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Touchard Polynomials and the Identified New Subclass of Analytic Functions

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Abstract. A new analytic function, which includes Touchard polynomials, is presented here as part of this work. Subsequently, we endeavour to derive appraisals for the $|d_2|$, $|d_3|$ Maclaurin coefficients with respect to this particular subfamily, as well as the Fekete-Szegö functional problem that is associated with it. Moreover, by elaborating on the parameters that were utilised in our primary findings, a multitude of new results are demonstrated below.

1. Preliminaries

In the year 1784, Legendre made the first discovery of orthogonal polynomials [1]. Since that time, a significant amount of research has been conducted specifically on these polynomials. In the process of solving mathematical models, orthogonal polynomials are frequently utilised primarily for the purpose of solving ordinary differential equations and satisfying model requirements. While it goes without saying that orthogonal polynomials are important for contemporary mathematics, it is also important to note that they have numerous applications in the fields of engineering and physics. It is common knowledge that these polynomials play an important role in matters that are associated with the theory of approximation. They are found in the field of differential equation theory as well as in mathematical statistics. Furthermore, it is worth noting that their

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applications in the fields of automated control, quantum physics, signal analysis, scattering theory, and axially symmetric potential theory have garnered a lot of attention [2,3].

In the field of combinatorics and the study of integer partitions, a family of polynomials known as Touchard polynomials presents themselves. These polynomials are named after the French mathematician Jacques Touchard (see [4]). A tight connection exists between these polynomials and Bell numbers (see [5] and [6]), which are a count of the number of ways in which a set can be partitioned. These polynomials are defined in terms of exponential generating functions. Probability theory, statistical mechanics, and the analysis of algorithms are only few of the fields that can benefit from the utilisation of Touchard polynomials. As well as being utilised in combinatorial issues that include set partitions, they offer a helpful instrument for comprehending the distribution of particles in certain physical systems. A more in-depth comprehension of the combinatorial structures and procedures in mathematics can be attained through the study of Touchard polynomials. Include a polynomial sequence of binomial type that, when X is a random variable that is distributed according to a Poisson distribution and has an expected value of ς, then the nth moment of this sequence is $E(X_\varsigma) = \mu(\varsigma, i)$, which leads to the following type:

$$\mu(\varsigma, i) = e^{\varsigma} \sum_{y=0}^{\infty} \frac{\varsigma^y y^i}{y!} \delta^y$$

For the purpose of presenting the result of the second force, the coefficients of Touchard polynomials are utilised

$$\Lambda_{\varsigma}^{i}(\delta) = \delta + \sum_{y=2}^{\infty} \frac{\varsigma^{y-1}(y-1)^i}{e^{\varsigma}(y-1)!} \delta^y, \quad (\delta \in \varsigma). \tag{1.1}$$

as long as i is greater than zero and ς is greater than zero, we may observe that the radius of convergence of the series mentioned above is infinite. Let Υ represent the class of functions that are analytic in the unit disc $\varsigma = \{\delta \in \mathbb{C} : |\delta| < 1\}$. The form of their expression is as follows:

$$h(\delta) = \delta + \sum_{y=2}^{\infty} d_y \delta^y, \quad (\delta \in \varsigma). \tag{1.2}$$

In addition, we consider Ω to be composed of functions that are normalised by the equation $f(0) = f'(0) - 1 = 0$ and are also univalent in the ς framework.

Use the symbol Υ to denote the subclass of Ω that is comprised of functions from the following form:

$$h(\delta) = \delta + \sum_{y=2}^{\infty} d_y \delta^y, \quad d_y \geq 0. \tag{1.3}$$

Within the context of functions $h(\delta) = \delta + \sum_{y=2}^{\infty} d_y \delta^y$ and $j(\delta) = \delta + \sum_{y=2}^{\infty} c_y \delta^y$, we establish the convolution of h and j by utilising the following equation:

$$(h * j)(\delta) = \delta + \sum_{y=2}^{\infty} d_y c_y \delta^y, \quad (\delta \in \varsigma).$$

We will define the linear operator $\varpi(i,\varsigma,\delta): \Upsilon \to \Upsilon$ of

$$\varpi_h(i,\varsigma,\delta) = \Lambda^i_\varsigma(\delta) * h(\delta) = \delta + \sum_{y=2}^{\infty} \frac{\varsigma^{y-1}(y-1)^i}{e^\varsigma (y-1)!} d_y \delta^y, \ \ (\delta \in \varsigma).$$

The first differential subordination problem was presented by Miller and Mocanu [7]; for further information, please refer to [8] and [11].

