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Analysis of Codimension-Two Bifurcations in a Discrete Modified Leslie-Gower System with Beddington-DeAngelis Response

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Abstract. This paper investigates a discrete-time modified Leslie-Gower prey-predator model featuring the Beddington-DeAngelis functional response. The primary aim is to delve into the intricate dynamics of this model. Initially, we assess the existence and local stability of the model's fixed points. Subsequently, we employ bifurcation theory and the normal form theorem to scrutinize the bifurcation behaviors of the model. Our research uncovers several co-dimension two bifurcations, such as 1:2 resonance, 1:3 resonance, and 1:4 resonance. Numerical simulations are performed to support the theoretical findings.

1. Introduction

In recent years, the predator-prey system has become increasingly significant across multiple areas of mathematical biology, economics and ecology. Grasping the dynamics of this system is essential because of its widespread occurrence and relevance [1]. The foundational studies by Lotka (1925) [2] and Volterra (1926) [3] established the groundwork for investigating predator-prey interactions. Since then, numerous researchers have delved into this relationship across various fields [4]- [8]. Mathematical models in this context can be divided into two categories: continuous-time models, which are expressed through differential equations, and discrete-time models, represented by difference equations. Recently, discrete-time models have gained favor for their practicality and suitability compared to continuous-time models. Furthermore, these discrete-time models exhibit more complex dynamics and can potentially lead to chaotic behaviors.

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Leslie was the first to propose the Leslie-Gower predator-prey model in his works [9] and [10]. The following equations describe this model:

$$\begin{aligned} \frac{dx}{dt} &= x(r_1 - b_1 x) - P(x)y, \\ \frac{dy}{dt} &= y\left(r_2 - \frac{a_2 y}{x}\right), \end{aligned} \tag{1.1}$$

In this model, the densities of prey and predator at time t are symbolized by x and y, respectively. The prey's growth rate is denoted by r_1, while competition strength among individuals is represented by b_1. The prey's maximum sustainable population, without predation, is $\frac{r_1}{b_1}$. The predator feeds on the prey following the function $P(x)$ and grows logistically at a rate r_2. The predator's carrying capacity scales with the prey population, with a factor of $\frac{r_2 x}{a_2}$, where a_2 indicates how much prey translates to predator offspring. The term $\frac{y}{x}$ captures the Leslie-Gower effect, showing predator decline due to food scarcity (per capita $\frac{y}{x}$). The Leslie model stresses that both prey x and predator y have growth limits, distinguishing it from the Lotka-Volterra model.

In 1965, Holling [11] introduced three categories of functional responses: Type I, Type II, and Type III. Among these, the Holling-type II functional response, expressed as $P(x) = \frac{a_1 x}{x+K_1}$, has become widely popular in predator-prey dynamics for its practicality [12]. This form is recognized as the most common representation of predation behavior. Various mathematical models, as discussed in the literature ([13]- [15]), have explored different functional responses to investigate the dynamics of predator-prey systems. Scholars [16] have indicated that when faced with significant scarcity, the predator population y might shift to alternative prey populations but is limited by the availability of its preferred food source x. To tackle this challenge, a modified predator-prey model integrating the Leslie-Gower framework and the Holling-type II functional response has been put forward [16]. The modified predator-prey model, considering both the Leslie-Gower structure and the Holling-type II functional response, can be represented as follows:

$$\begin{aligned} \frac{dx}{dt} &= x(r_1 - b_1 x) - \frac{a_1 x y}{x + K_1}, \\ \frac{dy}{dt} &= y\left(r_2 - \frac{a_2 y}{x + K_2}\right), \end{aligned} \tag{1.2}$$

where the parameters r_1, b_1, r_2, and a_2 retain their original interpretations from the previous system (1.1). Moreover, within this model setup, a_1 represents the maximum per capita reduction rate for the prey x. The values of K_1 and K_2" encapsulate the degree of protection that the environment affords to the prey x and the predator y, respectively. The researchers [16] have delved into the constraints and overall stability of the positive fixed point within the model (1.1). Numerous scholars have extensively explored the system (1.2), incorporating various extensions such as delays, impulses, harvesting, and other factors (refer to the works [17]- [27]).

In space-limited predation, predators have unrestricted movement but can only encounter prey within a small area. For instance, this may happen if predators could only catch prey along the edge of a safe haven, or if the only places prey could be vulnerable to predators were holes in the protective covering. In this scenario, at low predator densities, predators do not interfere with one another, but at large densities, they do so effectively through competition for foraging space. An elementary form of the functional response formula $P(x)$ with such characteristics is defined as $P(x) = \frac{\alpha x}{a+x+cy}$. Beddington [28] and DeAngelis *et al.* [29] outlined this type of functional response to explain predator feeding interference. Here, α represents the maximum predation rate; a represents the saturation constant; c scales the impact of the predator interference. Incorporating the Beddington-DeAngelis functional response into the model (1.2), we obtain the following modified form:

$$\begin{aligned} \frac{dx}{dt} &= x\left(r_1 - b_1 - \frac{\alpha y}{a+x+cy}\right), \\ \frac{dy}{dt} &= y\left(r_2 - \frac{\beta_1 y}{x+K_2}\right). \end{aligned} \tag{1.3}$$

Introducing the following rescaling set: $\bar{t} = \alpha r_1 t$, $\bar{x} = \frac{b_1}{r_1}x$, and $\bar{y} = \frac{1}{r_2}y$, and ignoring the bars, model (1.3) transformed into

$$\begin{aligned} \frac{dx}{dt} &= x\left(\eta - \eta x - \frac{y}{\gamma_1 + \gamma_2 x + \gamma_3 y}\right), \\ \frac{dy}{dt} &= y\left(\sigma - \frac{\omega y}{x+K}\right), \end{aligned} \tag{1.4}$$

where $\eta = \frac{1}{\alpha}$, $\gamma_1 = \frac{a r_1}{r_2}$, $\gamma_2 = \frac{r_1^2}{r_2 b_1}$, $\gamma_3 = c r_1$, $\sigma = \frac{r_2}{\alpha r_1}$, $\omega = \frac{\beta b_1 r_2}{\alpha r_1^2}$, and $K = \frac{b_1 K_2}{r_1}$. Employing the forward Euler's scheme to discretize system (1.4), resulting in the following discretized form:

$$\begin{aligned} x_{n+1} &= x_n + \delta x_n\left[\eta(1-x_n) - \frac{y_n}{\gamma_1 + \gamma_2 x_n + \gamma_3 y_n}\right], \\ y_{n+1} &= y_n + \delta y_n\left(\sigma - \frac{\omega y_n}{x_n + K}\right), \end{aligned} \tag{1.5}$$

where $\delta > 0$ is the integral step size.

The main goal of this study is to look into how a discrete modified Leslie-Gower predator-prey model that includes the Beddington-DeAngelis functional response changes over time. Moreover, our objective is to thoroughly examine the different co-dimension two bifurcation types that manifest in this model. Our approach entails the utilization of a normal form methodology to investigate resonance phenomena, encompassing 1:2 resonance, 1:3 resonance, and 1:4 resonance. Furthermore, we conduct computations to determine the non-degeneracy coefficients linked to each resonance form, thereby offering enhanced understanding of the system's dynamics.

The remainder of this paper is organized as follows: In Section 2, we explore the fixed points and assess their local stability within the context of the model presented in (1.5). Section 3 discusses various types of co-dimension two bifurcations identified at the positive fixed point. We conduct numerical simulations in Section 4 and Section 5 to corroborate our theoretical findings. Finally, Section 6 wraps up the paper with a summary of our key discoveries.

2. Analysis of Fixed Point Existence and Stability

In this section, our focus lies on investigating the existence and stability properties of fixed points within the model (1.5). We delve into a thorough examination of these fixed points, aiming to understand their presence and analyze their stability characteristics. By performing calculations, we deduce that the system (1.5) possesses the following non-negative fixed points:

(1) The fixed point $E_0 = (0, 0)$, known as the trivial fixed point.
(2) The two boundary fixed points $E_1 = (1, 0)$ and $E_2 = (0, \frac{\sigma K}{\omega})$.
(3) The coexistence fixed points are defined by the system

$$\begin{aligned}\frac{y}{\gamma_1 + \gamma_2 x + \gamma_3 y} =& \eta(1-x)\\ \omega y =& \sigma(x+K)\end{aligned}$$

Proposition 1. *Assume that* $\gamma_1\omega + \gamma_3\sigma K \geq \frac{\sigma K}{\eta}$. *Then, model (1.5) has a unique positive fixed point* $E_1^* = (x_1, y_1) := (x^*, y^*)$.

Proof. A coexistence fixed point $E^* = (x^*, y^*)$ of model (1.5) meets the conditions:

$$\begin{aligned}\frac{y^*}{\gamma_1 + \gamma_2 x^* + \gamma_3 y^*} =& \eta(1-x^*)\\ \omega y^* =& \sigma(x^*+K)\end{aligned} \tag{2.1}$$

Substitute the second equation of (2.1) in the first one; we obtain

$$(\gamma_2\omega + \gamma_3\sigma)x^{*2} + (\gamma_1\omega + \gamma_3\sigma K + \sigma - \gamma_2\omega - \gamma_3\sigma)x^* + \frac{\sigma K}{\eta} - \gamma_1\omega - \gamma_3\sigma K = 0. \tag{2.2}$$

The discriminant of equation (2.2) is

$$\begin{aligned}\Delta =& [(\gamma_1\omega + \gamma_3\sigma K + \sigma) - (\gamma_2\omega + \gamma_3\sigma)]^2 - 4(\gamma_2\omega + \gamma_3\sigma)\left(\frac{\sigma K}{\eta} - \gamma_1\omega - \gamma_3\sigma K\right)\\ =& (\gamma_1\omega + \gamma_3\sigma K + \sigma)^2 + (\gamma_2\omega + \gamma_3\sigma)^2 - 2(\gamma_1\omega + \gamma_3\sigma K + \sigma)(\gamma_2\omega + \gamma_3\sigma)\\ &+4(\gamma_2\omega + \gamma_3\sigma)\left(\gamma_1\omega + \gamma_3\sigma K - \frac{\sigma K}{\eta}\right)\\ =& (\gamma_1\omega + \gamma_3\sigma K + \sigma)^2 + (\gamma_2\omega + \gamma_3\sigma)^2 + 2(\gamma_1\omega + \gamma_3\sigma K + \sigma)(\gamma_2\omega + \gamma_3\sigma)\\ &-4(\gamma_2\omega + \gamma_3\sigma)\left[\sigma\left(\frac{K}{\eta} + 1\right)\right]\\ =& [(\gamma_1\omega + \gamma_3\sigma K + \sigma) + (\gamma_2\omega + \gamma_3\sigma)]^2 - 4(\gamma_2\omega + \gamma_3\sigma)\left[\sigma\left(\frac{K}{\eta} + 1\right)\right].\end{aligned}$$

Thus, if $\gamma_1\omega + \gamma_3\sigma K \geq \frac{\sigma K}{\eta}$, then

$$\begin{aligned}\Delta =& [(\gamma_1\omega + \gamma_3\sigma K + \sigma) + (\gamma_2\omega + \gamma_3\sigma)]^2 - 4(\gamma_2\omega + \gamma_3\sigma)\left[\sigma\left(\frac{K}{\eta} + 1\right)\right]\\ \geq& \left[\sigma\left(\frac{K}{\eta} + 1\right) + (\gamma_2\omega + \gamma_3\sigma)\right]^2 - 4(\gamma_2\omega + \gamma_3\sigma)\left[\sigma\left(\frac{K}{\eta} + 1\right)\right]\end{aligned} \tag{2.3}$$

$$= \left[\sigma\left(\frac{K}{\eta}+1\right)-(\gamma_2\omega+\gamma_3\sigma)\right]^2 \geq 0.$$

Consequently, model (1.5) has two coexistence fixed points, $E_1^*(x_1, y_1)$ and $E_2^*(x_2, y_2)$, which are defined as follows:

$$\begin{aligned} x_{1,2} =& \frac{[(\gamma_2\omega+\gamma_3\sigma)-(\gamma_1\omega+\gamma_3\sigma K+\sigma)] \pm \sqrt{\Delta}}{2(\gamma_2\omega+\gamma_3\sigma)}, \\ \omega y_{1,2} =& \sigma(x_{1,2}+K). \end{aligned} \tag{2.4}$$

Now, we show that if $\gamma_1\omega+\gamma_3\sigma K \geq \frac{\sigma K}{\eta}$, then one of these fixed points is in $\mathbb{R}_+^2$. Assume that

$$x_2 = \frac{[(\gamma_2\omega+\gamma_3\sigma)-(\gamma_1\omega+\gamma_3\sigma K+\sigma)] - \sqrt{\Delta}}{2(\gamma_2\omega+\gamma_3\sigma)}.$$

Then

$$2x_2 = 1 - \frac{\gamma_1\omega+\gamma_3\sigma K+\sigma+\sqrt{\Delta}}{\gamma_2\omega+\gamma_3\sigma}.$$

