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On Semi α-Lindelöf in Bitopological Spaces

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Abstract. This paper set up a Closure–operator scheme for semi–α–Lindelöfness in bitopological spaces to manage covering behavior generated by two interacting topologies. With the Čech–closure hull $H_{ij} = j\mathrm{cl}\ i\mathrm{int}\ j\mathrm{cl}\ i\mathrm{int}$, we reformulate ij–semi–α–open sets and obtain operator–level criteria for ij–semi–α–Lindelöfness. We prove a network estimate that bounds $L_{ij}^{S_\alpha}$ by the size of an ij–S_α–network, and a star criterion under ρ–discrete network decompositions of such networks. Structural consequences include hereditary and transfer over dense subsets, stability under countable sums, and a tube–type product when the second topology is discrete and the first factor is i–compact. Also, we introduce ij–S_α–perfect mappings and show preservation of ij–S_α–Lindelöfness with explicit cardinal bounds; images under ij–S_α– and ij–S_α^*–continuous maps are correspondingly controlled. Pairwise invariants are examined via $\widehat{L}_{\mathrm{pair}}^{S_\alpha}$, which lies between the one–sided quantities and equals their maximum whenever at least one is infinite.

1. Introduction

Contemporary scholarship on almost open sets begins with Levine's semi-open sets and Njastad's α-open sets [20,23]. Building on these, Navalagi developed semi-α-open sets, by compining the semi and α-open to represent a category that is solely amongst α-open and semi-open families [22]. Since then, the concept has spread in multiple directions: variants such as semi*α and simply*α sets and their associated mappings, compactness and Lindelöfness, and transfers to soft,

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neutrosophic, nano, and rough-set structure [5,6,9,11,13,14,26–28].

Through a conventional compactness to a dynamic covering foundation, the Lindelöf property has evolved. The evolution of it followed two stages: expanding the background classifications (broader topologies, soft and nano conditions, fuzzy and neutrosophic models, and bitopological spaces) and improving the covering classes (nearly open, α-open, semi-open, semi-α). Through the use of continuity and idealistic assumptions, scholars were able to extract Lindelöf, introduced cardinal invariants tied to bases and networks, and established stability under sums, images, and controlled products, often via continuity and perfect-type hypotheses. In this article we position the semi-α–Lindelöf property within bitopology, define and compare its cardinal invariants, give base and network tests, prove preservation and decomposition results, and supply examples that separate the main implications [2–4,7,8,10,19,24].

On identical set, bitopology records two conflicting different types of openness. Between ordinary i–open sets and the broader ij–pre– and ij–semi–open sets one finds the ij–α–open family [10,17]. In this paper, we examine the Lindelöf phenomena for the semi–α level in the bitopological context, with emphasis on cardinal invariants and stability under standard constructions.

We begin by transforming the family of ij–semi–α–open sets into preopen sets of a Čech–closure space using the hull $H_{ij} = j\text{cl } i\text{int } j\text{cl } i\text{int}$, The functional convenience of this operator approach is that it enables us to transfer many Lindelöf arguments that are expressed using the concept of reductions local bases.

We work throughout with $i \neq j \in \{1,2\}$; the topological index is used to indicate interiors and closures. In this article we introduce the ij–semi–α–Lindelöf number $L_{ij}^{S_\alpha}(\mathcal{M})$ and its pairwise version, and relate them to the classical i–Lindelöf number and to the α– and pre–Lindelöf numbers. Obtain network and star criteria: a countable ij–S_α–network implies ij–S_α–Lindelöf, and a ρ–locally finite ij–S_α–network yields the star variant, prove hereditary and dense-set transfer, and stability under countable sums, introduce ij–S_α–perfect functions and show they preserve ij–S_α–Lindelöfness with explicit cardinal bounds. For pairwise covers we establish sharp inequalities and identify conditions under which the pairwise number equals $\max\{L_{12}^{S_\alpha}, L_{21}^{S_\alpha}\}$, Additionally, simulations based on ordinal, Sorgenfrey, discrete, and co-countable structures demonstrate that our predictions are almost optimal by separating all concepts.

Our results refine the known relationships among ij–pre–, ij–semi–, and ij–α–openness [1,10,17] by isolating the covering behavior that is specific to the semi–α level. The H_{ij} foundation allows standard Lindelof proofs to adjust with little modification while maintaining the explicitness of truly bitopological consequences (the j–closure stages).

The hull H_{ij} is introduced in section 2, which also fixes notation and recalls the open-set classes. Section 3 develops the semi–α–Lindelöf number, network and star criteria, sums and a tube product, hereditary and dense-set transfer, image theorems, and ij–S_α–perfect maps together with cardinal bounds. The section closes with examples that separate implications and calibrate the sharpness of our assumptions.

2. Preliminaries

In this section we fix notation and basic tools for the bitopological spaces and recall the ij–pre, ij–semi, and ij–α–open classes and record their standard characterizations, then define the ij–semi–α–open family together with an equivalent hull form. We introduce the three ij–semi–α continuity variants and the pairwise Lindelöf perfect maps that will mediate our transfer results. Finally, we package the semi–α calculus into the Čech–closure hull H_{ij}, note its monotonicity and extensiveness, and identify ij–S_α–covers with covers by H_{ij}–preopen sets.

Definition 2.1. *Let $(\mathcal{M}, \gamma_1, \gamma_2)$ be bitopological and $A \subseteq \mathcal{M}$, with $i \neq j \in \{1, 2\}$. Then*

1. *ij–pre–open [17] if $A \subseteq i\text{-int}\Big(j\text{-cl}(A)\Big)$;*
2. *ij–semi–open [10] if $A \subseteq j\text{-cl}\Big(i\text{-int}(A)\Big)$;*
3. *ij–α–open [29] if $A \subseteq i\text{-int}\Big(j\text{-cl}(i\text{-int}(A))\Big)$.*

Proposition 2.1 ([21]). *A is ij–semi–open iff $\exists\, U \in \gamma_i$ with $U \subseteq A \subseteq j\text{-cl}(U)$.*

Proposition 2.2 ([15]). *A is ij–pre–open iff $\exists\, U \in \gamma_i$ with $A \subseteq U \subseteq j\text{-cl}(A)$.*

Theorem 2.1. *[1] A is ij–α–open iff A is both ij–semi–open and ij–pre–open.*

Definition 2.2 ([1]). *$A \subseteq \mathcal{M}$ is* ij–semi–α–open *if there exists $U \in ij\text{-}\alpha\mathrm{O}(\mathcal{M})$ with $U \subseteq A \subseteq j\text{-cl}(U)$. Denote the family by $ij\text{-}S_\alpha O(\mathcal{M})$.*

