} - 12P^n_{N_x-1,N_y-1} + 4P^n_{N_x-2,N_y-1}\right) \\ &+ \quad \frac{\Delta\tau.d_{N_y-1}}{3\Delta y^2}\left(8P^n_{N_x-1,N_y} - 12P^n_{N_x-1,N_y-1} + 8P^n_{N_x-1,N_y-2}\right) \\ &+ \quad \frac{4\Delta\tau.e_{N_x-1,N_y-1}}{9\Delta x\Delta y}\left(P^n_{N_x,N_y} - P^n_{N_x,N_y-2} - P^n_{N_x-2,N_y} + P^n_{N_x-2,N_y-2}\right) \\ &+ \quad \Delta\tau.S(P^n_{N_x-1,N_y-1}). \end{aligned} \tag{3.12}$$

5) For $V^n_{N_x-1,j}$ with $j = 2, \dots, N_y - 2$ and $n = 0, \dots, N_t - 1$:

$$\begin{aligned} P^{n+1}_{N_x-1,j} = P^n_{N_x-1,j} \quad &- \quad \frac{\Delta\tau}{\Delta x}\left(F(P^n_{N_x-1,j}) - F(P^n_{N_x-2,j})\right) - \frac{\Delta\tau}{\Delta y}\left(G(P^n_{N_x-1,j}) - G(P^n_{N_x-1,j-1})\right) \\ &+ \quad \frac{\Delta\tau.c_{N_x-1}}{3\Delta x^2}\left(8P^n_{N_x,j} - 12P^n_{N_x-1,j} + 4P^n_{N_x-2,j}\right) \\ &+ \quad \frac{\Delta\tau.d_j}{\Delta y^2}\left(P^n_{N_x-1,j+1} - 2P^n_{N_x-1,j} + P^n_{N_x-1,j-1}\right) \\ &+ \quad \frac{\Delta\tau.e_{N_x-1,j}}{3\Delta x\Delta y}\left(P^n_{N_x-1,j+1} - P^n_{N_x-1,j-1} - P^n_{N_x-2,j+1} + P^n_{N_x-2,j-1}\right) \\ &+ \quad \Delta\tau.S(P^n_{N_x-1,j}). \end{aligned} \tag{3.13}$$

6) For V^n_{i,N_y-1} with $i=2,\ldots,N_x-2$ and $n=0,\ldots,N_t-1$:

$$
\begin{aligned}
P^{n+1}_{i,N_y-1} = P^n_{i,N_y-1} &- \frac{\Delta\tau}{\Delta x}\left(F(P^n_{i,N_y-1}) - F(P^n_{i-1,N_y-1})\right) - \frac{\Delta\tau}{\Delta y}\left(G(P^n_{i,N_y-1}) - G(P^n_{i,N_y-2})\right)\\
&+ \frac{\Delta\tau.c_i}{\Delta x^2}\left(P^n_{i+1,N_y-1} - P^n_{i,N_y-1} + P^n_{i-1,N_y-1}\right)\\
&+ \frac{\Delta\tau.d_{N_y-1}}{3\Delta y^2}\left(8P^n_{i,N_y} - 12P^n_{i,N_y-1} + 8P^n_{i,N_y-2}\right)\\
&+ \frac{\Delta\tau.e_{i,N_y-1}}{3\Delta x\Delta y}\left(P^n_{i+1,N_y-1} - P^n_{i+1,N_y-2} - P^n_{i-1,N_y} + P^n_{i-1,N_y-2}\right)\\
&+ \Delta\tau.S(P^n_{i,N_y-1}).
\end{aligned}
\tag{3.14}
$$