Due to $\gamma_1\omega+\gamma_3\sigma K \geq \frac{\sigma K}{\eta}$, we obtain

$$2x_2 \leq 1 - \frac{\sigma\left(\frac{K}{\eta}+1\right)+\sqrt{\Delta}}{\gamma_2\omega+\gamma_3\sigma}.$$

Besides, from (2.3), we have

$$\sqrt{\Delta} \geq \left|\sigma\left(\frac{K}{\eta}+1\right)-(\gamma_2\omega+\gamma_3\sigma)\right|,$$

which implies that

$$2x_2 \leq 1 - \frac{\sigma\left(\frac{K}{\eta}+1\right)+\left|\sigma\left(\frac{K}{\eta}+1\right)-(\gamma_2\omega+\gamma_3\sigma)\right|}{\gamma_2\omega+\gamma_3\sigma}.$$

Thus, we have the following two cases:

(1) if $\sigma\left(\frac{K}{\eta}+1\right) \geq \gamma_2\omega+\gamma_3\sigma$, then

$$2x_2 \leq 1 - \frac{\sigma\left(\frac{K}{\eta}+1\right)+\sigma\left(\frac{K}{\eta}+1\right)-(\gamma_2\omega+\gamma_3\sigma)}{\gamma_2\omega+\gamma_3\sigma} = 2 - \frac{2\sigma\left(\frac{K}{\eta}+1\right)}{\gamma_2\omega+\gamma_3\sigma} \leq 0;$$

(2) if $\sigma\left(\frac{K}{\eta}+1\right) < \gamma_2\omega+\gamma_3\sigma$, then

$$2x_2 \leq 1 - \frac{\sigma\left(\frac{K}{\eta}+1\right)-\sigma\left(\frac{K}{\eta}+1\right)+(\gamma_2\omega+\gamma_3\sigma)}{\gamma_2\omega+\gamma_3\sigma} = 0.$$

It results that $E_2^*(x_2, y_2)$ is not in $\mathbb{R}_+^2$. Moreover, since

$$x_1x_2 = \frac{\frac{\sigma K}{\eta}-\gamma_1\omega-\gamma_3\sigma K}{\gamma_2\omega+\gamma_3\sigma} \leq 0,$$

then $E_1^*(x_1, y_1)$ is in $\mathbb{R}_+^2$. This completes the proof. □

Moving forward, we will now delve into the examination of local stability concerning the fixed points identified earlier. To determine the local stability of the discrete model (1.5), we calculate the eigenvalues of the Jacobian matrices, denoted as J, evaluated at each fixed point (x, y). This analysis allows us to gain insights into the stability properties of the system at these specific points. The Jacobian matrix of the system (1.5) at the point (x, y) is given by:

$$J(x,y) = \begin{pmatrix} 1+\delta[\eta(1-2x) - \frac{y(\gamma_1+\gamma_3 y)}{(\gamma_1+\gamma_2 x+\gamma_3 y)^2}] & -\frac{\delta x(\gamma_1+\gamma_2 x)}{(\gamma_1+\gamma_2 x+\gamma_3 y)^2} \\ \frac{\delta\omega y^2}{(x+K)^2} & 1+\delta[\sigma - \frac{2\omega y}{x+K}] \end{pmatrix}. \tag{2.5}$$

In our endeavor to analyze the stability of the fixed points within the system (1.5), we put forth the following lemma as a valuable tool for our investigation:

Lemma 2.1. *[30] Consider the function $F(\lambda) = \lambda^2 + P\lambda + Q$, where λ_1 and λ_2 are the two roots of $F(\lambda) = 0$. It is assumed that $F(1) > 0$. In light of these conditions, we proceed to investigate the following:*

1. *The condition $|\lambda_1| < 1$ and $|\lambda_2| < 1$ holds if and only if $F(-1) > 0$ and $Q < 1$;*
2. *The condition $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) is satisfied if and only if $F(-1) < 0$;*
3. *The condition $|\lambda_1| > 1$ and $|\lambda_2| > 1$ holds if and only if $F(-1) > 0$ and $Q > 1$;*
4. *The condition $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ is satisfied if and only if $F(-1) = 0$ and $P \neq 0, 2$.*
5. *The eigenvalues λ_1 and λ_2 are complex, and $|\lambda_1| = 1$ and $|\lambda_2| = 1$ if and only if $P^2 - 4Q < 0$ and $Q = 1$.*

To elucidate the various topological types associated with the fixed point $E^*(x^*, y^*)$, we put forth the following lemma:

Lemma 2.2. *[30] Suppose that $F(\lambda) = \lambda^2 + P\lambda + Q$ be the characteristic equation of the Jacobian matrix at the fixed point $E^*(x^*, y^*)$ with eigenvalues λ_1 and λ_2, then*

1. *The fixed point $E^*(x^*, y^*)$ is classified as a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. In this case, the sink is locally asymptotically stable;*
2. *The fixed point $E^*(x^*, y^*)$ is referred to as a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. In this situation, the source is locally unstable;*
3. *The fixed point $E^*(x^*, y^*)$ is known as a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$);*
4. *The fixed point $E^*(x^*, y^*)$ is considered non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.*

In this section, we present and discuss the following four theorems:

Theorem 2.1. *The trivial fixed point $E_0 = (0, 0)$ in the model (1.5) is characterized as a source point.*

Proof. By evaluating the Jacobian matrix J of the system (1.5) at the fixed point E_0, we can express it in the following form:

$$J(E_0) = \begin{pmatrix} 1+\delta\eta & 0 \\ 0 & 1+\delta\sigma \end{pmatrix}. \tag{2.6}$$

The eigenvalues of $J(E_0)$ are $\lambda_1 = 1 + \delta\eta$ and $\lambda_2 = 1 + \delta\sigma$. By applying Lemma 2.2, we can conclude that E_0 is classified as a source point. □

Theorem 2.2. *The axial fixed point $E_1 = (1,0)$ in the system (1.5) can exhibit various topological types. Specifically, E_1 is classified as a saddle if $\delta < \frac{2}{\eta}$. Conversely, if $\eta < \frac{2}{\delta}$, then E_1 is considered a saddle point. Moreover, E_1 is non-hyperbolic when $\eta = \frac{2}{\eta}$.*

Proof. By evaluating the Jacobian matrix J of the system (1.5) at the fixed point E_1, we can express it in the following form:

$$J(E_1) = \begin{pmatrix} 1-\delta\eta & -\frac{\delta}{\gamma_1+\gamma_2} \\ 0 & 1+\delta\sigma \end{pmatrix}. \tag{2.7}$$

Hence, the eigenvalues of the matrix $J(E_1)$ can be written as $\lambda_1 = 1-\delta\eta$ and $\lambda_2 = 1+\delta\sigma$. Therefore, we get the four types of topological by lemma 2.2. If $\delta < \frac{2}{\eta}$, then E_1 is referred to as a saddle point. If $\delta > \frac{2}{\eta}$, then E_1 is source point. Also, E_1 is a nonhyperbolic at $\delta = \frac{2}{\eta}$. □

Theorem 2.3. *The boundary fixed point $E_2 = (0, \frac{\sigma K}{\omega})$ is stable point when $\delta < \min\{\frac{2}{\sigma}, \frac{2(\gamma_1\omega+\gamma_3\sigma K)}{\sigma K-\eta(\gamma_1\omega+\gamma_3\sigma K)}\}$. Moreover, the equilibrium point E_2 is a saddle point at λ_1 is positive which implies $\eta(\gamma_1\omega+\gamma_3\sigma K) > \sigma K$. Also, E_1 is nonhyperbolic when $\sigma = \frac{2}{\delta}$.*

Proof. The Jacobian matrix J of the model (1.5) at point E_2 is defined as:

$$J(E_2) = \begin{pmatrix} 1+\delta(\eta-\frac{\sigma K}{\gamma_1\omega+\gamma_3\sigma K}) & 0 \\ \frac{\delta\sigma^2}{\omega} & 1-\delta\sigma \end{pmatrix}. \tag{2.8}$$

The eigenvalues of $J(E_2)$ can be written as $\lambda_1 = 1+\delta(\eta-\frac{\sigma K}{\gamma_1\omega+\gamma_3\sigma K})$ and $\lambda_2 = 1-\delta\sigma$. The equilibrium point E_2 is a sink (locally asymptotically stable) if $\delta < \min\{\frac{2}{\sigma}, \frac{2(\gamma_1\omega+\gamma_3\sigma K)}{\sigma K-\eta(\gamma_1\omega+\gamma_3\sigma K)}\}$ and $\sigma K > \eta(\gamma_1\omega+\gamma_3+\sigma K)$. Also, the point E_2 is called a saddle point when $\eta > \frac{\sigma K}{\gamma_1\omega+\gamma_3\sigma K}$ and $\delta > \frac{2}{\eta}$. E_2 is nonhyperbolic at $\delta = \frac{2}{\sigma}$. □

In addition, we explain the local stability analysis of the positive fixed point $E^*(x^*, y^*)$ by using the lemma 2.1. The Jacobian matrix of the system (1.5) at E^* take the form

$$J(x^*, y^*) = \begin{pmatrix} 1+\delta[\eta(1-2x^*)-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}] & -\frac{\delta x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\ \frac{\delta\sigma^2}{\omega} & 1-\delta\sigma \end{pmatrix}. \tag{2.9}$$

The auxiliary equation of the Jacobian matrix for the system (1.5), computed at the interior fixed point E^* can be written as

$$\lambda^2 - (2+\delta G)\lambda + 1 + \delta G + \delta^2 H = 0, \tag{2.10}$$

where

$$G = \eta(1-2x^*) - \frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} - \sigma,$$

and

$$H = \frac{\sigma^2}{\omega}\frac{x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} - \sigma\eta(1-2x^*) + \frac{\sigma y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}.$$

Let

$$F(\lambda) = \lambda^2 - (2+\delta G)\lambda + 1 + \delta G + \delta^2 H,$$

then

$$F(1) = \delta^2 H, \qquad F(-1) = 4 + 2\delta G + \delta^2 H.$$

Since

$$\begin{aligned}
H &= \sigma\left[\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} - \eta(1-2x^*)\right] + \frac{\sigma^2}{\omega}\frac{x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\
&= \frac{\sigma}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\left[y^*(\gamma_1+\gamma_3 y^*) + \eta(2x^*-1)(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2 + \frac{\sigma}{\omega}x^*(\gamma_1+\gamma_2 x^*)\right] \\
&= \frac{\sigma}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\
&\times\left[y^*(\gamma_1+\gamma_3 y^*) + \eta(x^*-1)(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2 + \eta x^*(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2 + \frac{\sigma}{\omega}x^*(\gamma_1+\gamma_2 x^*\right] \\
&= \frac{\sigma}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\
&\times\left[y^*(\gamma_1+\gamma_3 y^*) - y^*(\gamma_1+\gamma_2 x^*+\gamma_3 y^*) + \eta x^*(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2 + \frac{\sigma}{\omega}x^*(\gamma_1+\gamma_2 x^*\right] \\
&= \frac{\sigma}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\left[-\gamma_2 x^* y^* + \eta x^*(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2 + \frac{\sigma}{\omega}x^*(\gamma_1+\gamma_2 x^*\right] \\
&= \frac{\sigma}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\left[\eta x^*(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2 + \frac{x^*}{\omega}(\gamma_1\sigma+\gamma_2\sigma x^*-\gamma_2\omega y^*)\right] \\
&= \frac{\sigma}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\left[\eta x^*(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2 + \frac{x^*}{\omega}(\gamma_1\sigma-\gamma_2\sigma K)\right] \\
&= \frac{\sigma\eta x^*}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\left[(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2 + \frac{\sigma}{\eta\omega}(\gamma_1-\gamma_2 K)\right],
\end{aligned}$$

thus, $H > 0$ iff $\gamma_1 > \gamma_2 K$.

Based on the implications of Lemma 2.1, we can establish the local stability of the positive fixed point E^*.

Theorem 2.4. *Assume that $\gamma_1\omega + \gamma_3\sigma K \geq \frac{\sigma K}{\eta}$ and $\gamma_1 > \gamma_2 K$. The following cases hold:*

- *E^* is categorized as a sink if any of the following conditions is met:*
  (1) $G = -2\sqrt{H}$ *and* $0 < \delta < -\frac{G}{H}$,
  (2) $G < -2\sqrt{H}$ *and* $0 < \delta < \frac{-G-\sqrt{G^2-4H}}{H}$,
  Consequently, E^* exhibits local asymptotic stability.
- *E^* is considered a source if any of the following conditions is met:*
  (1) $G = -2\sqrt{H}$ *and* $\delta > -\frac{G}{H}$,
  (2) $G < -2\sqrt{H}$ *and* $\delta > \frac{-G+\sqrt{G^2-4H}}{H}$.
- *E^* is classified as a saddle if the following condition is met:*

$$G < -2\sqrt{H}, \qquad \textit{and} \qquad \frac{-G-\sqrt{G^2-4H}}{H} < \delta < \frac{-G+\sqrt{G^2-4H}}{H},$$

- *E^* is considered non-hyperbolic if any of the following conditions are met:*

$$G < -2\sqrt{H} \quad , \quad \delta = \frac{-G \pm \sqrt{G^2-4H}}{H} \quad \textit{and} \quad \delta \neq -\frac{2}{G}, -\frac{4}{G}, \tag{2.11}$$

or

$$-2\sqrt{H} < G < 0 \quad and \quad \delta = -\frac{G}{H}. \tag{2.12}$$

The non-hyperbolic conditions mentioned in Theorem 2.4 provide additional criteria for determining the local asymptotic stability of the positive fixed point. These conditions help us better understand the system's behavior and ensure a more comprehensive analysis of its stability.