Proposition 2.3 ([1]). *A is ij–semi–α–open iff*

$$A \subseteq j\text{-cl}\Big(i\text{-int}\Big(j\text{-cl}\Big(i\text{-int}(A)\Big)\Big)\Big).$$

Definition 2.3 ([2]). *A function $\Gamma : (M, \gamma_1, \gamma_2) \to (C, \rho_1, \rho_2)$ is pairwise Lindelöf perfect if Γ is closed, continuous, and $\Gamma^{-1}(c)$ is pairwise Lindelöf in the pairwise Lindelöf sense.*

Definition 2.4. *[1] Let $\Gamma : (\mathcal{M}, \gamma_1, \gamma_2) \to (\mathcal{C}, \rho_1, \rho_2)$ and $i \neq j \in \{1, 2\}$. Then*

1. *Γ is* ij–semi–α–continuous *iff $f^{-1}(V) \in ij\text{-}S_\alpha O(\mathcal{M})$ for every $V \in \rho_i$.*
2. *Γ is* ij–semi–α^*–continuous *iff $\Gamma^{-1}(W) \in ij\text{-}S_\alpha O(\mathcal{M})$ for every $W \in ij\text{-}S_\alpha O(\mathcal{C})$.*
3. *Γ is* ij–semi–α'–continuous *iff $\Gamma^{-1}(W) \in \gamma_i$ for every $W \in ij\text{-}S_\alpha O(\mathcal{C})$.*

Definition 2.5. *[22] For $A \subseteq \mathcal{M}$ set*

$$H_{ij}(A) := j\text{-cl}\Big(i\text{-int}(j\text{-cl}(i\text{-int}A))\Big).$$

Then A is ij–semi–α–open iff $A \subseteq H_{ij}(A)$.

Proposition 2.4. *[25] H_{ij} is monotone and extensive, i.e. $A \subseteq B \Rightarrow H_{ij}(A) \subseteq H_{ij}(B)$ and $A \subseteq H_{ij}(A)$. Hence $(\mathcal{M}, H_{ij})$ is a Čech closure space. ij–S_α–covers coincide with covers by H_{ij}–preopen sets.*

3. ij–Semi–α–Lindelöf in Bitopological Spaces

This section introduces the ij–S_α–Lindelöf property for sets and spaces and the cardinal invariants $L_{ij}^{S_\alpha}$ and $\widehat{L}_{\text{pair}}^{S_\alpha}$. We establish monotonicity and refinement rules, network and σ–discrete star criteria, and stability under closed subspaces, dense–set transfer, finite unions, countable sums, and a product when ρ_j is discrete and the second factor is i–compact.

Definition 3.1. *Let $(\mathcal{M}, \gamma_1, \gamma_2)$ be bitopological space and $A \subseteq \mathcal{M}$. A is ij–S_α–*Lindelöf *if every ij–S_α–open cover of A contains a countable subcover.*
Define

$$L_{ij}^{S_\alpha}(A) := \min\{\kappa : \textit{ every } ij\text{–}S_\alpha\textit{–open cover of } A \textit{ has a subcover of size } \leq \kappa\}.$$

And write $L_{ij}^{S_\alpha}(\mathcal{M})$ when $A = \mathcal{M}$.
The pairwise invariant is $L_{\text{pair}}^{S_\alpha}(A) = \max\{L_{12}^{S_\alpha}(A),\, L_{21}^{S_\alpha}(A)\}$.

Proposition 3.1. *For $A \subseteq B \subseteq \mathcal{M}$:*

$$L_{ij}^{S_\alpha}(A) \leq L_{ij}^{S_\alpha}(B) \leq L_{ij}^{S_\alpha}(\mathcal{M}).$$

If A is ij–S_α–Lindelöf then A is ij–α–Lindelöf.

Proof. Let $A \subseteq B \subseteq \mathcal{M}$. Any ij–S_α–cover of B also covers A. Hence subcover bounds transfer.

$$L_{ij}^{S_\alpha}(A) \leq L_{ij}^{S_\alpha}(B).$$

Applying the same argument to $B \subseteq \mathcal{M}$ gives $L_{ij}^{S_\alpha}(B) \leq L_{ij}^{S_\alpha}(\mathcal{M})$.
The implication follows since ij–α–open $\subseteq$ ij–S_α–open. □

Proposition 3.2. *If $\mathcal{W}$ is an ij–S_α–open cover of A and $\mathcal{U} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ refines $\mathcal{W}$, then $\mathcal{U}$ is an ij–S_α–open cover of A.*

Proof. Let $\mathcal{W}$ be an ij–S_α–open cover of A and let $\mathcal{U} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ refine $\mathcal{W}$ (relative to A). For each $a \in A$ choose $W_a \in \mathcal{W}$ with $a \in W_a$. By refinement, there exists $U_a \in \mathcal{U}$ with $a \in U_a \subseteq W_a$. Hence $a \in \bigcup \mathcal{U}$, so $\mathcal{U}$ covers A. Since every member of $\mathcal{U}$ is ij–S_α–open, $\mathcal{U}$ is an ij–S_α–open cover of A. □

Definition 3.2. *A bitopological space $(\mathcal{M}, \gamma_1, \gamma_2)$ is ij–*semi–α–Lindelöf *iff every ij–semi–α–open cover of $\mathcal{M}$ has a countable subcover. Define the cardinal invariant*

$$L_{ij}^{S_\alpha}(\mathcal{M}) := \min\Big\{\kappa : \textit{ every } ij\text{–}S_\alpha\textit{–open cover of } \mathcal{M} \textit{ has a subcover of size } \leq \kappa\Big\}.$$

The space $\mathcal{M}$ is pairwise S_α–Lindelöf *if it is both 12–S_α–Lindelöf and 21–S_α–Lindelöf; set $L_{\text{pair}}^{S_\alpha}(\mathcal{M}) := \max\{L_{12}^{S_\alpha}(\mathcal{M}), L_{21}^{S_\alpha}(\mathcal{M})\}$.*

Remark 3.1. *Every ij–semi–α–Lindelöf space is i–Lindelöf. Consequently $L_i(\mathcal{M}) \leq L_{ij}^{S_\alpha}(\mathcal{M})$.*

Definition 3.3. *A family $\mathcal{N}$ is an ij–S_α–network if for every ij–S_α–open W and every $m \in W$ there exists $N \in \mathcal{N}$ with $m \in N \subseteq W$.*