7) For V^n_{1,N_y-1} with $n=0,\ldots,N_t-1$:

$$
\begin{aligned}
P^{n+1}_{1,N_y-1} = P^n_{1,N_y-1} &- \frac{\Delta\tau}{\Delta x}\left(F(P^n_{2,N_y-1}) - F(P^n_{1,N_y-1})\right) - \frac{\Delta\tau}{\Delta y}\left(G(P^n_{1,N_y-1}) - G(P^n_{1,N_y-2})\right)\\
&+ \frac{\Delta\tau.c_1}{3\Delta x^2}\left(8P^n_{0,N_y-1} - 12P^n_{1,N_y-1} + 4P^n_{2,N_y-1}\right)\\
&+ \frac{\Delta\tau.d_{N_y-1}}{3\Delta y^2}\left(8P^n_{1,N_y} - 12P^n_{1,N_y-1} + 4P^n_{1,N_y-2}\right)\\
&+ \frac{4\Delta\tau.e_{1,N_y-1}}{9\Delta x\Delta y}\left(P^n_{2,N_y} - P^n_{2,N_y-2} - P^n_{0,N_y} + P^n_{0,N_y-2}\right)\\
&+ \Delta\tau.S(P^n_{1,N_y-1}).
\end{aligned}
\tag{3.15}
$$

8) For $V^n_{N_x-1,1}$ with $n=0,\ldots,N_t-1$:

$$
\begin{aligned}
P^{n+1}_{N_x-1,1} = P^n_{N_x-1,1} &- \frac{\Delta\tau}{\Delta x}\left(F(P^n_{N_x-1,1}) - F(P^n_{N_x-2,1})\right) - \frac{\Delta\tau}{\Delta y}\left(G(P^n_{N_x-1,2}) - G(P^n_{N_x-1,1})\right)\\
&+ \frac{\Delta\tau.c_{N_x-1}}{3\Delta x^2}\left(8P^n_{N_x,1} - 12P^n_{N_x-1,1} + 4P^n_{N_x-2,1}\right)\\
&+ \frac{\Delta\tau.d_1}{3\Delta y^2}\left(8P^n_{N_x-1,0} - 12P^n_{N_x-1,1} + 4P^n_{N_x-1,2}\right)\\
&+ \frac{4\Delta\tau.e_{N_x-1,1}}{9\Delta x\Delta y}\left(P^n_{N_x,2} - P^n_{N_x,0} - P^n_{N_x-2,2} + P^n_{N_x-2,0}\right)\\
&+ \Delta\tau.S(P^n_{N_x-1,1}).
\end{aligned}
\tag{3.16}
$$

4. Numerical examples

4.1. **Implementation and numerical setting.** In this section we present some numerical experiments of the FVM applied to the Black-Scholes two-dimensional model problem (2.4).

Regarding to the final and boundary conditions, many options with two assets can be considered. For our experiments, we only consider two main cases : basket option and spread option. The implementation and computations are performed with MATLAB software.

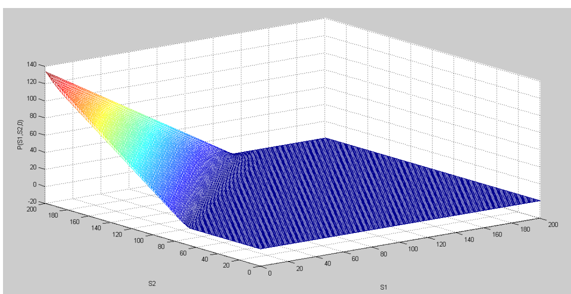
The model problem inner parameters are set to the following values : $q_1 = 0.03$, $q_2 = 0.02$, $r = 0.01$, $\rho = 0.025$, $\sigma_1 = 0.05$, $\sigma_2 = 0.1$, $S_{1\max} = S_{2\max} = 200$, $T = 1$ and $K = 70$. For the computational domain discretization,we use $N_x = 200$, $N_y = 130$ and $N_t = 400$.

4.2. **Spread option.** A spread option is a multi-asset option where the payoff is calculated from the difference betweem two underlying assets prices S_1 and S_2.