For every mathematical function h that belongs to the set of all functions Ω, there exists an inverse h^{-1}, which is defined by

$$h^{-1}(h(\delta)) = \delta \qquad (\delta \in \varsigma)$$

and

$$A = h(h^{-1}(A)) \qquad (|A| < r_0(h);\ r_0(h) \geq \frac{1}{4})$$

where

$$q(A) = h^{-1}(A) = A - d_2A^2 + (-d_3 + 2d_2^2)A^3 - (d_4 + 5d_2^3 - 5d_3d_2)A^4 + \cdots . \tag{1.4}$$

If both $h(\delta)$ and $h^{-1}(\delta)$ are univalent in ς, a function is said to be bi-univalent in ς.

Let us represent by the symbol Γ the class of bi-univalent functions in ς in accordance with the equation (1.2). With regard to the class Γ, an example can be found in the equation $h(\delta) = \frac{\delta}{1-\delta}$, whereas $h(\delta) = \frac{\delta}{1-\delta^2}$. are not members of the Gamma group. In the class Γ, there are some noteworthy function classes that may be found in ([12] and [13]).

Estimating coefficients has been the primary focus of several papers that have been conducted in recent years to investigate fundamental aspects of geometric function theory. Several subclasses of the class Γ were introduced, and non-sharp estimates on the coefficients $|d_2|$ and $|d_3|$ in the Maclaurin series expansion (1.2) were obtained in ([22], [10], [14]- [23]).

By utilising the Touchard polynomials, we establish a new subclass of the Γ space. Furthermore, we find bounds for the $|d_2|$ and $|d_3|$ Maclaurin coefficients, as well as Fekete–Szegö functional problems [24]. In addition to that, a collection of fresh findings is presented in the following.

2. Bounds of the class $\otimes^y(m,\varphi,\gamma,\backepsilon,\varsigma,i,F)$

At the beginning of this section, the new subclass $\otimes^y(m,\varphi,\gamma,\backepsilon,\varsigma,i,F)$ related to Touchard polynomials is defined.

Definition 2.1. *A function $h \in \Omega$ as stated in (1.2) is regarded as belonging to the class $\otimes^y(m,\varphi,\gamma,\backepsilon,\varsigma,i,F)$ if the following subordinations*

$$\left(1+me^{i\varphi}\right)\left\{1+\frac{1}{\gamma}\left[(\varpi_h(i,\varsigma,\delta))' + \backepsilon\delta\left(\varpi_h(i,\varsigma,\delta)\right)'' - 1\right]\right\} - me^{i\varphi} \prec F(\delta) \tag{2.1}$$

and

$$\left(1+me^{i\varphi}\right)\left\{1+\frac{1}{\gamma}\left[\left(\varpi_q(i,\varsigma,A)\right)' + \backepsilon A\left(\varpi_q(i,\varsigma,A)\right)'' - 1\right]\right\} - me^{i\varphi} \prec F(A), \tag{2.2}$$

where $\gamma \in \mathbb{C}\backslash\{0\}; m \geq 0; -\pi < \varphi \leq \pi; 0 \leq \eth \leq 1; \varsigma > 0; i \geq 0; \delta, A \in \mathbb{C}$, and the function $q = h^{-1}$ is given by (1.4).

Through the utilisation of certain values for the parameters γ, m and $\eth$, it is possible to discover a multitude of subclasses. For instance, we may provide the following instances; however, this list is not exhaustive.

Example 2.1. *When $\gamma = 1$, we obtain $\otimes^{y}(m, \varphi, \gamma, \eth, \varsigma, i, F)$. Here, $\otimes^{y}(m, \varphi, 1, \eth, \varsigma, i, F)$ is the set of functions $h \in \Omega$ that satisfy the following criteria and are provided by (1.4).*

$$\left(1 + me^{i\varphi}\right)\left\{(\varpi_h(i, \varsigma, \delta))' + \eth\delta\,(\varpi_h(i, \varsigma, \delta))''\right\} - me^{i\varphi} \prec F(\delta)$$

and

$$\left(1 + me^{i\varphi}\right)\left\{\left(\varpi_q(i, \varsigma, A)\right)' + \eth A\left(\varpi_q(i, \varsigma, A)\right)''\right\} - me^{i\varphi} \prec F(A),$$

where $m \geq 0; -\pi < \varphi \leq \pi; 0 \leq \eth \leq 1; \varsigma > 0; i \geq 0; \delta, A \in \mathbb{C}$ and the function $q = h^{-1}$.

