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Supermagic Triple Graphs

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Abstract. A graph is called supermagic if it admits a labelling of edges by pairwise different consecutive positive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. In this paper we apply some effective methods to construct supermagic labellings of some triple graphs. Also, the application of supermagic graphs to balanced logistics network design in support of the United Nations Sustainable Development Goals is presented.

1. Introduction

We consider finite graphs without loops and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G. The subgraph of a graph G induced by a set $Z \subseteq E(G)$ is denoted by $G[Z]$. For integers p and q, we denote by $[p,q]$ the set of all integers z satisfying $p \le z \le q$.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every} \quad v \in V(G), \tag{1.1}$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index-mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all} \quad v \in V(G). \tag{1.2}$$

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A magic labelling f of G is called a *supermagic labelling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) whenever there exists a supermagic (magic) labelling of G.

A bijection f from $E(G)$ into $[1,|E(G)|]$ is called a *degree-magic labelling* (or only *d-magic labelling*) of a graph G if its index-mapping f^* satisfies

$$f^*(v) = \frac{1}{2}(1+|E(G)|)\deg(v) \quad \text{for all} \quad v \in V(G). \tag{1.3}$$

We say that a graph G is *degree-magic* (or only *d-magic*) when there exists a d-magic labelling of G.

The concept of magic graphs was put forward by Sedl*áč*ek [1]. Supermagic graphs were introduced by Stewart [2]. There is by now a considerable number of papers published on magic and supermagic graphs; we sigle out [3], [4] and [5] as being more particularly relevant to the present paper, and refer to [6] for more comprehensive references. The notion of degree-magic graphs was then introduced in [7] and see increasingly in [8], [9], [10] and [11]. Degree-magic graphs extend supermagic regular graphs because the following result holds.

Proposition 1.1. [7] *Let G be a regular graph. Then G is supermagic if and only if it is degree-magic.*

For an integer $q \geq 2$, a spanning subgraph H of a graph G is called a $\frac{1}{q}$*-factor* of G whenever $\deg_H(v) = \frac{1}{q}\deg_G(v)$ for every vertex $v \in V(G)$. A bijection f from $E(G)$ onto $[1,|E(G)|]$ is called *q-gradual* if the set

$$F_q(f;i) := \Big\{e \in E(G) : \frac{1}{q}(i-1)|E(G)| < f(e) \leq \frac{1}{q}i|E(G)|\Big\}$$

induces a $\frac{1}{q}$-factor of G for each $i \in [1,q]$. We can see some results on q-gradual labellings in [5]. A graph G is called *balanced degree-magic* if there exists a 2-gradual d-magic labelling of G. Some findings of balanced d-magic graphs were described in [12], [13] and [14]. However, the concept of a q-gradual labelling seems to be useful also for $q > 2$.

Observation 1.1. [5] *Let* $f : E(G) \longrightarrow [1,|E(G)|]$ *be a q-gradual bijection and let* α *be a permutation of* $[1,q]$. *Then* $g : E(G) \longrightarrow [1,|E(G)|]$ *defined by*

$$g(e) = f(e) + \frac{1}{q}\Big(\alpha(i) - i\Big)|E(G)| \quad \textit{when} \quad e \in F_q(f;i),$$

is a q-gradual bijection satisfying

(i) $g^*(v) = f^*(v)$, *for every vertex* $v \in V(G)$,

(ii) $F_q(g;\alpha(i)) = F_q(f;i)$, *for each* $i \in [1,q]$.

The graph obtained by replacing each edge uv of a graph G with 2 edges joining u and v is denoted by ${}^2G$. Hence, $V({}^2G) = V(G)$ and $E({}^2G) = \bigcup_{e\in E(G)}\{(e,1),(e,2)\}$, where an edge $(e,i), i \in \{1,2\}$, is incident with a vertex v in ${}^2G$ whenever e is incident with v in G. In this case, $E_i({}^2G) := \bigcup_{e\in E(G)}\{(e,i)\}, i = 1,2$. Obviously, the subgraph of ${}^2G$ induced by $E_i({}^2G)$ is isomorphic to G (see [5]).

In this paper we will use some successful methods to construct degree-magic and supermagic labellings of some triple graphs and the capitalization of supermagic graphs is indicated.

