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Application of Fixed Point Results via Weakly Asymptotic Type Contractions

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Abstract. In this article, we present a new class of asymptotic contractions in the context of the usual metric spaces. We establish a few fixed point theorems related to our newly introduced weakly asymptotic Ciric type contractions. Also, we provide a constructive numerical example to support our main results. Moreover, we use our results to establish the existence of solutions for a specific kind of cantilever beam problem: a fourth-order two-point boundary value problem.

1. Introduction

The classical Banach contraction principle stands out as a highly valuable discovery in nonlinear analysis. Over the past century, a great deal of work has been done by mathematicians to obtain plenty of contractions, which they have used to discover a number of significant fixed points. One of the earliest versions of Banach's principle, credited to Caccioppoli [8], suggests the idea of "asymptotic contractions".

In 2003, Kirk [13] further developed this concept by introducing asymptotic contraction mappings, proving that such a mapping Ω on a complete metric space Z has a unique fixed point, provided that Ω has a bounded orbit, thereby broadening the scope of Banach contraction mapping.

Following this, Xu [17] introduced the notion of weak asymptotic contraction, demonstrating that such contractions, when operating on a complete metric space with a bounded orbit, ensure the existence of a unique fixed point using the ultra product technique. Numerous concepts associated with such types of contractions have been studied by renowned mathematicians in various ways, and consequently, many interesting results are provided in the literature (see [1,2,5,6,11,14–16]).

Later, Goyal [10] generalized Xu's result [17] by relaxing the continuity requirement. He demonstrated that the result remains valid when continuity is replaced by upper semicontinuity. A few

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years ago, Bera et al. [7] established the notion of asymptotic contraction in pairs, both for two mappings and for a finite family of mappings. Recently, Karmakar et al. [12] proposed two distinct forms of asymptotic contractions, referred to as Reich type and Chatterjea type weak asymptotic contractions.

Based on these motivations, we introduce a generalized type of asymptotic contraction: weakly asymptotic Ciric type contraction, which is more generalised version of other existing asymptotic type of contractions. We also establish the existence of a fixed point for this new contraction, supported by a relevant example. Finally, we apply our result to solve a special case of a cantilever beam problem, along with a numerical illustration.

2. Preliminaries

To present our findings, we need the following essential definitions and results.

Let (Z, d) be a metric space, where $d : Z \times Z \to [0, \infty)$ is a metric on the nonempty set Z. Let Ψ denote the collection of functions comprising all functions $\psi : [0, \infty) \to [0, \infty)$ which are upper semi-continuous and satisfying $\psi(w) < w$ for $w > 0$, and $\psi(0) = 0$.
Consider any function $\psi \in \Psi$, the function $\tilde{\psi}$ is defined as

$$\tilde{\psi}(t) = \max\Big\{\psi(w) : w \in [0, t] \cap \overline{R(d)}\Big\},$$

where $R(d) = \{d(i, j) : i, j \in Z\}$ denotes the range of d, and $\overline{R(d)}$ denotes its closure.
Following three properties of $\tilde{\psi}$ are due to Goyal [10]:

(1) $\tilde{\psi}$ is upper semi-continuous;
(2) $\tilde{\psi}(t) < t \;\forall\; t > 0$;
(3) $\tilde{\psi}$ is increasing.

Definition 2.1. *[9] In a metric space (Z, d), consider a mapping Ω from Z to itself. The orbit of Ω at a specific point $o \in Z$ is denoted as $O_o(\Omega)$ and defined as $O_o(\Omega) = \Big\{o, \Omega o, \Omega^2 o, \Omega^3 o, \cdots\Big\}$. If the set $O_o(\Omega)$ is bounded within the metric space (Z, d), then we say that Ω has a bounded orbit at $o \in Z$.*

Definition 2.2. *[9] Suppose $\Omega : Z \to Z$ is a self-mapping on the metric space (Z, d). If for any $o \in Z$ and for every sequence $\{p_r\}$ in the orbit $O_o(\Omega)$, such that $p_r \to s$ implies $\Omega p_r \to \Omega p$ as r approaches to infinity, then Ω is known as orbitally continuous.*

Definition 2.3. *Consider a metric space (Z, d) that is complete. A map $\Omega : Z \to Z$ is called a contraction if*

$$d(\Omega o, \Omega p) \leq \lambda d(o, p), \text{ for all } o, p \in Z, \tag{2.1}$$

where $\lambda \in [0, 1)$ is a constant.

Banach [3] established that every contraction on a complete metric space has a unique fixed point.

Definition 2.4. *[13] Suppose a metric space (Z, d) that is complete. A self-mapping $\Omega : Z \to Z$ is called an asymptotic contraction if there exists a family of functions $\{\psi_r\}_{r\geq 1}$ where each $\psi_r : [0, \infty) \to [0, \infty)$ for all $o, p \in Z$ and all $r \geq 1$*

$$d(\Omega^r o, \Omega^r p) \leq \psi_r(d(o, p)), \tag{2.2}$$

$\sum_{r=1}^{\infty} \psi_r < \infty$ and if $\psi_r \to \psi \in \Psi$ uniformly on the range of d.

Kirk [13] showed that an asymptotic contraction on a complete metric space has a unique fixed point, provided some orbit of Ω is bounded.

Definition 2.5. *[17] Consider a complete metric space (Z, d). A continuous mapping $\Omega : Z \to Z$ is said to be a weakly asymptotic contraction provided that, for any given $\delta > 0$, there exists an integer $r_\delta \geq 1$ such that*

$$d(\Omega^{r_\delta}(o), \Omega^{r_\delta}(p)) \leq \psi(d(o, p)) + \delta \tag{2.3}$$

for all $o, p \in Z$, where $\psi \in \Psi$.

Xu [17] proved that in a complete metric space, a weakly asymptotic contraction has unique fixed point if it has a bounded orbit.

Definition 2.6. *[4] A self-map Ω is called asymptotically regular if for each $r \in \Omega$,*

$$\lim_{r\to\infty} d(\Omega^{r+1} r, \Omega^r r) = 0.$$

Definition 2.7. *[17] Let (Z, d) be a complete metric space. A self-map $\Omega : Z \to Z$ is called a limiting asymptotic contraction if*

$$\limsup_{r\to\infty} d(\Omega^r r, \Omega^r s) \leq \psi(d(r, s)), \tag{2.4}$$

for all $r, s \in Z$, where $\psi \in \Psi$ is given function.

3. Result Discussion

In this section, we first introduce the concept of weakly asymptotic Ciric type contraction.

Definition 3.1. *Consider a metric space (Z, d) that is complete. A continuous map $\Omega : Z \to Z$ is termed a weakly asymptotic Ciric type contraction if for a given $\delta > 0$, there exists an integer $r_\delta \geq 1$ such that*

$$d(\Omega^{r_\delta} o, \Omega^{r_\delta} p) \leq \psi\Big[a_1 d(o, p) + a_2 d(o, \Omega o) + a_3 d(p, \Omega p) + a_4 d(o, \Omega p) + a_5 d(p, \Omega o)\Big] + \delta \tag{3.1}$$

for all $o, p \in Z$, where the constants satisfy $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $a_1 + a_2 + a_3 + 2a_4 + a_5 \leq 1$.

(1) If $a_1 = 1$ and all other coefficients are zero, then Ω reduces to the weakly asymptotic contraction.
(2) If $a_4 = a_5 = 0$, then Ω is a Reich type weakly asymptotic contraction (see [12]).
(3) If $a_2 = a_3 = 0$, then Ω is a Chatterjea type weakly asymptotic contraction (see [12]).

Before proving the main theorem, we establish the connection between the limiting asymptotic contraction and asymptotically regular properties.