Example 2.2. *When $\gamma = \eth = 1$, we obtain $\otimes^{y}(m, \varphi, \gamma, \eth, \varsigma, i, F)$. Here, $\otimes^{y}(m, \varphi, 1, 1, \varsigma, i, F)$ is the set of functions $h \in \Omega$ that satisfy the following criteria and are provided by (1.4).*

$$\left(1 + me^{i\varphi}\right)\left\{(\varpi_h(i, \varsigma, \delta))' + \delta\,(\varpi_h(i, \varsigma, \delta))''\right\} - me^{i\varphi} \prec F(\delta)$$

and

$$\left(1 + me^{i\varphi}\right)\left\{\left(\varpi_q(i, \varsigma, A)\right)' + A\left(\varpi_q(i, \varsigma, A)\right)''\right\} - me^{i\varphi} \prec F(A),$$

where $m \geq 0; -\pi < \varphi \leq \pi; \varsigma > 0; i \geq 0; \delta, A \in \mathbb{C}$, and the function $q = h^{-1}$.

Example 2.3. *When $\gamma = 1$ and $\eth = 0$, we obtain $\otimes^{y}(m, \varphi, \gamma, \eth, \varsigma, i, F)$. Here, $\otimes^{y}(m, \varphi, 1, 0, \varsigma, i, F)$ is the set of functions $h \in \Omega$ that satisfy the following criteria and are provided by (1.4).*

$$\left(1 + me^{i\varphi}\right)\left\{(\varpi_h(i, \varsigma, \delta))'\right\} - me^{i\varphi} \prec F(\delta)$$

and

$$\left(1 + me^{i\varphi}\right)\left\{\left(\varpi_q(i, \varsigma, A)\right)'\right\} - me^{i\varphi} \prec F(A),$$

where $m \geq 0; -\pi < \varphi \leq \pi; \varsigma > 0; i \geq 0; \delta, A \in \mathbb{C}$ and the function $q = h^{-1}$.

Example 2.4. *When $m = 0$, we obtain $\otimes^{y}(m, \varphi, \gamma, \eth, \varsigma, i, F)$. Here, $\otimes^{y}(0, \varphi, \gamma, \eth, \varsigma, i, F)$ is the set of functions $h \in \Omega$ that satisfy the following criteria and are provided by (1.4).*

$$1 + \frac{1}{\gamma}\left[(\varpi_h(i, \varsigma, \delta))' + \eth\delta\,(\varpi_h(i, \varsigma, \delta))'' - 1\right] \prec F(\delta)$$

and

$$1 + \frac{1}{\gamma}\left[\left(\varpi_q(i, \varsigma, A)\right)' + \eth A\left(\varpi_q(i, \varsigma, A)\right)'' - 1\right] \prec F(A),$$

where $\gamma \in \mathbb{C}\backslash\{0\}; 0; -\pi < \varphi \leq \pi; 0 \leq \eth \leq 1; \varsigma > 0; i \geq 0; \delta, A \in \mathbb{C}$, and the function $q = h^{-1}$.

Example 2.5. *When $\gamma = 1$ and $m = 0$, we obtain $\otimes^y(m,\varphi,\gamma,\eth,\varsigma,i,F)$. Here, $\otimes^y(0,\varphi,1,\eth,\varsigma,i,F)$ is the set of functions $h \in \Omega$ that satisfy the following criteria and are provided by (1.4).*

$$(\varpi_h(i,\varsigma,\delta))' + \eth\delta\,(\varpi_h(i,\varsigma,\delta))'' \prec F(\delta)$$

and

$$\left(\varpi_q(i,\varsigma,A)\right)' + \eth A\left(\varpi_q(i,\varsigma,A)\right)'' \prec F(A),$$

where $-\pi < \varphi \leq \pi; 0 \leq \eth \leq 1; \varsigma > 0; i \geq 0; \delta, A \in \mathbb{C}$ and the function $q = h^{-1}$.

Example 2.6. *When $\gamma = \eth = 1$ and $m = 0$, we obtain $\otimes^y(m,\varphi,\gamma,\eth,\varsigma,i,F)$. Here, $\otimes^y(0,\varphi,1,1,\varsigma,i,F)$ is the set of functions $h \in \Omega$ that satisfy the following criteria and are provided by (1.4).*

$$(\varpi_h(i,\varsigma,\delta))' + \delta\,(\varpi_h(i,\varsigma,\delta))'' \prec F(\delta)$$

and

$$\left(\varpi_q(i,\varsigma,A)\right)' + A\left(\varpi_q(i,\varsigma,A)\right)'' \prec F(A),$$

where $-\pi < \varphi \leq \pi; \varsigma > 0; i \geq 0; \delta, A \in \mathbb{C}$, and the function $q = h^{-1}$.

