Int. J. Anal. Appl. (2026), 24:9

International Journal of Analysis and Applications

Spherical Curves and Their Rigidity in Metric Spaces with Lower Curvature Bounds

Areeyuth Sama-Ae*

Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand

**Corresponding author:* areeyuth.s@psu.ac.th

Abstract. In this paper, we examine geometric relationships between metric spaces with curvature bounded below and their corresponding model spaces of constant curvature. Let γ be a closed spherical curve in a metric space whose curvature is bounded below by K, lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point. Let γ' denote the circle of radius r centered at the corresponding point in the model space of constant curvature K. Under suitable geometric equivalence conditions—namely, the preservation of pairwise distances between corresponding points of γ and γ', the isometry of convex hulls of corresponding geodesic triangles, and the equality of arc lengths or total curvature—we show that the geodesic surface enclosed by γ is isometric to the region bounded by γ'. This result offers a foundational geometric characterization of metric spaces with curvature bounded below through their model counterparts and provides a framework for further study of total curvature, convexity, and isometric embeddings in such spaces.

1. Introduction

The theory of metric spaces with curvature bounded below offers a powerful framework for doing geometry on non-smooth and singular spaces. It allows geometers to prove important theorems about the structure, topology, and geometry of a wide array of objects, unifying the study of smooth manifolds, convex surfaces, and various limit spaces under a common geometric principle. Burago et al. [13] provided an extensive account of the theory of metric spaces with curvature bounded below. Several studies have examined various geometric properties of metric spaces with curvature bounded below. Ambrosio et al. [7] studied metric measure spaces whose Riemannian Ricci curvature is bounded from below, while Cheeger and Colding [14] analyzed the resulting structural features. Petrunin [25] investigated parallel transport in Alexandrov spaces with lower curvature bounds. Naya and Innami [24] obtained comparison theorems for Steiner minimal trees on such surfaces. Halbeisen [19] focused on tangent cones in Alexandrov spaces of

Received: Nov. 20, 2025.

2020 *Mathematics Subject Classification.* 51F99, 51K05, 53C24, 53C45.

Key words and phrases. spherical curve; metric spaces with lower curvature bounds; total curvature; isometry.

https://doi.org/10.28924/2291-8639-24-2026-9

ISSN: 2291-8639

curvature bounded below, and Espínola [17] examined nearest and farthest point problems in these spaces. Sama-Ae et al. [30] studied distances between points and their nearest-point projections.

Research on the total curvature of closed curves is essential because it connects local bending with global geometric properties. It plays a key role in comparison geometry and supports rigidity theorems that relate curves to model spaces. The total curvature also provides a consistent framework for analyzing non-smooth curves and has applications in physics, biology, and material science, making it a fundamental tool in modern geometry. The total curvature of a curve serves as a key invariant connecting local geometry to global topology. Fenchel's Theorem shows that every closed curve has total curvature at least 2π, while the Fáry–Milnor Theorem reveals that knotted curves exceed 4π, demonstrating the interplay between geometry and topology. Defined without differential tools, the total curvature quantifies shape complexity and has broad applications, highlighting how a curve's global form arises from its accumulated local bending.

The curvature of a curve measures how quickly its tangent vector turns, indicating how much it deviates from a straight line. Pointwise curvature is this rate at a single point, while total curvature is the sum of all these changes along the entire curve, found by integrating pointwise curvature. For a smooth curve, curvature quantifies the curve's turn rate. The pointwise curvature shows the instantaneous rate of turning, and the total curvature is the integral of these local values over the curve's length. The study of total curvature dates back to the work of Fenchel [18], who in 1929 proved that any closed curve in three-dimensional Euclidean space possesses total curvature at least 2π, with equality holding precisely when the curve is a convex planar curve. Subsequently, Borsuk [9] and Milnor [23] generalized this result to n-dimensional Euclidean spaces. Alexandrov [4] further developed lower and upper curvature bounds for metric spaces lacking Riemannian structures, generalizing classical curvature concepts to arbitrary metric spaces [37]. Sasaki [32] established a strengthened form of Fenchel's theorem by applying Douglas's work on the Plateau problem [16] together with the Gauss–Bonnet theorem. In 1974, Tsukamoto [36] proved that in a complete simply connected Riemannian manifold with negative sectional curvature, the total curvature of a smooth closed curve exceeds 2π. Subsequent studies broadened this framework to encompass closed curves in Riemannian manifolds with non-positive curvature as well as in hyperbolic spaces [10, 21, 33]. A detailed exposition of this theory was presented in [5], where total curvature was introduced through the total rotation of geodesic polygons inscribed in the closed curve and approaching it arbitrarily closely. Additional commentary and development on the subject can be found in [6, 27]. Chern [15] addressed total curvature of immersed manifolds, and later works by van Rooij [37], Brickell and Hsiung [10]. Investigations in spherical geometries by Teufel [34, 35] connected total curvature with isoperimetric inequalities, while Honda et al. [20] extended these studies to curves with singularities.

This concept was subsequently extended to CAT(0) spaces by Alexander and Bishop [2, 3], who showed that the total curvature of any closed curve is bounded below by 2π, with equality occurring only when the curve forms a geodesic bigon or bounds a convex region isometric to a planar convex subset. Building on this foundation, Maneesawarng and Lenbury [22] further developed the theory of total curvature in CAT(K) spaces, establishing a framework for analyzing curves in spaces of bounded curvature. Within the setting of CAT(K) spaces, Sama-Ae and Maneesawarng [28] investigated the geometric properties of curves on spheres. Extending this work, Sama-Ae and Phon-On [29] characterized closed curves that bound geodesic surfaces isometric to disks in the model space, employing measures such as arc length, total curvature, and chord length. Later, Sama-Ae et al. [31] generalized these results to closed spherical curves enclosing geodesic surfaces isometric to disks in the model space. Phokeaw and Sama-Ae [26] demonstrated that in metric spaces with curvature bounded below, a totally geodesic surface bounded by a spherical curve is isometric to the corresponding region in $\mathbb{M}_K$, provided certain distance and angle conditions are satisfied. In the present paper, we introduce additional conditions that guarantee the totally geodesic surface enclosed by a closed spherical curve is isometric to the region bounded by a circle of the same radius. These enhanced criteria deepen our understanding of rigidity phenomena in spaces with curvature bounded below, extend classical results in spherical geometry, and offer new insights into the geometric structure of such spaces.

By combining the results obtained in Theorems 3.2 and 3.3, we derive the principal conclusion of this work, which unifies the local and global geometric properties of closed spherical curves in metric spaces whose curvature is bounded below. This theorem provides a rigidity criterion ensuring that a totally geodesic surface bounded by a closed spherical curve in such a space is isometric to the corresponding region in the model space of constant curvature. In particular, it shows that the equality of fundamental geometric quantities—such as length, total curvature, and chordal distance—between corresponding arcs is sufficient to guarantee a global isometry between the two geometric configurations.

Theorem 1.1. *Let (X, d) be a metric space of curvature bounded below by K. Let γ be a closed spherical curve in X, lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point $p \in X$, and let γ' be a circle of radius r centered at a point p' in the model space $(\mathbb{M}^2_K, d_K)$. Suppose that the following statements hold:*

1. *$d(x,z) = d_K(x',z')$, whenever $d(x,y) = d_K(x',y')$ and $d(y,z) = d_K(y',z')$, for all $x, y, z \in \gamma$ and $x', y', z' \in \gamma'$;*
2. *The convex hulls of $\triangle(x,y,z)$ and $\triangle(x',y',z')$ are isometric to each other if there exists a point $w \in [y,z]$ such that $d(x,w) = d_K(x',w')$, where $w' \in [y',z']$ is the comparison point of w;*
3. *$\ell(\gamma) = \ell(\gamma')$ or $\kappa_c(\gamma) = \kappa_c(\gamma')$; and*
4. *$d(x,y) = d_K(x',y')$, whenever $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ or $\kappa_c(\gamma_{xy}) = \kappa_c(\gamma'_{x'y'})$ for any subarc γ_{xy} with endpoints x, y of γ and any subarc $\gamma'_{x'y'}$ with endpoints x', y' of γ'.*

Then the convex hull $C(\gamma)$ bounded by γ is isometric to the convex hull $C(\gamma')$ bounded by γ'; that is, the totally geodesic surface enclosed by γ and the disk enclosed by γ' are mutually isometric.

2. Definitions and Preliminaires

Let (X, d) be a metric space, and let $\gamma : [a, b] \to X$ denote a curve in X. The length of γ, denoted by $\ell(\gamma)$, is defined as

$$\ell(\gamma) = \sup \sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all possible partitions $a = t_0 < t_1 < \cdots < t_k = b$ of the interval $[a, b]$. The *intrinsic metric* d^* induced by d is defined as

$$d^*(x, y) := \inf\{\ell(\gamma) \mid \gamma \text{ is a curve from } x \text{ to } y\},$$

for all $x, y \in X$. This defines a new distance function taking values in $[0, \infty]$. If the original metric d coincides with the intrinsic metric d^*, the space (X, d) is called a *length space*.

A *geodesic* in a metric space X is an isometric embedding of $\mathbb{R}$ into X, with its image also referred to as a geodesic. A *geodesic path* between two points p and q is a map $g : [0, \alpha] \subset \mathbb{R} \to X$ satisfying $g(0) = p$, $g(\alpha) = q$, and

$$d(g(s), g(t)) = |s - t| \quad \text{for all } s, t \in [0, \alpha].$$

The set $g([0, \alpha])$ is called a *geodesic segment* connecting p and q. If this segment is unique, it is denoted by $[p, q]$. A metric space (X, d) is a *geodesic space* if every pair of points in X can be joined by a geodesic segment.