Theorem 3.1. *Assume the metric space (Z, d) is complete. If a continuous map $\Omega : Z \to Z$ is a limiting asymptotic contraction, then Ω is asymptotically regular.*

Proof. Let $o \in Z$ be an arbitrary element. Since Ω is a limiting asymptotic contraction by Theorem 4 in [17], we get

$$\lim_{r\to\infty} d(\Omega^r o, \Omega^r p) = 0 \tag{3.2}$$

for all $o, p \in Z$. Now, we have,

$$d(\Omega^r o, \Omega^{r+1} o) = d(\Omega^r o, \Omega^r(\Omega o)).$$

Taking the limit as $r \to \infty$ and by (3.2) we get

$$\lim_{r\to\infty} d(\Omega^r o, \Omega^{r+1} o) = 0.$$

Since o is chosen arbitrarily, Ω is asymptotically regular. □

The following result establishes the existence and uniqueness of a fixed point for weakly asymptotic Ciric contractions.

Theorem 3.2. *Assume (Z, d) is a complete metric space. If a continuous map $\Omega : Z \to Z$ is a weakly asymptotic Ciric contraction and Ω is orbitally bounded, then it has exactly one fixed point.*

Proof. Let o be a arbitrary but fixed element of Z and let us consider the Picard sequence $o_1 = \Omega o, o_2 = \Omega o_1$ continuing in this way the r^{th} term of the sequence is $o_r = \Omega o_{r-1} = \Omega^r o$.
Since Ω is orbitally bounded we get the following:

$$A = \limsup_{r\to\infty} d(o_r, o_{r+1}) < \infty. \tag{3.3}$$

Since Ω is a weakly asymptotic Ciric type contraction we have

$$\begin{aligned} d(\Omega^{r_\delta} o_r, \Omega^{r_\delta} o_{r+1}) &\leq \psi\Big[a_1 d(o_r, o_{r+1}) + a_2 d(o_r, \Omega o_r) + a_3 d(o_{r+1}, \Omega o_{r+1}) + a_4 d(o_r, \Omega o_{r+1}) + a_5 d(o_{r+1}, \Omega o_r)\Big] + \delta \\ &= \tilde{\psi}\Big[a_1 d(o_r, o_{r+1}) + a_2 d(o_r, o_{r+1}) + a_3 d(o_{r+1}, o_{r+2}) + a_4 d(o_r, o_{r+2}) + a_5 d(o_{r+1}, o_{r+1})\Big] + \delta. \end{aligned}$$

By applying the triangle inequality and the increasing property of $\tilde{\psi}$ we get the following

$$d(\Omega^{r_\delta} o_r, \Omega^{r_\delta} o_{r+1}) \leq \tilde{\psi}\Big[a_1 d(o_r, o_{r+1}) + a_2 d(o_r, o_{r+1}) + a_3 d(o_{r+1}, o_{r+2}) + a_4\Big(d(o_r, o_{r+1}) + d(o_{r+1}, o_{r+2})\Big)\Big] + \delta.$$

In the previous equation, if we take $\limsup_{r\to\infty}$ on both sides, we obtain

$$\begin{aligned} \limsup_{r\to\infty} d(\Omega^{r_\delta} o_r, \Omega^{r_\delta} o_{r+1}) \leq \limsup_{r\to\infty} \tilde{\psi}\Big[&a_1 d(o_r, o_{r+1}) + a_2 d(o_r, o_{r+1}) + a_3 d(o_{r+1}, o_{r+2}) \\ &+ a_4\Big(d(o_r, o_{r+1}) + d(o_{r+1}, o_{r+2})\Big)\Big] + \delta. \end{aligned}$$

Thus,

$$A \leq \tilde{\psi}\Big[(a_1 + a_2 + a_3 + 2a_4)A\Big] + \delta.$$

But $\delta > 0$ is arbitrary, therefore

$$A \leq \tilde{\psi}\Big[(a_1 + a_2 + a_3 + 2a_4)A\Big].$$

Since $a_1 + a_2 + a_3 + 2a_4 \leq 1 - a_5 < 1$ and by the property of $\tilde{\psi}$ we have

$$\limsup_{r\to\infty} d(\Omega^{r_\delta} o_r, \Omega^{r_\delta} o_{r+1}) = 0. \tag{3.4}$$

Now, for all $r, s \geq 1$, let us define

$$d_{r,s} := d(\Omega^r o, \Omega^s o).$$

Define,

$$d_\infty = \limsup_{r,s\to\infty} d_{r,s} = \lim_{k\to\infty} \sup\Big\{d_{r,s} : \ r, s \geq k\Big\} < \infty. \tag{3.5}$$

Since Ω is a weakly asymptotic Ciric type contraction, therefore for a given $\delta > 0$ there exists a $r_\delta \in \mathbb{N}$ such that

$$\begin{aligned}
d_{r,s} = d(\Omega^r o, \Omega^s o) &= d(\Omega^{r_\delta}(\Omega^{r-r_\delta} o), \Omega^{r_\delta}(\Omega^{s-r_\delta} o)) \\
&\leq \psi\Big\{a_1 d(\Omega^{r-r_\delta} o, \Omega^{s-r_\delta} o) + a_2 d(\Omega^{r-r_\delta} o, \Omega(\Omega^{r-r_\delta} o)) + a_3 d(\Omega^{s-r_\delta} o, \Omega(\Omega^{s-r_\delta} o)) \\
&\qquad + a_4 d(\Omega^{r-r_\delta} o, \Omega(\Omega^{s-r_\delta} o)) + a_5 d(\Omega^{s-r_\delta} o, \Omega(\Omega^{r-r_\delta} o))\Big\} + \delta.
\end{aligned}$$

From the above equation we get,

$$\begin{aligned}
d(\Omega^{r_\delta}(\Omega^{r-r_\delta} o), \Omega^{r_\delta}(\Omega^{s-r_\delta} o)) &\leq \tilde{\psi}\Big\{a_1 d(\Omega^{r-r_\delta} o, \Omega^{s-r_\delta} o) + a_2 d(\Omega^{r-r_\delta} o, \Omega^{r-r_\delta+1} o) + a_3 d(\Omega^{s-r_\delta} o, \Omega^{s-r_\delta+1} o) \\
&\qquad + a_4 d(\Omega^{r-r_\delta} o, \Omega^{s-r_\delta+1} o) + a_5 d(\Omega^{s-r_\delta} o, \Omega^{r-r_\delta+1} o)\Big\} + \delta.
\end{aligned}$$

By triangle inequality and the property of $\tilde{\psi}$

$$\begin{aligned}
d(\Omega^{r_\delta}(\Omega^{r-r_\delta} o), \Omega^{r_\delta}(\Omega^{s-r_\delta} o)) &\leq \tilde{\psi}\Big\{a_1 d(\Omega^{r-r_\delta} o, \Omega^{s-r_\delta} o) + a_2 d(\Omega^{r-r_\delta} o, \Omega^{r-r_\delta+1} o) + a_3 d(\Omega^{s-r_\delta} o, \Omega^{s-r_\delta+1} o) \\
&\qquad + a_4 \Big[d(\Omega^{r-r_\delta} o, \Omega^{r-r_\delta+1} o) + d(\Omega^{r-r_\delta+1} o, \Omega^{s-r_\delta+1} o)\Big] \\
&\qquad + a_5 d(\Omega^{s-r_\delta} o, \Omega^{r-r_\delta+1} o)\Big\} + \delta.
\end{aligned}$$

Applying $\limsup\limits_{r,s\to\infty}$ on both sides, we have

$$\begin{aligned}
\limsup_{r,s\to\infty} d(\Omega^{r_\delta}(\Omega^{r-r_\delta} o), \Omega^{r_\delta}(\Omega^{s-r_\delta} o)) &\leq \limsup_{r,s\to\infty} \tilde{\psi}\Big\{a_1 d(\Omega^{r-r_\delta} o, \Omega^{s-r_\delta} o) + a_2 d(\Omega^{r-r_\delta} o, \Omega^{r-r_\delta+1} o) \\
&\qquad + a_3 d(\Omega^{s-r_\delta} o, \Omega^{s-r_\delta+1} o) + a_4\Big(d(\Omega^{r-r_\delta} o, \Omega^{r-r_\delta+1} o) \\
&\qquad + d(\Omega^{r-r_\delta+1} o, \Omega^{s-r_\delta+1} o)\Big) + a_5 d(\Omega^{s-r_\delta} o, \Omega^{r-r_\delta+1} o)\Big\} + \delta.
\end{aligned}$$