Example 2.7. *When $\gamma = 1$ and $\eth = m = 0$, we obtain $\otimes^y(m,\varphi,\gamma,\eth,\varsigma,i,F)$. Here, $\otimes^y(0,\varphi,1,0,\varsigma,i,F)$ is the set of functions $h \in \Omega$ that satisfy the following criteria and are provided by (1.4).*

$$(\varpi_h(i,\varsigma,\delta))' \prec F(\delta)$$

and

$$\left(\varpi_q(i,\varsigma,A)\right)' \prec F(A),$$

where $-\pi < \varphi \leq \pi; \varsigma > 0; i \geq 0; \delta, A \in \mathbb{C}$ and the function $q = h^{-1}$.

First, we give the coefficient estimates for the class $\otimes^y(m,\varphi,\gamma,\eth,\varsigma,i,F)$ given in Definition 2.1.

Theorem 2.1. *Let $f \in \Omega$ given by (1.2) belongs to the subclass $\otimes^y(m,\varphi,\gamma,\eth,\varsigma,i,F)$. Then*

$$|a_2| \leq \frac{\left|\gamma\right| \sqrt{E_1^3}}{\sqrt{\varsigma^2 e^{-\varsigma}\left|\left[3\cdot 2^{i-1}\gamma\,(2\eth+1)\,(1+me^{i\varphi})\,E_1^2 + 4e^{-\varsigma}\,(\eth+1)^2\,(1+me^{i\varphi})^2\,(E_1-E_2)\right]\right|}}$$

and

$$|a_3| \leq \frac{\left|\gamma\right| e^{\varsigma}E_1}{4}\left(\frac{1}{3\cdot 2^{i-3}\varsigma^2\,(2\eth+1)\,(1+me^{i\varphi})} + \frac{\left|\gamma\right| e^{\varsigma}E_1}{(\eth+1)^2\,(1+me^{i\varphi})^2}\right).$$

where $\gamma \in \mathbb{C}\backslash\{0\}; m \geq 0; -\pi < \varphi \leq \pi; 0 \leq \eth \leq 1; \varsigma > 0; i \geq 0.$

Proof. The function $f(\delta)$ is equal to $f(\delta) = \delta + \sum_{y=2}^{\infty} a_y\delta^y \in \otimes^y(m,\varphi,\gamma,\eth,\varsigma,i,F)$. We are able to take into consideration two functions $r, u : \varsigma \to \varsigma$ such that $r(0) = u(0) = 0$. Moreover, $|r(\delta)| < 1$ and $|u(A)| < 1$ for every $\delta, A \in \varsigma$. Based on Definition 2.1, it is possible to write

$$\left(1+me^{i\varphi}\right)\left\{1+\frac{1}{\gamma}\left[(\varpi_h(i,\varsigma,\delta))' + \eth\delta\,(\varpi_h(i,\varsigma,\delta))'' - 1\right]\right\} - me^{i\varphi} = F(r(\delta)) \qquad (2.3)$$

and

$$\left(1+me^{i\varphi}\right)\left\{1+\frac{1}{\gamma}\left[\left(\varpi_q(i,\varsigma,A)\right)'+\eth A\left(\varpi_q(i,\varsigma,A)\right)''-1\right]\right\}-me^{i\varphi}=F(u(A)), \tag{2.4}$$

where $g=f^{-1}$.

Define the function s and d as following:

$$s(\delta)=\frac{1+r(\delta)}{1-r(\delta)}=1+n_1\delta+t_2\delta^2+n_3\delta^3+\cdots,\ |n_j|\leq 2 \text{ for all } j\in\mathbb{N}. \tag{2.5}$$

and

$$d(A)=\frac{1+u(A)}{1-u(A)}=1+p_1A+p_2A^2+p_3A^3+\cdots,\ |p_j|\leq 2 \text{ for all } j\in\mathbb{N}. \tag{2.6}$$

or equivalently,

$$r(\delta)=\frac{s(\delta)-1}{s(\delta)+1}=\frac{n_1}{2}\delta+\frac{1}{2}\left(n_2-\frac{n_1^2}{2}\right)\delta^2+\frac{1}{2}\left(n_3+\frac{n_1}{2}\left(\frac{n_1^2}{2}-n_2\right)-\frac{n_1n_2}{2}\right)\delta^3+\cdots$$

amd

$$u(A)=\frac{d(A)-1}{d(A)+1}=\frac{p_1}{2}A+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)A^2+\frac{1}{2}\left(p_3+\frac{p_1}{2}\left(\frac{p_1^2}{2}-p_2\right)-\frac{p_1p_2}{2}\right)A^3+\cdots.$$

