

## Subsolutions, Discrete Regularity and $L^\infty$ - Error Estimates for a Class of Elliptic QVIs: A Unified Analysis

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**Abstract.** We provide a unified analysis for the standard finite element approximation of a class of elliptic quasi-variational inequalities (QVIs). We also derive  $L^\infty$ - Error estimates combining the concepts of subsolutions and discrete regularity.

### 1. INTRODUCTION

This paper is concerned with the finite element approximation of the class of quasi-variational inequalities (QVIs)

$$\begin{cases} a(u, v - u) \geq (f, v - u) \quad \forall v \in H^1(\Omega), \\ u \leq \psi + \Phi(u), \quad v \leq \psi + \Phi(u), \end{cases} \quad (1.1)$$

where  $\Omega$  is a bounded convex domain of  $\mathbb{R}^N$ , with boundary  $\Gamma$  sufficiently smooth,  $(\cdot, \cdot)$  denote the scalar product,  $a(\cdot, \cdot)$  a continuous and coercive bilinear form,  $f \geq 0$  and  $\psi \geq 0$  such that  $\partial\psi/\partial n \leq 0$  on  $\Gamma$ , are given smooth functions, and  $\Phi$  is a nondecreasing and Lipschitz nonlinear operator from  $L^\infty$  into itself.

Among many interesting examples is the QVI of impulse control [9]

$$\begin{cases} a(u, v - u) \geq (f, v - u) \quad \forall v \in H^1(\Omega), \\ u \leq \Phi(u), \quad v \leq \Phi(u), \end{cases} \quad (1.2)$$

where  $\Phi(u)$  is the obstacle defined by

$$\Phi(u) = k + \inf_{x + \xi \in \bar{\Omega}, \xi \geq 0} u(x + \xi), \quad (1.3)$$

and  $k > 0$  is a positive constant.

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Denoting by  $\mathbb{V}_h$  the finite element space consisting of continuous piecewise linear functions, and  $\{\varphi_i\}, i = 1, 2, \dots, m(h)$  the basis of  $\mathbb{V}_h$ , and  $r_h$  the usual interpolation operator, the discrete counterpart of (1.1) consists of seeking  $u_h \in \mathbb{V}_h$  such that

$$\begin{cases} a(u_h, v - u_h) \geq (f, v - u_h) & \forall v \in \mathbb{V}_h, \\ u_h \leq r_h(\psi + \Phi(u_h)), & v \leq r_h(\psi + \Phi(u_h)). \end{cases} \quad (1.4)$$

The finite element approximation in the  $L^\infty$  norm of elliptic QVIs of obstacle type problems was first studied in [1, 2]. It then gained intense activity in the last four decades (cf. e.g. [3–7]), where various methods were employed.

In this paper, we present a unified approximation approach to encompass a large class of QVIs. The method stands on qualitative properties of the solution in both the continuous and discrete cases: Lipschitz dependence with respect to the data, characterization of the solution as the least upper part of the set of subsolutions for QVIs, and the so-called discrete regularity of their approximate solutions.

More precisely, we construct a continuous subsolution  $\beta$  such that

$$\beta \leq u$$

and

$$\|\beta - u_h\|_\infty \leq Ch^p |\ln h|^q.$$

and a discrete subsolutions  $\gamma_h$  such that:

$$\gamma_h \leq u_h$$

and

$$\|u - \gamma_h\|_\infty \leq Ch^p |\ln h|^q.$$

In this situation, we establish the error estimate

$$\|u - u_h\|_\infty \leq Ch^p |\ln h|^q, \quad 0 < p \leq 2, \quad 0 \leq q \leq 2,$$

where the constants  $p$  and  $q$  depend on the regularity of the solution (cf. [12]).

## 2. THE CONTINUOUS PROBLEM

We begin by introducing some notations and assumptions.