Definition 2.1. [12] *Let K be a real number. The model space $\mathbb{M}^2_K$ is defined according to the sign of K as follows:*

$$\mathbb{M}^2_K = \begin{cases} \mathbb{R}^2, & \text{if } K = 0, \\ \text{the Euclidean sphere of radius } 1/\sqrt{K}, & \text{if } K > 0, \\ \text{the hyperbolic plane of constant curvature } K, & \text{if } K < 0. \end{cases}$$

Further details on the spaces $\mathbb{M}^2_K$ can be found in [1,8,11]. A geodesic triangle $\triangle(p, q, r)$ in a metric space X consists of the vertices p, q, and r, connected by geodesic segments $[p, q]$, $[q, r]$, and $[p, r]$. To such a triangle, one can associate a *comparison triangle* $\triangle(p', q', r')$ in the model space $(\mathbb{M}^2_K, d_K)$, where the side lengths match: $d(p, q) = d_K(p', q')$, $d(q, r) = d_K(q', r')$, and $d(p, r) = d_K(p', r')$. This comparison triangle exists whenever the perimeter satisfies

$$d(p, q) + d(q, r) + d(p, r) < \frac{2\pi}{\sqrt{K}},$$

and in this case it is unique up to isometries. When $K \leq 0$, we interpret $\frac{2\pi}{\sqrt{K}}$ as ∞.

Given a triangle $\triangle(p, q, r)$ in a metric space X and its corresponding comparison triangle $\triangle(p', q', r')$ in the model space $\mathbb{M}^2_K$, the *comparison point* for a point $x \in [q, r]$ is the point $x' \in [q', r']$ such that $d(q, x) = d_K(q', x')$. The *comparison angle* at q of the triangle $\triangle(p, q, r)$ is defined as the angle at q' in the comparison triangle $\triangle(p', q', r')$. The angle at p in $\triangle(p, q, r)$ within X is denoted by $\angle_p(q, r)$. The corresponding comparison angle at p' in $\mathbb{M}^2_K$ is denoted by $\widetilde{\angle}_p(q, r)$ or $\angle_{p'}(q', r')$.

Definition 2.2. [5, 13] *Let X be a length space. A locally complete space X is a* space with curvature bounded below by a real number K *if, for every point $x \in X$, there exists a neighborhood $U(x)$ such that the following condition holds:*

(A) For any four distinct points $p, q, r, s \in U(x)$, the inequality

$$\widetilde{\angle}_s(q, p) + \widetilde{\angle}_s(q, r) + \widetilde{\angle}_s(p, r) \leq 2\pi$$

is satisfied.

In spaces where, locally, any two points can be connected by a geodesic — in particular, in locally compact spaces — the condition (A) in Definition 2.2 can be replaced by the following condition:

(B) For every triangle $\triangle(p, q, r)$ within the neighborhood $U(x)$, and for any point s on the segment $[q, r]$, the inequality

$$d(p, s) \geq d_K(p', s')$$

holds, where s' is the corresponding point of s on the segment $[q', r']$ of the comparison triangle $\triangle(p', q', r')$ in $\mathbb{M}^2_K$.

Let (X, d) be a metric space with curvature bounded below by K, and let ρ and σ be two geodesics in X emanating from a common point $w \in X$. The angle between geodesics ρ and σ at w is defined by

$$\angle(\rho, \sigma) = \lim_{t \to 0} \cos^{-1}\left(\frac{d^2(w, \rho(t)) + d^2(w, \sigma(t)) - d^2(\rho(t), \sigma(t))}{2\, d(w, \rho(t))\, d(w, \sigma(t))}\right).$$

The angle at the vertex p of a geodesic triangle $\triangle(q, p, r)$ is then defined as the angle between the geodesic segments $[p, q]$ and $[p, r]$.

The condition (B) is equivalent to the following condition:

(C) for any triangle $\triangle(p, q, r)$ in $U(x)$,

$$\angle_p(q, r) \geq \angle_{p'}(q', r'), \quad \angle_q(p, r) \geq \angle_{q'}(p', r') \text{ and } \angle_r(p, q) \geq \angle_{r'}(p', q'),$$

where $\triangle(p', q', r')$ is a comparision triangle in $\mathbb{M}^2_K$ of the triangle $\triangle(p, q, r)$.

Spaces with curvature bounded below were introduced earlier through local conditions. For complete metric spaces, however, the corresponding global curvature bounds can be derived directly from these local assumptions. Since the space X considered in this work is a complete geodesic metric space, we refer to X as a metric space with curvature bounded below *in the large*.

Theorem 2.1. [12] *If X is a metric space with curvature bounded below by K, where $K > 0$, then* $\operatorname{diam}(X) \leq \pi/\sqrt{K}$ *and any triangle in X has perimeter no greater than* $2\pi/\sqrt{K}$.

A *closed curve* in a metric space (X, d) is defined as a continuous mapping from an oriented circle in the Euclidean plane. A *chain* V on a closed curve γ is a finite ordered collection of points corresponding to chosen parameter values, and the elements of V are referred to as the *vertices* of the chain. If the curve γ is composed of geodesic segments joining successive points of V, then the pair (γ, V) is called a *closed polygonal curve* with vertex chain V. Moreover, if there exists a point $u \in X$ and a real number $r > 0$ such that $d(x, u) = r$ for every point x on γ, then γ is referred to as a *spherical curve*, and r is its *radius*. For instance, a circle of radius $r > 0$ in $\mathbb{M}_K^2$ is a closed spherical curve whose points all lie at distance r from its center. A *circular arc* is simply a subarc of such a circle in $\mathbb{M}_K^2$.

If δ is a closed polygonal curve with a chain $\{\delta(t_0), \delta(t_1), \ldots, \delta(t_n) = \delta(t_0)\}$, then for δ inscribed in a closed curve γ, we define the *modulus* of δ associated with γ, denoted by $\mu_\gamma(\delta)$, as

$$\mu_\gamma(\delta) = \max\{\text{diam}(\gamma|_{[t_i,t_{i+1}]}) \mid 0 \le i \le n-1\},$$

where for each $0 \le i \le n-1$, the restriction $\gamma|_{[t_i,t_{i+1}]}$ denotes the subarc of γ with endpoints $\gamma(t_i)$ and $\gamma(t_{i+1})$, and $\ell(\gamma|_{[t_i,t_{i+1}]})$ is its length. Consider now a closed polygonal curve δ inscribed in γ. Let the vertices of δ be arranged as $p_1, p_2, \ldots, p_n, p_{n+1} = p_1$. For convenience, we denote by $\widehat{p_i}$ the angle formed by the vertices p_{i-1}, p_i, and p_{i+1}. The quantity $\kappa_c^*(\delta)$ denotes the *total rotation* of δ and is given by

$$\kappa_c^*(\delta) = \sum_{i=1}^{n} (\pi - \widehat{p_i}).$$

Finally, the *total curvature* $\kappa_c(\gamma)$ of a closed curve γ is defined by

$$\kappa_c(\gamma) = \lim_{\varepsilon \to 0} \sup_{\delta \in \Sigma_\varepsilon(\gamma)} \kappa_c^*(\delta),$$

where $\Sigma_\varepsilon(\gamma)$ denotes the family of all inscribed closed polygonal curves δ in γ with mesh $\mu_\gamma(\delta) < \varepsilon$. In the special case where γ itself is a closed polygonal curve, one has $\kappa_c^*(\gamma) = \kappa_c(\gamma)$.

A subset A of a metric space (X, d) is said to be *convex* if, for any two points $x, y \in A$, the geodesic segment connecting x and y is entirely contained in A. The *convex hull* of A, denoted $C(A)$, is the smallest convex set containing A. An *isometry* between two metric spaces (X, d) and (Y, d^*) is a function $i : X \to Y$ such that $d(x, y) = d^*(i(x), i(y))$ for all $x, y \in X$.

3. Spherical Curves and Their Characterizations

In this section, we first present a proposition demonstrating that the total curvature of a circular arc in the model space is the limit of the total rotation of a sequence of inscribed polysegments. We then introduce two properties of the space that will be assumed in the proofs of the subsequent lemmas and theorems, followed by the presentation of several lemmas, a remark, and the two main theorems. The following proposition states that if a curve—specifically, a circular arc—in the model space $\mathbb{M}_K^2$ is approximated by a sequence of progressively finer polysegments, the total rotations of these polysegments converge to the total curvature of the smooth circular arc. This result provides a key foundation for proving the main theorems of the paper.

For convenience, in what follows, we denote by γ_{ab} a spherical curve in a metric space with curvature bounded below, having endpoints a and b, and by $\gamma'_{a'b'}$ a circular arc in $\mathbb{M}_K^2$ with endpoints a' and b'.

Proposition 3.1. *Let γ' be a circular arc of a circle C with endpoints a and b in the space $(\mathbb{M}_K^2, d_K)$. Consider a sequence of polysegments δ_k whose ordered vertices are given by $a = p_0^{(k)}, p_1^{(k)}, \ldots, p_{n_k}^{(k)} \in C$.*

Assume that for some integer m with $1 \leq m \leq n_k$, the points $p_0^{(k)}, p_1^{(k)}, \ldots, p_m^{(k)}$ lie on the arc γ'. If the sequence satisfies

$$\ell(\delta_k) \to \ell(\gamma') \quad \textit{and} \quad \max_{0 \leq i \leq n_k - 1} d_K(p_i^{(k)}, p_{i+1}^{(k)}) \to 0 \quad \textit{as } k \to \infty,$$

then,

$$\lim_{k\to\infty} \kappa_c^*(\delta_k) = \kappa_c(\gamma').$$

Proof. We consider three possibilities:

Case 1. Assume that all points $p_{n_k}^{(k)}$ lie on the arc γ' for every k. Fix a polysegment $\delta = \delta_k$. Then, define a new polysegment $\delta' = \delta'_k$ inscribed in γ' with the ordered vertices $p_0^{(k)}, p_1^{(k)}, \ldots, p_{n_k}^{(k)}, b$ such that the total curvature satisfies $\kappa_c^*(\delta'_k) \geq \kappa_c^*(\delta_k)$, and the mesh of the subdivision satisfies $\mu_{\gamma'}(\delta'_k) \to 0 \quad$ as $k \to \infty$. Therefore, we conclude that

$$\kappa_c(\gamma') = \lim_{k\to\infty} \kappa_c^*(\delta'_k) \geq \lim_{k\to\infty} \kappa_c^*(\delta_k). \tag{3.1}$$

Next, we aim to establish the reverse inequality: $\kappa_c(\gamma') \leq \lim_{k\to\infty} \kappa_c^*(\delta_k)$. Let $\varepsilon > 0$ be arbitrary. Then, there exists a point $c \in \gamma'$ such that $\kappa_c\left(\gamma'_{cb}\right) = \varepsilon$. Assume k is sufficiently large. Since $\ell(\delta_k) \to \ell(\gamma')$ and $p_{n_k}^{(k)} \in \gamma'_{cb}$, there exists an index $m \leq n_k$ such that $p_m^{(k)} \in \gamma'_{ac}$.