Remark 2.1. *It is worth emphasizing that when neither condition (2.11) nor condition (2.12) is fulfilled, the positive fixed point of the system (1.5) can be classified as locally asymptotically stable.*

3. Codimension-Two Bifurcation Analysis

In this section, we have thoroughly examined the condition about the non-hyperbolic case. By using Eq. (2.10) to obtain the eigenvalues of $J(x^*, y^*)$ as

$$\lambda_{1,2} = \frac{2 - G\delta \pm \sqrt{G^2\delta^2 - 4\delta^2 H}}{2}.$$

When $G\delta = H\delta^2 = 4$, then $\lambda_1 = \lambda_2 = -1$, and the system (1.5) undergoes $1:2$ resonance at E^*. Also, if $G\delta = H\delta^2 = 3$, the eigenvalues are $\lambda_{1,2} = \frac{-1\pm\sqrt{3}i}{2}$. This clear that E^* is $1:3$ resonance point of system (1.5). Finally, if $G\delta = H\delta^2 = 2$, then $\lambda_{1,2} = \pm i$. The system (1.5) undergoes $1:4$ resonance at E^*. In the following, we take both δ and η as bifurcation parameters to introduce bifurcation analysis at $E^*(x^*, y^*)$. Furthermore, we define the three bifurcation types

$$F_{1j} = \left\{(\delta, \eta, \gamma_1, \gamma_2, \gamma_3, \sigma, \omega, K) \in \mathbb{R}^8_+ : G\delta = 6 - j, H\delta^2 = 6 - j,\ j = 2, 3, 4\right\},$$

at these specific values, resonances occur in the ratios of 1:2, 1:3, and 1:4, respectively.

3.1. Exploring the 1:2 resonance at $E^*(x^*, y^*)$.

Within this subsection, we will establish the occurrence of a 1:2 resonance at the positive fixed point E^* in model (1.5). To accomplish this, we will employ both bifurcation theory and normal form theory as outlined in the work of Kuznetsov [31]. Taking the parameters $(\tilde{\delta}, \tilde{\eta}, \gamma_1, \gamma_2, \gamma_3, \sigma, \omega, K)$ chosen at random from F_{12}, the model (1.5) becomes:

$$\begin{aligned} x_{n+1} &= x_n + \tilde{\delta} x_n \left(\tilde{\eta}(1 - x_n) - \frac{y_n}{\gamma_1 + \gamma_2 x_n + \gamma_3 y_n}\right), \\ y_{n+1} &= y_n + \tilde{\delta} y_n \left(\sigma - \frac{\omega y_n}{x_n + K}\right). \end{aligned} \tag{3.1}$$

At the positive fixed point $E^*(x^*, y^*)$ of model (1.5), the system exhibits two eigenvalues, namely $\lambda_1 = \lambda_2 = -1$. By considering $\tilde{\delta}$ and $\tilde{\eta}$ as the bifurcation parameters, we can introduce a small perturbation to the model (3.1), resulting in the following modified form:

$$\begin{aligned} x_{n+1} &= x_n + (\tilde{\delta} + \delta^*) x_n \Big[(\tilde{\eta} + \eta^*)(1 - x_n) - \frac{y_n}{\gamma_1 + \gamma_2 x_n + \gamma_3 y_n}\Big], \\ y_{n+1} &= y_n + (\tilde{\delta} + \delta^*) y_n \Big[\sigma - \frac{\omega y_n}{x_n + K}\Big], \end{aligned} \tag{3.2}$$

where $|\delta^*|, |\eta^*| \ll 1$ are small perturbation parameters.
Let $\tilde{u}(n) = x(n) - x^*$, $\tilde{v}(n) = y(n) - y^*$, $\delta = \tilde{\delta} + \delta^*$ and $h = \tilde{h} + h^*$, then we turned the fixed point E^* into the origin. Model (3.2) transformed to

$$\begin{aligned} \tilde{u}(n+1) &= a_1\tilde{u}(n) + a_2\tilde{v}(n) + a_{11}\tilde{u^2}(n) + a_{12}\tilde{u}(n)\tilde{v}(n) + a_{22}\tilde{v^2}(n) + O((|\tilde{u}(n)| + |\tilde{v}(n)|)^3), \\ \tilde{v}(n+1) &= b_1\tilde{u}(n) + b_2\tilde{v}(n) + b_{11}\tilde{u^2}(n) + b_{12}\tilde{u}(n)\tilde{v}(n) + b_{22}\tilde{v^2}(n) + O((|\tilde{u}(n)| + |\tilde{v}(n)|)^3), \end{aligned} \tag{3.3}$$

where

$$\begin{cases} a_1 = 1 + \delta[\eta - 2\eta x^* - \frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}],\ a_2 = \frac{-\delta x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}, \\ a_{11} = \delta[-\eta + \frac{\gamma_2 y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}],\ a_{12} = \frac{-\delta({\gamma_1}^2+\gamma_1\gamma_2 x^*+\gamma_1\gamma_3 y^*+2\gamma_2\gamma_3 x^* y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^3}, \\ a_{22} = \delta[\frac{\gamma_3 x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^3}],\ b_1 = \frac{\delta\omega {y^*}^2}{(x^*+K)^2},\ b_2 = 1 + \delta[\sigma - \frac{2\omega y^*}{x^*+K}], \\ b_{11} = \frac{-\delta\omega {y^*}^2}{(x^*+K)^3},\ b_{12} = \frac{2\delta\omega y^*}{(x^*+K)^2},\ b_{22} = \frac{-\delta\omega}{x^*+K}. \end{cases} \tag{3.4}$$

Let an invertible matrix T is given by

$$T = \begin{pmatrix} \frac{-a_2}{1+a_1} & \frac{-a_2}{(1+a_1)^2} \\ 1 & 0 \end{pmatrix},$$

by using the transformation:

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = T \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

Consequently, the mapping (3.3) can be transformed into:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \mapsto \begin{pmatrix} -1 + a_{10}(\delta,\eta) & 1 + a_{01}(\delta,\eta) \\ b_{10}(\delta,\eta) & -1 + b_{01}(\delta,\eta) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} \tilde{f}(\tilde{x},\tilde{y},\delta,\eta) \\ \tilde{g}(\tilde{x},\tilde{y},\delta,\eta) \end{pmatrix}, \tag{3.5}$$

where

$$\tilde{f}(\tilde{x},\tilde{y},\delta,\eta) = \sum_{2\le j+k\le 3} \tilde{f}_{j,k}(\delta,\eta)\,\tilde{x}^j\tilde{y}^k,$$

$$\tilde{g}(\tilde{x},\tilde{y},\delta,\eta) = \sum_{2\le j+k\le 3} \tilde{g}_{j,k}(\delta,\eta)\,\tilde{x}^j\tilde{y}^k,$$

and

$$\begin{cases} a_{10} = 2 + \delta[\sigma + \frac{\omega y^*(-2(x^*+K)(1+a_1)-a_2 y^*)}{(1+a_1)(x^*+K)^2}], \\ a_{01} = -1 - \frac{\delta\omega a_2 {y^*}^2}{(1+a_1)^2(x^*+K)^2}, \\ b_{10} = -1 + \frac{1}{1+a_1} - \frac{\delta\gamma_1 y^*}{(1+a_1)(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} + \frac{\delta x^*(\gamma_1+\gamma_2 x^*)}{a_2((\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2)} - \frac{\delta\gamma_3 {y^*}^2}{(1+a_1)(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} + \frac{\delta\eta}{1+a_1} - \frac{2\delta\eta x^*}{1+a_1} \\ \quad + \frac{{a_1}^2\delta x^*(\gamma_1+\gamma_2 x^*)}{a_2(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} + \frac{{a_1}^2(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2(1+\delta\eta-2\delta\eta x^*)-{a_1}^2\delta\gamma_1 y^*-{a_1}^2\delta\gamma_3 {y^*}^2}{(1+a_1)(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} - \delta\sigma + \frac{2\delta\omega y^*}{x^*+K} \\ \quad + \frac{\delta\omega {y^*}^2 a_2}{(1+a_1)(x^*+K)^2} + \frac{2a_1\delta x^*(\gamma_1+\gamma_2 x^*)}{a_2(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} - \frac{2a_1\delta\gamma_3 {y^*}^2}{(1+a_1)(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} - \frac{a_1(-1+a_1+a_1\delta\sigma+\delta(-2\eta+4x^*\eta+\sigma))}{1+a_1} \\ \quad + \frac{a_1 a_2\delta\omega {y^*}^2}{(1+a_1)(x^*+K)^2} - \frac{2a_1 y^*\delta\gamma_1}{(1+a_1)(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} + \frac{2a_1\delta\omega y^*}{x^*+K}, \\ b_{01} = \frac{1}{(1+a_1)^2}\Big[1 - \frac{\delta\gamma_1 y^*}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2+\delta\eta-2x^*\delta\eta} + {a_1}^2(1 - \frac{\delta(\gamma_1 y^*+\delta\gamma_3 {y^*}^2)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} + \delta\eta - 2x^*\delta\eta) \\ \quad + {y^*}^2(\frac{a_2\delta\omega}{(x^*+K)^2} - \frac{\delta\gamma_3}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}) + a_1(2 - \frac{2\delta\gamma_1 y^*}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} + 2\delta\eta - 4x^*\delta\eta) + \frac{2a_2\delta\omega {y^*}^2}{(x^*+K)^2} - \frac{2\delta\gamma_3 {y^*}^2}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\Big], \end{cases}$$

$$\begin{cases} \tilde{f}_{20} = -\frac{\delta\omega(K+a_1K+x^*+a_1x^*+a_2y^*)^2}{(1+a_1)^2(x^*+K)^3},\ \tilde{f}_{11} = -\frac{2\delta\omega a_1 y^*(K+a_1K+x^*+a_1x^*+a_2y^*)}{(1+a_1)^3(x^*+K)^3},\ \tilde{f}_{02} = \frac{\delta\omega {a_2}^2 y^{*2}}{(1+a_1)^4(x^*+K)^3}, \\ \tilde{f}_{30} = \tilde{f}_{12} = \tilde{f}_{21} = \tilde{f}_{03} = 0, \\ \tilde{g}_{20} = \delta\Big[-\frac{(1+a_1){\gamma_1}^2}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} - \frac{(1+a_1)^2\gamma_2\gamma_3x^{*2}}{a_2(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} - \frac{2x^*y^*\gamma_2\gamma_3}{(1+a_1)(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} - \frac{4a_1x^*y^*\gamma_2\gamma_3}{(1+a_1)(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} \\ \quad -\frac{2{a_1}^2x^*y^*\gamma_2\gamma_3}{(1+a_1)(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} - \frac{\gamma_1(x^*+a_1x^*+a_2y^*)(a_2\gamma_2+\gamma_3+a_1\gamma_3)}{a_3(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} + \frac{\omega}{x^*+K} + \frac{a_1\omega}{x^*+K} + \frac{{a_2}^2y^{*2}\omega}{(1+a_1)(x^*+K)^3} \\ \quad -\frac{a_2y^{*2}\gamma_2\gamma_3}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} + a_2\eta + \frac{2a_2y^*\omega}{(x^*+K)^2}\Big], \\ \tilde{g}_{11} = \frac{\delta}{(1+a_1)^3}\Big[-\frac{(1+a_1)^3{\gamma_1}^2}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} - \frac{(1+a_1)^2\gamma_1((1+a_1)\gamma_2x^*+y^*(2a_2\gamma_2+\gamma_3+a_1\gamma_3))}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} - \frac{2(1+a_1)^3\gamma_2\gamma_3x^*y^*}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} \\ \quad +2a_2(1+a_1)\Big((1+a_1)\eta + \frac{(1+a_1)\omega y^*}{(x^*+K)^2} - \frac{(1+a_1)^3\gamma_2\gamma_3y^{*2}}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} + \frac{a_2\omega y^{*2}}{(x^*+K)^3}\Big)\Big], \\ \tilde{g}_{02} = \frac{a_2\delta}{(1+a_1)^3}\Big[(1+a_1)\eta - \frac{(1+a_1)\gamma_1\gamma_2y^*}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} - \frac{(1+a_1)\gamma_2\gamma_3y^{*2}}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^3} + \frac{a_2\omega y^{*2}}{(x^*+K)^3}\Big], \\ \tilde{g}_{30} = \tilde{g}_{12} = \tilde{g}_{21} = \tilde{g}_{03} = 0. \end{cases}$$

The nonsingular linear coordinate transformation is given as

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} 1 + a_{01}(\delta,\eta) & 0 \\ -a_{10}(\delta,\eta) & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}.$$