Using (3.4) we get

$$\limsup_{r,s\to\infty} d(\Omega^{r_\delta}(\Omega^{r-r_\delta} o), \Omega^{r_\delta}(\Omega^{s-r_\delta} o)) \leq \limsup_{r,s\to\infty} \tilde{\psi}\Big\{a_1 d(\Omega^{r-r_\delta} o, \Omega^{s-r_\delta} o) + a_4\Big(d(\Omega^{r-r_\delta} o, \Omega^{r-r_\delta+1} o)$$

$$+ d(\Omega^{r-r_\delta+1}o, \Omega^{s-r_\delta+1}o)\Big) + a_5 d(\Omega^{s-r_\delta}o, \Omega^{r-r_\delta+1}o)\Big\} + \delta.$$

Therefore, by (3.5)

$$d_\infty \leq \tilde{\psi}\Big\{(a + a_4 + a_5)d_\infty\Big\} + \delta.$$

Since δ is arbitrary and $a_1 + a_2 + a_3 + 2a_4 + a_5 \leq 1$, we have

$$\limsup_{r\to\infty} d(\Omega^r o, \Omega^s o) = 0. \tag{3.6}$$

From the above, we can conclude that the sequence $\{\Omega^r o\}$ is a Cauchy. Given that (Z, d) is a complete metric space, the sequence $\{\Omega^r o\} \to \rho$ in Z. By the continuity of Ω, it implies that ρ is a fixed point of Ω. The uniqueness is easily follows from the contraction. □

Theorem 3.3. *Consider a metric space (Z, d) that is complete and ψ is a contractive gauge function. Assuming $\Omega : Z \to Z$ has an orbitally bounded at $o \in Z$ and orbitally continuous. Moreover, for each $\delta > 0$, there exists a $r_\delta \in \mathbb{N}$ satisfying*

$$d(\Omega^{r_\delta}o, \Omega^{r_\delta}p) \leq \psi\ \max\Big\{d(o,p), d(o,\Omega o), d(p,\Omega p), d(o,\Omega p), d(p,\Omega o)\Big\} + \delta, \tag{3.7}$$

for all $o, p \in Z$, then Ω has exactly one fixed point and sequence $\{\Omega^r o\}$ converges to it.

Proof. Consider an arbitrary fixed element $o \in Z$ and generate a sequence $\{o_r\}$ defined as $o_r = \Omega^r o$ for all $r \in \mathbb{N}$. Given that Ω is orbitally bounded, we get $A = \limsup_{r\to\infty} d(o_r, o_{r+1}) < \infty$. Let $\delta > 0$ be arbitrary. According to (3.7), for all $o, p \in Z$ there exists a $r_\delta \in \mathbb{N}$ such that

$$\begin{aligned} d(\Omega^{r_\delta}o_r, \Omega^{r_\delta}o_{r+1}) &\leq \psi\ \max\Big\{d(o_r,o_{r+1}), d(o_r,\Omega o_r), d(o_{r+1},\Omega o_{r+1}), d(o_r,\Omega o_{r+1}), d(o_{r+1},\Omega o_r)\Big\} + \delta \\ &\leq \psi\ \max\Big\{d(o_r,o_{r+1}), d(o_r,o_{r+2}), d(o_{r+1},o_{r+2})\Big\} + \delta. \end{aligned}$$

Now, we discuss the following cases:

Case 1: If $\max\Big\{d(o_r,o_{r+1}), d(o_r,o_{r+2}), d(o_{r+1},o_{r+2})\Big\} = d(o_r,o_{r+1})$, then we have

$$d(\Omega^{r_\delta}o_r, \Omega^{r_\delta}o_{r+1}) \leq \psi d(o_r,o_{r+1}) + \delta.$$

By increasing property of $\tilde{\psi}$ we get,

$$d(\Omega^{r_\delta}o_r, \Omega^{r_\delta}o_{r+1}) \leq \tilde{\psi} d(o_r,o_{r+1}) + \delta.$$

Taking $\limsup_{r\to\infty}$ on both sides, we obtain

$$\limsup_{r\to\infty} d(\Omega^{r_\delta}o_r, \Omega^{r_\delta}o_{r+1}) \leq \tilde{\psi} \limsup_{r\to\infty} d(o_r,o_{r+1}) + \delta,$$

which implies, $A \leq \tilde{\psi}(A) + \delta$. As $\delta > 0$, we have $A = 0$.

Case 2: If $\max\Big\{d(o_r,o_{r+1}), d(o_r,o_{r+2}), d(o_{r+1},o_{r+2})\Big\} = d(o_r,o_{r+2})$, then we have

$$d(\Omega^{r_\delta}o_r, \Omega^{r_\delta}o_{r+1}) \leq \psi d(o_r,o_{r+2}) + \delta.$$

By triangular inequality,

$$d(\Omega^{r_\delta}o_r,\Omega^{r_\delta}o_{r+1})\leq\psi(d(o_r,o_{r+1})+d(o_{r+1},o_{r+2}))+\delta.$$

By increasing property of $\tilde{\psi}$ we get,

$$d(\Omega^{r_\delta}o_r,\Omega^{r_\delta}o_{r+1})\leq\tilde{\psi}(d(o_r,o_{r+1})+d(o_{r+1},o_{r+2}))+\delta.$$

Taking $\limsup\limits_{r\to\infty}$ on both sides, we obtain

$$\limsup_{r\to\infty}d(\Omega^{r_\delta}o_r,\Omega^{r_\delta}o_{r+1})\leq\tilde{\psi}\limsup_{r\to\infty}(d(o_r,o_{r+1})+d(o_{r+1},o_{r+2}))+\delta,$$

which implies, $A\leq\tilde{\psi}(2A)+\delta$. As $\delta>0$, we have $A=0$.

Case 3: If $\max\Big\{d(o_r,o_{r+1}),d(o_r,o_{r+2}),d(o_{r+1},o_{r+2})\Big\}=d(o_{r+1},o_{r+2})$, then we have

$$d(\Omega^{r_\delta}o_r,\Omega^{r_\delta}o_{r+1})\leq\psi d(o_{r+1},o_{r+2})+\delta.$$

By increasing property of $\tilde{\psi}$ we get,

$$d(\Omega^{r_\delta}o_r,\Omega^{r_\delta}o_{r+1})\leq\tilde{\psi}d(o_{r+1},o_{r+2})+\delta.$$

By taking $\limsup\limits_{r\to\infty}$ in both sides we get

$$\limsup_{r\to\infty}d(\Omega^{r_\delta}o_r,\Omega^{r_\delta}o_{r+1})\leq\tilde{\psi}\limsup_{r\to\infty}d(o_{r+1},o_{r+2})+\delta,$$

which implies, $A\leq\tilde{\psi}(A)+\delta$. As $\delta>0$, we have $A=0$. From all the above cases, we get,

$$\limsup_{r\to\infty}d(\Omega^{r_\delta}o_r,\Omega^{r_\delta}o_{r+1})=0. \tag{3.8}$$

Now, we assume $d_{r,s}=d(\Omega^r o,\Omega^s o)\ \ \forall\, r,s\geq 1$ and

$$d_\infty=\limsup_{r,s\to\infty}d_{r,s}=\limsup_{k\to\infty}\Big\{d_{r,s}:r,s\geq k\Big\}.$$