**2.1. Notations and assumptions.** We are given functions

$$a_{jk}(x), b_k(x), a_0(x) \quad \text{sufficiently smooth}, \quad (2.1)$$

such that

$$\sum_{1 \leq j, k \leq N} a_{jk}(x) \xi_j \xi_k \geq \alpha |\xi|^2; \quad (x \in \bar{\Omega}, \xi \in R^N, \alpha > 0), \quad (2.2)$$

and

$$a_0(x) \geq c_0 \quad (x \in \bar{\Omega}; c_0 > 0). \quad (2.3)$$

We define the bilinear forms:  $\forall u, v \in H^1(\Omega)$

$$a(u, v) = \int_{\Omega} \left( \sum_{1 \leq j, k \leq N} a_{jk}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^N b_k(x) \frac{\partial u}{\partial x_k} v + a_0(x) uv \right) dx \tag{2.4}$$

We assume that  $a(., .)$  is coercive on  $H^1(\Omega)$  i.e.,

$$a(v, v) \geq \delta \|v\|_{H^1(\Omega)}^2, \quad \delta > 0, \quad \forall v \in H^1(\Omega)$$

Throughout the paper, we will introduce several variational inequalities (VI) problems. Let us, generically, denote by  $\kappa = \sigma(g, \psi)$ , the solution of the denote the solution of the VI

$$\begin{cases} a(\kappa, v - \kappa) \geq (g, v - \kappa) \forall v \in H^1(\Omega) \\ v \leq \phi, \kappa \leq \phi \end{cases} \tag{2.5}$$

and by  $\kappa_h = \sigma_h(g, \chi)$ , the solution of the corresponding discrete VI

$$\begin{cases} a(\kappa_h, v - \kappa_h) \geq (g, v - \kappa_h) \forall v \in \mathbb{V}_h \\ v \leq r_h \phi, \kappa_h \leq r_h \phi \end{cases} \tag{2.6}$$

**Theorem 2.1.** [12]

$$\|\kappa - \kappa_h\|_{\infty} \leq Ch^p |\ln h|^q, \quad 0 < p \leq 2, \quad 0 \leq q \leq 2.$$

**2.2. Existence and uniqueness.** We can, in the spirit of [9], establish existence of a unique solution to QVI (1.1). Indeed, consider the mapping

$$\begin{aligned} T : L^{\infty}(\Omega) &\rightarrow L^{\infty}(\Omega), \\ w &\rightarrow Tw = \zeta \end{aligned} \tag{2.7}$$

where  $\zeta$  is the solution of the variational inequality (VI)

$$\begin{cases} a(\zeta, v - \zeta) \geq (f, v - \zeta) & \forall v \in H^1(\Omega), \\ v \leq \psi + \Phi(w), & \zeta \leq \psi + \Phi(w). \end{cases} \tag{2.8}$$

Then, starting from  $u^0$ , solution of the equation

$$a(u^0, v) = (f, v) \quad \forall v \in H^1(\Omega), \tag{2.9}$$

we define the sequence  $(u^n)$  by

$$u^{n+1} = Tu^n, \quad n = 0, 1, \dots$$

where the iterate  $u^{n+1}$  solves the VI

$$\begin{cases} a(u^{n+1}, v - u^{n+1}) \geq (f, v - u^{n+1}) & \forall v \in H^1(\Omega), \\ v \leq \psi + \Phi(u^n), & u^{n+1} \leq \psi + \Phi(u^n). \end{cases} \tag{2.10}$$

Likewise, starting from  $u_0 = 0$ , we define the sequence  $(u_n)$  by

$$u_{n+1} = Tu_n, \quad n = 0, 1, \dots$$

where the iterate  $u_n$  solves the VI

$$\begin{cases} a(u_{n+1}, v - u_{n+1}) \geq (f, v - u_{n+1}) & \forall v \in H^1(\Omega), \\ v \leq \psi + \Phi(u_n), & u_{n+1} \leq \psi + \Phi(u_n). \end{cases} \quad (2.11)$$

And, as in [9], we have:

**Theorem 2.2.** *The sequences  $(u^n)$  and  $(u_n)$  converge decreasingly and increasingly, respectively, to the unique solution  $u$  of QVI (1.1).*

**2.3. A continuous Lipschitz dependence property.** Let  $\psi, \tilde{\psi}$  be in  $L^\infty(\Omega)$ , and  $u = \sigma(f, \psi + \Phi(u))$ ,  $\tilde{u} = \sigma(f, \tilde{\psi} + \Phi(\tilde{u}))$  be the corresponding solutions to (1.1), respectively.