Then, the equation (3.5) transformed to

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} -1 & 1 \\ \theta_1(\delta,\eta) & -1+\theta_2(\delta,\eta) \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} \hat{f}(\hat{x},\hat{y},\delta,\eta) \\ \hat{g}(\hat{x},\hat{y},\delta,\eta) \end{pmatrix}, \tag{3.6}$$

where

$$\hat{f}(\hat{x},\hat{y},\delta,\eta) = \sum_{2\le j+k\le 3} \hat{f}_{j,k}(\delta,\eta)\,\hat{x}^j\hat{y}^k,$$

$$\hat{g}(\hat{x},\hat{y},\delta,\eta) = \sum_{2\le j+k\le 3} \hat{g}_{j,k}(\delta,\eta)\,\hat{x}^j\hat{y}^k,$$

and

$$\begin{cases} \theta_1 = -a_{10}b_{01} + b_{10} + a_{01}b_{10}, \\ \theta_2 = a_{10} + b_{01}, \\ \hat{f}_{20} = \frac{1}{1+a_{01}}[a_{02}{a_{10}}^2 + (1+a_{01})(-a_{10}a_{11} + a_{20} + a_{01}a_{20})], \\ \hat{f}_{11} = \frac{-2a_{02}a_{10}+a_{11}+a_{01}a_{11}}{1+a_{01}},\ \hat{f}_{02} = \frac{a_{02}}{1+a_{01}}, \\ \hat{f}_{30} = \hat{f}_{12} = \hat{f}_{21} = \hat{f}_{03} = 0, \\ \hat{g}_{20} = \frac{a_{02}{a_{10}}^3+(1+a_{01})\left({a_{10}}^2(-a_{11}+b_{02})+a_{10}(1+a_{01})(a_{20}-b_{11})+b_{20}(1+a_{01})^2\right)}{1+a_{01}}, \\ \hat{g}_{11} = \frac{-2a_{02}{a_{10}}^2+(1+a_{01})(a_{10}a_{11}-2a_{10}b_{02}+b_{11}+a_{01}b_{11})}{1+a_{01}}, \\ \hat{g}_{02} = b_{02} + \frac{a_{02}a_{10}}{1+a_{01}}, \\ \hat{g}_{30} = \hat{g}_{12} = \hat{g}_{21} = \hat{g}_{03} = 0, \end{cases}$$

Finally, we offer the following transformation

$$\begin{aligned} \hat{x} &= \xi + \sum_{2\le j+k\le 3} \phi_{jk}\left(\tilde{\delta},\tilde{\eta}\right)\xi^j\mu^k, \\ \hat{y} &= \mu + \sum_{2\le j+k\le 3} \psi_{jk}\left(\tilde{\delta},\tilde{\eta}\right)\xi^j\mu^k, \end{aligned} \tag{3.7}$$

this transformation allows us to obtain the coefficients ϕ_{jk} and ψ_{jk} from the following relation:

$$\begin{cases} \phi_{20} = \frac{1}{2}\hat{f}_{20} + \frac{1}{4}\hat{g}_{20}, \\ \phi_{11} = \frac{1}{2}\hat{f}_{20} + \frac{1}{2}\hat{f}_{11} + \frac{1}{2}\hat{g}_{20} + \frac{1}{4}\hat{g}_{11}, \\ \phi_{02} = \frac{1}{4}\hat{f}_{11} + \frac{1}{2}\hat{f}_{02} + \frac{1}{8}\hat{g}_{20} + \frac{1}{4}\hat{g}_{11} + \frac{1}{4}\hat{g}_{02}, \\ \psi_{20} = \frac{1}{2}\hat{g}_{20}, \\ \psi_{11} = \frac{1}{2}\hat{g}_{20} + \frac{1}{2}\hat{g}_{11}, \\ \psi_{02} = \frac{1}{4}\hat{g}_{11} + \frac{1}{2}\hat{g}_{02} \end{cases}$$

By using the transformation (3.7) and its inverse transformation to compensate in map (3.6), we get

$$\begin{pmatrix} \xi \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} -1 & 1 \\ \theta_1(\delta,\eta) & 1+\theta_2(\delta,\eta) \end{pmatrix}\begin{pmatrix} \xi \\ \mu \end{pmatrix} + \begin{pmatrix} \Gamma(\hat{x},\hat{y},\delta,\eta) \\ \Sigma(\hat{x},\hat{y},\delta,\eta) \end{pmatrix}, \tag{3.8}$$

where

$$\Gamma(\hat{x},\hat{y},\delta,\eta) = \sum_{2\le j+k\le 3} \gamma_{jk}\xi^j\mu^k + O\left((|\xi|+|\mu|)^4\right),$$
$$\Sigma(\hat{x},\hat{y},\delta,\eta) = \sum_{2\le j+k\le 3} \sigma_{jk}\xi^j\mu^k + O\left((|\xi|+|\mu|)^4\right),$$

where γ_{jk} and σ_{jk} are certain functions of $\hat{f}_{jk}$, $\hat{g}_{jk}$ and ϕ_{jk}, ψ_{jk}.

$$\begin{cases} \gamma_{20}(\delta,\eta) = \hat{f_{20}} + \psi_{20} - 2\phi_{20} - \phi_{02}{\theta_1}^2 + \phi_{11}\theta_1, \\ \gamma_{11}(\delta,\eta) = \hat{f_{11}} + \psi_{11} - 2\phi_{02}\theta_1(1+\theta_2) + \phi_{11}(\theta_2-\theta_1) + 2\phi_{20}, \\ \gamma_{02}(\delta,\eta) = \hat{f_{02}} + \psi_{02} - \phi_{02}(1+(1+\theta_2)^2) + \phi_{11}(\theta_2+1) - \phi_{20}, \\ \sigma_{20}(\delta,\eta) = \hat{g_{20}} - \psi_{02}{\theta_1}^2 + \psi_{11}\theta_1 + \psi_{20}\theta_2 + \psi_{20}\theta_1, \\ \sigma_{11}(\delta,\eta) = \hat{g_{11}} - 2\psi_{02}\theta_1(\theta_2+1) + \psi_{11}(2-\theta_1+2\theta_2) + 2\psi_{20} + \phi_{11}\theta_1, \\ \sigma_{02}(\delta,\eta) = \hat{g_{02}} - \psi_{02}\theta_2(1+\theta_2) - \psi_{11}(1+\theta_2) - \psi_{20} + \phi_{02}\theta_1. \end{cases}$$

By taking ϕ_{jk}, ψ_{jk} with $j+k=2$, we can eliminate all quadratic terms. Indeed, all the quadratic terms are vanishing (see [32]).

$$\gamma_{20} = \gamma_{11} = \gamma_{02} = \sigma_{20} = \sigma_{11} = \sigma_{02} = 0.$$

Furthermore, at ϕ_{jk}, ψ_{jk} with $j+k=3$ all cubic terms are annihilate except those resonant terms. Namely the vanishing conditions

$$\gamma_{30} = \gamma_{12} = \gamma_{21} = \gamma_{03} = \sigma_{12} = \sigma_{03} = 0.$$

From the above transformations, map (3.8) turned into the normal form $1:2$ resonance as follows

$$\begin{pmatrix} \xi \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} -\xi+\mu \\ \theta_1(\delta,\eta)\,\xi + (-1+\theta_2(\delta,\eta))\,\mu + C(\delta,\eta)\,\xi^3 + D(\delta,\eta)\,\xi^2\mu \end{pmatrix} + O\left((|\xi|+|\mu|)\right)^4, \tag{3.9}$$

where $C(\delta,\eta) = \sigma_{30}$ and $D(\delta,\eta) = \sigma_{21}$. When $(\delta,\eta) = \left(\tilde{\delta},\tilde{\eta}\right)$, we get $\theta_1(\delta,\eta) = \theta_2(\delta,\eta) = 0$,

$$C\left(\tilde{\delta},\tilde{\eta}\right) = \hat{g}_{30} + \hat{f}_{20}\hat{g}_{20} + \frac{1}{2}\hat{g}_{20}^2 + \frac{1}{2}\hat{g}_{20}\hat{g}_{11},$$

$$D\left(\tilde{\delta},\tilde{\eta}\right)=\hat{g}_{21}+3\hat{f}_{30}+\frac{1}{2}\hat{f}_{20}\hat{g}_{11}+\frac{5}{4}\hat{g}_{20}\hat{g}_{11}$$
$$+\hat{g}_{20}\hat{g}_{02}+3\hat{f}_{20}+\frac{5}{2}\hat{f}_{20}\hat{g}_{20}+\frac{5}{2}\hat{f}_{11}\hat{g}_{20}+\hat{g}_{20}^{2}+\frac{1}{2}\hat{g}_{11}^{2}.$$

Based on the findings of Lemma 9.10 and Theorem 9.3 in [31], we can now present the following theorem as a conclusion.

Theorem 3.1. *Let $C\left(\tilde{\delta},\tilde{\eta}\right)\neq 0$ and $D\left(\tilde{\delta},\tilde{\eta}\right)+3C\left(\tilde{\delta},\tilde{\eta}\right)\neq 0$, then map (1.5) contain to the next one dynamical behaviors (see [32], [33])*

(1) *A flip bifurcation curve $F=(\theta_1,\theta_2):\theta_1=0$ exists, and there are non-trivial fixed points for $\theta_1<0$.*
(2) *A non-degenerate Neimark-Sacker bifurcation curve can be observed*
$$H=\{(\theta_1,\theta_2):\theta_1=-\theta_2+O\left((|\theta_1|+|\theta_2|)^2\right),\theta_1<0\};$$
(3) *there is a heteroclinic bifurcation curve*
$$HL=\{(\theta_1,\theta_2):\theta_1=-\frac{5}{3}\theta_2+O\left((|\theta_1|+|\theta_2|)^2\right),\theta_1<0\}.$$

3.2. **1:3 resonance of the model (1.5) at the positive fixed point $E^*(x^*,y^*)$.** Here, we discuss the dynamics which occur near 1:3 resonance of the model (1.5). Taking both $\left(\hat{\delta},\hat{\eta}\right)$ as bifurcation parameters from F_{13}.

The model (1.5) with parameters $(\hat{\delta},\hat{\eta},\gamma_1,\gamma_2,\gamma_3,\sigma,\omega,K)$ is given by

$$\begin{aligned} x_{n+1}&=x_n+\hat{\delta}x_n(\hat{\eta}(1-x_n)-\frac{y_n}{\gamma_1+\gamma_2x_n+\gamma_3y_n}),\\ y_{n+1}&=y_n+\hat{\delta}y_n(\sigma-\frac{\omega y_n}{x_n+K}). \end{aligned}\tag{3.10}$$

Model (3.10) at the positive fixed point E^* has two eigenvalues are $\lambda_{1,2}=\frac{-1\pm\sqrt{3}i}{2}$. Assume that $\hat{u}(n)=x(n)-x^*$, $\hat{v}(n)=y(n)-y^*$. Then, the model (3.10) is turned into

$$\begin{aligned} \hat{u}(n+1)&=a_1\hat{u}(n)+a_2\hat{v}(n)+a_{11}\hat{u^2}(n)+a_{12}\hat{u}(n)\hat{v}(n)+a_{22}\hat{v^2}(n)+O((|\hat{u}(n)|+|\hat{v}(n)|)^3),\\ \hat{v}(n+1)&=b_1\hat{u}(n)+b_2\hat{v}(n)+b_{11}\hat{u^2}(n)+b_{12}\hat{u}(n)\hat{v}(n)+b_{22}\hat{v^2}(n)+O((|\hat{u}(n)|+|\hat{v}(n)|)^3), \end{aligned}\tag{3.11}$$

where both $a_1,a_2,a_{11},a_{12},a_{22},b_1,b_2,b_{11},b_{12}$ and b_{22} are mentioned previously.