Definition 3.4. *A subfamily $\mathcal{P} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ is an ij–S_α–π–base if for every nonempty $V \in \gamma_i$ there exists $P \in \mathcal{P}$ with $P \subseteq H_{ij}(V)$.*

Theorem 3.1. *If $\mathcal{M}$ has an ij–S_α–network $\mathcal{N}$, then $L_{ij}^{S_\alpha}(\mathcal{M}) \leq |\mathcal{N}|$.*

Proof. Let $\mathcal{W} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ cover $\mathcal{M}$. Set $\mathcal{N}' := \{N_m : m \in \mathcal{M}\} \subseteq \mathcal{N}$. For each $N \in \mathcal{N}'$ select one $W_N \in \mathcal{W}$ with $N \subseteq W_N$. Then $\{W_N : N \in \mathcal{N}'\}$ covers $\mathcal{M}$, and

$$\Big|\{W_N : N \in \mathcal{N}'\}\Big| \leq |\mathcal{N}'| \leq |\mathcal{N}|.$$

Hence $L_{ij}^{S_\alpha}(\mathcal{M}) \leq |\mathcal{N}|$. □

Definition 3.5. *A space $\mathcal{M}$ is ij–S_α–star–Lindelöf if for every ij–S_α–open cover $\mathcal{U}$ there exists a countable subfamily $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}$ such that*

$$\mathrm{St}\Big(\{U_n : n \in \mathbb{N}\}, \mathcal{U}\Big) = \mathcal{M}.$$

Theorem 3.2. *If $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$ is an ij–S_α–network with each $\mathcal{N}_n$* discrete, *then $\mathcal{M}$ is ij–S_α–star–Lindelöf.*

Proof. Let $\mathcal{U} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ be a cover of $\mathcal{M}$. For each $N \in \mathcal{N}$ choose $U_N \in \mathcal{U}$ with $N \subseteq U_N$.

Fix $n \in \mathbb{N}$. Since $\mathcal{N}_n$ is discrete, the family

$$\Big\{\mathrm{St}(U_N, \mathcal{U}) : N \in \mathcal{N}_n\Big\}$$

covers $\bigcup \mathcal{N}_n$ and is pointwise finite. There exists a countable subfamily $\{U_{N_k^n}\}_{k\in\mathbb{N}}$ whose cover $\bigcup \mathcal{N}_n$.

Now set a countable $\mathcal{U}^* := \{U_{N_k^n} : n, k \in \mathbb{N}\}$. Fix $m \in \mathcal{M}$ and pick $W \in \mathcal{U}$ with $m \in W$. Because $\mathcal{N}$ is an ij–S_α–network, there exists $N \in \mathcal{N}$ with $m \in N \subseteq W$. This N lies in $\mathcal{N}_n$, hence $N \subseteq \mathrm{St}(U_{N_k^n}, \mathcal{U})$ for some k. Consequently W meets $U_{N_k^n}$ and therefore $m \in \mathrm{St}(\mathcal{U}^*, \mathcal{U})$. Since m was arbitrary, $\mathrm{St}(\mathcal{U}^*, \mathcal{U}) = \mathcal{M}$. □

Remark 3.2. *If $(\mathcal{M}, \gamma_1, \gamma_2)$ is ij–semi–α–compact, then it is ij–semi–α–Lindelöf and $L_{ij}^{S_\alpha}(\mathcal{M}) \leq \aleph_0$.*

Definition 3.6. *The invariant $\widehat{L}_{\text{pair}}^{S_\alpha}(\mathcal{M})$ is the least cardinal κ such that every pairwise S_α–open cover of $\mathcal{M}$ admits a subcover of size $\leq \kappa$; equivalently,*

$$\widehat{L}_{\text{pair}}^{S_\alpha}(X) := \min\Big\{\kappa : \text{ every pairwise } S_\alpha\text{–open cover of X has a subcover of size } \leq \kappa\Big\}.$$

Proposition 3.3.

$$\max\{L_{12}^{S_\alpha}(\mathcal{M}), L_{21}^{S_\alpha}(\mathcal{M})\} \leq \widehat{L}_{\text{pair}}^{S_\alpha}(\mathcal{M}) \leq L_{12}^{S_\alpha}(\mathcal{M}) + L_{21}^{S_\alpha}(\mathcal{M}).$$

Example 3.1. *Consider $\mathcal{M} = \mathbb{R}$, $\gamma_1 = \gamma_{\text{Euc}}$, $\gamma_2 = \gamma_{\text{disc}}$, and $(i, j) = (1, 2)$. Then $12\text{–}S_\alpha O(\mathcal{M}) = \gamma_1$ since for j discrete,*

$$A \subseteq 2\text{-cl}\,1\text{-int}\,2\text{-cl}\,1\text{-int}A = 1\text{-int}A \iff A \in \gamma_1.$$

Hence $L_{12}^{S_\alpha}(\mathcal{M}) = \aleph_0$. For $(i, j) = (2, 1)$, 2 is discrete, so for each $A \subseteq \mathcal{M}$,

$$2\text{-int}\,1\text{-cl}\,2\text{-int}A = 1\text{-cl}(A), \quad 1\text{-cl}(1\text{-cl}A) = 1\text{-cl}A,$$

and $A \subseteq$ 1-cl$A$*; thus every subset is* 21–S_α*–open. The cover by singletons yields* $L_{21}^{S_\alpha}(X) = 2^{\aleph_0}$*. By Proposition 3.3,*

$$\widehat{L}_{\text{pair}}^{S_\alpha}(\mathcal{M}) = 2^{\aleph_0}.$$

Example 3.2. *Consider* $\mathcal{M} = [0, \omega_1)$ *with* γ_1 *the order topology and* $\gamma_2 = \gamma_{\text{disc}}$*. As above,* 12–$S_\alpha O(\mathcal{M}) = \gamma_1$*, so*

$$L_{12}^{S_\alpha}(\mathcal{M}) = \omega_1$$

witnessed by the classical cover $\{[0, \alpha) : \alpha < \omega_1\}$*. For* $(i, j) = (2, 1)$*, every subset is* 21–S_α*–open, hence* $L_{21}^{S_\alpha}(\mathcal{M}) = |\mathcal{M}| = \omega_1$*. Therefore* $\widehat{L}_{\text{pair}}^{S_\alpha}(\mathcal{M}) = \omega_1$*.*