2. Triple Graphs

Let G be a graph and suppose that $Z \subseteq E(G)$. Define a graph $T(G;Z)$ by

$$V(T) = \bigcup_{v \in V(G)} \left\{v^1, v^2, v^3\right\}$$

and

$$E(T) = \bigcup_{uv \in Z} \left\{u^1v^1, u^2v^2, u^3v^3\right\} \cup \bigcup_{uv \in E(G)-Z} \left\{u^1v^2, u^2v^3, u^3v^1\right\}.$$

The graph $T(G;Z)$ is called a *triple graph* of a graph G. Note that $T(G;E(G))$ consists of three disjoint copies of G, i.e., it is isomorphic to $3G$.

Now we prove the vital results of this paper.

Lemma 2.1. *Let G be a graph such that* $\deg(v) \equiv 0 \pmod 3$ *for every vertex $v \in V(G)$. Suppose that the subgraph of G induced by $Z \subseteq E(G)$ has a $\frac{1}{3}$-factor. Then for any bijection $f : E(G) \longrightarrow [1, |E(G)|]$ there exists a* 3*-gradual bijection $g : E(T(G;Z)) \longrightarrow [1, 3|E(G)|]$ such that for every vertex $v \in V(G)$ it holds*

$$g^*(v^1) = g^*(v^2) = g^*(v^3) = f^*(v) + |E(G)| \deg(v).$$

Proof. Since the subgraph $G[Z]$ of a graph G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{3}$-factor, the degree of each vertex of $G[Z]$ has 3 as a factor. Thus, $G[Z]$ can be decomposed into 3 pairwise edge-disjoint $\frac{1}{3}$-factors. Similarly, the degree of each vertex of $H = G[E(G) - Z]$ also has 3 as a factor. Hence, H can be decomposed into 3 pairwise edge-disjoint $\frac{1}{3}$-factors. Let Z_1, Z_2, Z_3 be pairwise edge-disjoint $\frac{1}{3}$-factors of $G[Z]$ and let H_1, H_2, H_3 be pairwise edge-disjoint $\frac{1}{3}$-factors of H. Put $m := |E(G)|$ and $T := T(G;Z)$. Consider the bijection $g : E(T) \longrightarrow [1, 3m]$ given by

$$g(u^iv^j) = \begin{cases} f(uv) & \text{if} \quad i=1, j=1, uv \in E(Z_1), \\ f(uv)+m & \text{if} \quad i=2, j=2, uv \in E(Z_1), \\ f(uv)+2m & \text{if} \quad i=3, j=3, uv \in E(Z_1), \\ f(uv)+m & \text{if} \quad i=1, j=1, uv \in E(Z_2), \\ f(uv)+2m & \text{if} \quad i=2, j=2, uv \in E(Z_2), \\ f(uv) & \text{if} \quad i=3, j=3, uv \in E(Z_2), \\ f(uv)+2m & \text{if} \quad i=1, j=1, uv \in E(Z_3), \\ f(uv) & \text{if} \quad i=2, j=2, uv \in E(Z_3), \\ f(uv)+m & \text{if} \quad i=3, j=3, uv \in E(Z_3), \\ f(uv) & \text{if} \quad i=1, j=2, uv \in E(H_1), \\ f(uv)+m & \text{if} \quad i=2, j=3, uv \in E(H_1), \\ f(uv)+2m & \text{if} \quad i=3, j=1, uv \in E(H_1), \\ f(uv)+m & \text{if} \quad i=1, j=2, uv \in E(H_2), \\ f(uv)+2m & \text{if} \quad i=2, j=3, uv \in E(H_2), \\ f(uv) & \text{if} \quad i=3, j=1, uv \in E(H_2), \\ f(uv)+2m & \text{if} \quad i=1, j=2, uv \in E(H_3), \\ f(uv) & \text{if} \quad i=2, j=3, uv \in E(H_3), \\ f(uv)+m & \text{if} \quad i=3, j=1, uv \in E(H_3). \end{cases}$$