The Jacobian matrix of the model (3.10) at E^* is

$$A(\hat{\delta},\hat{\eta})=\begin{pmatrix} 1+\hat{\delta}[\hat{\eta}(1-2x^*)-\frac{y^*(\gamma_1+\gamma_3y^*)}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^2}] & -\frac{\hat{\delta}x^*(\gamma_1+\gamma_2x^*)}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^2}\\ \frac{\hat{\delta}\omega y^{*2}}{(x^*+K)^2} & 1+\hat{\delta}[\sigma-\frac{2\omega y^*}{x^*+K}] \end{pmatrix}.$$

In addition, we compute a pair of adjoint eigenvectors $q(\hat{\delta},\hat{\eta}),p(\hat{\delta},\hat{\eta})\in\mathbb{C}^2$ of $A(\hat{\delta},\hat{\eta})$ such at $A(\hat{\delta},\hat{\eta})q(\hat{\delta},\hat{\eta})=\frac{-1+\sqrt{3}i}{2}q(\hat{\delta},\hat{\eta})$, $A^T(\hat{\delta},\hat{\eta})p(\hat{\delta},\hat{\eta})=-\frac{1+\sqrt{3}i}{2}p(\hat{\delta},\hat{\eta})$,

$< p(\hat{\delta},\hat{\eta}), q(\hat{\delta},\hat{\eta}) >= 1$, where $< .,. >$ means the standard scalar product in $\mathbb{C}^2 :< p,q >= \bar{p_1}q_1 + \bar{p_2}q_2$. Then, we have

$$q(\hat{\delta},\hat{\eta}) = \begin{pmatrix} -\frac{\delta x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\ \frac{-3+\sqrt{3}i}{2} - \delta(\eta - 2\eta x^* - \frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*++\gamma_3 y^*)^2}) \end{pmatrix},$$

$$p(\hat{\delta},\hat{\eta}) = \begin{pmatrix} \frac{-\sqrt{3}i}{3}\Big(\frac{3+\sqrt{3}i}{2} + \delta(\sigma - \frac{2\omega y^*}{x^*+K})\Big)\Big(\frac{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}{y^*(\gamma_1+\gamma_3 y^*)}\Big) \\ \frac{-\sqrt{3}i}{3} \end{pmatrix}.$$

For any vector $X \in \mathbb{R}^2$, it is possible to represent it in the following form:

$$X = zq(\hat{\delta},\hat{\eta}) + \bar{z}\overline{q(\hat{\delta},\hat{\eta})}, z \in \mathbb{C}.$$

Therefore, the map (3.11) can be turned into the complex form as

$$z \to \frac{\sqrt{3}i-1}{2}z + \sum_{2\le l+k\le 3} \frac{1}{k!l!} g_{kl} z^k \bar{z}^l, \tag{3.12}$$

where

$$\begin{cases} g_{20}(\delta,\eta) = -\frac{1}{4\sqrt{3}i}\Bigg[\frac{-4\omega\delta^2 x^{*2} y^{*2}(\gamma_1+\gamma_2 x^*)^2}{(x^*+K)^3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^4} - \frac{-4\omega\delta^2 x^* y^*(\gamma_1+\gamma_2 x^*)\Big(-3+\sqrt{3}i-2\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big)}{(x^*+K)^2(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\ \quad -\frac{\omega\delta\Big(3-\sqrt{3}i+2\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big)}{(x^*+K)} + \frac{4\delta^2 x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\Big(\frac{\gamma_2 y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^3} - \eta\Big) \\ \quad +2\delta({\gamma_1}^2 + \gamma_1\gamma_2 x^* + \gamma_1\gamma_3 y^* + 2\gamma_2\gamma_3 x^* y^*)\Big(-3+\sqrt{3}i - 2\delta(\eta - 2x^*\eta - \frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big) \\ \quad +\gamma_3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2\Big(-3+\sqrt{3}i + 2\delta(\eta - 2x^*\eta - \frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big)^2(\frac{3}{2} + \frac{\sqrt{3}i}{2} + \delta\sigma - \frac{2\delta\omega y^*}{x^*+K})\Bigg], \\ g_{11}(\delta,\eta) = -\frac{1}{2\sqrt{3}i}\Bigg[\frac{-4\omega\delta^2 x^{*2} y^{*2}(\gamma_1+\gamma_2 x^*)^2}{(x^*+K)^3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^4} - \frac{-4\omega\delta^2 x^* y^*(\gamma_1+\gamma_2 x^*)\Big(-3+\sqrt{3}i-2\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big)}{(x^*+K)^2(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\ \quad -\frac{\omega\delta\Big(3-\sqrt{3}i+2\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big)}{(x^*+K)} + \frac{4\delta^2 x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\Big(\frac{\gamma_2 y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^3} - \eta\Big) \\ \quad +2\delta({\gamma_1}^2 + \gamma_1\gamma_2 x^* + \gamma_1\gamma_3 y^* + 2\gamma_2\gamma_3 x^* y^*)\Big(-3+\sqrt{3}i - 2\delta(\eta - 2x^*\eta - \frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big) \\ \quad +\gamma_3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2\Big(-3+\sqrt{3}i + 2\delta(\eta - 2x^*\eta - \frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big)^2(\frac{3}{2} + \frac{\sqrt{3}i}{2} + \delta\sigma - \frac{2\delta\omega y^*}{x^*+K})\Bigg], \\ g_{02}(\delta,\eta) = -\frac{1}{4\sqrt{3}i}\Bigg[\frac{-4\omega\delta^2 x^{*2} y^{*2}(\gamma_1+\gamma_2 x^*)^2}{(x^*+K)^3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^4} - \frac{-4\omega\delta^2 x^* y^*(\gamma_1+\gamma_2 x^*)\Big(-3+\sqrt{3}i-2\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big)}{(x^*+K)^2(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\ \quad -\frac{\omega\delta\Big(3-\sqrt{3}i+2\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big)}{(x^*+K)} + \frac{4\delta^2 x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\Big(\frac{\gamma_2 y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^3} - \eta\Big) \\ \quad +2\delta({\gamma_1}^2 + \gamma_1\gamma_2 x^* + \gamma_1\gamma_3 y^* + 2\gamma_2\gamma_3 x^* y^*)\Big(-3+\sqrt{3}i - 2\delta(\eta - 2x^*\eta - \frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big) \\ \quad +\gamma_3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2\Big(-3+\sqrt{3}i + 2\delta(\eta - 2x^*\eta - \frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\Big)^2(\frac{3}{2} + \frac{\sqrt{3}i}{2} + \delta\sigma - \frac{2\delta\omega y^*}{x^*+K})\Bigg], \\ g_{30}(\delta,\eta) = g_{12}(\delta,\eta) = g_{21}(\delta,\eta) = g_{03}(\delta,\eta) = 0. \end{cases}$$

To remove all quadratic terms, we introduce the following transformation

$$z = \beta + \frac{1}{2}h_{20}\beta^2 + h_{11}\beta\bar{\beta} + \frac{1}{2}h_{02}\bar{\beta}^2, \tag{3.13}$$

by using the transformation (3.13) and its inverse transformation. Map (3.12) turned into

$$\beta \to \frac{\sqrt{3}i-1}{2}\beta + \sum_{2\le l+k\le 3}\frac{1}{k!l!}\varrho_{kl}\beta^k\overline{\beta}^l + O(|\beta|^4), \tag{3.14}$$

where

$$\varrho_{20} = g_{20} + \sqrt{3}ih_{20},\ \varrho_{11} = 2g_{11} + (\sqrt{3}i-3)h_{11} \textit{ and } \varrho_{02} = g_{02}.$$

By putting

$$h_{20} = \frac{\sqrt{3}i}{3}g_{20},\ h_{11} = \frac{3+\sqrt{3}i}{6}g_{11} \textit{ and } h_{02} = 0.$$

According to the previous values of ϱ and h, we get $\varrho_{20} = \varrho_{11} = 0$ and $\varrho_{02} = g_{02}$, and $\varrho_{30}, \varrho_{21}, \varrho_{12}, \varrho_{03}$ can be simplified as

$$\begin{cases} \varrho_{30} = \frac{3-\sqrt{3}i}{2}g_{11}\overline{g}_{02} + \sqrt{3}ig_{20}^2 + g_{30}, \\ \varrho_{21} = \frac{3+2\sqrt{3}i}{2}g_{11}g_{20} + \frac{3-\sqrt{3}i}{3}|g_{11}|^2 + g_{21}, \\ \varrho_{12} = \frac{3+\sqrt{3}i}{6}g_{20}g_{02} + \frac{3-\sqrt{3}i}{3}\overline{g}_{11}g_{02} + \frac{3+\sqrt{3}i}{3}g_{11}^2 - \frac{\sqrt{3}i}{3}\overline{g}_{20}g_{11}, \\ \varrho_{03} = \sqrt{3}ig_{11}g_{02} - \sqrt{3}ig_{02}\overline{g}_{20} + g_{03}. \end{cases}$$

For simplified the equation (3.14), we offer the following transformation

$$\beta = \xi + \frac{1}{6}h_{30}\xi^3 + \frac{1}{2}h_{21}\xi^2\overline{\xi} + \frac{1}{2}h_{12}\xi\overline{\xi}^2 + \frac{1}{6}h_{03}\overline{\xi}^3. \tag{3.15}$$

After using the equation (3.15) and its inverse, map (3.14) take the form:

$$\xi = \frac{\sqrt{3}i-1}{2}\xi + \frac{1}{2}g_{02}\overline{\xi}^2 + \sum_{l+k=3}\frac{1}{k!l!}\tilde{\varrho}_{kl}\xi^k\overline{\xi}^l + O(|\xi|^4), \tag{3.16}$$

where

$$\tilde{\varrho}_{30} = \varrho_{30} + \frac{\sqrt{3}i-3}{2}h_{30},\ \tilde{\varrho}_{21} = \varrho_{21},$$
$$\tilde{\varrho}_{12} = \varrho_{12} + \sqrt{3}ih_{12} \textit{ and } \tilde{\varrho}_{03} = \varrho_{03} + \frac{\sqrt{3}i-3}{2}h_{03}.$$

By setting

$$h_{30} = \frac{3+\sqrt{3}i}{6}\varrho_{30},\ h_{12} = \frac{\sqrt{3}i}{3}\varrho_{12}, h_{03} = \frac{3+\sqrt{3}i}{6}\varrho_{03} \textit{ and } h_{21} = 0,$$

then we obtain $\tilde{\varrho}_{30} = \tilde{\varrho}_{21} = \tilde{\varrho}_{03} = 0$.

Finally, we demonstrate the normal form representation of the bifurcation exhibiting 1:3 resonance as follows:

$$\xi \to \frac{\sqrt{3}i-1}{2}\xi + B(\hat{\delta},\hat{\eta})\overline{\xi} + C(\hat{\delta},\hat{\eta})\xi|\xi|^2 + O(|\xi|^4), \tag{3.17}$$

where

$$B(\hat{\delta},\hat{\eta}) = \frac{1}{2}g_{02}(\delta,\eta),\ C(\hat{\delta},\hat{\eta}) = \frac{(3+\sqrt{3}i)g_{20}g_{11}}{6} + \frac{(3-\sqrt{3}i)|g_{11}|^2}{6} + \frac{1}{2}g_{21}.$$

If

$$B_1(\hat{\delta},\hat{\eta}) = \frac{-3}{2}(\sqrt{3}i+1)B(\hat{\delta},\hat{\eta}),$$
$$C_1(\hat{\delta},\hat{\eta}) = -3|B(\hat{\delta},\hat{\eta})|^2 - \frac{3}{2}(\sqrt{3}i+1)C(\hat{\delta},\hat{\eta}).$$

By Theorem (2) in ([34]) and Lemma (9.13) in ([31]) can be obtained the following theorem

Theorem 3.2. *If $B_1(\hat{\delta},\hat{\eta}) \neq 0$ and $Re(C_1(\hat{\delta},\hat{\eta})) \neq 0$, at the fixed point E^*, the mapping (1.5) displays the following complex dynamical behaviors:*

(a) A Neimark-Sacker bifurcation occurs at the trivial fixed point E_1 of the map (3.17);
(b) A saddle cycle of period-three, corresponding to the saddle fixed points E_k (where $k = 1,2,3$), is present in the map (3.17);
(c) The stable and unstable invariant manifolds of the period-three cycle intersect transversally, creating a narrow parameter region where a homoclinic structure forms.In order to remove certain cubic terms, we introduce the following transformation:

3.3. **1:4 resonance of the model (1.5) at the positive fixed point $E^*(x^*,y^*)$.** In this subsection, we argue the 1:4 resonance of the system (1.5) at the positive fixed point $E^*(x^*,y^*)$ when the parameters δ and η varying in small neighborhood of F_{14}. Also, we take parameters $(\tilde{\delta},\tilde{\eta},\gamma_1,\gamma_2,\gamma_3,\sigma,\omega,K)$ arbitrarily from F_{14}, taking $\tilde{\delta}$ and $\tilde{\eta}$ as bifurcation parameters.
Model (1.5) with $(\tilde{\delta},\tilde{\eta},\gamma_1,\gamma_2,\gamma_3,\sigma,\omega,K)$ is given by

$$\begin{aligned} x_{n+1} &= x_n + \tilde{\delta}x_n(\tilde{\eta}(1-x_n) - \frac{y_n}{\gamma_1+\gamma_2x_n+\gamma_3y_n}), \\ y_{n+1} &= y_n + \tilde{\delta}y_n(\sigma - \frac{\omega y_n}{x_n+K}), \end{aligned} \tag{3.18}$$

where $|\delta-\tilde{\delta}|, |\eta-\tilde{\eta}| \ll 1$, the eigenvalues of model (3.18) at the positive point are $\lambda_{1,2} = \pm i$.
Assume that $\tilde{u} = x - x^*$ and $\tilde{v} = y - y^*$. Then, we turned the point $E^*(x^*,y^*)$ into the origin point.
Model (3.18) transformed to

$$\begin{aligned} \tilde{u}(n+1) &= a_1\tilde{u}(n) + a_2\tilde{v}(n) + a_{11}\tilde{u^2}(n) + a_{12}\tilde{u}(n)\tilde{v}(n) + a_{22}\tilde{v^2}(n) + O((|\tilde{u}(n)|+|\tilde{v}(n)|)^3), \\ \tilde{v}(n+1) &= b_1\tilde{u}(n) + b_2\tilde{v}(n) + b_{11}\tilde{u^2}(n) + b_{12}\tilde{u}(n)\tilde{v}(n) + b_{22}\tilde{v^2}(n) + O((|\tilde{u}(n)|+|\tilde{v}(n)|)^3), \end{aligned} \tag{3.19}$$

where $a_1,a_2,a_{11},a_{12},a_{22}$ and $b_1,b_2,b_{11},b_{12},b_{22}$ are given in Eq. (3.4).
The Jacobian matrix of the model (3.18) at E^* is given by

$$A(\tilde{\delta},\tilde{\eta}) = \begin{pmatrix} 1+\tilde{\delta}[\tilde{\eta}(1-2x^*) - \frac{y^*(\gamma_1+\gamma_3y^*)}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^2}] & -\frac{\tilde{\delta}x^*(\gamma_1+\gamma_2x^*)}{(\gamma_1+\gamma_2x^*+\gamma_3y^*)^2} \\ \frac{\tilde{\delta}\omega y^{*2}}{(x^*+K)^2} & 1+\tilde{\delta}[\sigma-\frac{2\omega y^*}{x^*+K}] \end{pmatrix}.$$