Theorem 3.3. *For every bitopological space* $\mathcal{M}$*,*

$$\max\{L_{12}^{S_\alpha}(\mathcal{M}), L_{21}^{S_\alpha}(\mathcal{M})\} \le \widehat{L}_{\text{pair}}^{S_\alpha}(\mathcal{M}) \le L_{12}^{S_\alpha}(\mathcal{M}) + L_{21}^{S_\alpha}(\mathcal{M}).$$

If at least one of $L_{12}^{S_\alpha}(\mathcal{M})$ *or* $L_{21}^{S_\alpha}(\mathcal{M})$ *is infinite, then*

$$\widehat{L}_{\text{pair}}^{S_\alpha}(\mathcal{M}) = \max\{L_{12}^{S_\alpha}(\mathcal{M}), L_{21}^{S_\alpha}(\mathcal{M})\}.$$

Proof. For the lower bound, note that a pairwise S_α–open cover may consist solely of 12–S_α–open sets (or solely of 21–S_α–open sets). Hence any universal bound for pairwise covers must dominate both $L_{12}^{S_\alpha}(\mathcal{M})$ and $L_{21}^{S_\alpha}(\mathcal{M})$, giving

$$\max\{L_{12}^{S_\alpha}(\mathcal{M}),\ L_{21}^{S_\alpha}(\mathcal{M})\} \le \widehat{L}_{\text{pair}}^{S_\alpha}(\mathcal{M}).$$

For the upper bound, let $\mathcal{W}$ be a pairwise S_α–open cover. Decompose

$$\mathcal{W} = \mathcal{W}_{12} \cup \mathcal{W}_{21}, \quad \mathcal{W}_{12} \subseteq 12\text{-}S_\alpha O(\mathcal{M}), \ \ \mathcal{W}_{21} \subseteq 21\text{-}S_\alpha O(\mathcal{M}).$$

Set $A = \bigcup \mathcal{W}_{12}$ and $B = \mathcal{M} \setminus A$. Then $B \subseteq \bigcup \mathcal{W}_{21}$. By the definitions of $L_{12}^{S_\alpha}$ and $L_{21}^{S_\alpha}$ there exist

$$\mathcal{V}_{12} \subseteq \mathcal{W}_{12}, \quad |\mathcal{V}_{12}| \le L_{12}^{S_\alpha}(\mathcal{M}), \text{ covering } A,$$

$$\mathcal{V}_{21} \subseteq \mathcal{W}_{21}, \quad |\mathcal{V}_{21}| \le L_{21}^{S_\alpha}(\mathcal{M}), \text{ covering } B.$$

Thus $\mathcal{V}_{12} \cup \mathcal{V}_{21}$ covers $\mathcal{M}$ and

$$\widehat{L}_{\text{pair}}^{S_\alpha}(\mathcal{M}) \le |\mathcal{V}_{12} \cup \mathcal{V}_{21}| \le L_{12}^{S_\alpha}(\mathcal{M}) + L_{21}^{S_\alpha}(\mathcal{M}).$$

If at least one of $L_{12}^{S_\alpha}(\mathcal{M})$ or $L_{21}^{S_\alpha}(\mathcal{M})$ is infinite, then for cardinals

$$L_{12}^{S_\alpha}(\mathcal{M}) + L_{21}^{S_\alpha}(\mathcal{M}) = \max\{L_{12}^{S_\alpha}(\mathcal{M}),\ L_{21}^{S_\alpha}(\mathcal{M})\}.$$

Combining with the lower bound yields the stated equality. □

Proposition 3.4. *Every* ij*–semi–*α*–closed subset of an* ij*–semi–*α*–Lindelöf space is* ij*–semi–*α*–Lindelöf.*

Proof. Let $A \subseteq \mathcal{M}$ be ij–S_α–closed and assume $\mathcal{M}$ is ij–S_α–Lindelöf. Given an ij–S_α–open cover $\mathcal{W}$ of A, the family $\mathcal{W} \cup \{\mathcal{M} \setminus A\}$ is an ij–S_α–open cover of $\mathcal{M}$. By ij–S_α–Lindelöfness, it has a countable subcover $\{W_n : n \in \mathbb{N}\} \cup \{\mathcal{M} \setminus A\}$. Dropping $\mathcal{M} \setminus A$ (if selected) leaves a countable subfamily of $\mathcal{W}$ that covers A. Hence A is ij–S_α–Lindelöf. □

Definition 3.7. *A subset $D \subseteq \mathcal{M}$ is ij–S_α–dense if its ij–S_α–hull equals $\mathcal{M}$, i.e., $H_{ij}(D) = \mathcal{M}$.*

Proposition 3.5. *If $D \subseteq \mathcal{M}$ is ij–S_α–dense and ij–S_α–Lindelöf, then $\mathcal{M}$ is ij–S_α–Lindelöf and*

$$L_{ij}^{S_\alpha}(\mathcal{M}) \leq L_{ij}^{S_\alpha}(D).$$

Proof. Let $\mathcal{W} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ cover $\mathcal{M}$. The trace

$$\mathcal{W}\restriction D := \{W \cap D : W \in \mathcal{W}\}$$

is an ij–S_α–open cover of D. By ij–S_α–Lindelöfness of D there exists a subfamily $\{W_\gamma : \gamma < \kappa\} \subseteq \mathcal{W}$ with $\kappa \leq L_{ij}^{S_\alpha}(D)$ such that $D \subseteq \bigcup_{\gamma<\kappa}(W_\gamma \cap D)$. For ij–S_α–open U,

$$\mathcal{M} = H_{ij}(D) \subseteq H_{ij}\left(\bigcup_{\gamma<\kappa}(W_\gamma \cap D)\right) \subseteq \bigcup_{\gamma<\kappa} H_{ij}(W_\gamma \cap D) \subseteq \bigcup_{\gamma<\kappa} H_{ij}(W_\gamma) = \bigcup_{\gamma<\kappa} W_\gamma.$$

Thus $\{W_\xi : \xi < \kappa\}$ is a subcover of $\mathcal{M}$ of size $\leq L_{ij}^{S_\alpha}(D)$. Hence $\mathcal{M}$ is ij–S_α–Lindelöf and $L_{ij}^{S_\alpha}(X) \leq L_{ij}^{S_\alpha}(D)$. □

Example 3.3. *Consider $\mathcal{M} = \mathbb{R}$, $\gamma_1 = \gamma_{\text{Euc}}$, $\gamma_2 = \gamma_{\text{coc}}$, and $(i, j) = (1, 2)$. Set $D = (0, 1)$. Then $1\text{–}int(D) = D \neq \emptyset$ and $2\text{–}cl(D) = \mathcal{M}$. Hence*

$$H_{12}(D) = \text{2-cl}\Big(\text{1-int}\,\text{2-cl}(\text{1-int}D)\Big) = \text{2-cl}\Big(\text{1-int}\,\mathcal{M}\Big) = \text{2-cl}(\mathcal{M}) = \mathcal{M},$$

so D is 12–S_α–dense. But D is not 1–dense in $(\mathcal{M}, \gamma_1)$, since $1\text{–}cl(D) = [0, 1] \neq \mathcal{M}$.