By applying the final two equalities found in (2.3) and (2.4), we have

$$\begin{aligned}&\left(1+me^{i\varphi}\right)\left\{1+\frac{1}{\gamma}\left[(\varpi_h(i,\varsigma,\delta))'+\eth\delta\,(\varpi_h(i,\varsigma,\delta))''-1\right]\right\}-me^{i\varphi}\\&=1+\frac{1}{2}E_1n_1\delta+\left(\frac{1}{2}E_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}E_2n_1^2\right)\delta^2+\cdots\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&\left(1+me^{i\varphi}\right)\left\{1+\frac{1}{\gamma}\left[\left(\varpi_q(i,\varsigma,A)\right)'+\eth A\left(\varpi_q(i,\varsigma,A)\right)''-1\right]\right\}-me^{i\varphi}\\&=1+\frac{1}{2}E_1p_1A+\left(\frac{1}{2}E_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}E_2p_1^2\right)A^2+\cdots.\end{aligned} \tag{2.8}$$

On the basis of a comparison between the pertinent coefficients in (2.7) and (2.8), it may be concluded that

$$\frac{2\varsigma e^{-\varsigma}}{\gamma}\left(\eth+1\right)\left(1+me^{i\varphi}\right)d_2=\frac{1}{2}E_1n_1, \tag{2.9}$$

$$\frac{3\cdot 2^i\varsigma^2e^{-\varsigma}}{2!\gamma}\left(2\eth+1\right)\left(1+me^{i\varphi}\right)d_3=\frac{1}{2}E_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}E_2n_1^2, \tag{2.10}$$

$$-\frac{2\varsigma e^{-\varsigma}}{\gamma}\left(\eth+1\right)\left(1+me^{i\varphi}\right)d_2=\frac{1}{2}E_1p_1, \tag{2.11}$$

and

$$\frac{3\cdot 2^i\varsigma^2e^{-\varsigma}}{2!\gamma}\left(2\eth+1\right)\left(1+me^{i\varphi}\right)\left(2d_2^2-d_3\right)=\frac{1}{2}E_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}E_2p_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$n_1=-p_1 \tag{2.13}$$

and

$$2^5\varsigma^2e^{-2\varsigma}\left(\circlearrowleft+1\right)^2\left(1+me^{i\varphi}\right)^2d_2^2=\gamma^2E_1^2(n_1^2+p_1^2). \tag{2.14}$$

If we add (2.10) to (2.12), we get

$$\frac{3\cdot2^i\varsigma^2e^{-\varsigma}}{\gamma}\left(2\circlearrowleft+1\right)\left(1+me^{i\varphi}\right)d_2^2=\frac{1}{2}E_1\left(n_2+p_2\right)+\frac{1}{4}(E_2-E_1)\left(n_1^2+p_1^2\right). \tag{2.15}$$

Taking into consideration both equations (2.14) and (2.15), we may conclude that

$$d_2^2=\frac{\gamma^2E_1^3\left(n_2+p_2\right)}{\varsigma^2e^{-\varsigma}\left[3\cdot2^{i+1}\gamma E_1^2\left(2\circlearrowleft+1\right)\left(1+me^{i\varphi}\right)+2^4e^{-\varsigma}\left(\circlearrowleft+1\right)^2\left(1+me^{i\varphi}\right)^2\left(E_1-E_2\right)\right]} \tag{2.16}$$

Furthermore, by subtracting (2.12) from (2.10) and by viewing (2.13), we are able to establish the following:

$$d_3=\frac{\gamma e^{\varsigma}E_1\left(n_2-p_2\right)}{3\cdot2^{i+1}\varsigma^2\left(2\circlearrowleft+1\right)\left(1+me^{i\varphi}\right)}+\frac{\gamma^2e^{2\varsigma}E_1^2n_1^2}{2^4\varsigma^2\left(\circlearrowleft+1\right)^2\left(1+me^{i\varphi}\right)^2}. \tag{2.17}$$