Also, we can got a pair of adjoint eigenvectors $q(\tilde{\delta},\tilde{\eta}), p(\tilde{\delta},\tilde{\eta}) \in \mathbb{C}^2$ such that
$A(\tilde{\delta},\tilde{\eta})q(\tilde{\delta},\tilde{\eta}) = iq(\tilde{\delta},\tilde{\eta})$,

$A^T(\tilde{\delta},\tilde{\eta})p(\tilde{\delta},\tilde{\eta}) = -ip(\tilde{\delta},\tilde{\eta})$,
$< p(\tilde{\delta},\tilde{\eta}), q(\tilde{\delta},\tilde{\eta}) >= 1$. According to the previous relations, we have

$$q(\tilde{\delta},\tilde{\eta}) = \begin{pmatrix} \frac{\delta x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\ 1+\delta(\eta-2\eta x^* - \frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*++\gamma_3 y^*)^2}) - i \end{pmatrix},$$

$$p(\tilde{\delta},\tilde{\eta}) = \begin{pmatrix} \left(\frac{-i(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}{2\delta x^*(\gamma_1+\gamma_2 x^*)}\right)(1+\delta(\sigma - \frac{2\omega y^*}{x^*+K}) + i) \\ -\frac{i}{2} \end{pmatrix},$$

For any vector $\mathrm{Y_n} = (\tilde{u},\tilde{v})^T \in \mathbb{R}^2$ can be written as

$$\mathrm{Y_n} = zq(\tilde{\delta},\tilde{\eta}) + \bar{z}\bar{q}(\tilde{\delta},\tilde{\eta}), z \in \mathbb{C}.$$

Then, the model (3.19) transformed to the following complex relation

$$z \longmapsto iz + \sum_{2\le k+l\le 3} \frac{1}{k!l!} g_{kl} z^k \bar{z}^l, \tag{3.20}$$

where

$$\begin{cases} g_{20} = -\frac{i}{2}\Big[\frac{-\omega\delta^2 x^{*2} y^{*2}(\gamma_1+\gamma_2 x^*)^2}{(x^*+K)^3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^4} + \frac{2\omega\delta^2 x^* y^*(\gamma_1+\gamma_2 x^*)\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right)}{(x^*+K)^2(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\ \quad -\frac{\omega\delta\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right)^2}{(x^*+K)} + \frac{\delta^2 x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\left(\frac{\gamma_2 y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^3} - \eta\right) \\ \quad -\delta({\gamma_1}^2+\gamma_1\gamma_2 x^*+\gamma_1\gamma_3 y^*+2\gamma_2\gamma_3 x^* y^*)\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right) \\ \quad +\gamma_3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right)^2(1+i+\delta\sigma-\frac{2\delta\omega y^*}{x^*+K})\Big], \\ g_{11} = -i\Big[\frac{-\omega\delta^2 x^{*2} y^{*2}(\gamma_1+\gamma_2 x^*)^2}{(x^*+K)^3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^4} + \frac{2\omega\delta^2 x^* y^*(\gamma_1+\gamma_2 x^*)\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right)}{(x^*+K)^2(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\ \quad -\frac{\omega\delta\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right)^2}{(x^*+K)} + \frac{\delta^2 x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\left(\frac{\gamma_2 y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^3} - \eta\right) \\ \quad -\delta({\gamma_1}^2+\gamma_1\gamma_2 x^*+\gamma_1\gamma_3 y^*+2\gamma_2\gamma_3 x^* y^*)\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right) \\ \quad +\gamma_3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right)^2(1+i+\delta\sigma-\frac{2\delta\omega y^*}{x^*+K})\Big], \\ g_{02} = -\frac{i}{2}\Big[\frac{-\omega\delta^2 x^{*2} y^{*2}(\gamma_1+\gamma_2 x^*)^2}{(x^*+K)^3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^4} + \frac{2\omega\delta^2 x^* y^*(\gamma_1+\gamma_2 x^*)\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right)}{(x^*+K)^2(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2} \\ \quad -\frac{\omega\delta\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right)^2}{(x^*+K)} + \frac{\delta^2 x^*(\gamma_1+\gamma_2 x^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2}\left(\frac{\gamma_2 y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^3} - \eta\right) \\ \quad -\delta({\gamma_1}^2+\gamma_1\gamma_2 x^*+\gamma_1\gamma_3 y^*+2\gamma_2\gamma_3 x^* y^*)\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right) \\ \quad +\gamma_3(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2\left(1-i+\delta(\eta-2x^*\eta-\frac{y^*(\gamma_1+\gamma_3 y^*)}{(\gamma_1+\gamma_2 x^*+\gamma_3 y^*)^2})\right)^2(1+i+\delta\sigma-\frac{2\delta\omega y^*}{x^*+K})\Big], \\ g_{30} = g_{12} = g_{21} = g_{03} = 0. \end{cases}$$

To remove the quadratic terms, we introduce the following equation

$$z = \beta + \frac{1}{2}h_{20}\beta^2 + h_{11}\beta\bar{\beta} + \frac{1}{2}h_{02}\bar{\beta}^2, \tag{3.21}$$

where coefficients h_{kl} with $k+l=2$ will be presented in the following relation.
By using the transformation (3.21) and its inverse transformation, the model (3.20) can be turned into the form

$$\beta \longmapsto i\beta + \sum_{2\leq +l\leq 3} \frac{1}{k!l!}\varrho_{kl}\beta^k\bar{\beta}^l + O((|\beta|+|\bar{\beta}|)^4), \tag{3.22}$$

where

$$\begin{cases} \varrho_{20} = \frac{1}{2}(g_{20} + (\lambda - \lambda^2)h_{20}), \varrho_{11} = (g_{11} + (\lambda - |\lambda|^2)h_{11}), \\ \varrho_{20} = \frac{1}{2}(g_{02} + (\lambda - \bar{\lambda}^2)h_{02}), \varrho_{30} = \frac{1}{6}(g_{30} + (\lambda - \lambda^2)h_{30}), \\ \varrho_{21} = \frac{1}{2}(g_{21} + (\lambda - \lambda|\lambda|^2)h_{21}), \varrho_{12} = \frac{1}{2}(g_{12} + (\lambda - \bar{\lambda}|\lambda|^2)h_{12}), \\ \varrho_{03} = \frac{1}{6}(g_{03} + (\lambda - \bar{\lambda}^3)h_{03}). \end{cases}$$

Then, we conclude that $\varrho_{20} = \varrho_{11} = \varrho_{02} = 0$.
In order to eliminate specific cubic terms, we introduce the following transformation:

$$\beta = \xi + \frac{1}{6}h_{30}\xi^3 + \frac{1}{2}h_{21}\xi^2\bar{\xi} + \frac{1}{2}h_{12}\xi\bar{\xi}^2 + \frac{1}{3}h_{03}\bar{\xi}^3. \tag{3.23}$$

By employing equation (3.23) and its inverse transformation, we can derive:

$$\beta = i\xi + \sum_{n+l=3} \frac{1}{k!l!}\bar{\varrho}_{kl}\xi^k\bar{\xi}^l + O(|\omega|)^4, \tag{3.24}$$

where

$$\bar{\varrho}_{30} = \varrho_{30} + 2ih_{30},\ \bar{\varrho}_{21} = \varrho_{21},$$

$$\bar{\varrho}_{12} = \varrho_{12} + 2ih_{12},\ \bar{\varrho}_{03} = \varrho_{03}.$$

Let

$$h_{30} = \frac{i}{2}\varrho_{30},\ h_{12} = \frac{i}{2}\varrho_{12},\ h_{03} = h_{21} = 0,$$

from the previous relation we obtain $\bar{\varrho}_{30} = \bar{\varrho}_{12} = 0$.
Moreover, when approaching the 1:4 resonance, the map (3.24) can be modified to:

$$\xi \to i\xi + C(\bar{\delta},\bar{\eta})\xi|\xi|^2 + C(\bar{\delta},\bar{\eta})|\bar{\xi}|^3 + O(|\xi|)^4, \tag{3.25}$$

where

$$C(\bar{\delta},\bar{\eta}) = ig_{11}^2 - \frac{1}{2}g_{11}\bar{g_{20}}(1+i) + \bar{g}_{11}g_{02} + g_{02}g_{11}(i-1) - \frac{1}{2}g_{11}g_{20}(1-2i),$$
$$D(\bar{\delta},\bar{\eta}) = \frac{i-1}{4}g_{02}g_{11} - \frac{i+1}{4}g_{20}g_{11} + \frac{1}{6}g_{03}.$$

Let $C_1(\bar{\delta},\bar{\eta}) = -4iC(\bar{\delta},\bar{\eta})$, $D_1(\bar{\delta},\bar{\eta}) = -4iD(\bar{\delta},\bar{\eta})$. At $D_1 \neq 0$, we obtain
$A(\bar{\delta},\bar{\eta}) = \frac{C_1(\bar{\delta},\bar{\eta})}{|D_1(\bar{\delta},\bar{\eta})|}$.

Theorem 3.3. *Suppose that $(\bar{\delta},\bar{\eta}) \in F_{14}$. If $ReA(\bar{\delta},\bar{\eta}) \neq 0$ and $ImA(\bar{\delta},\bar{\eta}) \neq 0$, the model (1.5) has many dynamical behaviors (see [31])*

(1) *At the trivial fixed point E_1 of (3.25), a Neimark-Sacker bifurcation curve emerges. Additionally, when $\lambda = -i$, there exists an invariant circle, while at $\lambda = i$, the invariant circle vanishes.*

(2) *When* $|A| > 1$*, the system (3.25) exhibits a total of eight non-trivial equilibrium points, identified as* S_k *and* E_k *for* $k = 1, 2, 3, 4$*. These eight fixed points undergo appearance or disappearance via a fold bifurcation occurring at their respective parameter values.*

(3) *A Neimark-Sacker bifurcation occurs at each of the fixed points* E_k *for* $k = 1, 2, 3, 4$*.*

4. Numerical Continuation

This section utilizes the MATLAB package MATCONTM [35] to conduct numerical bifurcation analysis, demonstrating the qualitative behaviors discussed in the preceding theorems and lemmas for the system (1.5). Also, to control the occurence both 1:2, 1:3 and 1:4 resonances. Moreover, The bifurcation analysis is depend on continuation methods that describe the solution manifolds of fixed points when the parameters values of the map vary [36]. But, in a two parameter case the boundaries of stability domains of cycles are curves of codimension 1 bifurcations. For the reason, we apply a numerical continuation methods to compute.

Now, we perform a numerical continuation of the positive fixed point $E^*(x^*, y^*)$ by using MATCONTM. By fixing $\eta = 0.25, \gamma_1 = 11, \gamma_2 = 2.2, \gamma_3 = 11, \sigma = 0.18182, \omega = 0.04545, K = 2$ and $\delta = 3.5$, This values of parameters is mentioned in both (see [37], [38]). The MATCONTM report is

```
label = PD, x = (0.874292 11.498431 0.656134)
Normal form coefficient of PD = 1.165652e+00
label = NS, x = (0.671257 10.686203 11.357375)
normal form coefficient of NS = -5.219139e-02
label = BP, x = (-0.000000 0.000880 0.080809)
```

Fig. 1 illustrates the presence of period doubling and Neimark-Sacker bifurcations at the positive fixed point E^*, identified as PD and NS, respectively. Additionally, a transcritical bifurcation occurs at E_0 and is denoted as a branch point (BP) in the provided reports.

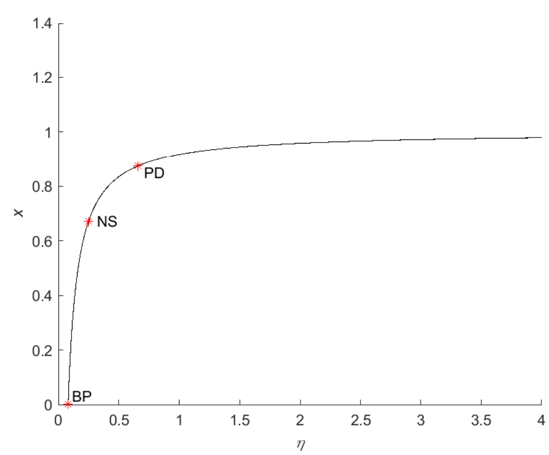


Figure 1. The continuation of E^* in (η, x)-plane. The branch point (BP), period doubling point (PD) and Neimark-Sacker point (NS).