**Lemma 2.1.** *If  $f \geq \tilde{f}$  and  $\psi \geq \tilde{\psi}$  then  $u \geq \tilde{u}$ .*

*Proof.* Let  $u^n = \sigma(f, \psi + \Phi(u^{n-1}))$ , and  $\tilde{u}^n = \sigma(\tilde{f}, \tilde{\psi} + \Phi(\tilde{u}^{n-1}))$ . As  $f \geq \tilde{f}$  and  $\psi \geq \tilde{\psi}$ , then, making use of standard comparison results in elliptic variational inequalities, we get  $u^n \geq \tilde{u}^n$ . Finally, passing to the limit, the desired result follows.  $\square$

**Proposition 2.1.** *Under conditions of lemma 2.1, we have*

$$\|u - \tilde{u}\|_\infty \leq \|\psi - \tilde{\psi}\|_\infty.$$

*Proof.* For sake of notational simplicity, we will write  $u = \sigma(f, \psi)$  instead of  $\sigma(f, \psi + \Phi(u))$ .

Set

$$\gamma = \|\psi - \tilde{\psi}\|_\infty.$$

As

$$\psi \leq \tilde{\psi} + \gamma,$$

thanks to lemma 2.1 we have

$$\sigma(f, \psi) \leq \sigma(f, \tilde{\psi} + \gamma)$$

But

$$\sigma(f, \tilde{\psi}) + \gamma = \tilde{u} + \gamma = \sigma(f + a_0\gamma, \tilde{\psi} + \gamma)$$

$$f \leq f + a_0\gamma$$

So, applying standard comparison results in VIs, we get

$$\sigma(f, \tilde{\psi} + \gamma) \leq \sigma(f + a_0\gamma, \tilde{\psi} + \gamma) = \sigma(f, \tilde{\psi}) + \gamma$$

Hence,

$$\sigma(f, \psi) \leq \sigma(f, \tilde{\psi} + \gamma) \leq \sigma(f, \tilde{\psi}) + \gamma$$

or

$$\sigma(f, \psi) \leq \sigma(f, \tilde{\psi}) + \gamma$$

And, as the roles of  $\psi$  and  $\tilde{\psi}$  are symmetric, we also have

$$\sigma(f, \tilde{\psi}) \leq \sigma(f, \psi) + \gamma$$

Thus

$$\|u - \tilde{u}\|_\infty \leq \|\psi - \tilde{\psi}\|_\infty.$$

□

### 2.4. Continuous subsolution.

**Definition 2.1.**  $\omega \in H^1(\Omega)$  is said to be a subsolution for QVI (1.1) if

$$\begin{cases} a(\omega, v) \leq (f, v) & \forall v \in H^1(\Omega), v \geq 0, \\ \omega \leq \psi + \Phi(\omega) \end{cases} \quad (2.12)$$

Let  $\mathbb{X}$  denote the set of such subsolutions. Then, we have:

**Theorem 2.3.** The solution of QVI (1.1) is the maximum element of  $\mathbb{X}$ .

*Proof.* It is an adaptation of [9].

□

## 3. THE DISCRETE PROBLEM

For the sake of simplicity we suppose that  $\Omega$  is polyhedral. We then consider a regular and quasi-uniform triangulation  $\tau_h$  of  $\bar{\Omega}$ , consisting of  $n$  simplices  $K$ . Denote by  $h = \max_{K \in \tau_h} h_K$ , the meshsize of  $\tau_h$  with  $h_K$  being the diameter of  $K$ . For each  $K \in \tau_h$ , denote by  $P_1(K)$  the set of polynomials on  $K$  with degree no more than 1. The  $P_1$ -conforming finite element space is given by

$$\mathbb{V}_h = \{v : v \in H^1(\Omega) \cap C(\bar{\Omega}), v|_K \in P_1(K), \forall K \in \tau_h\}$$

Let  $M_i, 1 \leq i \leq m(h)$ , denote the the vertices of the triangulation  $\tau_h$ , and let  $\varphi_i, 1 \leq i \leq m(h)$ , denote the functions of  $\mathbb{V}_h$  which satisfy

$$\varphi_i(M_j) = \delta_{ij}, 1 \leq i, j \leq m(h)$$

so that the functions  $\varphi_i$  form a basis of  $V_h$ . For every  $v \in H^1(\Omega) \cap C(\bar{\Omega})$ , the function

$$r_h v(x) = \sum_{i=1}^{m(h)} v(M_i) \varphi_i(x)$$

represents the interpolate of  $v$  over  $\tau_h$ .