If $\delta>0$ is arbitrarily chosen, then for all $r,s\in\mathbb{N}$ there exists a $r_\delta\in\mathbb{N}$ such that

$$\begin{aligned}
d_{r,s}&=d(\Omega^r o,\Omega^s o)=d(\Omega^{r_\delta o}(\Omega^{r-r_\delta}o),\Omega^{r_\delta}(\Omega^{s-r_\delta}))\\
&\leq\psi\max\Big\{d(\Omega^{r-r_\delta}o,\Omega^{s-r_\delta}o),d(\Omega^{r-r_\delta}o,\Omega(\Omega^{r-r_\delta}o)),d(\Omega^{s-r_\delta}o,\Omega(\Omega^{s-r_\delta}o)),d(\Omega^{r-r_\delta}o,\Omega(\Omega^{s-r_\delta}o)),\\
&\qquad d(\Omega^{s-r_\delta}o,\Omega(\Omega^{r-r_\delta}o))\Big\}+\delta\\
&=\psi\max\Big\{d(o_{r-r_\delta},o_{s-r_\delta}),d(o_{r-r_\delta},o_{r-r_\delta+1}),d(o_{s-r_\delta},o_{s-r_\delta+1}),d(o_{r-r_\delta},o_{s-r_\delta+1}),d(o_{s-r_\delta},o_{r-r_\delta+1})\Big\}+\delta\\
&\leq\tilde{\psi}\max\Big\{d(o_{r-r_\delta},o_{s-r_\delta}),d(o_{r-r_\delta},o_{r-r_\delta+1}),d(o_{s-r_\delta},o_{s-r_\delta+1}),d(o_{r-r_\delta},o_{s-r_\delta+1}),d(o_{s-r_\delta},o_{r-r_\delta+1})\Big\}+\delta\\
d_{r,s}&\leq\tilde{\psi}\max\Big\{d_{r-r_\delta,s-r_\delta},d_{r-r_\delta,r-r_\delta+1},d_{s-r_\delta,s-r_\delta+1},d_{r-r_\delta,s-r_\delta+1},d_{s-r_\delta,r-r_\delta+1}\Big\}+\delta.
\end{aligned}$$

Consider,

$$\limsup_{r,s\to\infty}\ \tilde{\psi}\max\Big\{d_{r-r_\delta,s-r_\delta},d_{r-r_\delta,r-r_\delta+1},d_{s-r_\delta,s-r_\delta+1},d_{r-r_\delta,s-r_\delta+1},d_{s-r_\delta,r-r_\delta+1}\Big\}$$

$$\leq \tilde{\psi}\Big(\limsup_{r,s\to\infty} \max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\}\Big)$$

which implies,

$$\limsup_{r,s\to\infty}\ \tilde{\psi}\max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\}$$

$$\leq \tilde{\psi}\ \limsup_{k\to\infty}\ \max\begin{Bmatrix} d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, \\ d_{r-r_\delta,r-r_\delta+1} + d_{r-r_\delta+1,s-r_\delta+1}, \\ d_{s-r_\delta,s-r_\delta+1} + d_{s-r_\delta+1,r-r_\delta+1} : \\ r-r_\delta+1, s-r_\delta+1 \geq k \end{Bmatrix}$$

Let $\limsup_{r,s\to\infty} \max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,r-r_\delta+1} + d_{r-r_\delta+1,s-r_\delta+1},$

$d_{s-r_\delta,s-r_\delta+1} + d_{s-r_\delta+1,r-r_\delta+1}\Big\} = S,$

then

$$\limsup_{k\to\infty}\ \max\begin{Bmatrix} d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, \\ d_{r-r_\delta,r-r_\delta+1} + d_{r-r_\delta+1,s-r_\delta+1}, \\ d_{s-r_\delta,s-r_\delta+1} + d_{s-r_\delta+1,r-r_\delta+1} : \\ r-r_\delta+1, s-r_\delta+1 \geq k \end{Bmatrix} = S. \tag{3.9}$$

Since $\tilde{\psi}$ is upper semi-continuous, we get

$$\lim_{k\to\infty} \tilde{\psi}\Big(\sup\max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\}$$

$$: m-r_\delta+1, r-r_\delta \geq k\Big) \leq \tilde{\psi}(S).$$

Obviously,

$$\sup\Big\{\tilde{\psi}\Big(\max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1} : m-r_\delta+1, r-r_\delta \geq k\Big\}\Big)\Big\}$$

$$\leq \tilde{\psi}\Big(\sup\Big\{\max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}$$

$$: m-r_\delta+1, r-r_\delta \geq k\Big\}\Big\}\Big)$$

Hence, $\limsup_{k\to\infty} \tilde{\psi}\Big(\max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}$

$: m-r_\delta+1, r-r_\delta \geq k\Big\}\Big) \leq \tilde{\psi}(S)$

which implies,

$$\limsup_{r,s\to\infty}\ \tilde{\psi}\Big(\max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\}\Big) \leq \tilde{\psi}(S). \tag{3.10}$$

Substituting (3.9) in (3.10),

$$\limsup_{r,s\to\infty} \tilde{\psi}\Big(\max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\}\Big)$$
$$\leq \tilde{\psi}\Big(\limsup_{r,s\to\infty} \max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\}\Big)$$

Now, we discuss the following subcases:

Subcase 1:

$$\text{If } \max\left\{\begin{array}{l} d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, \\ d_{r-r_\delta,r-r_\delta+1} + d_{r-r_\delta+1,s-r_\delta+1}, \\ d_{s-r_\delta,s-r_\delta+1} + d_{s-r_\delta+1,r-r_\delta+1} : \\ \qquad\qquad r-r_\delta+1, s-r_\delta+1 \geq k \end{array}\right\} = d_{r-r_\delta,s-r_\delta}$$

then we get

$$\limsup_{r,s\to\infty} \max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\} = d_\infty.$$

Hence, we obtain

$$\limsup_{r,s\to\infty} d_{r,s} \leq \tilde{\psi}\Big(d_\infty\Big) + \delta$$

implies $d_\infty \leq \tilde{\psi}\Big(d_\infty\Big) + \delta$ it is applicable to all $\delta > 0$, so we get $d_\infty = 0$. Therefore,

$$\lim_{r,s\to\infty} d(o_r, o_s) = 0.$$

Subcase 2:

$$\text{If } \max\left\{\begin{array}{l} d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, \\ d_{r-r_\delta,r-r_\delta+1} + d_{r-r_\delta+1,s-r_\delta+1}, \\ d_{s-r_\delta,s-r_\delta+1} + d_{s-r_\delta+1,r-r_\delta+1} : \\ \qquad\qquad r-r_\delta+1, s-r_\delta+1 \geq k \end{array}\right\} = d_{r-r_\delta,r-r_\delta+1}$$

then from (3.8) we get

$$\limsup_{r,s\to\infty} \max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\} = 0.$$

Hence, we obtain

$$\limsup_{r,s\to\infty} d_{r,s} \leq \tilde{\psi}\Big(d_\infty\Big) + \delta$$

implies $d_\infty \leq \tilde{\psi}\Big(d_\infty\Big) + \delta$, it is applicable to all $\delta > 0$, so we get $d_\infty = 0$. Therefore,

$$\lim_{r,s\to\infty} d(o_r, o_s) = 0.$$

Subcase 3:

$$\text{Similarly if }\max\left\{\begin{matrix} d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, \\ d_{r-r_\delta,r-r_\delta+1}+d_{r-r_\delta+1,s-r_\delta+1}, \\ d_{s-r_\delta,s-r_\delta+1}+d_{s-r_\delta+1,r-r_\delta+1}: \\ r-r_\delta+1, s-r_\delta+1\geq k \end{matrix}\right\} = d_{s-r_\delta,s-r_\delta+1}$$

then from (3.8) we get

$$\limsup_{r,s\to\infty}\max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\} = 0.$$

Hence, we obtain

$$\limsup_{r,s\to\infty} d_{r,s} \leq \tilde{\psi}\Big(d_\infty\Big)+\delta$$

implies $d_\infty \leq \tilde{\psi}\Big(d_\infty\Big)+\delta$, it is applicable to all $\delta > 0$, so we get $d_\infty = 0$. Therefore,

$$\lim_{r,s\to\infty} d(o_r, o_s) = 0.$$

Subcase 4:

$$\text{If }\max\left\{\begin{matrix} d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, \\ d_{r-r_\delta,r-r_\delta+1}+d_{r-r_\delta+1,s-r_\delta+1}, \\ d_{s-r_\delta,s-r_\delta+1}+d_{s-r_\delta+1,r-r_\delta+1}: \\ r-r_\delta+1, s-r_\delta+1\geq k \end{matrix}\right\} = d_{r-r_\delta,r-r_\delta+1}+d_{r-r_\delta+1,s-r_\delta+1}$$

then we get

$$\limsup_{r,s\to\infty}\max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\} = d_\infty.$$