Example 3.4. *For each $\alpha < \omega_1$ let $I_\alpha = [0, 1]$ with the Euclidean topology. Define*

$$\mathcal{M} = \bigsqcup_{\alpha<\omega_1} I_\alpha, \quad \gamma_1 = \textit{topological sum of Euclidean topologies},$$

$$\gamma_2 = \textit{topological sum of co–countable topologies}.$$

Fix $(i, j) = (1, 2)$ and define $D = \bigsqcup_{\alpha<\omega_1}(0, \frac{1}{2}) \subseteq \mathcal{M}$. Then for each α, $1\text{–}int(D \cap I_\alpha) = (0, \frac{1}{2}) \neq \emptyset$ and $2\text{–}cl(D \cap I_\alpha) = I_\alpha$. Therefore $2\text{–}cl(1\text{–}int\,D) = \mathcal{M}$, and as in Example 3.3 we get $H_{12}(D) = X$. Thus D is 12–S_α–dense in X. Yet D is not 1–dense, since in each component $1\text{–}cl(D \cap I_\alpha) = [0, \frac{1}{2}]$ is a proper subset of I_α.

Theorem 3.4. *Let $(\mathcal{M}, \gamma_1, \gamma_2)$ be bitopological and fix $i \neq j$. For $D \subseteq \mathcal{M}$ the following are equivalent:*

(1) *D is ij–S_α–dense, i.e. $H_{ij}(D) = \mathcal{M}$.*
(2) *Every nonempty ij–S_α–open set meets D.*
(3) *$j\text{–}cl\Big(i\text{–}int(D)\Big) = \mathcal{M}$.*

Moreover, for every $A \subseteq \mathcal{M}$,

$$H_{ij}^{A}(D \cap A) = A \cap H_{ij}^{\mathcal{M}}(D),$$

so if D is ij–S_α–dense in $\mathcal{M}$ and A is any subspace, then $D \cap A$ is ij–S_α–dense in A.

Proof. (1)⇒(2): If $H_{ij}(D) = \mathcal{M}$ and W is nonempty ij–S_α–open with $W \cap D = \varnothing$, this contradicts the hull–based characterization of ij–S_α–openness. Hence every nonempty ij–S_α–open set meets D.

(2)⇒(3): Suppose j–cl$(i$–int $D) \neq \mathcal{M}$. Then some $c \in \mathcal{M}$ admits a j–open $V \ni c$ with $V \cap i$–int $D = \varnothing$. By definition, i–int V is ij–S_α–open and misses D, contradicting (2). Thus j–cl$(i$–int $D) = \mathcal{M}$.

(3)⇒(1): Using the standard formula for the ij–S_α–hull,

$$H_{ij}(D) = j\text{-cl}\Big(i\text{-int}\, j\text{-cl}(i\text{-int}D)\Big) = j\text{-cl}\Big(i\text{-int}\,\mathcal{M}\Big) = j\text{-cl}(\mathcal{M}) = \mathcal{M}.$$

For the subspace identity, use the subspace rules

$$i\text{-int}_A(E) = A \cap i\text{-int}_{\mathcal{M}}(E), \qquad j\text{-cl}_A(E) = A \cap j\text{-cl}_{\mathcal{M}}(E),$$

and compute directly:

$$\begin{aligned} H_{ij}^A(D \cap A) &= j\text{-cl}_A\Big(i\text{-int}_A\, j\text{-cl}_A\Big(i\text{-int}_A(D \cap A)\Big)\Big) \\ &= A \cap j\text{-cl}_{\mathcal{M}}\Big(i\text{-int}_{\mathcal{M}}\, j\text{-cl}_{\mathcal{M}}\Big(i\text{-int}_{\mathcal{M}} D\Big)\Big) = A \cap H_{ij}^{\mathcal{M}}(D). \end{aligned}$$

This also yields the final statement. □

Proposition 3.6. *If A and B are ij–semi–α–Lindelöf subsets of $(\mathcal{M}, \gamma_1, \gamma_2)$, then $A \cup B$ is ij–semi–α–Lindelöf.*

Proof. Let $\mathcal{W} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ cover $A \cup B$. Since A is ij–S_α–Lindelöf, there exists a countable subfamily $\{W_n\}_{n \in \mathbb{N}} \subseteq \mathcal{W}$ covering A. Similarly, there is a countable subfamily $\{V_n\}_{n \in \mathbb{N}} \subseteq \mathcal{W}$ covering B. The union $\{W_n : n \in \mathbb{N}\} \cup \{V_n : n \in \mathbb{N}\}$ is countable and covers $A \cup B$. Hence $A \cup B$ is ij–S_α–Lindelöf. □

Corollary 3.1. *Finite unions of ij–semi–α–Lindelöf subsets are ij–semi–α–Lindelöf.*

Remark 3.3. *For $A, B \subseteq \mathcal{M}$,*

$$L_{ij}^{S_\alpha}(A \cup B) \;\le\; L_{ij}^{S_\alpha}(A) \;+\; L_{ij}^{S_\alpha}(B).$$

In particular, if both are $\le \aleph_0$, then $A \cup B$ is ij–S_α–Lindelöf.

Lemma 3.1. *Let $(\mathcal{M}, \gamma_1, \gamma_2)$, (C, ρ_1, ρ_2) be bitopological and fix $i \neq j$. Assume ρ_j is discrete. If $W \in (i, j)$–$S_\alpha O(\mathcal{M} \times C)$ and $(m, c) \in W$, then there exist $U \in ij$–$S_\alpha O(\mathcal{M})$ and $V \in \rho_i$ with $(m, c) \in U \times V \subseteq W$.*

Theorem 3.5. *If $\{\mathcal{M}_k\}_{k \in \mathbb{N}}$ are ij–S_α–Lindelöf, then the topological sum $\bigoplus_k \mathcal{M}_k$ with componentwise (γ_1, γ_2) is ij–S_α–Lindelöf and*

$$L_{ij}^{S_\alpha}\Big(\bigoplus_k \mathcal{M}_k\Big) = \sup_k L_{ij}^{S_\alpha}(\mathcal{M}_k).$$

Proof. Let $\mathcal{W} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ cover $\mathcal{M}$. For each k, the trace

$$\mathcal{W}_k := \{W \cap \mathcal{M}_k : W \in \mathcal{W}\}$$

is an ij–S_α–open cover of $\mathcal{M}_k$. Since $\mathcal{M}_k$ is ij–S_α–Lindelöf, choose a countable subfamily $\mathcal{V}_k \subseteq \mathcal{W}$ with $\bigcup \mathcal{V}_k \supseteq \mathcal{M}_k$. Then $\mathcal{V} := \bigcup_{k\in\mathbb{N}} \mathcal{V}_k$ is countable and covers $\mathcal{M}$, so $\mathcal{M}$ is ij–S_α–Lindelöf.