**3.1. The discrete maximum principle (d.m.p.)** Now, in order to establish existence and uniqueness of a solution to QVI (1.4), the stiffness matrix is required to be an M-matrix.

**Definition 3.1.** A real matrix  $d \times d$  matrix  $C = (c_{ij})$  with  $c_{ij} \leq 0, \forall i \neq j, 1 \leq i, j \leq d$ , is called an M-Matrix if  $C$  is nonsingular and  $C^{-1} \geq 0$  (i.e., all entries of its inverse are nonnegative).

3.1.1. *The stiffness matrix.* Denote by  $A$  the matrix with generic coefficient

$$A_{ij} = a(\varphi_i, \varphi_j), \quad 1 \leq i, j \leq m(h). \quad (3.1)$$

As the bilinear form  $a(\cdot, \cdot)$  is coercive, we have

$$A \text{ is positive definite,} \quad (3.2)$$

and

$$A_{ii} > 0 \quad \forall i = 1, \dots, m(h). \quad (3.3)$$

Furthermore, if the matrix  $(a_{jk}(x))$  involved in the bilinear form (2.4) is symmetric ( $a_{jk} = a_{kj}$ ), then, mesh conditions for which the off-diagonal entries of  $A$  satisfy

$$a_{ij} \leq 0, \quad \forall i \neq j, 1 \leq i, j \leq m(h), \quad (3.4)$$

can be found (see [10]). Therefore, combining (3.2), (3.3) and (3.4), we have the following lemma.

**Lemma 3.1.** *The matrix  $A$  is an M-matrix.*

*Proof.* See [10, 11]. □

3.2. **Existence and uniqueness.** As in the continuous case, we consider the mapping

$$\begin{aligned} T_h : L^\infty(\Omega) &\rightarrow \mathbb{V}_h, \\ w &\rightarrow T_h w = \zeta_h, \end{aligned} \quad (3.5)$$

where  $\zeta_h$  is the solution of the variational inequality VI

$$\begin{cases} a(\zeta_h, v - \zeta_h) \geq (g, v - \zeta_h) & \forall v \in \mathbb{V}_h, \\ v \leq r_h(\psi + \Phi(w)), & \zeta_h \leq r_h(\psi + \Phi(w)). \end{cases} \quad (3.6)$$

Then, starting from  $u_h^0$ , solution of the equation

$$a(u_h^0, v) = (f, v) \quad \forall v \in \mathbb{V}_h, \quad (3.7)$$

we construct the sequence  $(u_h^n)$  by

$$u_h^{n+1} = T_h u_h^n, \quad n = 0, 1, \dots, \quad (3.8)$$

such that  $\hat{u}_h^{n+1}$  solves the VI

$$\begin{cases} a(u_h^{n+1}, v - u_h^{n+1}) \geq (f, v - u_h^{n+1}) & \forall v \in \mathbb{V}_h, \\ v \leq r_h(\psi + \Phi(u_h^n)), & u_h^{n+1} \leq r_h(\psi + \Phi(u_h^n)). \end{cases} \quad (3.9)$$

Likewise, starting from  $u_{0,h} = 0$ , we define the sequence  $(u_{n,h})$  by

$$u_{n+1,h} = T u_{n,h}, \quad n = 0, 1, \dots$$

where the iterate  $u_{n+1,h}$  solves the VI

$$\begin{cases} a(u_{n+1,h}, v - u_{n+1,h}) \geq (f, v - u_{n+1,h}) & \forall v \in \mathbb{V}_h, \\ v \leq r_h(\psi + \Phi(u_{n,h})), & u_{n+1,h} \leq r_h(\psi + \Phi(u_{n,h})). \end{cases} \quad (3.10)$$

**Theorem 3.1.** Under the d.mp assumption, the sequences  $(u_h^n)$  and  $(u_{n,h})$  converge decreasingly and increasingly, respectively, to the unique solution of QVI (1.4).

**3.3. A discrete Lipschitz dependence property.** Let  $\psi, \tilde{\psi}$  be in  $L^\infty(\Omega)$ , and  $u_h = \sigma_h(f, \psi + \Phi(u_h))$ ,  $\tilde{u}_h = \sigma_h(f, \tilde{\psi} + \Phi(\tilde{u}_h))$  be the corresponding solutions to QVI (1.4), respectively.