Now, construct a new polysegment $\delta'' = \delta''_k$ by replacing all vertices of δ_k following $p_m^{(k)}$ with the point c. Then, δ''_k is inscribed in the subarc γ'_{ac}, and we have $\kappa_c^*(\delta_k) \geq \kappa_c^*(\delta''_k)$, with $\mu_{\gamma'_{ac}}(\delta''_k) \to 0$ as $k \to \infty$. Hence,

$$\kappa_c\left(\gamma'_{ac}\right) = \lim_{k\to\infty} \kappa_c^*(\delta''_k) \leq \lim_{k\to\infty} \kappa_c^*(\delta_k).$$

Since γ' is smooth, its total curvature decomposes additively as

$$\kappa_c(\gamma') = \kappa_c\left(\gamma'_{ac}\right) + \kappa_c\left(\gamma'_{cb}\right) = \kappa_c\left(\gamma'_{ac}\right) + \varepsilon.$$

Therefore, we conclude that

$$\kappa_c(\gamma') \leq \lim_{k\to\infty} \kappa_c^*(\delta_k) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\kappa_c(\gamma') \leq \lim_{k\to\infty} \kappa_c^*(\delta_k). \tag{3.2}$$

Combining the inequalities (3.1) and (3.2), we obtain

$$\kappa_c(\gamma') = \lim_{k\to\infty} \kappa_c^*(\delta_k),$$

as required.

Case 2. Assume that all points $p_{n_k}^{(k)}$ lie outside γ' for every k. Fix a polysegment $\delta = \delta_k$. Then, there exists an index $m \leq n_k$ such that $p_m^{(k)} \in \gamma'$. Construct a new polysegment $\delta' = \delta'_k$ by replacing all vertices of δ following $p_m^{(k)}$ with the point b. As a result, δ'_k is inscribed in γ', and it satisfies

$$\kappa_c^*(\delta'_k) \leq \kappa_c^*(\delta_k) \quad \text{and} \quad \mu_{\gamma'}(\delta'_k) \to 0 \text{ as } k \to \infty.$$

Therefore, we have

$$\kappa_c(\gamma') = \lim_{k\to\infty} \kappa_c^*(\delta_k') \le \lim_{k\to\infty} \kappa_c^*(\delta_k). \tag{3.3}$$

We now aim to show the reverse inequality: $\lim_{k\to\infty} \kappa_c^*(\delta_k) \le \kappa_c(\gamma')$. Suppose, for contradiction, that $\kappa_c(\gamma') < \lim_{k\to\infty} \kappa_c^*(\delta_k)$. Let $\varepsilon > 0$ be given. Then, we can find a subarc γ'_{ad} of the circle C such that

$$\kappa_c\Big(\gamma'_{ad}\Big) = \kappa_c(\gamma') + \kappa_c\Big(\gamma'_{bd}\Big) = \kappa_c(\gamma') + \varepsilon.$$

Since $p_{n_k}^{(k)} \to b$ as $k \to \infty$, for sufficiently large k, the point $p_{n_k}^{(k)}$ lies in γ'_{ad}. Modify the polysegment δ_k by replacing all vertices lying on γ'_{bd} with b, resulting in a new polysegment $\delta = \delta_k$ inscribed in γ'. For large k, we then have

$$\kappa_c^*(\delta_k') + \varepsilon > \kappa_c^*(\delta_k), \quad \text{and} \quad \mu_{\gamma'}(\delta_k') \to 0 \text{ as } k \to \infty.$$

Thus, we obtain that

$$\kappa_c(\gamma') = \lim_{k\to\infty} \kappa_c^*(\delta_k') + \varepsilon \ge \lim_{k\to\infty} \kappa_c^*(\delta_k),$$

which leads to a contradiction. Therefore, it must be that

$$\lim_{k\to\infty} \kappa_c^*(\delta_k) \le \kappa_c(\gamma'). \tag{3.4}$$

Combining inequalities (3.3) and (3.4), we conclude that

$$\lim_{k\to\infty} \kappa_c^*(\delta_k) = \kappa_c(\gamma').$$

Case 3. The points $p_{n_k}^{(k)}$ lie on γ' for some values of k, and do not lie on γ' for others. In this situation, we select a subsequence $\delta_{s(k)}$ of δ_k such that either all the points $p_{n_{s(k)}}^{(s(k))}$ lie on γ' or none of them do. Consequently, the argument reduces to either Case 1 or Case 2, and the result follows accordingly.

This completes the proof. □

Let (X, d) be a metric space of curvature bounded below by K. Let γ be a spherical curve in X lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p, and γ' be a circular arc of radius r centered at a point p' in the model space $(\mathbb{M}_K^2, d_K)$. Let q, r, s and q', r', s' be consecutive points on γ and γ', respectively such that

$$d(q, r) = d_K(q', r'), \quad d(r, s) = d_K(r', s').$$

If $d(q, s) = d_K(q', s')$, then the comparison triangle $\triangle(q', r', s')$ corresponds to the triangle $\triangle(q, r, s)$, and by the angle condition of X, we obtain

$$\angle_r(q, s) \ge \angle_{r'}(q', s'). \tag{3.5}$$

This inequality becomes strict, that is $\angle_r(q, s) > \angle_{r'}(q', s')$, whenever $d(q, s) > d_K(q', s')$. On the other hand, if $d(q, s) < d_K(q', s')$, then the triangle $\triangle(q', r', s')$ may no longer correspond to $\triangle(q, r, s)$, and in such a situation the inequality (3.5) may fail to hold.

Suppose that $\triangle(p',q',r')$ and $\triangle(p',r',s')$ are comparison triangles for the geodesic triangles $\triangle(p,q,r)$ and $\triangle(p,r,s)$ in X, respectively. A characteristic property of X implies the angle comparisons:

$$\angle_r(p,q) \geq \angle_{r'}(p',q') \quad \text{and} \quad \angle_r(p,s) \geq \angle_{r'}(p',s').$$

Adding these inequalities yields

$$\angle_r(p,q) + \angle_r(p,s) \geq \angle_{r'}(p',q') + \angle_{r'}(p',s') = \angle_{r'}(q',s'). \tag{3.6}$$

However, by the triangle inequality for angles in X, we have

$$\angle_r(q,s) \leq \angle_r(p,q) + \angle_r(p,s). \tag{3.7}$$

Due to the inequalites (3.6) and (3.7), a direct comparison between $\angle_r(q,s)$ and $\angle_{r'}(q',s')$ cannot be made. Since the proofs of the lemmas and theorems in this section require a comparison between $\angle_r(q,s)$ and $\angle_{r'}(q',s')$, their statements must include certain additional assumptions.

Next, we present two fundamental properties that will be assumed in the forthcoming theorems. These properties establish essential geometric constraints on the metric space under consideration and play a crucial role in ensuring the validity of subsequent results related to the behavior of curves, their lengths, and total curvature in metric spaces with curvature bounded below.

Property A. Let (X,d) be a metric space of curvature bounded below by K. Let γ be a spherical curve in X lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point, and γ' be a circular arc of radius r centered at a point in the model space $(\mathbb{M}^2_K, d_K)$. For all $x,y,z \in \gamma$ and $x',y',z' \in \gamma'$,

$$d(x,z) = d_K(x',z'),$$

whenever

$$d(x,y) = d_K(x',y') \text{ and } d(y,z) = d_K(y',z')$$

Property B. Let (X,d) be a metric space of curvature bounded below by K. Let $\triangle(x,y,z)$ be a triangle in X, and let $\triangle(x',y',z')$ be the comparison triangle of $\triangle(x,y,z)$ in the model space $(\mathbb{M}^2_K, d_K)$. The convex hulls of $\triangle(x,y,z)$ and $\triangle(x',y',z')$ are isometric to each other if there exists a point $w \in [y,z]$ such that

$$d(x,w) = d_K(x',w'),$$

where $w' \in [y',z']$ is the comparison point of w.

In support of the proof of Theorem 3.1, we now present a lemma that supplies a key comparison between points on a spherical curve in the space X and points on a corresponding circular arc in the model space $\mathbb{M}^2_K$.