For its index-mapping we obtain

$$\begin{aligned} g^*(v^1) &= \sum_{vw \in E(Z_1)} g(v^1w^1) + \sum_{vw \in E(Z_2)} g(v^1w^1) + \sum_{vw \in E(Z_3)} g(v^1w^1) \\ &+ \sum_{vw \in E(H_1)} g(v^1w^2) + \sum_{vw \in E(H_2)} g(v^1w^2) + \sum_{vw \in E(H_3)} g(v^1w^2) \\ &= \sum_{vw \in E(Z_1)} f(vw) + \sum_{vw \in E(Z_2)} \big(f(vw)+m\big) + \sum_{vw \in E(Z_3)} \big(f(vw)+2m\big) \\ &+ \sum_{vw \in E(H_1)} f(vw) + \sum_{vw \in E(H_2)} \big(f(vw)+m\big) + \sum_{vw \in E(H_3)} \big(f(vw)+2m\big) \\ &= \sum_{vw \in E(G)} f(vw) + m \cdot \frac{1}{3}\deg(v) + 2m \cdot \frac{1}{3}\deg(v) = f^*(v) + m\deg(v) \end{aligned}$$

for every vertex $v^1 \in V(T)$. Similarly, $g^*(v^2) = g^*(v^3) = f^*(v) + m\deg(v)$ for all vertices $v^2, v^3 \in V(T)$. Since Z_1, Z_2, Z_3 are $\frac{1}{3}$-factors of $G[Z]$ and H_1, H_2, H_3 are $\frac{1}{3}$-factors of H, the sets

$$\begin{aligned} F_3(g;1) &= \left\{u^1v^1; uv \in E(Z_1)\right\} \cup \left\{u^3v^3; uv \in E(Z_2)\right\} \cup \left\{u^2v^2; uv \in E(Z_3)\right\} \\ &\cup \left\{u^1v^2; uv \in E(H_1)\right\} \cup \left\{u^3v^1; uv \in E(H_2)\right\} \cup \left\{u^2v^3; uv \in E(H_3)\right\}, \end{aligned}$$

$$\begin{aligned} F_3(g;2) &= \left\{u^2v^2; uv \in E(Z_1)\right\} \cup \left\{u^1v^1; uv \in E(Z_2)\right\} \cup \left\{u^3v^3; uv \in E(Z_3)\right\} \\ &\cup \left\{u^2v^3; uv \in E(H_1)\right\} \cup \left\{u^1v^2; uv \in E(H_2)\right\} \cup \left\{u^3v^1; uv \in E(H_3)\right\} \end{aligned}$$

and

$$\begin{aligned} F_3(g;3) &= \left\{u^3v^3; uv \in E(Z_1)\right\} \cup \left\{u^2v^2; uv \in E(Z_2)\right\} \cup \left\{u^1v^1; uv \in E(Z_3)\right\} \\ &\cup \left\{u^3v^1; uv \in E(H_1)\right\} \cup \left\{u^2v^3; uv \in E(H_2)\right\} \cup \left\{u^1v^2; uv \in E(H_3)\right\} \end{aligned}$$

induce $\frac{1}{3}$-factors of T. □

We say that a q-gradual bijection $f : E(G) \longrightarrow [1, |E(G)|]$ *respects* a set $Z(Z \subseteq E(G))$ if for each $i \in [1, q]$ either $F_q(f;i) \subseteq Z$ or $F_q(f;i) \subseteq E(G) - Z$. Evidently, a q-gradual bijection f respects a set Z if and only if there exists a subset $I \subset [1, q]$ such that $Z = \bigcup_{i \in I} F_q(f;i)$ (see [5]).

Lemma 2.2. *Let $q \geq 2$ be a positive integer and let G be a graph such that $\deg(v) \equiv 0 \pmod{3q}$ for every vertex $v \in V(G)$. If f is a q-gradual bijection from $E(G)$ onto $[1, |E(G)|]$ which respects a set $Z \subseteq E(G)$, then there exists a bijection g from $E(T(G;Z))$ onto $[1, 3|E(G)|]$ such that for every vertex $v \in V(G)$ it holds*

$$g^*(v^1) = g^*(v^2) = g^*(v^3) = f^*(v) + |E(G)| \deg(v).$$

Proof. Since f respects the set Z, according to Observation 1.1, we can assume that there is an integer $k \in [1, q]$ such that $Z = \bigcup_{i=1}^{k} F_q(f;i)$. As $\deg(v) \equiv 0 \pmod{3q}$ for every vertex $v \in V(G)$, the degree of each vertex of $G[F_q(f;i)]$ has 3 as a factor. Thus, $G[F_q(f;i)]$ can be decomposed into 3 pairwise edge-disjoint $\frac{1}{3}$-factors. Let H_{i1}, H_{i2}, H_{i3} be pairwise edge-disjoint $\frac{1}{3}$-factors of $G[F_q(f;i)]$. We can see that $\bigcup_{i=1}^{k} H_{i1}, \bigcup_{i=1}^{k} H_{i2}, \bigcup_{i=1}^{k} H_{i3}$ are $\frac{1}{3}$-factors of $G[Z]$. According to Lemma 2.1, there exists a desired bijection $g : E(T(G;Z)) \longrightarrow [1, 3|E(G)|]$. □

3. Degree-Magic and Supermagic Graphs

In this section we present some sufficient conditions for triple graphs to be degree-magic or supermagic.