**Lemma 3.2.** Let the d.mp hold. If  $\psi \geq \tilde{\psi}$ , then  $u_h \geq \tilde{u}_h$ .

**Proposition 3.1.** Under the conditions of lemma 3.2, we have

$$\|u_h - \tilde{u}_h\|_\infty \leq \|\psi - \tilde{\psi}\|_\infty.$$

**3.4. The discrete regularity.** This concept of “discrete regularity”, can be regarded as the discrete counterpart of the Lewy-Stampaccia estimate  $\|\mathcal{A}u\|_\infty \leq C$  ( $\mathcal{A}$  being the operator associated with bilinear form (2.4)), extended to the variational form through the  $L^1 - L^\infty$  duality. The main role it plays in the present paper is in the “regularization” of the obstacles appearing in the discrete QVI (1.4).

**Theorem 3.2** (cf. [2]). Assume that  $\mathcal{A}\psi$  and  $\mathcal{A}(\Phi(u))$  are bounded in  $L^\infty(\Omega)$ . Then, there exists a constant  $C$  independent of  $h$  such that

$$|a(u_h, \varphi_s)| \leq C \|\varphi_s\|_{L^1(\Omega)} \quad \forall s = 1, 2, \dots, M. \tag{3.11}$$

Moreover, there exists a family of right-hand side  $g^{(h)}$  bounded in  $L^\infty(\Omega)$  such that  $u_h$ , the solution of the discrete QVI (1.2) satisfies the equation

$$a(u_h, v) = (g^{(h)}, v) \quad \forall v \in \mathbb{V}_h \tag{3.12}$$

Let  $u^{(h)}$  be the corresponding continuous counterpart of (3.12), that is,

$$a(u^{(h)}, v) = (g^{(h)}, v) \quad \forall v \in H^1(\Omega) \tag{3.13}$$

then, we have, uniformly with respect to  $h$ ,

$$\|u^{(h)}\|_{W^{2,p}(\Omega)} \leq C(\text{independent of } h) \tag{3.14}$$

and

$$\|u^{(h)} - u_h\|_\infty \leq Ch^2 |\ln h|^2. \tag{3.15}$$

*Proof.* It is an adaptation of [2]. □

**3.5. Discrete subsolution.**

**Definition 3.2.**  $\omega_h \in \mathbb{V}_h$  is said to be a subsolution for the system of QVIs (1.4) if

$$\begin{cases} a(\omega_h, \varphi_s) \leq (f, \varphi_s) & \forall s = 1, \dots, m(h), \\ \omega_h \leq r_h(\psi + \Phi(\omega_h)). \end{cases} \tag{3.16}$$

Let  $\mathbb{X}_h$  be the set of discrete subsolutions.

**Theorem 3.3.** Under the d.m.p., the solution of QVI (1.4) is the maximum element of the set  $\mathbb{X}_h$ .

#### 4. THE FINITE ELEMENT ERROR ANALYSIS

This section is devoted to deriving the error estimate of the approximation. As announced earlier, the method consists of constructing an appropriate subsolution for each of the continuous and discrete QVIs (1.1) and (1.4), respectively. Let us begin with the discrete subsolution; the continuous one will follow the same way but is somehow more delicate, as it requires the regularization of the obstacles.

**4.1. Construction of a discrete subsolution.** From now on,  $C$  denotes a constant independent of  $h$ .

Let us introduce the following discrete VI:

$$\begin{cases} a(\bar{u}_h, v - \bar{u}_h) \geq (f, v - \bar{u}_h) & \forall v \in \mathbb{V}_h, \\ \bar{u}_h \leq r_h(\psi + \Phi(u)), & v \leq r_h(\psi + \Phi(u)). \end{cases} \quad (4.1)$$

where  $u$  is the solution of (1.1). So,  $\bar{u}_h = \sigma_h(f, \psi + \Phi(u))$  is nothing but the approximation of  $u = \sigma(f, \psi + \Phi(u))$  as a solution of a V.I. Therefore, thanks to [12], we have

$$\|u - \bar{u}_h\|_\infty \leq Ch^p |\ln h|^q \quad (4.2)$$

**Theorem 4.1.** *There exists a function  $\alpha_h$  such that*

$$\alpha_h \leq u_h \quad \text{and} \quad \|\alpha_h - u\|_\infty \leq Ch^p |\ln h|^q.$$