Additionally, by the utilisation of (2.16) and (2.17) as well as by observing (2.5) and (2.6), we have discovered that

$$|d_2|\leq\frac{\left|\gamma\right|\sqrt{E_1^3}}{\sqrt{\varsigma^2e^{-\varsigma}\left|\left[3\cdot2^{i-1}\gamma\left(2\circlearrowleft+1\right)\left(1+me^{i\varphi}\right)E_1^2+4e^{-\varsigma}\left(\circlearrowleft+1\right)^2\left(1+me^{i\varphi}\right)^2\left(E_1-E_2\right)\right]\right|}}$$

and

$$|d_3|\leq\frac{\left|\gamma\right|e^{\varsigma}E_1}{4}\left(\frac{1}{3\cdot2^{i-3}\varsigma^2\left(2\circlearrowleft+1\right)\left(1+me^{i\varphi}\right)}+\frac{\left|\gamma\right|e^{\varsigma}E_1}{\left(\circlearrowleft+1\right)^2\left(1+me^{i\varphi}\right)^2}\right).$$

All of these assertions are supported by the Theorem 2.1.. □

Fekete and Szegö found a sharp bound on the functional $\left|d_3-\epsilon d_2^2\right|$ for a function $f\in\Omega$ in 1933 [24].

We prove the functional $\left|d_3-\epsilon d_2^2\right|$ for class functions $\otimes^y(m,\varphi,\gamma,\circlearrowleft,\varsigma,i,F)$ using the values of d_2^2 and d_3.

Theorem 2.2. *Let $f\in\Omega$ given by (1.2) belongs to the subclass $\otimes^y(m,\varphi,\gamma,\circlearrowleft,\varsigma,i,F)$. Then*

$$\left|d_3-\epsilon d_2^2\right|\leq\begin{cases}\frac{|\gamma|e^{\varsigma}E_1}{3\cdot2^{i-1}\varsigma^2(2\circlearrowleft+1)(1+me^{i\varphi})} & 0\leq\left|\Theta(\epsilon)\right|\leq\frac{e^{\varsigma}}{3\cdot2^{i+1}\varsigma^2(2\circlearrowleft+1)(1+me^{i\varphi})},\\ \\ 4\left|\gamma\right|\left|\Theta(\epsilon)\right|E_1 & \left|\Theta(\epsilon)\right|\geq\frac{e^{\varsigma}}{3\cdot2^{i+1}\varsigma^2(2\circlearrowleft+1)(1+me^{i\varphi})}.\end{cases}$$

where

$$\Theta(\epsilon)=\frac{\gamma e^{\varsigma}E_1^2\left(1-\epsilon\right)}{2^4\varsigma^2\left[3\cdot2^{i-3}\gamma E_1^2\left(2\circlearrowleft+1\right)\left(1+me^{i\varphi}\right)+e^{-\varsigma}\left(\circlearrowleft+1\right)^2\left(1+me^{i\varphi}\right)^2\left(E_1-E_2\right)\right]}$$

where $\gamma\in\mathbb{C}\backslash\{0\}$; $m\geq0$; $-\pi<\varphi\leq\pi$; $0\leq\circlearrowleft\leq1$; $\varsigma>0$; $i\geq0$.

Proof. We have obtained the following by subtracting (2.12) from (2.10) and by viewing (2.13):

$$d_3 = \frac{\gamma E_1 (n_2 - p_2)}{3 \cdot 2^{i+1} \varsigma^2 e^{-\varsigma} (2Ↄ + 1)(1 + me^{i\varphi})} + d_2^2$$

which ultimately results in writing

$$d_3 - \epsilon d_2^2 = \frac{\gamma E_1 (n_2 - p_2)}{3 \cdot 2^{i+1} \varsigma^2 e^{-\varsigma} (2Ↄ + 1)(1 + me^{i\varphi})} + (1-\epsilon) d_2^2$$

$$d_3 - \epsilon d_2^2 = \gamma E_1 \Bigg(\left[\Theta(\epsilon) + \frac{1}{3 \cdot 2^{i+1} \varsigma^2 e^{-\varsigma} (2Ↄ + 1)(1 + me^{i\varphi})} \right] n_2 + \left[\Theta(\epsilon) - \frac{1}{3 \cdot 2^{i+1} \varsigma^2 e^{-\varsigma} (2Ↄ + 1)(1 + me^{i\varphi})} \right] p_2 \Bigg), \tag{2.18}$$

where

$$\Theta(\epsilon) = \frac{\gamma e^{\varsigma} E_1^2 (1-\epsilon)}{2^4 \varsigma^2 \left[3 \cdot 2^{i-3} \gamma E_1^2 (2Ↄ + 1)(1 + me^{i\varphi}) + e^{-\varsigma} (Ↄ + 1)^2 (1 + me^{i\varphi})^2 (E_1 - E_2) \right]},$$