Hence, we obtain

$$\limsup_{r,s\to\infty} d_{r,s} \leq \tilde{\psi}\Big(d_\infty\Big)+\delta$$

implies $d_\infty \leq \tilde{\psi}\Big(d_\infty\Big)+\delta$, it is applicable to all $\delta > 0$, so we get $d_\infty = 0$. Therefore,

$$\lim_{r,s\to\infty} d(o_r, o_s) = 0.$$

Subcase 5:

$$\text{If }\max\left\{\begin{matrix} d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, \\ d_{r-r_\delta,r-r_\delta+1}+d_{r-r_\delta+1,s-r_\delta+1}, \\ d_{s-r_\delta,s-r_\delta+1}+d_{s-r_\delta+1,r-r_\delta+1}: \\ r-r_\delta+1, s-r_\delta+1\geq k \end{matrix}\right\} = d_{s-r_\delta,s-r_\delta+1}+d_{s-r_\delta+1,r-r_\delta+1}$$

then we get

$$\limsup_{r,s\to\infty}\max\Big\{d_{r-r_\delta,s-r_\delta}, d_{r-r_\delta,r-r_\delta+1}, d_{s-r_\delta,s-r_\delta+1}, d_{r-r_\delta,s-r_\delta+1}, d_{s-r_\delta,r-r_\delta+1}\Big\} = d_\infty.$$

Hence, we obtain that

$$\limsup_{r,s\to\infty} d_{r,s} \le \tilde{\psi}\Big(d_\infty\Big) + \delta$$

which implies that $d_\infty \le \tilde{\psi}\Big(d_\infty\Big) + \delta$, for all $\delta > 0$,. So we get $d_\infty = 0$. Therefore,

$$\lim_{r,s\to\infty} d(o_r, o_s) = 0.$$

After discussing all possible cases, we obtain that $\lim_{r,s\to\infty} d(o_r, o_s) = 0$. Hence the sequence $\{o_r\}$ is a Cauchy and as metric space (Z, d) is a complete, $\{o_r\} \to \rho$ in Z. Given that, Ω is orbitally continuous, we get $\Omega\rho = \rho$. □

4. Numerical Example

This section presents an example that illustrates our main result.

Example 4.1. *Suppose* $Z = \left[\frac{1}{50}, 1\right]$ *provided with the usual metric* d. *A self-map* $\Omega : Z \to Z$ *is defined by*

$$\Omega o = \begin{cases} \frac{o+\frac{2}{7}}{2} & \text{if } \frac{1}{50} \le o \le \frac{1}{2} \\ \frac{o}{3} & \text{if } \frac{1}{2} < o \le 1 \end{cases}$$

Furthermore, for all $t \in [0,\infty)$ *we choose* $\psi(t) = \frac{t}{6}$. *The possible three cases are discussed as follows:*
Case 1: *If* $o \in \left[\frac{1}{50}, \frac{1}{2}\right)$ *and* $p \in \left(\frac{1}{2}, 1\right]$, *then we have the following:*
Since $\frac{1}{2^r} \to \infty$, *for given* $\delta = \frac{1}{2}$, *there exists* $s \in \mathbb{N}$ *such that for all* $s \le r$

$$\frac{1}{2^r} < \frac{1}{2} \text{ implies } \frac{p}{2^r} \le \frac{1}{2^r} < \frac{1}{2}$$

In general, for any $s \le r$, *we have,*

$$\begin{aligned}
\Omega^r o &= \frac{o + \frac{2}{7}(2^r - 1)}{2^r} = \frac{o+\frac{2}{7}}{2^r} + \frac{2(2^r-2)}{2^r \cdot 7} \\
\Omega^r p &= \frac{\frac{p}{3^s} + \frac{2}{7}(2^{r-s}-1)}{2^{r-s}} = \frac{p+\frac{2}{7}}{2^{r-s}\cdot 3^s} + \frac{2}{7} - \left(\frac{2}{2^{r-s}\cdot 7} + \frac{2}{2^{r-s}\cdot 3^s \cdot 7}\right)
\end{aligned}$$

Consider,

$$\begin{aligned}
\left|\Omega^r o - \Omega^r p\right| &= \left|\frac{o+\frac{2}{7}}{2^r} + \frac{2(2^r-2)}{2^r\cdot 7} - \frac{p+\frac{2}{7}}{2^{r-s}\cdot 3^s} - \frac{2}{7} + \left(\frac{2}{2^{r-s}\cdot 7} + \frac{2}{2^{r-s}\cdot 3^s\cdot 7}\right)\right| \\
&= \left|\frac{o+\frac{2}{7}}{2^r} + \frac{2(2^r-2)}{2^r\cdot 7} - \frac{p}{2^{r-s}\cdot 3^s} - \frac{\frac{2}{7}}{2^{r-s}\cdot 3^s} - \frac{2}{7} + \left(\frac{2}{2^{r-s}\cdot 7} + \frac{\frac{2}{7}}{2^{r-s}\cdot 3^s}\right)\right| \\
&= \left|\frac{o+\frac{2}{7}}{2^r} + \frac{2(2^r-2)}{2^r\cdot 7} - \frac{p}{2^{r-s}\cdot 3^s} - \frac{2(2^{r-s}-1)}{2^{r-s}\cdot 7}\right| \\
&= \left|\frac{o+\frac{2}{7}}{2^r} - \frac{p}{2^{r-s}\cdot 3^s} + \frac{2}{7}\left(\frac{2^r-2}{2^r} - \frac{2^{r-s}-1}{2^{r-s}}\right)\right| \\
&= \left|\frac{o+\frac{2}{7}}{2^r} - \frac{p}{2^{r-s}\cdot 3^s} + \frac{2}{7}\left(\frac{1}{2^{r-s}} - \frac{1}{2^{r-1}}\right)\right|
\end{aligned}$$

By adding and subtracting some terms we get,

$$
\begin{aligned}
\left|\Omega^r o - \Omega^r p\right| &= \left| \frac{o+\frac{2}{7}}{2^r} - \frac{o}{2^{r-1}} + \frac{o}{2^{r-1}} - \frac{p}{2^{r-1}} + \frac{p}{2^{r-1}} - \frac{p}{2^{r-s}\cdot 3^{s-1}} + \frac{p}{2^{r-s}\cdot 3^{s-1}} - \frac{p}{2^{r-s}\cdot 3^{s}} \right.\\
&\qquad \left. + \frac{u}{2^{r-s+1}} - \frac{o}{2^{r-s+1}} + \frac{3}{2^{r-s}\cdot 7} - \frac{1}{2^{r-s}\cdot 7} - \frac{2}{2^{r-1}\cdot 7} \right|\\
&= \left| \frac{o+\frac{2}{7}}{2^r} - \frac{o}{2^{r-1}} + \frac{o}{2^{r-1}} - \frac{p}{2^{r-1}} + \frac{p}{2^{r-1}} - \frac{p}{2^{r-s}\cdot 3^{s-1}} + \frac{p}{2^{r-s}\cdot 3^{s-1}} - \frac{\frac{p}{3}}{2^{r-s}\cdot 3^{s-1}} \right.\\
&\qquad \left. + \frac{o}{2^{r-s+1}} - \frac{o}{2^{r-s+1}} - \frac{2}{2^{r-s+1}\cdot 7} + \frac{3}{2^{r-s}\cdot 7} - \frac{2}{2^{r-1}\cdot 7} \right|\\
&= \left| \frac{o+\frac{2}{7}}{2^r} - \frac{o}{2^{r-1}} + \frac{o}{2^{r-1}} - \frac{p}{2^{r-1}} + \frac{p}{2^{r-1}} - \frac{p}{2^{r-s}\cdot 3^{s-1}} + \frac{p}{2^{r-s}\cdot 3^{s-1}} - \frac{\frac{p}{3}}{2^{r-s}\cdot 3^{s-1}} \right.\\
&\qquad \left. + \frac{o}{2^{r-s+1}} - \frac{o}{2^{r-s+1}} - \frac{2}{2^{r-s+1}\cdot 7} + \frac{p}{2^{r-s}} - \frac{p}{2^{r-s}} + \frac{3}{2^{r-s}\cdot 7} - \frac{2}{2^{r-1}\cdot 7} \right|\\
&= \left| \frac{o+\frac{2}{7}}{2^r} - \frac{o}{2^{r-1}} + \frac{o}{2^{r-1}} - \frac{p}{2^{r-1}} + \frac{p}{2^{r-1}} - \frac{p}{2^{r-s}\cdot 3^{s-1}} + \frac{p}{2^{r-s}\cdot 3^{s-1}} - \frac{\frac{p}{3}}{2^{r-s}\cdot 3^{s-1}} \right.\\
&\qquad \left. + \frac{o}{2^{r-s+1}} - \frac{o}{2^{r-s+1}} - \frac{\frac{5y}{6}}{2^{r-s}} + \frac{3}{2^{r-s}\cdot 7} - \frac{2}{2^{r-1}\cdot 7} \right|
\end{aligned}
$$