Lemma 3.1. *Let (X,d) be a metric space of curvature bounded below by K that satisfies Property A. Let γ be a spherical curve in X lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p, with endpoints x and y. Let γ' be a circular arc of radius r centered at a point p' in the model space $(\mathbb{M}^2_K, d_K)$, with endpoints x' and y'.*

If $\ell(\gamma) = \ell(\gamma') \leq \frac{\pi}{\sqrt{K}}$ and $d(x,y) = d_K(x',y')$, then

$$d(x,z) \leq d_K(x',z') \quad \textit{and} \quad d(z,y) \leq d_K(z',y'),$$

where $z \in \gamma$ and $z' \in \gamma'$ such that

$$d(x,z) = d(z,y) \quad \text{and} \quad d_K(x',z') = d_K(z',y').$$

Proof. Let $\ell(\gamma) = \ell(\gamma') \leq \frac{\pi}{\sqrt{K}}$ and $d(x,y) = d_K(x',y')$. Assume, for contradiction, that either $d(x,z) > d_K(x',z')$ or $d(z,y) > d_K(z',y')$. Then we can choose points $u', v' \in \gamma'$ such that

$$d(x,z) = d_K(u',z') \quad \text{and} \quad d(y,z) = d_K(v',z').$$

By Property A, this implies that

$$d(x,y) = d_K(u',v').$$

However, with this choice of u' and v' on γ', we obtain

$$d(x',y') < d_K(u',v'),$$

which is a contradiction. Therefore, we must have $d(x,z) \leq d_K(x',z')$ and $d(z,y) \leq d_K(z',y')$. □

The following theorem presents several fundamental comparison results involving curve length, chord length, and total curvature for spherical curves in metric spaces with curvature bounded below. These results form an essential part of the analytical framework required for the proofs of the main theorems. In particular, the theorem provides precise inequalities that relate geometric quantities of a given spherical curve in the ambient metric space to those of its corresponding comparison curve in the model space of constant curvature. Such estimates play a crucial role in establishing rigidity phenomena and ensuring that local geometric constraints propagate to global geometric conclusions.

Theorem 3.1. *Let (X,d) be a metric space of curvature bounded below by K that satisfies Property A. Let γ be a spherical curve in X, lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p with endpoints a and b. Let γ' denote the circular arc of radius r centered at a point p' in the model space $(\mathbb{M}_K^2, d_K)$, with endpoints a' and b'. Assume further that*

$$\ell(\gamma) = \ell(\gamma') \leq \frac{\pi}{\sqrt{K}} \quad \text{and} \quad d(a,b) = d_K(a',b').$$

(1) *Let δ_n is a polygonal curve consisting of consecutive points $a = p_1, p_2, \dots, p_n = b$, where $p_i \in \gamma$ for all $i \in \{2,3,...,n-1\}$, and similarly, let δ'_n be a polygonal curve with consecutive points $a' = p'_1, p'_2, \dots, p'_n = b'$, where $p'_i \in \gamma'$, for all $i \in \{2,3,...,n-1\}$. Suppose $d(p_i,p_{i+1}) = d_K(p'_i,p'_{i+1})$, for all $i \in \{2,3,...,n-1\}$. Then*

$$\kappa_c^*(\delta_n) \leq \kappa_c^*(\delta'_n).$$

(2) *If $\ell(\gamma) \leq \ell(\gamma')$, then $\kappa_c(\gamma) \leq \kappa_c(\gamma')$.*

(3) *If $d(a,b) \leq d_K(a',b')$, then $\ell(\gamma) \leq \ell(\gamma')$.*

(4) *If $d(a,b) \leq d_K(a',b')$, then $\kappa_c(\gamma) \leq \kappa_c(\gamma')$.*

Proof. (1): By assumption, for each i, the triangle $\triangle(p', p'_{i-1}, p'_i)$ serves as a comparison triangle for triangle $\triangle(p, p_{i-1}, p_i)$. By the angle property in X, it follows that $\widehat{p_i} \geq \widehat{p'_i}$. Hence,

$$\kappa_c^*(\delta_n) = \sum_{i=2}^{n-1}(\pi - \widehat{p_i}) \leq \sum_{i=2}^{n-1}(\pi - \widehat{p'_i}) = \kappa_c^*(\delta'_n).$$

(2): First, we consider the case when $\ell(\gamma) = \ell(\gamma')$. Let $\{\delta_k\}_{k=1}^{\infty}$ be a sequence of polysegments inscribed in γ, each consisting of geodesic segments of equal length, such that $\mu_\gamma(\delta_k) \to 0$ as $k \to \infty$. Fix $\delta = \delta_k$ with vertices $a = p_1, p_2, \ldots, p_n = b$. On the model space $\mathbb{M}_K^2$, we construct a corresponding polysegment $\tau = \tau_k$ with ordered vertices $a' = p'_1, p'_2, \ldots, p'_n = b'$, where each p'_i lies on a circle of radius r centered at p', and

$$d(p_i, p_{i+1}) = d_K(p'_i, p'_{i+1}), \quad \text{for all } i \in \{1, 2, \ldots, n-1\}.$$

Without loss of generality, we may assume that τ and γ' wind around p' in the same direction. Then there exists an integer $m \leq n$ such that $p'_i \in \gamma'$ for all $i \in \{1, 2, \ldots, m\}$. By Property A, for each $i \in \{2, 3, \ldots, n-1\}$, the triangle $\triangle(p'_{i-1}, p'_i, p'_{i+1})$ in $\mathbb{M}_K^2$ is a comparison triangle for $\triangle(p_{i-}, p_i, p_{i+1})$ in X. Consequently, we have

$$\widehat{p_i} \geq \widehat{p'_i} \quad \text{for all } i.$$

We therefore have that

$$\kappa_c^*(\delta_k) = \kappa_c^*(\delta) = \sum_{i=2}^{n-1}(\pi - \widehat{p_i}) \leq \sum_{i=2}^{n-1}(\pi - \widehat{p'_i}) = \kappa_c^*(\delta'_k),$$

and consequently

$$\kappa_c(\gamma) = \lim_{k\to\infty} \kappa_c^*(\delta_k) \leq \lim_{k\to\infty} \kappa_c^*(\delta'_k). \tag{3.8}$$

By Proposition 3.1, we have

$$\lim_{k\to\infty} \kappa_c^*(\delta'_k) = \kappa_c(\gamma'). \tag{3.9}$$

The inequality $\kappa_c(\gamma) \leq \kappa_c(\gamma')$ follows from (3.8) and (3.9).

Now if $\ell(\gamma) < \ell(\gamma')$, there is a subarc γ^* of γ' such that $\ell(\gamma^*) = \ell(\gamma)$, and hence

$$\kappa_c(\gamma) \leq \kappa_c(\gamma^*) < \kappa_c(\gamma'),$$

by the result just obtained. Hence, (2) is completely proved.

(3): We construct inductively a sequence $\{\delta_n\}$ of polysegments inscribed in γ and a sequence $\{\delta'_n\}$ of polysegments inscribed in γ' as follows. Let $\delta_1 = [a, b]$ and $\delta'_1 = [a', b']$. It is then clear that

$$\ell(\delta_1) \leq \ell(\delta'_1).$$

Next, we construct δ_2 to be a polysegment with ordered vertices a, p_1, b, by choosing a point $p_1 \in \gamma$ such that

$$d(a, p_1) = d(p_1, b).$$

Similarly, we construct a polysegment δ'_2 with ordered vertices a', p'_1, b' by choosing a point $p'_1 \in \gamma'$ satisfying

$$d_K(a', p'_1) = d_K(p'_1, b').$$

By Lemma 3.1, we obtain

$$d(a, p_1) \leq d_K(a', p'_1) \quad \text{and} \quad d(p_1, b) \leq d_K(p'_1, b'),$$

which implies that

$$\ell(\delta_2) \leq \ell(\delta'_2).$$

To construct δ_3, on γ, we insert points p_{21} and p_{22} between each pair of consecutive vertices of δ_2 such that

$$d(a, p_{21}) = d(p_{21}, p_1) = d(p_1, p_{22}) = d(p_{22}, b).$$

Thus, δ_3 is a polysegment with ordered vertices $a, p_{21}, p_1, p_{22}, b$. Similarly, we construct δ'_3 in the same manner as δ_3, obtaining a polysegment with ordered vertices $a', p'_{21}, p'_1, p'_{22}, b'$. By the same argument as above, it follows that

$$\ell(\delta_3) \leq \ell(\delta'_3).$$

For $n \geq 4$, we construct δ_n and δ'_n in the same manner as in the previous steps. Therefore, we have

$$\ell(\delta_n) \leq \ell(\delta'_n), \quad \text{for all } n \geq 4.$$

Moreover, by Proposition 3.1, we have that

$$\ell(\delta_n) \to \ell(\gamma) \quad \text{and} \quad \ell(\delta'_n) \to \ell(\gamma') \quad \text{as } n \to \infty,$$

and hence

$$\ell(\gamma) = \lim_{n\to\infty} \ell(\delta_n) \leq \lim_{n\to\infty} \ell(\delta'_n) = \ell(\gamma').$$

Now, if $d(a, b) < d_K(a', b')$, there exists a subarc γ^* of γ' with endpoints a' and b'' such that $d(a, b) = d_K(a', b'')$. Hence,

$$\ell(\gamma) \leq \ell(\gamma^*) < \ell(\gamma'),$$

by the result just obtained. Therefore, (3) is completely proved.

(4): If $d(a, b) \leq d_K(a', b')$, then by (2) and (3), we have that $\kappa_c(\gamma) \leq \kappa_c(\gamma')$. □

Based on the results of Theorem 3.1, we now formulate a conclusion that will serve as an essential component in the proof of Theorem 3.2. This conclusion refines the comparison framework established earlier and provides a direct link between the geometric behavior of spherical curves in a metric space with curvature bounded below and their corresponding comparison curves in the model space.