Set $\kappa_k := L_{ij}^{S_\alpha}(\mathcal{M}_k)$ and $\kappa := \sup_k \kappa_k$. For the lower bound, $\mathcal{M}_k$ is a clopen subspace of $\mathcal{M}$, so by monotonicity

$$\kappa_k \leq L_{ij}^{S_\alpha}(\mathcal{M}) \quad \text{for all } k, \qquad \Rightarrow \quad \sup_k \kappa_k \leq L_{ij}^{S_\alpha}(\mathcal{M}).$$

For the upper bound, from the construction above we may for each k select $\mathcal{U}_k \subseteq \mathcal{W}$ with $|\mathcal{U}_k| \leq \kappa_k$ and $\bigcup \mathcal{U}_k \supseteq \mathcal{M}_k$. Then

$$\mathcal{U} := \bigcup_{k\in\mathbb{N}} \mathcal{U}_k$$

covers $\mathcal{M}$ and satisfies $|\mathcal{U}| \leq \kappa \cdot \aleph_0$. If κ is infinite, cardinal arithmetic gives $\kappa \cdot \aleph_0 = \kappa$, hence $L_{ij}^{S_\alpha}(\mathcal{M}) \leq \kappa$. Combining with the lower bound yields

$$L_{ij}^{S_\alpha}(\mathcal{M}) = \sup_k L_{ij}^{S_\alpha}(\mathcal{M}_k).$$

□

Theorem 3.6. *Assume ρ_j on C is discrete. If $(\mathcal{M}, \gamma_1, \gamma_2)$ is ij–S_α–Lindelöf and (C, ρ_1, ρ_2) is i–compact, then $(\mathcal{M} \times C, \gamma_i \times \rho_i, \gamma_j \times \rho_j)$ is (i, j)–S_α–Lindelöf.*

Proof. Let $\mathcal{W} \subseteq (i, j)$–$S_\alpha O(\mathcal{M} \times C)$ cover $\mathcal{M} \times C$. For each $(m, c) \in \mathcal{M} \times C$ pick $W_{m,c} \in \mathcal{W}$ with $(m, c) \in W_{m,c}$. By Lemma 3.1 there exist $U_{m,c} \in ij$–$S_\alpha O(\mathcal{M})$ and $V_{m,c} \in \rho_i$ with $(m, c) \in U_{m,c} \times V_{m,c} \subseteq W_{m,c}$.

Fix $m \in \mathcal{M}$. Then $\{V_{m,c} : c \in C\}$ is an i–open cover of C. By i–compactness of C choose $c_1, \ldots, c_{g(m)}$ with $C = \bigcup_{\ell=1}^{g(m)} V_{m,c_\ell}$. Set $U_m := \bigcap_{\ell=1}^{g(m)} U_{m,c_\ell} \in ij$–$S_\alpha O(\mathcal{M})$ (finite intersections are allowed since $U_m \subseteq U_{m,c_\ell}$ will only be used to form rectangles), and note that $\{U_m : m \in \mathcal{M}\} \subseteq ij$–$S_\alpha O(\mathcal{M})$ covers $\mathcal{M}$ because for any m we have $m \in U_m$.

Since $\mathcal{M}$ is ij–S_α–Lindelöf, select a countable set $I \subseteq \mathcal{M}$ such that $\{U_m : m \in I\}$ covers $\mathcal{M}$. Then the countable family of rectangles

$$\mathcal{R} := \left\{ U_m \times V_{m,c_\ell} \ : \ m \in I,\ 1 \leq \ell \leq g(m) \right\}$$

covers $\mathcal{M} \times C$, and each member of $\mathcal{R}$ is contained in some $W_{m,c_\ell} \in \mathcal{W}$. Hence $\mathcal{W}$ has a countable subcover. □

Definition 3.8. *A bitopological space $\mathcal{M}$ is locally ij–S_α–Lindelöf if every $m \in \mathcal{M}$ admits $U \in ij$-$S_\alpha O(\mathcal{M})$ with $m \in U$ and U ij–S_α–Lindelöf.*

Proposition 3.7. *If $\mathcal{M} = \bigcup_n U_n$ with each U_n ij–S_α–Lindelöf and $\{U_n\}$ ij–S_α–locally finite, then $\mathcal{M}$ is ij–S_α–Lindelöf.*

Proof. Let $\mathcal{W} \subseteq ij$-$S_\alpha O(\mathcal{M})$ cover $\mathcal{M}$. For each n, the trace

$$\mathcal{W} \restriction U_n := \{W \cap U_n : W \in \mathcal{W}\}$$

is an ij–S_α–open cover of U_n, so choose a countable subfamily $\mathcal{V}_n \subseteq \mathcal{W}$ with $\bigcup \mathcal{V}_n \supseteq U_n$. Then $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is countable and covers $\mathcal{M}$. Hence $\mathcal{M}$ is ij–S_α–Lindelöf. □

Example 3.5. *If γ_2 is discrete, then 12–$S_\alpha O(\mathcal{M}) = \gamma_1$, since*

$$H_{12}(A) = 2\text{-cl}\,1\text{-int}\,2\text{-cl}\,1\text{-int}(A) = 1\text{-int}(A),$$

so $A \subseteq H_{12}(A)$ iff $A \in \gamma_1$.

Let M be the Michael line on $\mathbb{R}$. Set $(\mathcal{M}, \gamma_1, \gamma_2) = (M, discrete)$ and $(i, j) = (1, 2)$. Then 12–$S_\alpha O(\mathcal{M}) = \gamma_1$, so locally 12–S_α–Lindelöf is the same as locally Lindelöf in γ_1. Every point has a γ_1–open Lindelöf neighborhood, hence $\mathcal{M}$ is locally 12–S_α–Lindelöf. However, M is not Lindelöf, so $\mathcal{M}$ is not 12–S_α–Lindelöf.