By triangular inequality,

$$
\begin{aligned}
\left|\Omega^r o - \Omega^r p\right| &\leq \left| \frac{\frac{o+\frac{2}{7}}{2}}{2^{r-1}} - \frac{o}{2^{r-1}} \right| + \left| \frac{o}{2^{r-1}} - \frac{p}{2^{r-1}} \right| + \left| \frac{p}{2^{r-1}} - \frac{p}{2^{r-s}\cdot 3^{s-1}} - \frac{\frac{5y}{6}}{2^{r-s}} \right| + \left| \frac{p}{2^{r-s}\cdot 3^{s-1}} - \frac{\frac{p}{3}}{2^{r-s}\cdot 3^{s-1}} \right|\\
&\qquad + \left| \frac{o}{2^{r-s+1}} - \frac{\frac{p}{3}}{2^{r-s+1}} \right| + \left| \frac{p}{2^{r-s}} - \frac{\frac{o+\frac{2}{7}}{2}}{2^{r-s}} \right| + \left| \frac{3}{2^{r-s}\cdot 7} - \frac{2}{2^{r-1}\cdot 7} \right|\\
&\leq \left| \frac{\frac{o+\frac{2}{7}}{2}}{2^{r-1}} - \frac{o}{2^{r-1}} \right| + \left| \frac{o}{2^{r-1}} - \frac{1}{2^{r-1}} \right| + |p| \left| \frac{1}{2^{r-1}} - \frac{1}{2^{r-s}\cdot 3^{s-1}} - \frac{5}{2^{r-s}\cdot 6} \right|\\
&\qquad + \left| \frac{p}{2^{r-s}\cdot 3^{s-1}} - \frac{\frac{p}{3}}{2^{r-s}\cdot 3^{s-1}} \right| + \left| \frac{o}{2^{r-s+1}} - \frac{\frac{p}{3}}{2^{r-s+1}} \right| + \left| \frac{p}{2^{r-s}} - \frac{\frac{o+\frac{2}{7}}{2}}{2^{r-s}} \right| + \left| \frac{3}{2^{r-s}\cdot 7} - \frac{2}{2^{r-1}\cdot 7} \right|
\end{aligned}
$$

From this,

$$
\begin{aligned}
d(\Omega^r o, \Omega^r p) \leq \frac{d(\Omega o, o)}{2^{r-1}} + \frac{d(o,p)}{2^{r-1}} + \left| \frac{1}{2^{r-1}} - \frac{1}{2^{r-s}\cdot 3^{s-1}} - \frac{5}{2^{r-s}\cdot 6} \right| + \frac{d(p,\Omega p)}{2^{r-s}\cdot 3^{s-1}} + \frac{d(o,\Omega p)}{2^{r-s+1}} + \frac{d(p,\Omega o)}{2^{r-s}}\\
+ \frac{1}{7}\left| \frac{3}{2^{r-s}} - \frac{2}{2^{r-1}} \right|
\end{aligned}
$$

$$
\begin{aligned}
d(\Omega^r o, \Omega^r p) \leq \frac{d(o,p)}{2^{r-1}} + \frac{d(o,\Omega o)}{2^{r-1}} + \frac{d(p,\Omega p)}{2^{r-s}\cdot 3^{s-1}} + \frac{d(o,\Omega p)}{2^{r-s+1}} + \frac{d(p,\Omega o)}{2^{r-s}}\\
+ \left| \frac{1}{2^{r-1}} - \frac{1}{2^{r-s}\cdot 3^{s-1}} - \frac{5}{2^{r-s}\cdot 6} \right| + \frac{1}{7}\left| \frac{3}{2^{r-s}} - \frac{2}{2^{r-1}} \right|
\end{aligned}
\tag{4.1}
$$

Assume,

$$o_r = \left|\frac{1}{2^{r-1}} - \frac{1}{2^{r-s}\cdot 3^{s-1}} - \frac{5}{2^{r-s}\cdot 6}\right| + \frac{1}{7}\left|\frac{3}{2^{r-s}} - \frac{2}{2^{r-1}}\right|.$$

As $o_r \to 0$ when $r \to \infty$, there exists a natural number k such that $\|o_r\| < \delta$ for all $r \geq k$. Take $k' = \max\{k, s+5\}$ and therefore, for all $r \geq k'$, by (4.1), we have

$$\begin{aligned}
d(\Omega^r o, \Omega^r p) &\leq \frac{d(o,p)}{2^{r-1}} + \frac{d(o,\Omega o)}{2^{r-1}} + \frac{d(p,\Omega p)}{2^{r-s}\cdot 3^{s-1}} + \frac{d(o,\Omega p)}{2^{r-s+1}} + \frac{d(p,\Omega o)}{2^{r-s}} \\
&\leq \frac{1}{9}\Big[d(o,p) + d(o,\Omega o) + d(p,\Omega p) + d(o,\Omega p) + d(p,\Omega o)\Big] \\
&\leq \frac{2\psi}{3}\Big[d(o,p) + d(o,\Omega o) + d(p,\Omega p) + d(o,\Omega p) + d(p,\Omega o)\Big] + \delta
\end{aligned}$$

Case 2: *If $o, p \in \left[\frac{1}{50}, \frac{1}{2}\right)$, then we have the following:*

$$\begin{aligned}
\left|\Omega^r o - \Omega^r p\right| &= \left|\frac{o + \frac{2}{7}(2^r - 1)}{2^r} - \frac{p + \frac{2}{7}(2^r-1)}{2^r}\right| = \left|\frac{o + \frac{2}{7}}{2^r} - \frac{p + \frac{2}{7}}{2^r}\right| \\
&= \left| -\frac{o}{2^r} + \frac{p}{2^r} - \frac{o}{2^r} + \frac{\frac{o+\frac{2}{7}}{2}}{\frac{2^r}{2}} - \frac{\frac{p+\frac{2}{7}}{2}}{\frac{2^r}{2}} + \frac{p}{2^r} + \frac{o}{2^r} - \frac{p}{2^r} + \frac{\frac{o+\frac{2}{7}}{2}}{\frac{2^r}{2}} - \frac{\frac{p+\frac{2}{7}}{2}}{\frac{2^r}{2}} \right| \\
&= \left| -\frac{o}{2^r} + \frac{p}{2^r} - \frac{o}{2^r} + \frac{\frac{o+\frac{2}{7}}{2}}{2^{r-1}} - \frac{\frac{p+\frac{2}{7}}{2}}{2^{r-1}} + \frac{p}{2^r} + \frac{o}{2^r} - \frac{p}{2^r} + \frac{\frac{o+\frac{2}{7}}{2}}{2^{r-1}} - \frac{\frac{p+\frac{2}{7}}{2}}{2^{r-1}} \right| \\
&\leq \left| -\frac{o}{2^r} + \frac{p}{2^r}\right| + \left| -\frac{o}{2^r} + \frac{\frac{o+\frac{2}{7}}{2}}{2^{r-1}}\right| + \left| -\frac{\frac{p+\frac{2}{7}}{2}}{2^{r-1}} + \frac{p}{2^r}\right| + \left|\frac{o}{2^r} - \frac{\frac{p+\frac{2}{7}}{2}}{2^{r-1}}\right| + \left| -\frac{p}{2^r} + \frac{\frac{o+\frac{2}{7}}{2}}{2^{r-1}}\right|
\end{aligned}$$