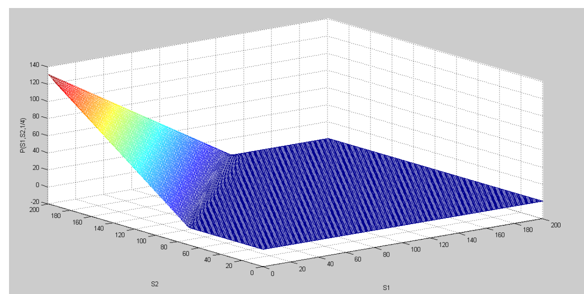
4.2.1. *Call spread option.* The payoff of the call spread option is given by $\phi(S_1, S_2, T) = \max(S_1 - S_2 - K, 0)$ and the boundary conditions are

$$\begin{aligned} P(0, S_2, t) &= \max(-S_2 - Ke^{-r(T-t)}, 0), \\ P(S_1, 0, t) &= \max(S_1 - Ke^{-r(T-t)}, 0), \\ P(S_{1\max}, S_2, t) &= \max(S_{1\max} - S_2 - Ke^{-r(T-t)}, 0), \\ P(S_1, S_{2\max}, t) &= \max(S_1 - S_{2\max} - Ke^{-r(T-t)}, 0). \end{aligned} \tag{4.1}$$

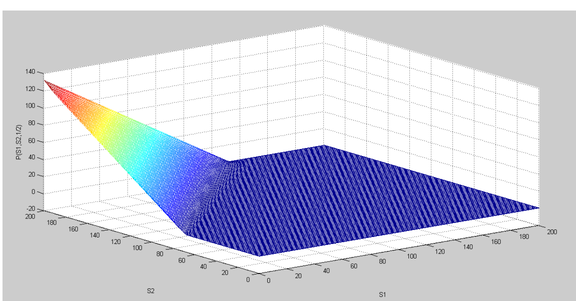
FVM numerical approximation of the call spread option values at different times are presented on figure (1).



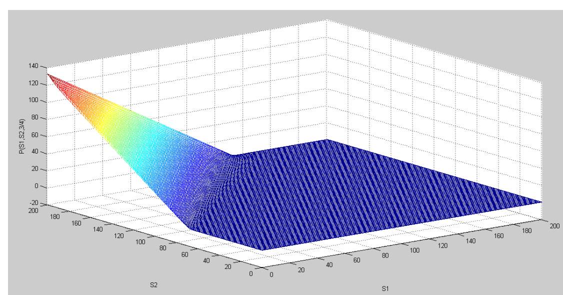
(a) Call spread option at $t = 0$



(b) Call spread option at $t = 0.25$



(c) Call spread option at $t = 0.5$



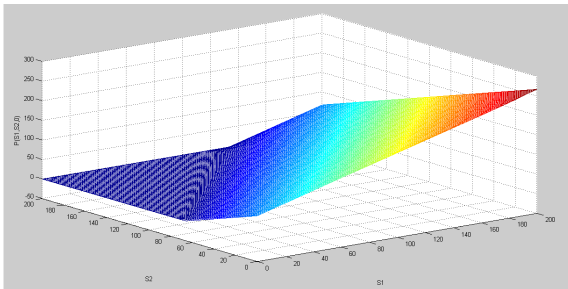
(d) Call spread option at $t = 0.75$

Figure 1. FVM numerical solution to the Black-Scholes two dimensional model problem for a call spread option.

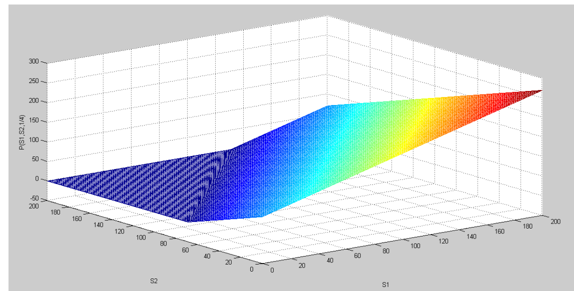
4.2.2. *Put spread option.* The payoff of the put spread option is given by $\phi(S_1, S_2, T) = \max(K - (S_1 - S_2), 0)$ and the boundary conditions are

$$\begin{aligned} P(0, S_2, t) &= \max(Ke^{-r(T-t)} + S_2, 0) = 0, \\ P(S_1, 0, t) &= \max(Ke^{-r(T-t)} - S_1, 0), \\ P(S_{1\max}, S_2, t) &= \max(Ke^{-r(T-t)} - (S_{1\max} - S_2), 0), \\ P(S_1, S_{2\max}, t) &= \max(Ke^{-r(T-t)} - (S_1 - S_{2\max}), 0). \end{aligned} \tag{4.2}$$