Remark 3.1. By the assertion of Theorem 3.1(3), if $d(a,b) \leq d_K(a',b')$, then $\ell(\gamma) \leq \ell(\gamma')$. Consequently, when $d(a,b) < d_K(a',b')$, we have $\ell(\gamma) < \ell(\gamma')$ by applying the theorem to a circular arc γ' having the same chord length as γ. Hence, if $\ell(\gamma) = \ell(\gamma')$, it follows that $d(a,b) \geq d_K(a',b')$. Moreover, if $d(a,b) \leq d_K(a',b')$ and $\ell(\gamma) = \ell(\gamma')$, then $d(a,b) = d_K(a',b')$. Similarly, if $d(a,b) \leq d_K(a',b')$ and $\kappa_c(\gamma) = \kappa_c(\gamma')$, then $d(a,b) = d_K(a',b')$. Theorem 3.1(2) states that if $\ell(\gamma) \leq \ell(\gamma')$, then $\kappa_c(\gamma) \leq \kappa_c(\gamma')$. Hence, if $\kappa_c(\gamma) = \kappa_c(\gamma')$, it follows that $\ell(\gamma) \geq \ell(\gamma')$. Moreover, assuming that $d(a,b) = d_K(a',b')$, Theorem 3.1(3) gives $\ell(\gamma) \leq \ell(\gamma')$. Therefore, if $d(a,b) = d_K(a',b')$ and $\kappa_c(\gamma) = \kappa_c(\gamma')$, we conclude that $\ell(\gamma) = \ell(\gamma')$.

The following lemma shows that when a spherical curve in a metric space whose curvature is bounded below by K satisfies certain length and distance conditions, the convex hull determined by its principal points is isometric to the corresponding region in the model space.

Lemma 3.2. *Let (X,d) be a metric space of curvature bounded below by K that satisfies both Property A and Property B. Let γ be a spherical curve in X lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p with endpoints a and b. Let γ' be a circular arc of radius r centered at a point p' in the model space $(\mathbb{M}^2_K, d_K)$ with endpoints a' and b'. Assume that $d(a,b) = d_K(a',b')$ and $\ell(\gamma) = \ell(\gamma') \leq \frac{\pi}{\sqrt{K}}$. Choose points $e \in \gamma$ and $e' \in \gamma'$ satisfying*

$$\ell(\gamma_{ae}) = \ell(\gamma'_{a'e'}).$$

Then the geodesic segment $[p,e]$ intersects the geodesic segment $[a,b]$ at some point. Moreover, the convex hulls

$$C(\{a,e,b,p\}) \quad \text{and} \quad C(\{a',e',b',p'\})$$

are isometric to each other.

Proof. First, we show that the geodesic segment $[a,b]$ intersects the geodesic segment $[p,e]$ at a point. Let m' denote the intersection point of $[a',b']$ and $[p',e']$. Along the geodesic segment $[e,p]$, let m be the point satisfying

$$d(m,e) = d_K(m',e').$$

Then the point m' corresponds to the point m. Since $\triangle(a',e',p')$ corresponds to $\triangle(a,e,p)$ and $\triangle(b',e',p')$ corresponds to $\triangle(b,e,p)$, we have

$$d(a,m) \geq d_K(a',m') \quad \text{and} \quad d(b,m) \geq d_K(b',m'). \tag{3.10}$$

By triangle inequality,

$$d(a,b) \leq d(a,m) + d(m,b). \tag{3.11}$$

Using assumption, and combining (3.10) and (3.11), we have

$$d_K(a',b') = d(a,b) \leq d(a,m) + d(m,b) \geq d_K(a',m') + d_K(b',m') \geq d_K(a',b'). \tag{3.12}$$

That implies

$$d(a,m) = d_K(a',m') \quad \text{and} \quad d(b,m) = d_K(b',m'),$$

therefore the equality (3.12) forces

$$d_K(a',b') = d(a,b) = d(a,m) + d(m,b) = d_K(a',m') + d_K(b',m') = d_K(a',b').$$

Consequently, the geodesic segment $[a,b]$ intersects the segment $[p,e]$ at the point m.

Since X satisfies Property B, the convex hulls of the triangles $\triangle(a,e,p)$ and $\triangle(b,e,p)$ in X are isometric to the convex hulls of the triangles $\triangle(a',e',p')$ and $\triangle(b',e',p')$ in $\mathbb{M}^2_K$, respectively. Furthermore, the convex hulls of the triangles $\triangle(a,e,b)$ and $\triangle(a,p,b)$ in X are isometric to the convex hulls of the triangles $\triangle(a',e',b')$ and $\triangle(a',p',b')$ in $\mathbb{M}^2_K$, respectively.

The next step is to verify that $C(\{a,e,b,p\})$ and $C(\{a',e',b',p'\})$ are isometric. By the definition of the convex hull, $C(\{a,e,b,p\})$ exists and is uniquely determined, as previously noted. Define the maps

$$i_1 : C(\{a,e,b\}) \to C(\{a',e',b'\}) \quad \text{and} \quad i_2 : C(\{a,b,p\}) \to C(\{a',b',p'\})$$

such that $i_1(a) = a'$, $i_1(b) = b'$, $i_1(e) = e'$, $i_2(a) = a'$, $i_2(b) = b'$, and $i_2(p) = p'$. Let

$$i : C(\{a,e,b,p\}) \longrightarrow C(\{a',e',b',p'\}) = C(\{a',e',b'\}) \cup C(\{a',b',p'\})$$

be the map defined by $i|_{C(\{a,e,b\})} = i_1$ and $i|_{C(\{a,b,p\})} = i_2$. We shall prove that i is an isometry by verifying that:

(1) i is an isometry onto its image; and
(2) $C(\{a,e,b,p\}) = C(\{a,e,b\}) \cup C(\{a,b,p\})$.

It is clear that i is surjective. Injectivity follows from the properties of intersecting geodesic segments and the isometry of convex hulls. To prove (1), let $z_1, z_2 \in C(\{a,e,b,p\})$, and set $z_1' = i(z_1)$, $z_2' = i(z_2)$. We must verify that $d(z_1,z_2) = d_K(z_1',z_2')$. If both z_1 and z_2 belong to the same convex hull, i.e. $C(\{a,e,b\})$, $C(\{a,e,p\})$, $C(\{a,b,p\})$, or $C(\{e,b,p\})$, the result is immediate.

Without loss of generality, assume $z_1 \in C(\{a,e,m\})$ and $z_2 \in C(\{b,m,p\})$. Suppose that $[z_1',z_2']$ meets $[a',b']$ at z_3' and $[e',p']$ at z_4' such that $z_3' \in [z_1',z_4']$ (the case $z_4' \in [z_1',z_3']$ can be treated analogously). Let z_3 and z_4 be points satisfying $i_1(z_3) = z_3'$ and $i_2(z_4) = z_4'$. In $\mathbb{M}^2_K$, we have

$$[z_1',z_4'] = [z_1',z_3'] \cup [z_3',z_4'] \quad \text{and} \quad [z_3',z_2'] = [z_3',z_4'] \cup [z_4',z_2'].$$

Since $C(\{a,e,p\})$ is isometric to $C(\{a',e',p'\})$ and $[z_1',z_4'] \subset C(\{a',b',p'\})$, we obtain

$$[z_1,z_4] = [z_1,z_3] \cup [z_3,z_4] \subset C(\{a,b,p\}),$$

with $d(z_1,z_3) = d_K(z_1',z_3')$ and $d(z_3,z_4) = d_K(z_3',z_4')$, and hence

$$d(z_1,z_4) = d(z_1,z_3) + d(z_3,z_4) = d_K(z_1',z_3') + d_K(z_3',z_4') = d_K(z_1',z_4').$$

Similarly, because $C(\{a,b,p\})$ is isometric to $C(\{a',b',p'\})$ and $[z_3',z_2'] \subset C(\{a',b',p'\})$, we have

$$[z_3,z_2] = [z_3,z_4] \cup [z_4,z_2] \subset C(\{a,b,p\}),$$

with $d(z_3,z_4) = d_K(z_3',z_4')$ and $d(z_4,z_2) = d_K(z_4',z_2')$, and therefore

$$d(z_3,z_2) = d(z_3,z_4) + d(z_4,z_2) = d_K(z_3',z_4') + d_K(z_4',z_2') = d_K(z_3',z_2').$$

Hence,

$$[z_1, z_2] = [z_1, z_3] \cup [z_3, z_4] \cup [z_4, z_2]$$

forms a geodesic segment, and thus

$$\begin{aligned} d(z_1, z_2) &= d(z_1, z_3) + d(z_3, z_4) + d(z_4, z_2) \\ &= d_K(z_1', z_3') + d_K(z_3', z_4') + d_K(z_4', z_2') \\ &= d_K(z_1', z_2'). \end{aligned}$$

We now prove (2). Let $M = \{a, e, b\}$ and $N = \{a, b, p\}$. We first show that $C(M) \cup C(N)$ is convex. Let $u_1, u_2 \in C(M) \cup C(N)$. We must show that the geodesic segment $[u_1, u_2]$ is contained in $C(M) \cup C(N)$. If both u_1 and u_2 belong to $C(M)$ or both to $C(N)$, there is nothing to prove. Otherwise, assume $u_1 \in C(M)$ and $u_2 \in C(N)$. Let u_1' and u_2' be the corresponding points to u_1 and u_2, respectively, and let t' denote the intersection point of $[u_1', u_2']$ with $[a', b']$. Set $t = i^{-1}(t')$. Then,

$$d(u_1, u_2) = d_K(u_1', u_2') = d_K(u_1', t') + d_K(t', u_2') = d(u_1, t) + d(t, u_2),$$

which implies that $[u_1, t]$ and $[t, u_2]$ together form a geodesic segment joining u_1 and u_2. Hence,

$$[u_1, u_2] = [u_1, t] \cup [t, u_2] \subset C(M) \cup C(N),$$

and therefore $C(M) \cup C(N)$ is convex.