Thus,

$$\begin{aligned}
d(\Omega^r o, \Omega^r p) &\leq \frac{1}{2^r}\Big[d(o,p) + d(o,\Omega o) + d(p,\Omega p) + d(o,\Omega p) + d(p,\Omega o)\Big] \\
&\leq \frac{1}{9}\Big[d(o,p) + d(o,\Omega o) + d(p,\Omega p) + d(o,\Omega p) + d(p,\Omega o)\Big]
\end{aligned}$$

Thus, we get

$$d(\Omega^r o, \Omega^r p) \leq \frac{2\psi}{3}\Big[d(o,p) + d(o,\Omega o) + d(p,\Omega p) + d(o,\Omega p) + d(p,\Omega o)\Big] + \delta. \tag{4.2}$$

Case 3: *If $o, p \in \left(\frac{1}{2}, 1\right]$, then we have the following:*

$$\begin{aligned}
\left|\Omega^r o - \Omega^r p\right| &= \left|\frac{\frac{o}{3^s} + \frac{2}{7}(2^{r-s}-1)}{2^{r-s}} - \frac{\frac{p}{3^s} + \frac{2}{7}(2^{r-s}-1)}{2^{r-s}}\right| \\
&\leq \left|\frac{-o}{2^{r-s}} + \frac{p}{2^{r-s}} - \frac{o}{2^{r-s}} + \frac{\frac{o}{3^s} + \frac{2}{7}(2^{r-s}-1)}{2^{r-s}} + \frac{p}{2^{r-s}} - \frac{\frac{p}{3^s} + \frac{2}{7}(2^{r-s}-1)}{2^{r-s}}\right. \\
&\qquad\qquad \left. + \frac{o}{2^{r-s}} - \frac{p}{2^{r-s}} + \frac{\frac{o}{3^s} + \frac{2}{7}(2^{r-s}-1)}{2^{r-s}} - \frac{\frac{p}{3^s} + \frac{2}{7}(2^{r-s}-1)}{2^{r-s}}\right|
\end{aligned}$$

This implies,

$$\begin{aligned} d(\Omega^r o, \Omega^r p) &\leq \frac{1}{2^{r-s}}\Big[d(o,p) + d(o,\Omega o) + d(p,\Omega p) + d(o,\Omega p) + d(p,\Omega o)\Big] \\ &\leq \frac{1}{9}\Big[d(o,p) + d(o,\Omega o) + d(p,\Omega p) + d(o,\Omega p) + d(p,\Omega o)\Big] \end{aligned}$$

Thus, we get

$$d(\Omega^r o, \Omega^r p) \leq \frac{2\psi}{3}\,[d(o,p) + d(o,\Omega o) + d(p,\Omega p) + d(o,\Omega p) + d(p,\Omega o)] + \delta \tag{4.3}$$

for all $o, p \in Z$. Finally, from all the 3 cases we can conclude that, Ω is a weakly asymptotic Ciric type contraction on Z.

5. Application

In this section, we use our result into cantilever beam problem. Let us consider the following two-point boundary value problem (BVP):

$$\left.\begin{aligned} &\alpha^{(4)}(t) = \lambda\, K(t, \alpha(t)),\ x < t < y; \\ &\alpha(x) = \alpha'(x) = \alpha''(y) = \alpha'''(y) = 0 \end{aligned}\right\} \tag{5.1}$$

where $K : [x,y] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $\lambda \geq 0$. The above-mentioned BVP represents a beam problem with a uniform load distribution, commonly referred to as a cantilever beam problem. Equation (5.1) can be expressed equivalently as the subsequent integral equation:

$$\alpha(t) = \lambda \int_x^y G(t,o)K(o,\alpha(o))do, \tag{5.2}$$

for $t \in [x,y]$, where Green's function, $G : [x,y] \times [x,y] \to \mathbb{R}$ is defined by

$$G(t,o) = \begin{cases} o^2\Big(t^2 - \frac{t}{2} - \frac{t^3}{2}\Big) + o^3\Big(\frac{1}{6} - \frac{t^2}{2} + \frac{t^3}{3}\Big), & \text{if } x \leq o \leq t \leq y; \\ t^2\Big(o^2 - \frac{o}{2} - \frac{o^3}{2}\Big) + t^3\Big(\frac{1}{6} - \frac{o^2}{2} + \frac{o^3}{3}\Big), & \text{if } x \leq t \leq o \leq y. \end{cases}$$

Now we demonstrate that the boundary value problem (5.1) has a unique solution.

Theorem 5.1. *Let us assume the following for each $\alpha, \beta \in C[x,y]$:*

$$\begin{aligned} \sup_{o\in[x,y]} \Big|K(o,\alpha(o)) - K(o,\beta(o))\Big| \leq \max\Big\{ &\sup_{o\in[x,y]} \Big|\alpha(o) - \beta(o)\Big|, \sup_{o\in[x,y]} \Big|\alpha(t) - \lambda\int_x^y G(t,o)K(o,\alpha(o))do\Big|, \\ &\sup_{o\in[x,y]} \Big|\beta(t) - \lambda\int_x^y G(t,o)K(o,\beta(o))do\Big|, \\ &\sup_{o\in[x,y]} \Big|\alpha(t) - \lambda\int_x^y G(t,o)K(o,\beta(o))do\Big|, \\ &\sup_{o\in[x,y]} \Big|\beta(t) - \lambda\int_x^y G(t,o)K(o,\alpha(o))do\Big| \Big\} \end{aligned}$$

If the function K is bounded and

$$\sup_{o\in[x,y]} \int_x^y G(t,o)do < \frac{1}{\lambda},\ \lambda \geq 0,$$

then the boundary value problem (5.1) has exactly one solution in $C[x,y]$.

Proof. Let $(C[x,y],d)$ be the complete metric space, with a sup metric d on $C[x,y]$. Initially, we define Ω on $C[x,y]$ by

$$\Omega(\alpha)(t) = \lambda \int_x^y G(t,o)K(o,\alpha(o))do,$$

for all $o \in C[x,y]$ and $t \in [x,y]$. Then it becomes clear that Ω is a self-map on $C[x,y]$ which is also continuous. Given that K is bounded, Ω has bounded orbits at all $o \in C[x,y]$. For every $\alpha, \beta \in C[x,y]$ and any $x \leq t \leq y$, we have:

$$\begin{aligned}
\left|(\Omega\alpha)(t) - (\Omega\beta)(t)\right| &= \left|\lambda \int_x^y G(t,o)K(o,\alpha(o))do - \lambda \int_x^y G(t,o)K(o,\beta(o))do\right| \\
&= \lambda \left| \int_x^y G(t,o)\{K(o,\alpha(o)) - K(o,\beta(o))\}do\right| \\
&\leq \lambda \int_x^y G(t,o)\left|K(o,\alpha(o)) - K(o,\beta(o))\right|do \\
&\leq \lambda \sup_{o\in[x,y]} \left|K(o,\alpha(o)) - K(o,\beta(o))\right| \int_x^y G(t,o)do. \qquad (5.3)
\end{aligned}$$

Now, we discuss the following three possible cases:

Case 1: Suppose,

$$\begin{aligned}
\max\Bigg\{ &\sup_{o\in[x,y]} \left|\alpha(o) - \beta(o)\right|, \\
&\sup_{o\in[x,y]} \left|\alpha(t) - \lambda \int_x^y G(t,o)K(o,\alpha(o))do\right|, \\
&\sup_{o\in[x,y]} \left|\beta(t) - \lambda \int_x^y G(t,o)K(o,\beta(o))do\right|, \\
&\sup_{o\in[x,y]} \left|\alpha(t) - \lambda \int_x^y G(t,o)K(o,\beta(o))do\right|, \\
&\sup_{o\in[x,y]} \left|\beta(t) - \lambda \int_x^y G(t,o)K(o,\alpha(o))do\right|\Bigg\} = \sup_{o\in[x,y]} \left|\alpha(o) - \beta(o)\right|
\end{aligned}$$

by applying the given condition and (5.3), we can obtain

$$\left|(\Omega\alpha)(t) - (\Omega\beta)(t)\right| \leq \lambda \sup_{o\in[x,y]} \left|\alpha(o) - \beta(o)\right| \int_x^y G(t,o)do = \lambda\, d(\alpha,\beta) \int_x^y G(t,o)do. \qquad (5.4)$$