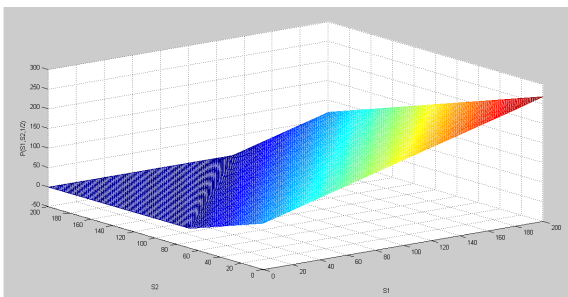
FVM numerical approximation of the put spread option values at different times are presented on figure (2).



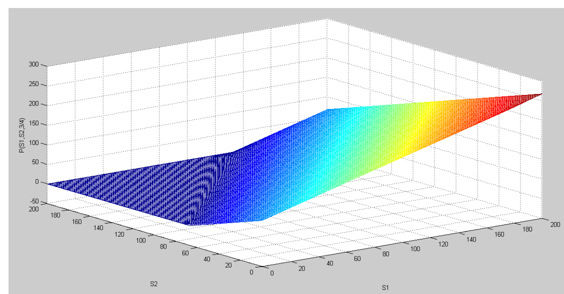
(a) Put spread option at $t = 0$



(b) Put spread option at $t = 0.25$



(c) Put spread option at $t = 0.5$



(d) Put spread option at $t = 0.75$

FIGURE 2. FVM numerical solution to the Black-Scholes two dimensional model problem for a put spread option.

4.2.3. *Analysis of spread options results.* Figures 1(a), 1(b), 1(c) and 1(d) show the FVM approximations of the call spread option respectively at times $t = 0$, $t = 0.25$, $t = 0.5$ and $t = 0.75$. We observe that the call spread option price increases when S_1 drecreases and S_2 increases (see table 4.2.3). This is consistent with a behavior of a call spread option which is based on the difference between S_1 and S_2.

Moreover, figures 2(a), 2(b), 2(c) and 2(d) show the FVM approximations of the put spread option respectively at times $t = 0$, $t = 0.25$, $t = 0.5$ and $t = 0.75$. We can notice that the put spread option price increases when S_1 increases and S_2 decreases (see table 4.2.3).

Times	$t = 0$	$t = 0.25$	0.50	$t = 0.75$
Maximum value of call spread option	133.4139	132.5743	131.7240	130.8631
Maximum value of put spread option	266.5861	267.4257	268.2760	269.1369

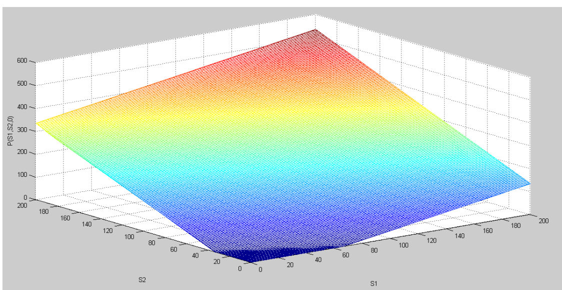
TABLE 1. Maximum values of call and put spread options computed by FVM at times 0, 0.25, 0.5 and 0.75.

4.3. **Basket option.** A basket option is more specifically an exotic option where the payoff is based on a weighted sum or average of the values of a portfolio or basket of underlying assets.