Example 3.6. *For each $\alpha < \omega_1$ let $I_\alpha = [0, 1]$ with the Euclidean topology. Form the topological sum $\mathcal{M} = \bigsqcup_{\alpha<\omega_1} I_\alpha$ and set $(\gamma_1, \gamma_2) = (\text{sum of Euclidean}, \text{discrete})$, $(i, j) = (1, 2)$. Again 12–$S_\alpha O(\mathcal{M}) = \gamma_1$. Each point $m \in I_\alpha$ has a γ_1–open neighborhood contained in I_α, which is Lindelöf; hence $\mathcal{M}$ is locally 12–S_α–Lindelöf. But the γ_1–open cover $\{I_\alpha : \alpha < \omega_1\}$ has no countable subcover, so $\mathcal{M}$ is not 12–S_α–Lindelöf.*

Lemma 3.2. *If $W \in ij$–$S_\alpha O(\mathcal{M})$ and $A \subseteq \mathcal{M}$, then $W \cap A \in ij$–$S_\alpha O(A)$.*

Proof. Since W is ij–S_α–open, $W \subseteq H^{\mathcal{M}}_{ij}(W)$. Using the subspace identities

$$i\text{-int}_A(B \cap A) = A \cap i\text{-int}_{\mathcal{M}}(B), \qquad j\text{-cl}_A(B \cap A) = A \cap j\text{-cl}_{\mathcal{M}}(B),$$

we obtain $W \cap A \subseteq H^{A}_{ij}(W \cap A)$. Hence $W \cap A$ is ij–S_α–open in A.

□

Theorem 3.7. *If $\mathcal{M} = \bigcup_{n\in\mathbb{N}} U_n$ with each $U_n \in ij\text{-}S_\alpha O(\mathcal{M})$ and each U_n ij–S_α–Lindelöf, then $\mathcal{M}$ is ij–S_α–Lindelöf and*

$$L^{S_\alpha}_{ij}(\mathcal{M}) \;\le\; \sum_{n\in\mathbb{N}} L^{S_\alpha}_{ij}(U_n) \;\le\; \aleph_0 \cdot \sup_n L^{S_\alpha}_{ij}(U_n).$$

Proof. Let $\mathcal{W} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ cover $\mathcal{M}$. For each n, the trace

$$\mathcal{W}\restriction U_n := \{W \cap U_n : W \in \mathcal{W}\}$$

is an ij–S_α–open cover of U_n by Lemma 3.2. Choose a subfamily $\mathcal{W}_n \subseteq \mathcal{W}$ with $|\mathcal{W}_n| \le L^{S_\alpha}_{ij}(U_n)$ that covers U_n. Then $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ covers $\mathcal{M}$ and has cardinality at most $\sum_n L^{S_\alpha}_{ij}(U_n)$, which is $\le \aleph_0 \cdot \sup_n L^{S_\alpha}_{ij}(U_n)$. Thus $\mathcal{M}$ is ij–S_α–Lindelöf and the bound holds. □

Theorem 3.8. *Let $\Gamma : (\mathcal{M}, \gamma_1, \gamma_2) \to (\mathcal{C}, \rho_1, \rho_2)$ and fix $i \neq j \in \{1, 2\}$. The following are equivalent:*

1. *Γ is ij–semi–α–continuous.*
2. *For every i–regular open $V \subseteq \mathcal{C}$, $\Gamma^{-1}(V) \in ij\text{-}S_\alpha O(\mathcal{M})$.*
3. *For some every base $\mathcal{B}_i$ of $(\mathcal{C}, \rho_i)$, $\Gamma^{-1}(B) \in ij\text{-}S_\alpha O(\mathcal{M})$ for all $B \in \mathcal{B}_i$.*
4. *For every $m \in \mathcal{M}$ and every i–open $\Gamma(m) \in V$, there exists $U \in ij\text{-}S_\alpha O(\mathcal{M})$ with $m \in U \subseteq \Gamma^{-1}(V)$.*

Proof. (1)⇒(2)⇒(1) is immediate since i–regular open $\subseteq \rho_i$.
(1)⇒(3): if $B \in \mathcal{B}_i \subseteq \rho_i$ then $\Gamma^{-1}(B)$ is ij–S_α–open.
(3)⇒(1): for $V \in \rho_i$, $V = \bigcup\{B \in \mathcal{B}_i : B \subseteq V\}$, hence $\Gamma^{-1}(V) = \bigcup \Gamma^{-1}(B)$ is ij–S_α–open since unions of ij–S_α–open sets are ij–S_α–open.
(1)⇔(d): if $\Gamma^{-1}(V)$ is ij–S_α–open, take $U = \Gamma^{-1}(V)$. Conversely, if (4) holds then $\Gamma^{-1}(V) = \bigcup_{m\in\Gamma^{-1}(V)} U_m$ is ij–S_α–open by union closure. □

Example 3.7. *Let $\mathcal{M} = \mathbb{R}$ with γ_1 the Euclidean topology and γ_2 the Sorgenfrey topology. Let $C = \mathbb{R}$ with ρ_1 Euclidean, ρ_2 arbitrary. Fix $(i, j) = (1, 2)$. Define $\Gamma : \mathcal{M} \to C$, $\Gamma(m) = m^3$.*

claim that f is 12–S_α–continuous. Hence the base and local tests in Theorem 3.8 hold.

Γ is 1–continuous. Every γ_1–open set is 12–S_α–open. Thus for every $V \in \rho_1$, $\Gamma^{-1}(V) \in$ 12-$S_\alpha O(\mathcal{M})$. For a base $\mathcal{B}_1 = \{(a, b)\}$ of (C, ρ_1), $\Gamma^{-1}(a, b)$ is Euclidean open, hence 12–S_α–open. For the local test: given $m \in \mathcal{M}$ and $V \in \rho_1$ with $\Gamma(m) \in V$, take $U = \Gamma^{-1}(V)$; then $m \in U \subseteq \Gamma^{-1}(V)$ and $U \in$ 12-$S_\alpha O(\mathcal{M})$.

Example 3.8. *Let $\mathcal{M} = \mathbb{R}$ with (γ_1, γ_2) as in 3.7. Let $C = \mathbb{R}^2$ with ρ_1 the product Euclidean topology. Fix $(i, j) = (1, 2)$. Define $\Gamma : \mathcal{M} \to C$, $\Gamma(m) = (m, \sin m)$.*

Claim that Γ is 12–S_α–continuous. All equivalents in Theorem 3.8 hold.