Since $C(M \cup N)$ is the smallest convex set containing $M \cup N$, we have

$$C(M \cup N) \subset C(M) \cup C(N).$$

Conversely, as both $C(M)$ and $C(N)$ are subsets of $C(M \cup N)$, it follows that

$$C(M) \cup C(N) \subset C(M \cup N).$$

Therefore, $C(M) \cup C(N) = C(M \cup N)$ is convex. □

From Lemma 3.2, we can deduce the following lemma, which serves as an essential intermediary step in establishing the geometric comparison results needed for the proof of Theorem 3.2. In particular, this lemma refines the conclusions of Lemma 3.2 by applying them to a more specific configuration of points along the spherical curve and its corresponding model curve.

Lemma 3.3. *Let (X, d) be a metric space of curvature bounded below by K that satisfies both Property A and Property B. Let γ be a spherical curve in X, lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p with endpoints a and b, and let γ' be a circular arc of radius r centered at a point p' in the model space $(\mathbb{M}_K^2, d_K)$ with endpoints a' and b'. Assume that $d(a, b) = d_K(a', b')$ and $\ell(\gamma) = \ell(\gamma') \leq \frac{\pi}{\sqrt{K}}$. Let $a, e_1, e_2, \ldots, e_n, b$ and $a', e_1', e_2', \ldots, e_n', b'$ be ordered vertices in γ and γ', respectively, such that*

$$\ell(\gamma_{ae_i}) = \ell(\gamma'_{a'e_i'}), \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

Then the convex hulls $C(\{a, e_1, e_2, \ldots, e_n, b, p\})$ and $C(\{a', e_1', e_2', \ldots, e_n', b'\})$ are isometric to each other.

The following lemma is crucial in guaranteeing that the totally geodesic surface enclosed by a closed spherical curve in a metric space of curvature bounded below is isometric to the disk bounded by a circle in the model space.

Lemma 3.4. *Let (X, d) be a metric space of curvature bounded below by K that satisfies both Property A and Property B. Let γ be a spherical curve in X, lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p, with endpoints a and b, and let γ' be a circular arc of radius r centered at a point p' in the model space $(\mathbb{M}^2_K, d_K)$, with endpoints a' and b'. Suppose that $d(a,b) = d_K(a',b')$ and $\ell(\gamma) = \ell(\gamma') \leq \frac{\pi}{\sqrt{K}}$. Then the following statements hold:*

(1) $\bigcup_{e\in\gamma}[p,e] = C(\gamma \cup \{p\})$*; and*
(2) $C(\gamma \cup \{p\})$ *is isometric to* $C(\gamma' \cup \{p'\})$*.*

Proof. By assumption, each geodesic segment $[p,e]$, with $e \in \gamma$, intersects the geodesic segment $[a,b]$.

(1): By the definition of $C(\gamma \cup \{p\})$, we have

$$\bigcup_{e\in\gamma}[p,e] \subset C(\gamma \cup \{p\}).$$

Next, we show that $\bigcup_{e\in\gamma}[p,e]$ is a convex set. Let $x, y \in \bigcup_{e\in\gamma}[p,e]$. Then $x \in [p,e_1]$ and $y \in [p,e_2]$ for some e_1, e_2 on γ. If $e_1 = e_2$, the statement is clear. Suppose instead that $e_1 \neq e_2$. Let e'_1 and e'_2 be two points on γ' such that

$$\ell(\gamma'_{a'e'_1}) = \ell(\gamma_{a,e_1}) \quad \text{and} \quad \ell(\gamma'_{a'e'_2}) = \ell(\gamma_{ae_2}).$$

Without loss of generality, assume that e'_1 lies between a' and e'_2 on γ'. By Lemma 3.3, we obtain $[x,y] \subseteq C(\{p,a,e_1,e_2,b\})$. Furthermore, it is easy to see that every point $z \in [x,y]$ lies on a segment $[p,e^*]$ for some $e^* \in \gamma_{ab}$. Hence, the set $\bigcup_{e\in\gamma_{ab}}[p,e]$ is convex.

Since $C(\gamma \cup \{p\})$ is the smallest convex set containing $\gamma \cup \{p\}$, it follows that

$$C(\gamma \cup \{p\}) \subseteq \bigcup_{e\in\gamma}[p,e].$$

Consequently,

$$\bigcup_{e\in\gamma}[p,e] = C(\gamma \cup \{p\}).$$

(2): Define a map $j : C(\gamma' \cup \{p'\}) \to C(\gamma \cup \{p\})$ such that for each segment $[p',c']$, where c' is a point on γ' is mapped isometrically onto the geodesic segment $[p,c]$, where c is a point on γ satisfying $\ell(\gamma_{ac}) = \ell(\gamma'_{a'c'})$. By (1), it is easy to see that j is a bijection. To show that j is an isometry from $C(\gamma' \cup \{p'\})$ onto $C(\gamma \cup \{p\})$, we verify that j preserves distances between points. Let x' and y' be points on the segments $[p',e']$ and $[p',f']$, respectively, where e' and f' lie on γ'. On the corresponding geodesic segments $[p,e]$ and $[p,f]$, let x and y be the points corresponding to x' and y', respectively. We shall show that $d(x,y) = d_K(x',y')$. By Lemma 3.2, the sets $\{p',a',e',f',b'\}$ and $\{p,a,e,f,b\}$ determine corresponding isometric convex hulls. Hence, the equality $d(x,y) = d_K(x',y')$ follows immediately. □

This subsequent theorem constitutes the first main result of this paper. It establishes that in a metric space with curvature bounded below, if a closed spherical curve and a corresponding circle in the model space share the same radius, have equal lengths, and possess identical distance relations between their boundary points, then the convex hull enclosed by the curve is isometric to the model disk. Consequently, both surfaces exhibit identical geometric structures.

Theorem 3.2. *Let (X, d) be a metric space of curvature bounded below by K. Let γ be a closed spherical curve in X, lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p, and let γ' be a circle of radius r centered at a point p' in the model space $(\mathbb{M}^2_K, d_K)$. Suppose that the following statements hold:*

(1) *X satisfies both Property A and Property B;*
(2) *$\ell(\gamma) = \ell(\gamma')$;*
(3) *$d(a,b) = d_K(a', b')$, whenever $\ell(\gamma_{ab}) = \ell(\gamma'_{a'b'})$ for any subarc γ_{ab} of γ and any subarc $\gamma'_{a'b'}$ of γ'.*

Then $C(\gamma)$ is isometric to $C(\gamma')$, that is, the totally geodesic surface bounded by γ and the disk bounded by γ' are isometric to each other.

Proof. Let $x, y \in \gamma$ and $x', y' \in \gamma'$ be such that $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'}) = \ell(\gamma)/2$. By condition (3), it follows that $d(x,y) = d_K(x', y')$. By Lemma 3.4, we obtain

$$\bigcup_{e \in \gamma_{xy}} [p, e] = C(\gamma_{xy} \cup \{p\}) \quad \text{and} \quad \bigcup_{f \in \gamma_{yx}} [p, f] = C(\gamma_{yx} \cup \{p\}).$$

Moreover,

$$\bigcup_{e' \in \gamma'} [p', e'] = C(\gamma' \cup \{p'\}).$$

We now define a map

$$j_1 : C(\{p'\} \cup \gamma'_{x'y'}) \longrightarrow C(\{p\} \cup \gamma_{xy})$$

such that each segment $[p', z']$, where $z' \in \gamma'_{x'y'}$, is mapped to the geodesic segment $[p, z]$, where $z \in \gamma_{xy}$ satisfies $\ell(\gamma_{xz}) = \ell(\gamma'_{x'z'})$. Similarly, we define

$$j_2 : C(\{p'\} \cup \gamma'_{y'x'}) \longrightarrow C(\{p\} \cup \gamma_{yx})$$

in the same manner as j_1. Lemma 3.4 ensures that both j_1 and j_2 are isometries.

We now demonstrate that $C(\gamma')$ and $C(\gamma)$ are isometric to each other. By the definition of convex hull, we observe that $C(\gamma)$ exists and is unique. We define a map

$$i : C(\gamma') = C(\gamma'_{x'y'}) \cup C(\gamma'_{y'x'}) \to C(\gamma)$$

such that the restriction of i to $C(\{p'\} \cup \gamma'_{x'y'})$ is j_1, and its restriction to $C(\{p'\} \cup \gamma'_{y'x'})$ is j_2. To prove that i is an isometry from $C(\gamma')$ to $C(\gamma)$, it suffices to verify that i is an isometry onto its image and that $C(\gamma) = C(\gamma_{xy}) \cup C(\gamma_{yx})$.

It is clear that i is surjective. Moreover, as established in Lemma 3.3, the injectivity of i follows from the fact that the corresponding geodesic segments intersect uniquely and the associated convex hulls are isometric. To verify that i preserves distances between any two points, let $z'_1, z'_2 \in C(\gamma')$, and set $z_1 = i(z'_1)$ and $z_2 = i(z'_2)$. If $z'_1, z'_2 \in C(\gamma'_{x'y'})$ or $z'_1, z'_2 \in C(\gamma'_{y'x'})$, then the distance preservation follows directly. Hence, it remains to consider the mixed case. Assume that $z'_1 \in C(\gamma'_{x'y'})$ and $z'_2 \in C(\gamma'_{y'x'})$. Hence, $z'_1 \in [p', v'_1]$ and $z'_2 \in [p', v'_2]$ for some $v'_1 \in \gamma'_{x'y'}$ and $v'_2 \in \gamma'_{y'x'}$.

On X, we let $[p, v_1]$ be the geodesic segment containing z_1 and let $[p, v_2]$ be the geodesic segment containing z_2 where $v_1 \in \gamma_{xy}$ and $v_2 \in \gamma_{yx}$.