In the PD point $x = (0.874292, 11.498431, 0.656134)$ for the equation (2.10) there exist two-cycle founded by using continuation method. During the supercritical PD bifurcation the cycles becomes unstable. The MATCONTM report of The continuation of the 2-cycles is given by

```
label = PD, x = (0.555513 11.521091 0.783599)
Normal form coefficient of PD = 7.126824e+00
label = PD, x = (1.071504 10.590277 0.783599)
Normal form coefficient of PD = 5.588433e+00
```

The continuation 4-cycle of the PD point is happened when the value of a is greater than 0.783599. The MATCONTM report is formulated as

```
label = PD, x = (0.455719 11.718551 0.810624)
Normal form coefficient of PD = 2.786561e+02
label = PD, x = (0.656300 11.268111 0.810624)
Normal form coefficient of PD = 6.896115e+01
label = PD, x = (1.026801 10.280370 0.810624)
Normal form coefficient of PD = 8.75237e+01
label = PD, x = (1.106514 10.835061 0.810624)
Normal form coefficient of PD = 5.187714e+01
```

According to the previous reports, we deduce that the PD point contain the 2 and 4 cycles of the second and fourth iterates illustrated in Fig. 2.

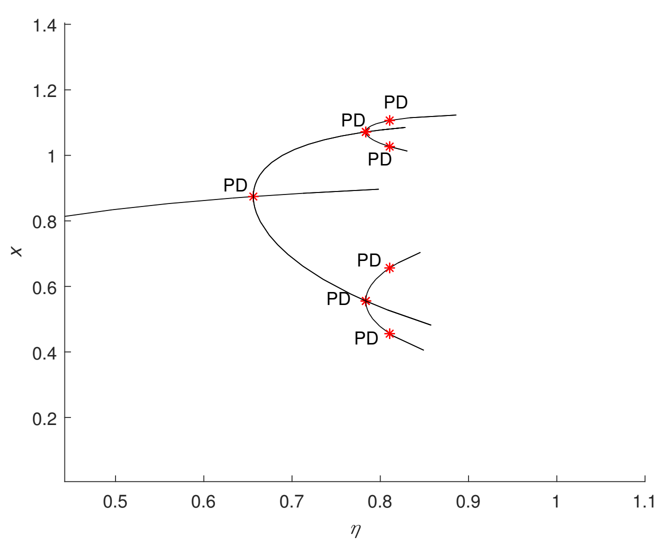
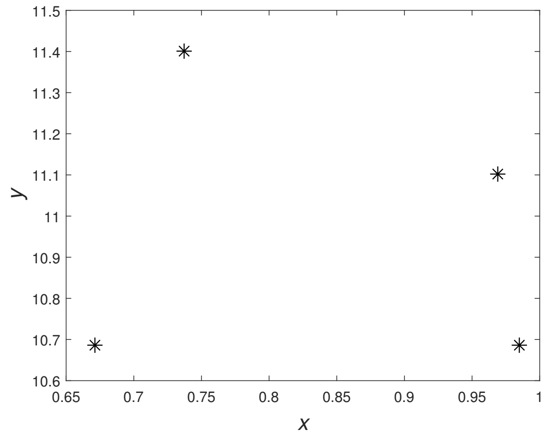
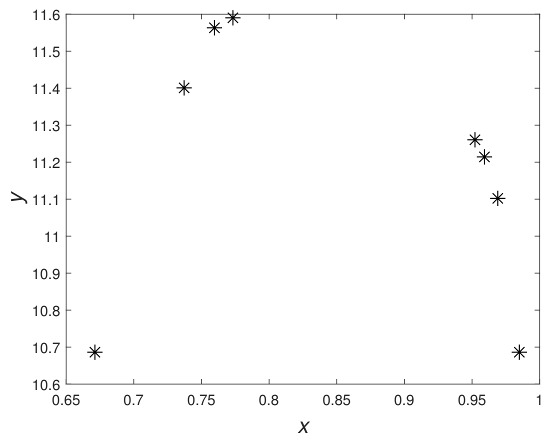


Figure 2. A series of PD points in the plane (η, x) iterated first, second and fourth.

In Fig. 3, a stable 4-cycle is observed at $\eta = 0.656134$, while a stable 8-cycle appears when η exceeds 0.856240.



(a)



(b)

Figure 3. (a) A stable 4-cycle of (1.5) for $a = 0.656134$. (b) A stable 8-cycle of (1.5) for $a = 0.856240$.

In addition to, here we show the codimension-two bifurcation analysis. Firstly, we use the previously given PD point to deduce the period doubling curve at the parametric values as $\gamma_1 = 11, \gamma_2 = 2.2, \gamma_3 = 11, \sigma = 0.18182, \omega = 0.04545, K = 2$ and assume η and δ as free parameters. The MATCONTM report is as follows

```
label = R_{2}, x=(0.729414 10.918856 9.935135 0.304061)
Normal form coefficient for R_{2}:[c,d] = 2.334444e-02, -2.974846e-01
label = GPD, x=(0.733720 10.936085 9.354096 0.309003)
Normal form coefficient of GPD = 8.115912e-03
```

According to the MATCONTM report, we observed that the period doubling curve contain 1 : 2 resonance (R2) and generalized period doubling (GPD) shown in Fig. 4.

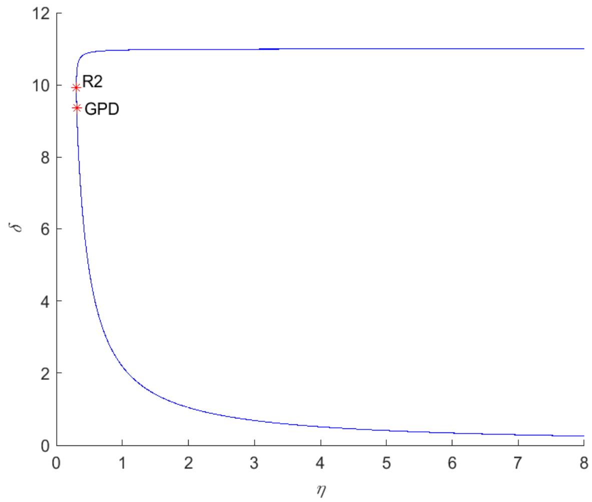


Figure 4. Period doubling curve including *R*2 point and *GPD* point.

Secondly, we select the NS point to compute the Neimark-Sacker curve. The MATCONTM report is given as

```
label = R_{2}, x = (0.653875 22.350463 11.600697 4.933126 -1.000000)
Normal form coefficient for R_{2}:[c,d] = -2.851703e-03, -4.994401e-02
label = R_{3}, x = (0.823860 1.094225 7.800859 -0.550334 -0.500000)
Normal form coefficient of R_{3}:Re(c_1) = -7.061466e-01
label = R_{4}, x = (0.971817 0.100343 4.720136 -0.946734 0.000000)
Normal form coefficient of R_{4}:A=-3.646602e+00+-3.615025e+00i.
```

The Neimark-Sacker curve is concluded to have 1 : 2 resonance (R2), 1 : 3 resonance (R3), and 1 : 4 resonance (R4). Fig. 5 depicts this curve.

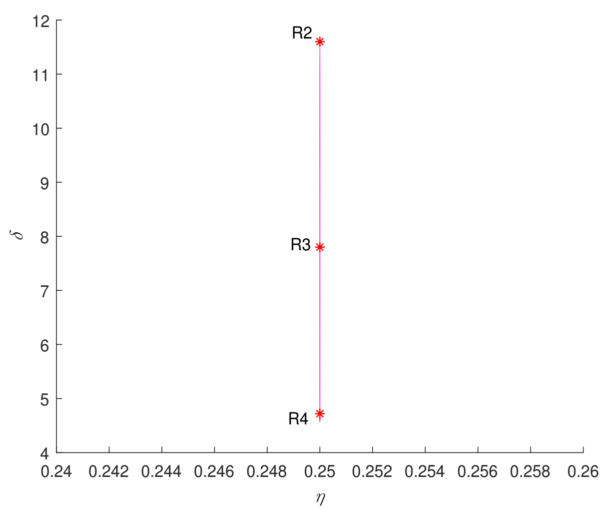


Figure 5. The Neimark-Sacker curve includes the *R*2, *R*3, and *R*4 points.

In Fig. 6, we present 4-cycle of the $R4$ point for $\eta = 0.25, \gamma_1 = 11, \gamma_2 = 2.2, \gamma_3 = 11, \sigma = 0.18182, \omega = 0.04545, K = 2$ and $\delta = 11.3574$, this cycle is occured when $C_4 = \{X_1, X_2, X_3, X_4\}$, where $X_1 = (0.671257, 10.6862)$.

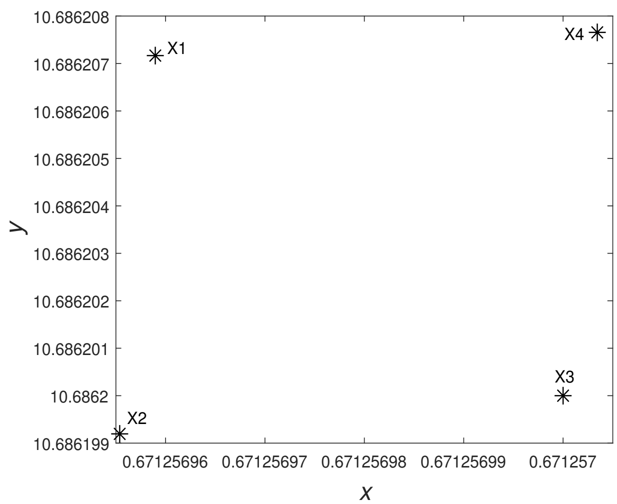


Figure 6. A stable 4-cycle near $R4$ point for $\delta = 11.3574$.

5. Numerical Simulation

In this section, we present some numerical simulation results to validate our analytical results and extract many dynamic behaviors from the model (1.5). The qualitative dynamical behaviors of map (1.5) for the fixed values of the parameters $\eta = 0.25, \gamma_1 = 11, \gamma_2 = 2.2, \gamma_3 = 11, \sigma = 0.18182, \omega = 0.04545, K = 2$, and different values of δ near the calculated Neimark-Sacker point corresponding to $\delta = 11.3574$ are established by simulations.

Fig. 7 (a) shows that E^* is a stable attractor for $\delta = 11.35$ of the map (1.5). The behavior of the map (1.5) before the NS point at $\delta = 11.39$ is depicted in Fig. 7 (b). Figures 7 (c) and 7 (d) are demonstrates the behavior of the model after the NS bifurcation when $\delta = 11.44$ and $\delta = 11.48$. From Figures 7b and 7c we deduce that that the positive fixed point E^* loses its stability via NS bifurcation if δ different from 011.6 to 12.95. Since the normal form coefficient of NS is negative, the dynamics of model (1.5) out the NS point is a stable closed invariant curve bifurcates from E_* which coexists with these the unstable fixed point.

Fig. 8 (a) represents the collapse of the closed curve when $\delta = 11.6$. The strange attractor of the map (1.5) if $\delta = 12.95$ is shown in Fig. 8 (b), which exhibit a fractal structure.

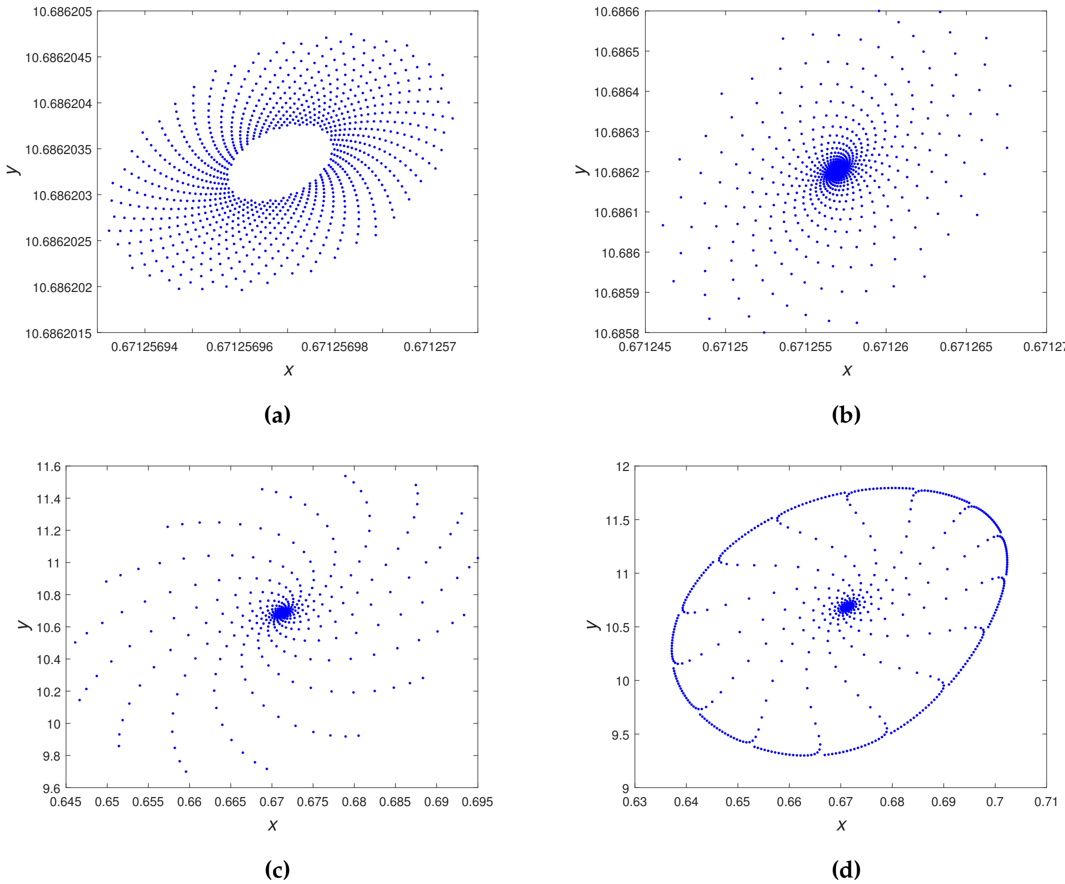


Figure 7. Phase portrait of the map (1.5) near the NS point with different values of δ. (a) Attracting fixed point at $\delta = 0.0064$. (b) A phase portrait for $\delta = 0.006564$. (c) A phase portrait for $\delta = 0.0067$. (d) The closed invariant curve.