*Proof.* Indeed,  $\bar{u}_h$  being solution to the discrete VI (4.1), it is also a subsolution, that is,

$$\begin{cases} a(\bar{u}_h, \varphi_s) \leq (f, \varphi_s) & \forall \varphi_s, s = 1, 2, \dots, m(h), \\ \bar{u}_h \leq r_h(\psi + \Phi(u)). \end{cases}$$

But, as  $r_h$  and  $\Phi$  are Lipschitz, we have

$$\begin{aligned} \bar{u}_h &\leq \psi + \Phi(u) - \Phi(\bar{u}_h) + \Phi(\bar{u}_h) \\ &\leq \psi + \|\Phi(u) - \Phi(\bar{u}_h)\|_\infty + \Phi(\bar{u}_h) \\ &\leq \psi + C\|u - \bar{u}_h\|_\infty + \Phi(\bar{u}_h). \end{aligned}$$

Then

$$\begin{cases} a(\bar{u}_h, \varphi_s) \leq (f, \varphi_s) & \forall \varphi_s, s = 1, 2, \dots, m(h) \\ \bar{u}_h \leq r_h(\psi + C\|u - \bar{u}_h\|_\infty + \Phi(\bar{u}_h)) \end{cases}$$

That is,  $\bar{u}_h$  is a subsolution for the QVI with data  $f$  and  $\tilde{\psi} = \psi + C\|u - \bar{u}_h\|_\infty$ .

Let  $\bar{U}_h = \sigma_h(f, \tilde{\psi})$  be the solution of such a QVI. Then, making use of proposition 3.1, we have

$$\|u_h - \bar{U}_h\|_\infty \leq \|\psi - \tilde{\psi}\|_\infty \leq C\|u - \bar{u}_h\|_\infty$$

and, due to Theorem 3.3, we have

$$\bar{u}_h \leq \bar{U}_h \leq u_h + \|u - \bar{u}_h\|_\infty$$

Now putting

$$\alpha_h = \bar{u}_h - \|u - \bar{u}_h\|_\infty$$

we clearly have

$$\alpha_h \leq u_h$$

and, using (4.2), we get

$$\|\alpha_h - u\|_\infty \leq \|\bar{u}_h - u\|_\infty + \|u - \bar{u}_h\|_\infty \leq Ch^p |\ln h|^q.$$

□

**4.2. Construction of a continuous subsolution.** Let  $\bar{u} = \sigma(f, \psi + \Phi(u^{(h)}))$  be the solution of the following continuous V.I:

$$\begin{cases} a(\bar{u}, v - \bar{u}) \geq (f, v - \bar{u}) \quad \forall v \in H^1(\Omega), \\ \bar{u}^i \leq \psi + \Phi(u^{(h)}), \quad v \leq \psi + \Phi(u^{(h)}). \end{cases} \quad (4.3)$$

**Lemma 4.1.**

$$\|\bar{u} - u_h\|_\infty \leq Ch^p |\ln h|^q \quad (4.4)$$

*Proof.* Denote by  $\bar{\omega}_h = \sigma_h(f, \psi + \Phi(u^{(h)}))$  the approximation of

$$\bar{u} = \sigma(f, \psi + \Phi(u^{(h)}))$$

Then, using standard result on  $L^\infty$  error estimate for VIs [12], we have

$$\|\bar{u} - \bar{\omega}_h\|_\infty \leq Ch^p |\ln h|^q$$

On the other hand, since  $u_h = \sigma_h(f, \psi + \Phi(u_h))$  and  $\Phi$  is Lipschitz, combining standard results of Lipschitz dependence of the solution of VI with respect to the obstacle with estimate (4.2), we get

$$\begin{aligned} \|\bar{\omega}_h - u_h\|_\infty &\leq \left\| \psi + \Phi(u^{(h)}) - (\psi + \Phi(u_h)) \right\|_\infty \\ &\leq C \left\| \Phi(u^{(h)}) - \Phi(u_h) \right\|_\infty \\ &\leq C \|u^{(h)} - u_h\|_\infty. \end{aligned}$$

Hence

$$\begin{aligned} \|\bar{u} - u_h\|_\infty &\leq \|\bar{u} - \bar{\omega}_h\|_\infty + \|\bar{\omega}_h - u_h\|_\infty \\ &\leq Ch^p |\ln h|^q. \end{aligned}$$