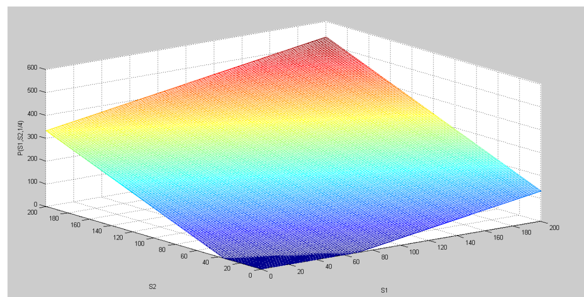
4.3.1. *Call basket option.* The payoff of the call basket option is given by $\phi(S_1, S_2, T) = \max(\alpha S_1 + \beta S_2 - K, 0)$ and the boundary conditions are

$$\begin{aligned} P(0, S_2, t) &= \max(\beta S_2 - Ke^{-r(T-t)}, 0), \\ P(S_1, 0, t) &= \max(\alpha S_1 - Ke^{-r(T-t)}, 0), \\ P(S_{1\,\max}, S_2, t) &= \max(\alpha S_{1\,\max} + \beta S_2 - Ke^{-r(T-t)}, 0), \\ P(S_1, S_{2\,\max}, t) &= \max(\alpha S_1 + \beta S_{2\,\max} - Ke^{-r(T-t)}, 0), \end{aligned} \tag{4.3}$$

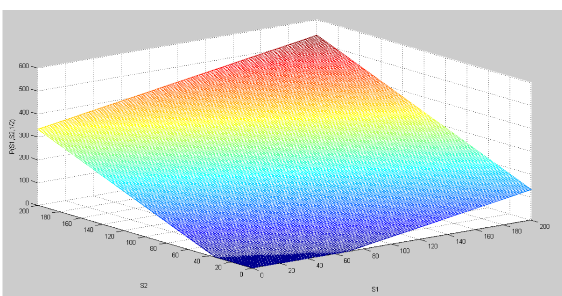
where α and β are positive reals and represent respectively the weights of assets S_1 and S_2. For the numerical experiments we set $\alpha = 1, \beta = 2$. FVM numerical approximation of the call basket option values at different times are computed and presented on figure (3).



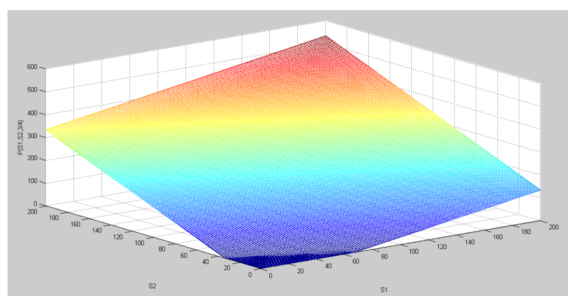
(a) Call basket option at $t = 0$



(b) Call basket option at $t = 0.25$



(c) Call basket option at $t = 0.5$



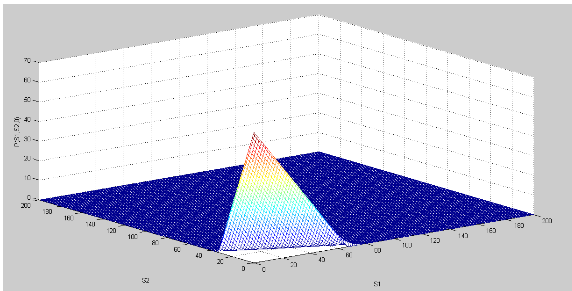
(d) Call basket option at $t = 0.75$

FIGURE 3. FVM numerical solution to the Black-Scholes two dimensional model problem for a call basket option.

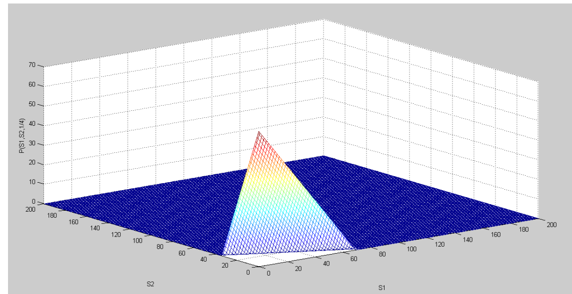
4.3.2. *Put basket option.* The payoff of the put basket option is given by $\phi(S_1, S_2, T) = \max(K - (\alpha S_1 + \beta S_2), 0)$ and the boundary conditions are