If $\ell(\gamma'_{v'_1v'_2}) \leq \ell(\gamma')/2$, we then have $\gamma'_{v'_1v'_2} = \gamma'_{v'_1y'} \cup \gamma'_{y'v'_2}$. By applying (3) together with Lemma 3.4, we obtain that $C(\gamma'_{v'_1y'})$ is isometric to $C(\gamma_{v_1y})$ via j_1, and similarly $C(\gamma'_{v'_2y'})$ is isometric to $C(\gamma_{v_2y})$ via j_2. Consequently, it follows that $C(\gamma'_{v'_1v'_2})$ is isometric to $C(\gamma_{v_1v_2})$ under the map i, and therefore

$$d(z_1, z_2) = d_K(z'_1, z'_2).$$

Additionally, we also have $d(z_1, z_2) = d_K(z'_1, z'_2)$ if $\ell(\gamma'_{v'_2v'_1}) \leq \ell(\gamma')/2$.

We now demonstrate that

$$C(\gamma) = C(\gamma_{xy} \cup \gamma_{yx}) = C(\gamma_{xy}) \cup C(\gamma_{yx}).$$

It is necessary to demonstrate that the set $C(\gamma_{xy} \cup \gamma_{yx})$ is convex. Without losing generality, we suppose that $u_1 \in C(\gamma_{xy})$ and $u_2 \in C(\gamma_{yx})$. Thus $u_1 \in [p, w_1]$ and $u_2 \in [p, w_2]$, for some $w_1 \in \gamma_{xy}$ and $w_2 \in \gamma_{yx}$. Since j_1 is an isometry from $C(\gamma'_{x'y'})$ to $C(\gamma_{xy})$ and j_2 is an isometry from $C(\gamma'_{y'x'})$ to $C(\gamma_{yx})$, we may choose points $w'_1 \in \gamma_{x'y'}$ and $w'_2 \in \gamma_{y'x'}$ corresponding to w_1 and w_2, respectively, and similarly select points u'_1 and u'_2 corresponding to u_1 and u_2, respectively.

If $\ell(\gamma'_{w'_1w'_2}) \leq \ell(\gamma')/2$, then $\gamma'_{w'_1w'_2} = \gamma'_{w'_1y'} \cup \gamma'_{y'w'_2}$ is the result. As $C(\gamma'_{w'_1y'})$ is isometric to $C(\gamma_{w_1y})$ by j_1 and $C(\gamma'_{w'_2y'})$ is isometric to $C(\gamma_{w_2y})$ by j_2, we thus obtain that $C(\gamma'_{w'_1w'_2})$ is isometric to $C(\gamma_{w_1w_2})$ by i. Consequently, we obtain

$$d(u_1, u_2) = d_K(u'_1, u'_2).$$

Let s' be the point where $[u'_1, u'_2]$ intersects $[x', y']$, and let

$$s = j_1(s') = j_2(s') = i(s').$$

Hence,

$$d(u_1, u_2) = d_K(u'_1, u'_2) = d_K(u'_1, s') + d_K(s', u'_2) = d(u_1, s) + d(s, u_2).$$

This implies that

$$[u_1, u_2] = [u_1, s] \cup [s, u_2] \subset C(\gamma_{xy}) \cup C(\gamma_{yx}).$$

Therefore, $C(\gamma_{xy}) \cup C(\gamma_{yx})$ is a convex set.

If $\ell(\gamma'_{w'_2w'_1}) \leq \ell(\gamma')/2$, the same argument applies, just as in the case $\ell(\gamma'_{w'_1w'_2}) \leq \ell(\gamma')/2$, and we again obtain that $C(\gamma_{xy}) \cup C(\gamma_{yx})$ is a convex set.

Accordingly, we conclude that $C(\gamma')$ is isometric to $C(\gamma)$. The proof of the theorem is now complete. □

Before presenting the second main theorem, we first state the following lemma, which provides a geometric comparison: in a metric space with curvature bounded below, a spherical curve and its corresponding model arc have equal lengths if and only if their total curvatures coincide, provided that their endpoints are equidistant in both spaces.

Lemma 3.5. *Let (X,d) be a metric space of curvature bounded below by K that satisfies both Property A and Property B. Let γ be a spherical curve in X, lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p, with endpoints a and b, and let γ' be a circular arc of radius r centered at a point p' in the model space $(\mathbb{M}^2_K, d_K)$, with endpoints a' and b'. If that $d(a,b) = d_K(a',b')$, then $\ell(\gamma) = \ell(\gamma')$ if and only if $\kappa_c(\gamma) = \kappa_c(\gamma')$.*

Proof. Assume first that $\ell(\gamma) = \ell(\gamma')$. Since $d(a,b) = d_K(a',b')$, it follows from Lemma 3.4 that the convex hulls $C(\gamma \cup \{p\})$ and $C(\gamma' \cup \{p'\})$ are isometric. Consequently, we obtain $\kappa_c(\gamma) = \kappa_c(\gamma')$.

Conversely, suppose that $\kappa_c(\gamma) = \kappa_c(\gamma')$. Then, by Theorem 3.1(2), we have $\ell(\gamma) \geq \ell(\gamma')$. Since $d(a,b) = d_K(a',b')$, it follows that $\ell(\gamma) \leq \ell(\gamma')$, and thus $\ell(\gamma) = \ell(\gamma')$. □

Another lemma, which will be used in the proof of the second main result, states that if a closed spherical curve in a metric space with curvature bounded below preserves the same pairwise distances as a corresponding circle in the model space, then their total lengths are equal if and only if their total curvatures coincide.

Lemma 3.6. *Let (X,d) be a metric space of curvature bounded below by K that satisfies both Property A and Property B. Let γ be a closed spherical curve in X, lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p, and let γ' be a circle of radius r centered at a point p' in the model space $(\mathbb{M}^2_K, d_K)$. Assume that $d(a,b) = d_K(a',b')$, whenever $\ell(\gamma_{ab}) = \ell(\gamma'_{a'b'})$ for every subarc γ_{ab} of γ with endpoints a, b and subarc $\gamma'_{a'b'}$ of γ' with endpoints a', b'. Then, $\ell(\gamma) = \ell(\gamma')$ if and only if $\kappa_c(\gamma) = \kappa_c(\gamma')$.*

Proof. We now prove the sufficiency part. Assume that $\ell(\gamma) = \ell(\gamma')$. Let p_1, p_2, and p_3 be three consecutive points on γ such that

$$\ell(\gamma_{p_1p_2}) = \ell(\gamma_{p_2p_3}) = \ell(\gamma_{p_3p_1}) = \frac{\ell(\gamma)}{3}.$$

On γ', choose three consecutive points p'_1, p'_2, p'_3 such that

$$\ell(\gamma'_{p'_1p'_2}) = \ell(\gamma'_{p'_2p'_3}) = \ell(\gamma'_{p'_3p'_1}) = \frac{\ell(\gamma')}{3}.$$

By the assumption, we have that

$$d(p_1,p_2) = d_K(p'_1,p'_2), \quad d(p_2,p_3) = d_K(p'_2,p'_3), \quad \text{and} \quad d(p_3,p_1) = d_K(p'_3,p'_1).$$

From Lemma 3.5, it follows that

$$\kappa_c(\gamma_{p_1p_2}) = \kappa_c(\gamma'_{p'_1p'_2}), \quad \kappa_c(\gamma_{p_2p_3}) = \kappa_c(\gamma'_{p'_2p'_3}), \quad \text{and} \quad \kappa_c(\gamma_{p_3p_1}) = \kappa_c(\gamma'_{p'_3p'_1}).$$

Therefore, we conclude that

$$\kappa_c(\gamma) = \kappa_c(\gamma').$$

The proof of necessity proceeds in a similar manner. □

The following theorem constitutes the second main theorem of this paper. It demonstrates that in a metric space with curvature bounded below, if a closed spherical curve and a corresponding circle in the model space have the same radius, equal total curvatures, and identical pairwise distances between points, then the convex hull enclosed by the curve is isometric to the model disk. Consequently, both surfaces share the same geometric structure.

Theorem 3.3. *Let X be a metric space whose curvature is bounded below by K. Let γ be a closed spherical curve in X, lying at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p, and let γ' be a circle of radius r centered at a point p' in the model space $(\mathbb{M}_K^2, d_K)$. Suppose that the following statements hold:*

(1) *X satisfies both Property A and Property B;*
(2) $\kappa_c(\gamma) = \kappa_c(\gamma')$;
(3) $d(a,b) = d_K(a',b')$, *whenever $\kappa_c(\gamma_{ab}) = \kappa_c(\gamma'_{a'b'})$ for any subarc γ_{ab} of γ and any subarc $\gamma'_{a'b'}$ of γ'.*

Then $C(\gamma)$ is isometric to $C(\gamma')$, that is, the totally geodesic surface bounded by γ and the disk bounded by γ' are isometric to each other.

Proof. By (2), we can choose three consecutive points p_1, p_2, and p_3 on γ such that

$$\kappa_c(\gamma_{p_1p_2}) = \kappa_c(\gamma_{p_2p_3}) = \kappa_c(\gamma_{p_3p_1}) = \tfrac{\kappa_c(\gamma)}{3}.$$

Similarly, on γ', select three consecutive points p'_1, p'_2, p'_3 satisfying

$$\kappa_c(\gamma'_{p'_1p'_2}) = \kappa_c(\gamma'_{p'_2p'_3}) = \kappa_c(\gamma'_{p'_3p'_1}) = \tfrac{\kappa_c(\gamma')}{3}.$$

By (3), we get that

$$d(p_1,p_2) = d_K(p'_1,p'_2), \quad d(p_2,p_3) = d_K(p'_2,p'_3), \quad \text{and} \quad d(p_3,p_1) = d_K(p'_3,p'_1).$$

Then, by Remark 3.1, it follows that

$$\ell(\gamma_{p_1p_2}) = \ell(\gamma'_{p'_1p'_2}), \quad \ell(\gamma_{p_2p_3}) = \ell(\gamma'_{p'_2p'_3}), \quad \text{and} \quad \ell(\gamma_{p_3p_1}) = \ell(\gamma'_{p'_3p'_1}).$$

Applying Lemma 3.6, we obtain $\ell(\gamma) = \ell(\gamma')$. Finally, by Theorem 3.2, the desired result follows. □

4. Conclusion

In this work, we have identified geometric conditions that allow a closed spherical curve in a metric space with curvature bounded below to be accurately compared with a circle in the model space of constant curvature. In particular, if the space satisfies the distance and curvature equivalence conditions given in (1)–(4) of Theorem 1.1, then the convex hull enclosed by the spherical curve γ is isometric to the convex hull enclosed by the corresponding circle γ' in the model space. Hence, the totally geodesic surface bounded by γ and the disk bounded by γ' are isometric to each other.