Then, taking into consideration references (2.5) and (2.6), we come to the conclusion that

$$\left| d_3 - \epsilon d_2^2 \right| \le \begin{cases} \frac{|\gamma| e^{\varsigma} E_1}{3 \cdot 2^{i-1} \varsigma^2 (2Ↄ+1)(1+me^{i\varphi})} & 0 \le \left|\Theta(\epsilon)\right| \le \frac{e^{\varsigma}}{3 \cdot 2^{i+1} \varsigma^2 (2Ↄ+1)(1+me^{i\varphi})}, \\ 4 \left|\gamma\right| \left|\Theta(\epsilon)\right| E_1 & \left|\Theta(\epsilon)\right| \ge \frac{e^{\varsigma}}{3 \cdot 2^{i+1} \varsigma^2 (2Ↄ+1)(1+me^{i\varphi})}. \end{cases}$$

Which are asserted by the Theorem 2.2. □

3. Corollaries

I) If we set $\gamma = e^{i\phi} \cos\phi$, $(-\frac{\pi}{2} < \phi < \frac{\pi}{2})$, $m = 0$ and

$$F(\delta) = \frac{1 + (1-2\varepsilon)\delta}{1-\delta} = 1 + 2(1-\varepsilon)\delta + 2(1-\varepsilon)\delta^2 + \cdots \quad (0 \le \varepsilon < 1)$$

which gives $E_1 = E_2 = 2(1-\varepsilon)$; in Theorems 2.1 and Theorem 2.2, we get the following corollaries.

Corollary 3.1. *Let $f \in \Omega$ given by (1.2) belongs to the class $\otimes^{y}(0, \varphi, e^{i\phi}\cos\phi, Ↄ, \varsigma, i, \frac{1+(1-2\varepsilon)\delta}{1-\delta})$. Then*

$$|d_2| \le \sqrt{\frac{e^{\varsigma}(1-\varepsilon)\cos\phi}{3 \cdot 2^{i-2} \varsigma^2 (2Ↄ + 1)}}, \quad |d_3| \le e^{\varsigma}(1-\varepsilon) \left(\frac{1}{3 \cdot 2^{i-2} \varsigma^2 (2Ↄ + 1)} + \frac{e^{\varsigma}(1-\varepsilon)\cos\phi}{(Ↄ + 1)^2} \right) \cos\phi$$

and

$$\left| d_3 - \epsilon d_2^2 \right| \le \begin{cases} \frac{e^{\varsigma}(1-\varepsilon)\cos\phi}{3 \cdot 2^{i} \varsigma^2 (2Ↄ+1)} & 0 \le 1 - \epsilon \le 1, \\ \frac{e^{\varsigma}(1-\varepsilon)|1-\epsilon|\cos\phi}{3 \cdot 2^{i-2} \varsigma^2 (2Ↄ+1)} & \epsilon \le 0. \end{cases}$$

For $Ↄ = 0$; Corollary 3.1 simplifies to the following Corollary.

Corollary 3.2. *Let $f \in \Omega$ given by (1.2) belongs to the class $\otimes^y(0, \varphi, e^{i\phi}\cos\phi, 0, \varsigma, i, \frac{1+(1-2\varepsilon)\delta}{1-\delta})$. Then*

$$|d_2| \leq \sqrt{\frac{e^{\varsigma}(1-\varepsilon)\cos\phi}{3\cdot 2^{i-2}\varsigma^2}}, \quad |d_3| \leq e^{\varsigma}(1-\varepsilon)\left(\frac{1}{3\cdot 2^{i-2}\varsigma^2} + e^{\varsigma}(1-\varepsilon)\cos\phi\right)\cos\phi$$

and

$$\left|d_3 - \epsilon d_2^2\right| \leq \begin{cases} \frac{e^{\varsigma}(1-\varepsilon)\cos\phi}{3\cdot 2^{i}\varsigma^2} & 0 \leq 1-\epsilon \leq 1, \\ \frac{e^{\varsigma}(1-\varepsilon)|1-\epsilon|\cos\phi}{3\cdot 2^{i-2}\varsigma^2} & \epsilon \leq 0. \end{cases}$$

II) If we set $\gamma = 1$, $(-\frac{\pi}{2} < \phi < \frac{\pi}{2})$, $m = 0$ and

$$F(\delta) = \left(\frac{1+\delta}{1-\delta}\right)^a = 1 + 2a\delta + 2a^2\delta^2 + \cdots \quad (0 < a \leq 1)$$

which gives $E_1 = 2a$ and $E_2 = 2a^2$, in Theorems 2.1 and Theorem 2.2, we get the following corollaries.