Theorem 3.1. *Let G be a degree-magic graph such that $\deg(v) \equiv 0 \pmod 3$ for every vertex $v \in V(G)$. If the subgraph of G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{3}$-factor, then the graph $T(G;Z)$ is degree-magic.*

Proof. As G is a d-magic graph, there is a d-magic labelling f from $E(G)$ onto $[1, |E(G)|]$. According to Lemma 2.1, there exists a 3-gradual bijection $g : E(T(G;Z)) \longrightarrow [1, 3|E(G)|]$ satisfying

$$g^*(v^1) = g^*(v^2) = g^*(v^3) = f^*(v) + |E(G)| \deg(v)$$

for every vertex $v \in V(G)$. Since f is a d-magic labelling, $f^*(v) = \frac{1}{2}(1+|E(G)|)\deg(v)$. Hence,

$$\begin{aligned} g^*(v^1) &= g^*(v^2) = g^*(v^3) = \frac{1}{2}(1+|E(G)|)\deg(v) + |E(G)|\deg(v) \\ &= \frac{1}{2}(1+3|E(G)|)\deg(v) = \frac{1}{2}(1+|E(T(G;Z))|)\deg(v). \end{aligned}$$

Therefore, g is a d-magic labelling of $T(G;Z)$. □

Combining Proposition 1.1 and Theorem 3.1, we suddenly have

Corollary 3.1. *Let G be a supermagic regular graph such that* $\deg(v) \equiv 0 \pmod 3$ *for every vertex* $v \in V(G)$. *If the subgraph of G induced by a set* $Z \subseteq E(G)$ *has a* $\frac{1}{3}$*-factor, then the graph* $T(G;Z)$ *is supermagic.*

Corollary 3.1 provides a method to construct supermagic graphs. For example, the complete bipartite graph $K_{6,6}$ is supermagic and contains non-isomorphic subgraphs having a $\frac{1}{3}$-factor. By Corollary 3.1, the graph $T(K_{6,6};E(H))$ is supermagic for each such subgraph H.

A totally disconnected graph has a $\frac{1}{3}$-factor. By using Corollary 3.1, we get

Corollary 3.2. *Let G be a supermagic regular graph such that* $\deg(v) \equiv 0 \pmod 3$ *for every vertex* $v \in V(G)$. *Then the triple graph* $T(G;\emptyset)$ *of a graph G is supermagic.*

Since the graph $3G$ is isomorphic to $T(G;E(G))$, by using Corollary 3.1, we have the next corollary.

Corollary 3.3. *Let G be a supermagic regular graph such that* $\deg(v) \equiv 0 \pmod 3$ *for every vertex* $v \in V(G)$. *Then the graph* $3G$ *is supermagic.*

For the graph ^2G, we get the next result.

Corollary 3.4. *Let G be a graph having a* $\frac{1}{3}$*-factor. Then the graph* $T(^2G;E_1(^2G))$ *of a graph G is degree-magic.*

Proof. Let h be a bijection from $E(G)$ onto $[1,|E(G)|]$. Consider a mapping $f : E(^2G) \longrightarrow [1,2|E(G)|]$ given by

$$f((e,j)) = \begin{cases} h(e) & \text{if} \quad j=1, \\ 1+2|E(G)|-h(e) & \text{if} \quad j=2. \end{cases}$$

Evidently, f is a bijection. Moreover, $f((e,1)) + f((e,2)) = 1+2|E(G)|$ for every edge $e \in E(G)$. Therefore,

$$f^*(v) = (1+2|E(G)|)\deg_G(v) = \frac{1}{2}(1+|E(^2G)|)\deg_{^2G}(v).$$

Hence, f is a degree-magic labelling of ^2G. Since the subgraph of ^2G induced by $E_1(^2G)$ is isomorphic to G, it contains a $\frac{1}{3}$-factor. By Theorem 3.1, $T(^2G;E_1(^2G))$ is a d-magic graph. □

Combining Proposition 1.1 and Corollary 3.4, we suddenly have

Corollary 3.5. *Let G be a regular graph having a $\frac{1}{3}$-factor. Then the triple graph $T(^2G; E_1(^2G))$ of a graph G is supermagic.*

In Theorem 3.1, $G[Z]$ is any subgraph of a graph G. However, $G[Z]$ must be a spanning subgraph of G in the following result.