$$\begin{aligned} P(0, S_2, t) &= \max(Ke^{-r(T-t)} - \beta S_2, 0) = 0, \\ P(S_1, 0, t) &= \max(Ke^{-r(T-t)} - \alpha S_1, 0), \\ P(S_{1\,\max}, S_2, t) &= \max(Ke^{-r(T-t)} - (\alpha S_{1\,\max} + \beta S_2), 0), \\ P(S_1, S_{2\,\max}, t) &= \max(Ke^{-r(T-t)} - (\alpha S_1 + \beta S_{2\,\max}), 0), \end{aligned} \tag{4.4}$$

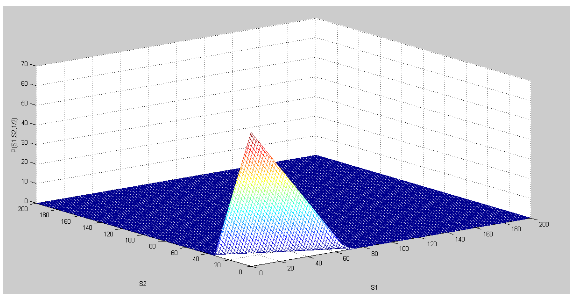
where α and β are positive reals and represent respectively the weights of assets S_1 and S_2. For the numerical experiments we set $\alpha = 1, \beta = 2$. FVM numerical approximation of the put basket option values at different times are presented on figure (4).



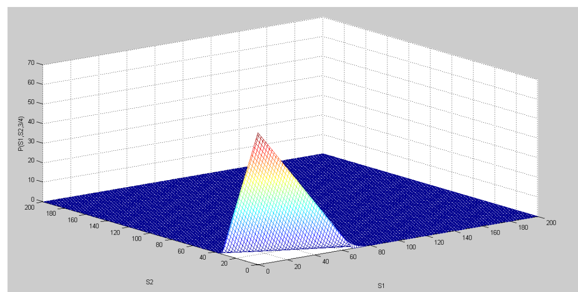
(a) Put basket option at $t = 0$



(b) Put basket option at $t = 0.25$



(c) Put basket option at $t = 0.5$



(d) Put basket option at $t = 0.75$

Figure 4. FVM numerical solution to the Black-Scholes two dimensional model problem for a put basket option.

4.3.3. *Analysis of basket options results.* Figures 3(a), 3(b), 3(c) and 3(d) show the FVM approximations of the call basket option respectively at times $t = 0$, $t = 0.25$, $t = 0.5$ and $t = 0.75$. We observe that the call basket option price increases rapidly when S_1 and S_2 increase (see table 4.3.3) and that is consistent with a behavior of a call basket option which is based on the weighted average of the underlying assets S_1 and S_2.

Moreover, figures 4(a), 4(b), 4(c) and 4(d) show the FVM approximations of the put basket option respectively at times $t = 0$, $t = 0.25$, $t = 0.5$ and $t = 0.75$. The put basket option has nul price for when the values of S_1 and S_2 are near the origin.

Times	$t = 0$	$t = 0.25$	0.50	$t = 0.75$
Maximum value of call basket option	533.413	532.5764	531.7283	530.8696
Maximum value of put basket option	66.58.5861	67.4257	68.2760	69.1369

Table 2. Maximum values of call and put basket options computed by FVM at times 0, 0.25, 0.5 and 0.75.

5. Conclusion

In this paper we present the multi-assets Black-Scholes equation. We then apply the finite volume method to discretize the two-dimensional model problem. Numerical tests have been achieved and provided satisfactory results. This shows that the FVM is an efficient method to approximate numerical solutions to multi-dimensional Black-Scholes equation.

However, stability issues has not been considered in this work. Thus, future work should pursue more investigations on the FVM schemes stability and more generally, extend FVM to others models in finance.

Data Availability Statement: The MATLAB codes used to computes the results of this study are available from the corresponding author upon request.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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