Corollary 3.3. *Let $f \in \Omega$ given by (1.2) belongs to the class $\otimes^y(0, \varphi, 1, \eth, \varsigma, i, \left(\frac{1+\delta}{1-\delta}\right)^a)$. Then*

$$|d_2| \leq \frac{2a}{\sqrt{\varsigma^2 e^{-\varsigma}\left(3\cdot 2^{i-2}(2\eth+1)a + e^{-\varsigma}(\eth+1)^2(1-a)\right)}},$$

$$|d_3| \leq e^{\varsigma}a\left(\frac{1}{3\cdot 2^{i-2}\varsigma^2(2\eth+1)} + \frac{e^{\varsigma}a}{(\eth+1)^2}\right)$$

and

$$\left|d_3 - \epsilon d_2^2\right| \leq \begin{cases} \frac{e^{\varsigma}a}{3\cdot 2^{i-2}\varsigma^2(2\eth+1)} & 0 \leq \left|\Theta(\epsilon)\right| \leq \frac{e^{\varsigma}}{3\cdot 2^{i+1}\varsigma^2(2\eth+1)}, \\ \frac{2e^{\varsigma}a^2|1-\epsilon|}{\varsigma^2\left(3\cdot 2^{i-1}(2\eth+1)a + e^{-\varsigma}(\eth+1)^2(1-a)\right)} & \left|\Theta(\epsilon)\right| \geq \frac{e^{\varsigma}}{3\cdot 2^{i+1}\varsigma^2(2\eth+1)}. \end{cases}$$

where

$$\Theta(\epsilon) = \frac{e^{\varsigma}(1-\epsilon)a}{2^3\varsigma^2\left(3\cdot 2^{i-2}(2\eth+1)a + e^{-\varsigma}(\eth+1)^2(1-a)\right)}.$$

For $\eth = 0$; Corollary 3.1 simplifies to the following Corollary.

Corollary 3.4. *Let $f \in \Omega$ given by (1.2) belongs to the class $\otimes^y(0, \varphi, 1, 0, \varsigma, i, \left(\frac{1+\delta}{1-\delta}\right)^a)$. Then*

$$|d_2| \leq \frac{2a}{\sqrt{\varsigma^2 e^{-\varsigma}\left(3\cdot 2^{i-2}a + e^{-\varsigma}(1-a)\right)}}, \quad |d_3| \leq e^{\varsigma}a\left(\frac{1}{3\cdot 2^{i-2}\varsigma^2} + e^{\varsigma}a\right)$$

and

$$\left|d_3 - \epsilon d_2^2\right| \leq \begin{cases} \frac{e^{\varsigma}a}{3\cdot 2^{i-2}\varsigma^2} & 0 \leq \left|\Theta(\epsilon)\right| \leq \frac{1}{3\cdot 2^{i+1}\varsigma^2 e^{-\varsigma}}, \\ \frac{2e^{\varsigma}a^2|1-\epsilon|}{\varsigma^2(3\cdot 2^{i-1}a + e^{-\varsigma}(1-a))} & \left|\Theta(\epsilon)\right| \geq \frac{1}{3\cdot 2^{i+1}\varsigma^2 e^{-\varsigma}}. \end{cases}$$

where

$$\Theta(\epsilon) = \frac{e^{\varsigma}a(1-\epsilon)}{2^3\varsigma^2\left(3\cdot 2^{i-2}a + e^{-\varsigma}(1-a)\right)}.$$

4. Concluding Remark

In the current investigation, we have presented and investigated the coefficient-related issues that are associated with each. These subclasses are all included in the class of bi-univalent functions that are defined in the open unit disc ς. For these bi-univalent function classes, the definitions that correspond to them are supplied in Definition 2.1. In the course of our study, we have included the computation of estimates for the Fekete-Szegö functional issues, as well as the Maclaurin coefficients $|d_2|$ and $|d_3|$ for functions that belong to each of these bi-univalent function classes. When we become more specific about the parameters that were involved in our primary findings, we discover a significant number of additional novel findings.

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