Theorem 3.2. *Let $q \geq 2$ be a positive integer and let G be a graph such that $\deg(v) \equiv 0 \pmod{3q}$ for every vertex $v \in V(G)$. If G admits a q-gradual degree-magic labelling which respects a set $Z \subseteq E(G)$, then the graph $T(G; Z)$ is degree-magic.*

Proof. Suppose that f is a q-gradual d-magic labelling of G which respects a set Z. According to Lemma 2.2, there exists a bijection g from $E(T(G; Z))$ onto $[1, 3|E(G)|]$ satisfying

$$g^*(v^1) = g^*(v^2) = g^*(v^3) = f^*(v) + |E(G)| \deg(v),$$

for every vertex $v \in V(G)$. Since f is a d-magic labelling, $f^*(v) = \frac{1}{2}(1 + |E(G)|) \deg(v)$. Thus,

$$\begin{aligned} g^*(v^1) &= g^*(v^2) = g^*(v^3) = \frac{1}{2}(1 + |E(G)|) \deg(v) + |E(G)| \deg(v) \\ &= \frac{1}{2}(1 + 3|E(G)|) \deg(v) = \frac{1}{2}(1 + |E(T(G; Z))|) \deg(v). \end{aligned}$$

Therefore, g is a d-magic labelling of $T(G; Z)$. □

For the graph ^2G, we have the next finding.

Corollary 3.6. *Let $q \geq 2$ be a positive integer and let G be a graph such that $\deg(v) \equiv 0 \pmod 3$ for every vertex $v \in V(G)$. If G can be decomposed into q pairwise edge-disjoint $\frac{1}{q}$-factors, then the triple graph $T(^2G; E_1(^2G))$ of a graph G is degree-magic.*

Proof. Evidently, $\deg(v) \equiv 0 \pmod{3q}$ for every vertex $v \in V(G)$. Let $H_1, H_2, ..., H_q$ be q pairwise edge-disjoint $\frac{1}{q}$-factors of G. Put $m = \frac{1}{q}|E(G)|$. Clearly, the subgraph $H_i, i \in [1, q]$, has m edges. Suppose that h_i is a bijection from $E(H_i)$ onto $[1, m]$ for $i \in [1, q]$. Consider a mapping $f : E(^2G) \longrightarrow \big[1, 2qm\big]$ given by

$$f((e, j)) = \begin{cases} h_i(e) + (i-1)m & \text{if} \quad j = 1 \quad \text{and} \quad e \in E(H_i), \\ 1 + (1 + 2q - i)m - h_i(e) & \text{if} \quad j = 2 \quad \text{and} \quad e \in E(H_i). \end{cases}$$

Obviously, f is a bijection and $f((e, 1)) + f((e, 2)) = 1 + 2qm$ for every edge $e \in E(G)$. Therefore,

$$f^*(v) = (1 + 2qm) \deg(v) = \frac{1}{2}(1 + |E(^2G)|) \deg_{^2G}(v).$$

Moreover, for $i \in [1, q]$, we have $F_{2q}(f; i) = \{(e, 1) \in E(^2G) : e \in E(H_i)\}$ and $F_{2q}(f; q + i) = \{(e, 2) \in E(^2G) : e \in E(H_{1+q-i})\}$. Thus, the mapping f is a $2q$-gradual degree-magic labelling of ^2G which respects the set $E_1(^2G)$. By Theorem 3.2, $T(^2G; E_1(^2G))$ is a d-magic graph. □

Since any regular graph of degree d, where $6 \leq d \equiv 0 \pmod 3$, can be decomposed into $\frac{d}{3}$ pairwise edge-disjoint $\frac{1}{d/3}$-factors, we immediately have

Corollary 3.7. *Let G be a regular graph of degree d, where $6 \leq d \equiv 0 \pmod 3$. Then the triple graph $T(^2G; E_1(^2G))$ of a graph G is supermagic.*