□

**Theorem 4.2.** *There exists a function  $\beta^{(h)}$  such that:*

$$\beta^{(h)} \leq u$$

and

$$\|\beta^{(h)} - u_h\|_\infty \leq Ch^p |\ln h|^q.$$

*Proof.* Indeed,  $\bar{u}$  being the solution of the VI (4.3), it is also a subsolution for the same VI, that is,

$$\begin{cases} a(\bar{u}, v) \leq (f, v) & \forall v \in H^1(\Omega), v \geq 0, \\ \bar{u} \leq \psi + \Phi(u^{(h)}). \end{cases}$$

Then, as

$$\begin{aligned} \bar{u} &\leq \psi + \Phi(u^{(h)}) - \Phi(\bar{u}) + \Phi(\bar{u}) \\ &\leq \psi + \left\| \Phi(u^{(h)}) - \Phi(\bar{u}) \right\|_{\infty} + \Phi(\bar{u}) \end{aligned}$$

we have

$$\begin{cases} a(\bar{u}, v) \leq (f, v) & \forall v \in H^1(\Omega), v \geq 0, \\ \bar{u} \leq \psi + \left\| \Phi(u^{(h)}) - \Phi(\bar{u}) \right\|_{\infty} + \Phi(\bar{u}). \end{cases}$$

That is,  $\bar{u}$  is a subsolution for the QVI with data  $f$  and  $\tilde{\psi} = \psi + \left\| \Phi(u^{(h)}) - \Phi(\bar{u}) \right\|_{\infty}$ .

Let  $\bar{U}$  be the solution of such a system. Then we have

$$\bar{u} \leq \bar{U} = \sigma(f, \tilde{\psi})$$

On the other hand, since  $\Phi$  is Lipschitz, making use of Proposition 2.1 and estimates (3.15), (4.4), we have

$$\begin{aligned} \|u - \bar{U}\|_{\infty} &\leq \|\psi - \tilde{\psi}\|_{\infty} \\ &\leq \left\| \Phi(u^{(h)}) - \Phi(\bar{u}) \right\|_{\infty} \\ &\leq \left\| \Phi(u^{(h)}) - \Phi(u_h) \right\|_{\infty} + \left\| \Phi(u_h) - \Phi(\bar{u}) \right\|_{\infty} \\ &\leq C \|u^{(h)} - u_h\|_{\infty} + C \|u_h - \bar{u}\|_{\infty} \\ &\leq Ch^p |\ln h|^q. \end{aligned}$$

Hence, making use of Theorem 2.3, we obtain

$$\bar{u} \leq \bar{U} \leq u + Ch^p |\ln h|^q$$

and putting

$$\beta^{(h)} = \bar{u} - Ch^p |\ln h|^q$$

we clearly have

$$\beta^{(h)} \leq u$$

Finally, using to (3.15) and (4.4), we obtain

$$\begin{aligned} \left\| \beta^{(h)} - u_h \right\|_{\infty} &\leq \|\bar{u} - u_h\|_{\infty} + Ch^p |\ln h|^q \\ &\leq Ch^p |\ln h|^q \end{aligned}$$

□

Now, we are in a position to derive the error estimate for QVI (1.1).

**Theorem 4.3.**

$$\|u - u_h\|_\infty \leq Ch^p |\ln h|^q.$$

*Proof.* Indeed, making use of Theorems 4.1 and 4.2, we have

$$\begin{aligned} u_h &\leq \beta^{(h)} + Ch^p |\ln h|^q \\ &\leq u + Ch^p |\ln h|^q \\ &\leq \alpha_h + Ch^p |\ln h|^q \end{aligned}$$

Thus

$$\|u - u_h\|_\infty \leq Ch^p |\ln h|^q.$$

□

## 5. CONCLUSION

We have presented a unified analysis for the finite element approximation of a class of coercive elliptic quasi-variational inequalities. Our approach stands mainly on functional analysis tools which are the concepts of subsolutions and discrete regularity. We believe that the findings of this paper can be exploited to extend the study to the noncoercive situation. This will be our focus in a future work.

**Conflicts of Interest:** The author declares that there are no conflicts of interest regarding the publication of this paper.

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