This result offers an important geometric characterization connecting metric spaces with curvature bounded below to their corresponding constant-curvature model spaces, forming a basis for further investigations into total curvature, convexity, and isometric embeddings within such metric spaces.

For further research, these results can be extended to the study of higher-dimensional analogues, such as totally geodesic hypersurfaces and convex bodies in Alexandrov or CAT(K) spaces. Another promising direction is the exploration of applications in geometric analysis and global differential geometry, where curvature comparison theorems play a central role in the study of manifolds with curvature bounds. Moreover, potential applications can be found in geometric modeling, computer graphics, and structural design, where curvature-preserving transformations are essential. Future studies may also focus on relaxing the assumptions of curvature boundedness or exploring similar isometric relationships in non-smooth or discrete metric geometries.

Acknowledgment: The author sincerely thanks all individuals who contributed to this research. Their guidance, collaboration, and support were vital to the completion of this work. This research was supported by the National Science, Research and Innovation Fund (NSRF) and Prince of Songkla University (Grant No. SAT6801325S).

Conflicts of Interest: The author declares that there are no conflicts of interest related to the publication of this paper.

References

[1] D.V. Alekseevskij, E.B. Vinberg, A.S. Solodovnikov, Geometry of Spaces of Constant Curvature, in: Encyclopaedia of Mathematical Sciences, Springer, Berlin, Heidelberg, 1993: pp. 1–138. https://doi.org/10.1007/978-3-662-02901-5_1.

[2] S.B. Alexander, R.L. Bishop, Comparison Theorems for Curves of Bounded Geodesic Curvature in Metric Spaces of Curvature Bounded Above, Differ. Geom. Appl. 6 (1996), 67–86. https://doi.org/10.1016/0926-2245(96)00008-3.

[3] S.B. Alexander, R.L. Bishop, The Fary-Milnor Theorem in Hadamard Manifolds, Proc. Am. Math. Soc. 126 (1998), 3427–3436. https://doi.org/10.1090/s0002-9939-98-04423-2.

[4] A.D. Alexandrov, Theory of Curves Based on the Approximation by Polygonal Lines, Thesis, Nauchnaya Sessiya Leningradskogo University, Tezisy Dokladov, 1946.

[5] A.D. Aleksandrov, V.N. Berestovskii, I.G. Nikolaev, Generalized Riemannian Spaces, Russ. Math. Surv. 41 (1986), 1–54. https://doi.org/10.1070/rm1986v041n03abeh003311.

[6] A.D. Alexandrov, Y.G. Reshetnyak, General Theory of Irregular Curves, Springer, Dordrecht, 1989. https://doi.org/10.1007/978-94-009-2591-5.

[7] L. Ambrosio, N. Gigli, G. Savaré, Metric Measure Spaces with Riemannian Ricci Curvature Bounded from Below, Duke Math. J. 163 (2014), 1405–1490. https://doi.org/10.1215/00127094-2681605.

[8] W. Ballmann, Lectures on Spaces of Nonpositive Curvature, Birkhäuser, Basel, 1995. https://doi.org/10.1007/978-3-0348-9240-7.

[9] K. Borsuk, Sur la Courbure Totale des Lignes Fermées, Ann. Soc. Polon. Math. 20 (1947), 251–265.

[10] F. Brickell, C.C. Hsiung, The Total Absolute Curvature of Closed Curves in Riemannian Manifolds, J. Differ. Geom. 9 (1974), 177–193. https://doi.org/10.4310/jdg/1214432100.

[11] M.R. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer, Heidelberg, 1999. https://doi.org/10.1007/978-3-662-12494-9.

[12] D. Burago, Y. Burago, S. Ivanov, A Course in Metric Geometry, American Mathematical Society, Providence, 2001. https://doi.org/10.1090/gsm/033.

[13] Y. Burago, M. Gromov, G. Perel'man, A.D. Alexandrov Spaces with Curvature Bounded Below, Russ. Math. Surv. 47 (1992), 1–58. https://doi.org/10.1070/rm1992v047n02abeh000877.

[14] J. Cheeger, T.H. Colding, On the Structure of Spaces with Ricci Curvature Bounded Below. I, J. Differ. Geom. 46 (1997), 406–480. https://doi.org/10.4310/jdg/1214459974.

[15] S.S. Chern, R.K. Lashof, On the Total Curvature of Immersed Manifolds, Am. J. Math. 79 (1957), 306–318. https://doi.org/10.2307/2372684.

[16] J. Douglas, Solution of the Problem of Plateau, Trans. Am. Math. Soc. 33 (1931), 263–321. https://doi.org/10.2307/1989472.

[17] R. Espínola, C. Li, G. López, Nearest and Farthest Points in Spaces of Curvature Bounded Below, J. Approx. Theory 162 (2010), 1364–1380. https://doi.org/10.1016/j.jat.2010.02.007.

[18] W. Fenchel, Über Krümmung und Windung Geschlossener Raumkurven, Math. Ann. 101 (1929), 238–252.

[19] S. Halbeisen, On Tangent Cones of Alexandrov Spaces with Curvature Bounded Below, Manuscripta Math. 103 (2000), 169–182. https://doi.org/10.1007/s002290070018.

[20] A. Honda, C. Tanaka, Y. Yamauchi, The Total Absolute Curvature of Closed Curves with Singularities, Adv. Geom. 25 (2025), 93–104. https://doi.org/10.1515/advgeom-2024-0024.

[21] M.C. López, V.F. Mateos, J.M. Masqué, Total Curvature of Curves in Riemannian Manifolds, Differ. Geom. Appl. 28 (2010), 140–147. https://doi.org/10.1016/j.difgeo.2009.10.008.

[22] C. Maneesawarng, Y. Lenbury, Total Curvature and Length Estimate for Curves in CAT(K) Spaces, Differ. Geom. Appl. 19 (2003), 211–222. https://doi.org/10.1016/s0926-2245(03)00031-7.

[23] J.W. Milnor, On the Total Curvature of Knots, Ann. Math. 52 (1950), 248–257. https://doi.org/10.2307/1969467.

[24] S. Naya, N. Innami, A Comparison Theorem for Steiner Minimum Trees in Surfaces with Curvature Bounded Below, Tohoku Math. J. 65 (2013), 131–157. https://doi.org/10.2748/tmj/1365452629.

[25] A. Petrunin, Parallel Transportation for Alexandrov Space with Curvature Bounded Below, Geom. Funct. Anal. 8 (1998), 123–148. https://doi.org/10.1007/s000390050050.

[26] C. Phokaew, A. Sama-ae, Some Rigidity Theorems of Closed Geodesic Polygons and Spherical Curves in Metric Spaces with Curvature Bounded Below, Eur. J. Pure Appl. Math. 17 (2024), 3932–3944. https://doi.org/10.29020/nybg.ejpam.v17i4.5571.

[27] Y.G. Reshetnyak, Inextensible Mappings in a Space of Curvature No Greater Than K, Sov. Math. J. 9 (1968), 683–689. https://doi.org/10.1007/bf02199105.

[28] A. Sama-Ae, C. Maneesawarng, Geometry of Curves on Spheres in CAT(k) Spaces, Southeast Asian Bull. Math. 32 (2008), 767–778.

[29] A. Sama-Ae, A. Phon-on, Total Curvature and Some Characterizations of Closed Curves in CAT(k) Spaces, Geom. Dedicata 199 (2018), 281–290. https://doi.org/10.1007/s10711-018-0349-y.

[30] A. Sama-Ae, A. Phon-on, N. Makaje, A. Hazanee, A Distance Between Two Points and Nearest Points in a Metric Space of Curvature Bounded Below, Thai J. Math. Special Issue (2022), 229–239.

[31] A. Sama-Ae, A. Phon-on, N. Makaje, A. Hazanee, P. Riyapan, Some Characterizations of a Closed Geodesic Polygon and a Closed Spherical Curve in a CAT(k) Space, Thai J. Math. 23 (2025), 185–196.

[32] S. Sasaki, On the Total Curvature of a Closed Curve, Jpn. J. Math.: Trans. Abstr. 29 (1959), 118–125. https://doi.org/10.4099/jjm1924.29.0_118.

[33] J. Szenthe, On the Total Curvature of Closed Curves in Riemannian Manifolds, Publ. Math. Debr. 15 (2022), 99–105. https://doi.org/10.5486/pmd.1968.15.1-4.15.

[34] E. Teufel, On the Total Absolute Curvature of Closed Curves in Spheres, Manuscripta Math. 57 (1986), 101–108. https://doi.org/10.1007/bf01172493.

[35] E. Teufel, The Isoperimetric Inequality and the Total Absolute Curvature of Closed Curves in Spheres, Manuscripta Math. 75 (1992), 43–48. https://doi.org/10.1007/bf02567070.

[36] Y. Tsukamoto, On the Total Absolute Curvature of Closed Curves in Manifolds of Negative Curvature, Math. Ann. 210 (1974), 313–319. https://doi.org/10.1007/bf01434285.

[37] A.C.M. van Rooij, The Total Curvature of Curves, Duke Math. J. 32 (1965), 313–324. https://doi.org/10.1215/s0012-7094-65-03232-1.