4. The Application

Transportation and logistics systems play a fundamental role in modern supply chains because they enable the continuous movement of goods between production facilities, storage warehouses, and end customers. Effective collaboration among logistics hubs, such as distribution centers, delivery vehicles, and cross-dock facilities, is essential in order to reduce transportation costs, minimize delivery delays, and maintain a consistently high level of service quality. One of the primary challenges in these systems is the *issue of load balancing*. In many cases, certain hubs or vehicles may become overloaded with excessive delivery assignments, while other hubs or vehicles remain significantly under-utilized. This imbalance creates serious operational inefficiencies, including traffic congestion, increased fuel consumption, longer delivery times, and higher overall operating expenses. Consequently, the development of a mathematical framework that can support fair and equitable distribution of workload among all participating hubs is crucial for achieving resilient, reliable, and optimally functioning logistics networks.

Graph theory provides an effective and structured foundation for modeling collaboration within transportation and logistics systems, where each vertex represents a logistics hub or transport unit and each edge signifies a delivery route or task-sharing connection. Within this framework, a supermagic graph is defined by a distinct edge-labelling scheme using consecutive positive integers such that the sum of all incident edge labels at every vertex is identical. When applied to logistics operations, these numerical labels can represent practical workload indicators such as shipment quantities, transportation costs, or delivery frequencies, ensuring that each hub manages an equal share of operational responsibilities. This balanced workload distribution prevents both overload and underutilization, leading to improved fairness, reduced congestion, and enhanced overall efficiency in the network. Furthermore, the uniformity enforced by supermagic labelling promotes structural stability, supports adaptive and resource-efficient planning, and contributes to long-term sustainability in supply chain infrastructure. As a result, supermagic graph–based network designs align closely with several United Nations Sustainable Development Goals, including SDG 9 on resilient infrastructure, SDG 11 on sustainable urban mobility, SDG 12 on responsible resource utilization, and SDG 13 related to climate action. Thus, integrating supermagic graph principles into logistics and supply chain modeling not only advances theoretical research in graph labelling but also provides a valuable pathway toward environmentally sustainable and socially equitable global logistics networks.

To demonstrate the real-world meaning of a supermagic graph, consider a *regional parcel delivery network* consisting of six distribution hubs. These hubs are divided into two groups of three, for example: three main city hubs and three suburban hubs. Goods are transported only between hubs in different groups, forming a bipartite logistics network. This system can be modeled mathematically by the complete bipartite graph $K_{3,3}$. Each transportation route (edge) between hubs is assigned a unique workload value from 1 to 9, representing measures such as shipping volume, transport cost, or freight capacity. If the labels are assigned so that the total workload of all routes incident to each hub is the same such as a balanced sum of 15 units per hub, then the network satisfies the definition of a supermagic graph. In this case, every hub participates in the logistics system equally, preventing any single location from becoming overburdened while others remain underutilized. Such balanced freight distribution strategies reflect real practices in logistics collaboration to maintain fair workload distribution, reduce congestion, and ensure fast and reliable service. Moreover, when delivery conditions change over time, supermagic-based labelling can be adapted dynamically, enabling logistics planners to rebalance hub workloads while preserving efficiency and preventing future bottlenecks.

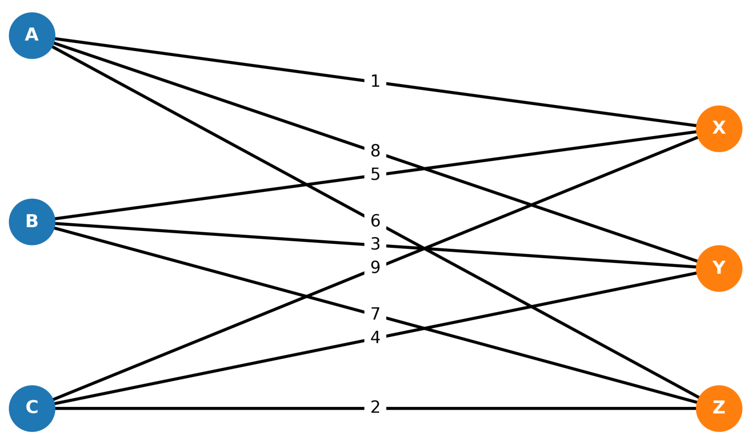

Figure 1. This figure shows a supermagic edge labelling of the graph $K_{3,3}$ used to model a balanced logistics network. Each edge label represents a unique route workload, and every hub receives a total workload of 15, ensuring perfect balance across all hubs.

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