Case 2: Suppose,

$$\begin{aligned}
&\max\Bigg\{\sup_{o\in[x,y]}\Big|\alpha(o)-\beta(o)\Big|,\\
&\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|,\\
&\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|,\\
&\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|,\\
&\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|\Bigg\}=\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|
\end{aligned}$$

by applying the given condition and (5.3), we can obtain

$$\begin{aligned}
\Big|(\Omega\alpha)(t)-(\Omega\beta)(t)\Big| &\leq \lambda\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|\int_x^y G(t,o)do\\
&=\lambda\, d(\alpha,\Omega\alpha)\int_x^y G(t,o)do. \qquad (5.5)
\end{aligned}$$

Case 3: Suppose,

$$\begin{aligned}
&\max\Bigg\{\sup_{o\in[x,y]}\Big|\alpha(o)-\beta(o)\Big|,\\
&\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|,\\
&\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|,\\
&\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|,\\
&\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|\Bigg\}=\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|
\end{aligned}$$

by applying the given condition and (5.3), we can obtain

$$\begin{aligned}
\Big|(\Omega\alpha)(t)-(\Omega\beta)(t)\Big| &\leq \lambda\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|\int_x^y G(t,o)do\\
&=\lambda\, d(\beta,\Omega\beta)\int_x^y G(t,o)do \qquad (5.6)
\end{aligned}$$

Case 4: Suppose,

$$\max\left\{\sup_{o\in[x,y]}\left|\alpha(o)-\beta(o)\right|,\right.$$
$$\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|,$$
$$\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|,$$
$$\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|,$$
$$\left.\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|\right\}=\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|$$

by applying the given condition and (5.3), we can obtain

$$\begin{aligned}\left|(\Omega\alpha)(t)-(\Omega\beta)(t)\right| &\le \lambda\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|\int_x^y G(t,o)do\\ &=\lambda\, d(\alpha,\Omega\beta)\int_x^y G(t,o)do. \end{aligned}\tag{5.7}$$

Case 5: Similarly, if

$$\max\left\{\sup_{o\in[x,y]}\left|\alpha(o)-\beta(o)\right|,\right.$$
$$\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|,$$
$$\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|,$$
$$\sup_{o\in[x,y]}\left|\alpha(t)-\lambda\int_x^y G(t,o)K(o,\beta(o))do\right|,$$
$$\left.\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|\right\}=\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|$$

by applying the given condition and (5.3), we can obtain

$$\begin{aligned}\left|(\Omega\alpha)(t)-(\Omega\beta)(t)\right| &\le \lambda\sup_{o\in[x,y]}\left|\beta(t)-\lambda\int_x^y G(t,o)K(o,\alpha(o))do\right|\int_x^y G(t,o)do\\ &=\lambda_1 d(\beta,\Omega\alpha)\int_x^y G(t,o)do. \end{aligned}\tag{5.8}$$

Thus using (5.4)-(5.8) in (5.3), we get

$$\left|(\Omega\alpha)(t)-(\Omega\beta)(t)\right|\le\lambda\max\Big\{d(\alpha,\beta),d(\alpha,\Omega\alpha),d(\beta,\Omega\beta),d(\alpha,\Omega\beta),d(\beta,\Omega\alpha)\Big\}\int_x^y G(t,o)do,$$

which implies,

$$d(\Omega\alpha,\Omega\beta) = \sup_{t\in[x,y]} \left|(\Omega\alpha)(t) - (\Omega\beta)(t)\right|$$
$$\leq \lambda \ \max\Big\{d(\alpha,\beta), d(\alpha,\Omega\alpha), d(\beta,\Omega\beta), d(\alpha,\Omega\beta), d(\beta,\Omega\alpha)\Big\} \sup_{t\in[x,y]} \int_x^y G(t,o)do$$

for all $t \in [x,y]$. Hence, by the given condition, we have

$$d(\Omega\alpha,\Omega\beta) \leq \max\{d(\alpha,\beta), d(\alpha,\Omega\alpha), d(\beta,\Omega\beta), d(\alpha,\Omega\beta), d(\beta,\Omega\alpha)\}.$$

Consider the function $\psi \in \Psi$, which is defined as $\psi(t) = t \ \forall \ t \in [x,y]$. By choosing $i_\delta = 1$ for every given $\delta > 0$ yields

$$d(\Omega^{r_\delta}\alpha, \Omega^{r_\delta}\beta) \leq \psi \max\{d(\alpha,\beta), d(\alpha,\Omega\alpha), d(\beta,\Omega\beta), d(\alpha,\Omega\beta), d(\beta,\Omega\alpha)\} + \delta,$$

for all $\alpha, \beta \in C[x,y]$. So, according to Theorem 3.3, Ω has only one fixed point in $C[x,y]$. This means that problem (5.1) has only one fixed point in $C[x,y]$. □

Example 5.1. *In the boundary value problem (5.1), consider the function*

$$K(t,\alpha(t)) = \frac{t(1-t)(x+2)}{2x^2+3}$$

with $t \in [0,1]$, *and* $x \in [0,\infty)$. *Theorem 3.3 shows that BVP (5.1) possesses a unique fixed point* ρ, *for each* $\lambda \geq 0$. *Using MATLAB 2024a, we can plot graphs for BVP (5.1) with* $\lambda = 100, 80, 60, 40, 20,$ *and* 1, *as*

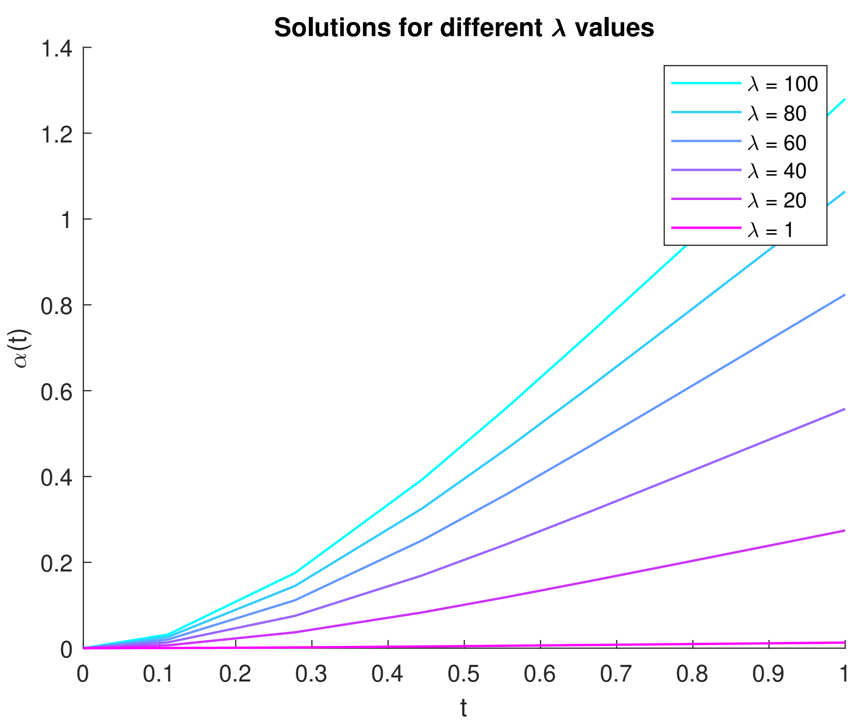

Figure 1. Solutions for BVP (5.1) with $K(t,\alpha(t)) = \dfrac{t(1-t)(x+2)}{2x^2+3}$

shown in Figure (1).

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