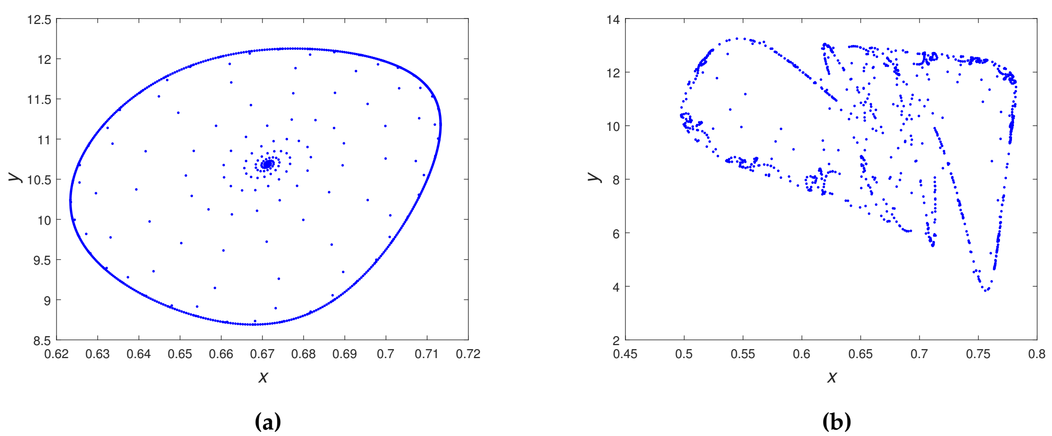


Figure 8. Phase portrait of the map (1.5) with different values of δ. (a) The typical the closed invariant curve for $\delta = 0.007$. (b) Chaotic attractor for $\delta = 0.0089$.

6. Conclusions

We present the nonlinear dynamics of a discrete modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response, given by system (1.5), obtained by applying a forward Euler discretization method to the continuous system (1.4). We have discussed the existence of fixed points in our model and provided a detailed analysis of their local stability. Moreover, we have established the necessary and sufficient conditions for the asymptotic stability of the positive fixed point. Additionally, we have explored various co-dimension two bifurcations, including 1:2 resonance, 1:3 resonance, and 1:4 resonance, by considering both δ and η as bifurcation parameters. The discrete modified Leslie–Gower predator-prey model exhibits rich and complex dynamics. Finally, we have conducted numerical simulations to validate our theoretical findings.

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References

[1] A.A. Berryman, The Orgins and Evolution of Predator-Prey Theory, Ecology 73 (1992), 1530–1535. https://doi.org/10.2307/1940005.

[2] A.J. Lotka, Elements of Physical Biology, Williams and Wilkins, Baltimore, (1925).

[3] V. Volterra, Variazioni e Fluttuazioni del Numero di Individui in Specie Animali Conviventi, Mem. Acad. Lincei 2 (1926), 31–113.

[4] S. Pal, N. Pal, S. Samanta, J. Chattopadhyay, Effect of Hunting Cooperation and Fear in a Predator-Prey Model, Ecol. Complex. 39 (2019), 100770. https://doi.org/10.1016/j.ecocom.2019.100770.

[5] S. Kumar, H. Kharbanda, Chaotic Behavior of Predator-Prey Model with Group Defense and Non-Linear Harvesting in Prey, Chaos, Solitons Fractals 119 (2019), 19–28. https://doi.org/10.1016/j.chaos.2018.12.011.

[6] Y. Zhou, W. Sun, Y. Song, Z. Zheng, J. Lu, et al., Hopf Bifurcation Analysis of a Predator–Prey Model with Holling-Ii Type Functional Response and a Prey Refuge, Nonlinear Dyn. 97 (2019), 1439–1450. https://doi.org/10.1007/s11071-019-05063-w.

[7] S. Akhtar, R. Ahmed, M. Batool, N.A. Shah, J.D. Chung, Stability, Bifurcation and Chaos Control of a Discretized Leslie Prey-Predator Model, Chaos, Solitons Fractals 152 (2021), 111345. https://doi.org/10.1016/j.chaos.2021.111345.

[8] H. Deng, F. Chen, Z. Zhu, Z. Li, Dynamic Behaviors of Lotka–Volterra Predator–Prey Model Incorporating Predator Cannibalism, Adv. Differ. Equ. 2019 (2019), 359. https://doi.org/10.1186/s13662-019-2289-8.

[9] P.H. Leslie, Some Further Notes on the Use of Matrices in Population Mathematics, Biometrika 35 (1948), 213–245. https://doi.org/10.2307/2332342.

[10] P.H. Leslie, a Stochastic Model for Studying the Properties of Certain Biological Systems by Numerical Methods, Biometrika 45 (1958), 16–31. https://doi.org/10.1093/biomet/45.1-2.16.

[11] C.S. Holling, The Functional Response of Predators to Prey Density and Its Role in Mimicry and Population Regulation, Mem. Entomol. Soc. Can. 97 (1965), 5–60. https://doi.org/10.4039/entm9745fv.

[12] J. Roy, S. Alam, Study on Autonomous and Nonautonomous Version of a Food Chain Model with Intraspecific Competition in Top Predator, Math. Methods Appl. Sci. 43 (2019), 3167–3184. https://doi.org/10.1002/mma.6109.

[13] S. Khajanchi, S. Banerjee, Role of Constant Prey Refuge on Stage Structure Predator–Prey Model with Ratio Dependent Functional Response, Appl. Math. Comput. 314 (2017), 193–198. https://doi.org/10.1016/j.amc.2017.07.017.

[14] S. Khajanchi, Dynamic Behavior of a Beddington–Deangelis Type Stage Structured Predator–Prey Model, Appl. Math. Comput. 244 (2014), 344–360. https://doi.org/10.1016/j.amc.2014.06.109.

[15] F. Chen, L. Chen, X. Xie, On a Leslie–Gower Predator–Prey Model Incorporating a Prey Refuge, Nonlinear Anal.: Real World Appl. 10 (2009), 2905–2908. https://doi.org/10.1016/j.nonrwa.2008.09.009.

[16] M. Aziz-Alaoui, M. Daher Okiye, Boundedness and Global Stability for a Predator-Prey Model with Modified Leslie-Gower and Holling-Type II Schemes, Appl. Math. Lett. 16 (2003), 1069–1075. https://doi.org/10.1016/s0893-9659(03)90096-6.

[17] S. Yu, Global Asymptotic Stability of a Predator-Prey Model with Modified Leslie-Gower and Holling-Type II Schemes, Discret. Dyn. Nat. Soc. 2012 (2012), 208167. https://doi.org/10.1155/2012/208167.

[18] R. Yafia, F. El Adnani, H.T. Alaoui, Limit Cycle and Numerical Similations for Small and Large Delays in a Predator–Prey Model with Modified Leslie–Gower and Holling-Type II Schemes, Nonlinear Anal.: Real World Appl. 9 (2008), 2055–2067. https://doi.org/10.1016/j.nonrwa.2006.12.017.

[19] H. Guo, X. Song, An Impulsive Predator–Prey System with Modified Leslie–Gower and Holling Type II Schemes, Chaos Solitons Fractals 36 (2008), 1320–1331. https://doi.org/10.1016/j.chaos.2006.08.010.

[20] Y. Zhu, K. Wang, Existence and Global Attractivity of Positive Periodic Solutions for a Predator–Prey Model with Modified Leslie–Gower Holling-Type II Schemes, J. Math. Anal. Appl. 384 (2011), 400–408. https://doi.org/10.1016/j.jmaa.2011.05.081.

[21] M. Liu, K. Wang, Dynamics of a Leslie–Gower Holling-Type II Predator–Prey System with Lévy Jumps, Nonlinear Anal.: Theory Methods Appl. 85 (2013), 204–213. https://doi.org/10.1016/j.na.2013.02.018.

[22] T. Kar, A. Ghorai, Dynamic Behaviour of a Delayed Predator–Prey Model with Harvesting, Appl. Math. Comput. 217 (2011), 9085–9104. https://doi.org/10.1016/j.amc.2011.03.126.

[23] H. Huo, X. Wang, C. Castillo-Chavez, Dynamics of a Stage-Structured Leslie-Gower Predator-Prey Model, Math. Probl. Eng. 2011 (2011), 149341. https://doi.org/10.1155/2011/149341.

[24] Z. Li, M. Han, F. Chen, Global Stability of a Stage-Structured Predator–prey Model With Modified Leslie–gower and Holling-Type II Schemes, Int. J. Biomath. 05 (2012), 1250057. https://doi.org/10.1142/s179352451250057x.

[25] C. Liu, Q. Zhang, J. Huang, Stability Analysis of a Harvested Prey-Predator Model with Stage Structure and Maturation Delay, Math. Probl. Eng. 2013 (2013), 329592. https://doi.org/10.1155/2013/329592.

[26] R. Gupta, P. Chandra, Bifurcation Analysis of Modified Leslie–Gower Predator–Prey Model with Michaelis–Menten Type Prey Harvesting, J. Math. Anal. Appl. 398 (2013), 278–295. https://doi.org/10.1016/j.jmaa.2012.08.057.

[27] C. Ji, D. Jiang, N. Shi, Analysis of a Predator–Prey Model with Modified Leslie–Gower and Holling-Type II Schemes with Stochastic Perturbation, J. Math. Anal. Appl. 359 (2009), 482–498. https://doi.org/10.1016/j.jmaa.2009.05.039.

[28] J.R. Beddington, Mutual Interference Between Parasites or Predators and Its Effect on Searching Efficiency, J. Anim. Ecol. 44 (1975), 331–340. https://doi.org/10.2307/3866.

[29] D.L. DeAngelis, R.A. Goldstein, R.V. O'Neill, A Model for Tropic Interaction, Ecology 56 (1975), 881–892. https://doi.org/10.2307/1936298.

[30] A.C.J. Luo, Regularity and Complexity in Dynamical Systems, Springer, New York, 2012. https://doi.org/10.1007/978-1-4614-1524-4.

[31] Y.A. Kuznetsov, Elements of Applied Bifurcation Theory, Springer, New York, 1998. https://doi.org/10.1007/b98848.

[32] B. Li, Z. He, 1:2 and 1:4 Resonances in a Two-Dimensional Discrete Hindmarsh–Rose Model, Nonlinear Dyn. 79 (2014), 705–720. https://doi.org/10.1007/s11071-014-1696-3.

[33] Y.A. Kuznetsov, Elements of Applied Bifurcation Theory, Springer, New York, 2004. https://doi.org/10.1007/978-1-4757-3978-7.

[34] B. Li, Z. He, 1 : 3 Resonance and Chaos in a Discrete Hindmarsh-Rose Model, J. Appl. Math. 2014 (2014), 896478. https://doi.org/10.1155/2014/896478.

[35] W. Govaerts, R.K. Ghaziani, Y.A. Kuznetsov, H.G.E. Meijer, Numerical Methods for Two-Parameter Local Bifurcation Analysis of Maps, SIAM J. Sci. Comput. 29 (2007), 2644–2667. https://doi.org/10.1137/060653858.

[36] E.L. Allgower, K. Georg, Numerical Continuation Methods, Springer, Berlin, Heidelberg, 1990. https://doi.org/10.1007/978-3-642-61257-2.

[37] D. Bai, J. Li, W. Zeng, Global Stability of the Boundary Solution of a Nonautonomous Predator-Prey System with Beddington-Deangelis Functional Response, J. Biol. Dyn. 14 (2020), 421–437. https://doi.org/10.1080/17513758.2020.1772999.

[38] Q. Yue, Dynamics of a Modified Leslie–Gower Predator–Prey Model with Holling-Type II Schemes and a Prey Refuge, SpringerPlus 5 (2016), 461. https://doi.org/10.1186/s40064-016-2087-7.