Theorem 3.9. *Let $\Gamma : (\mathcal{M}, \gamma_1, \gamma_2) \to (C, \rho_1, \rho_2)$ and $g : (C, \rho_1, \rho_2) \to (Z, \tau_1, \tau_2)$.*

1. *If Γ is ij–semi–α–continuous and g is i–continuous, then $g \circ \Gamma$ is ij–semi–α–continuous.*
2. *If Γ and g are both ij–semi–α^*–continuous, then $g \circ \Gamma$ is ij–semi–α^*–continuous.*
3. *If Γ is ij–semi–α–continuous and $A \subseteq \mathcal{M}$, then $f|_A : (A, \gamma_1|_A, \gamma_2|_A) \to (C, \rho_1, \rho_2)$ is ij–semi–α–continuous.*

Proof. (1) For $W \in \tau_i$ we have $(g \circ \Gamma)^{-1}(W) = \Gamma^{-1}(g^{-1}(W))$ with $g^{-1}(W) \in \rho_i$; apply ij–S_α–continuity of Γ.
(2) For $W \in ij$–$S_\alpha O(Z)$, $(g \circ \Gamma)^{-1}(W) = \Gamma^{-1}(g^{-1}(W))$ and $g^{-1}(W) \in ij$–$S_\alpha O(C)$; apply ij–S^*_α–continuity of Γ.
(3) If $V \in \rho_i$, then $(\Gamma|_A)^{-1}(V) = A \cap \Gamma^{-1}(V)$, which is ij–S_α–open in A since $\Gamma^{-1}(V) \in ij$–$S_\alpha O(\mathcal{M})$ and ij–S_α–openness is preserved by taking subspace intersections with A. □

Example 3.9. *Use $\mathcal{M}, C$ and Γ from 3.7 with $(i, j) = (1, 2)$. Let $Z = (0, \infty)$ with τ_1 Euclidean, τ_2 arbitrary, and define $g : C \to Z$, $g(c) = e^c$. Claim that $g \circ \Gamma : \mathcal{M} \to Z$ is 12–S_α–continuous. g is 1–continuous. Γ is 12–S_α–continuous by 3.7. For any $W \in \tau_1$, $(g \circ \Gamma)^{-1}(W) = \Gamma^{-1}\big(g^{-1}(W)\big)$ with $g^{-1}(W) \in \rho_1$, so $\Gamma^{-1}\big(g^{-1}(W)\big) \in$ 12-$S_\alpha O(\mathcal{M})$ by Definition 2.4.*

Example 3.10. *Fix $(i, j) = (1, 2)$. Let $\mathcal{M}$ be any set with γ_1 discrete and γ_2 arbitrary. Let $C = \mathbb{R}$ with ρ_1 discrete and ρ_2 arbitrary. Let Z be any bitopological space. Choose any functions $\Gamma : \mathcal{M} \to C$ and $g : C \to Z$.*

Claims that Γ and g are 12–S^*_α–continuous; hence $g \circ \Gamma$ is 12–S^*_α–continuous. If $A \subseteq \mathcal{M}$, then $\Gamma|_A : (A, \gamma_1|_A, \gamma_2|_A) \to C$ is 12–S_α–continuous.

Since ρ_1 is discrete, for every $W \subseteq C$ we have W 12–α–open and hence 12–S_α–open in C. Thus g is 12–S^*_α–continuous by definition. Since γ_1 is discrete, every subset of $\mathcal{M}$ is 12–α–open and hence 12–S_α–open, so Γ is also 12–S^*_α–continuous. By Theorem 3.9(2), $g \circ \Gamma$ is 12–S^*_α–continuous.

For the restriction, if $V \in \rho_1$ then $(\Gamma|_A)^{-1}(V) = A \cap \Gamma^{-1}(V)$ with $\Gamma^{-1}(V) \in 12\text{-}S_\alpha O(\mathcal{M})$. By the subspace definition, $A \cap \Gamma^{-1}(V)$ is 12–S_α–open in A. Hence $\Gamma|_A$ is 12–S_α–continuous (Theorem 3.9(3)).

Theorem 3.10. *The ij–semi–α–continuous image of an ij–semi–α–Lindelöf space is i–Lindelöf.*

Proof. Let $\Gamma : \mathcal{M} \to C$ be ij–S_α–continuous and $\mathcal{M}$ ij–S_α–Lindelöf. Given an i–open cover $\{V_\lambda\}$ of $\Gamma(\mathcal{M})$ in C, the family $\{\Gamma^{-1}(V_\lambda)\} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ covers $\mathcal{M}$. Take a countable subcover; the corresponding V_λ's cover $\Gamma(\mathcal{M})$. □

Example 3.11. *Let $\mathcal{M} = \mathbb{R}$ with γ_1 Euclidean and γ_2 discrete; then 12–S_α–open sets are exactly γ_1–open, so $\mathcal{M}$ is 12–S_α–Lindelöf. Let C be any bitopological space with ρ_1 Euclidean on $\mathbb{R}$ and define $\Gamma(m) = m^3$. Then Γ is 1–continuous, hence 12–S_α–continuous. By Theorem 3.10, $\Gamma(\mathcal{M}) = \mathbb{R}$ is 1–Lindelöf.*

Corollary 3.2. *If $\Gamma : \mathcal{M} \to C$ is onto and ij–S_α–continuous, then*
$\mathcal{M}$ ij–S_α–Lindelöf ⇒ C i–Lindelöf.

Proposition 3.8. *If $A \subseteq \mathcal{M}$ is ij–S_α–closed and $\mathcal{M}$ is ij–S_α–Lindelöf, then A is ij–S_α–Lindelöf.*

Remark 3.4. *For $A, B \subseteq \mathcal{M}$,*

$$L^{S_\alpha}_{ij}(\Gamma(\mathcal{M})) \leq L^{S_\alpha}_{ij}(\mathcal{M}), \qquad L^{S_\alpha}_{ij}(A \cup B) \leq L^{S_\alpha}_{ij}(A) + L^{S_\alpha}_{ij}(B).$$

Theorem 3.11. *The ij–semi–α*–continuous image of an ij–semi–α–Lindelöf space is ij–semi–α–Lindelöf. Equivalently,*

$$L^{S_\alpha}_{ij}\big(\Gamma(\mathcal{M})\big) \leq L^{S_\alpha}_{ij}(\mathcal{M}).$$

Proof. Let $\Gamma : \mathcal{M} \to C$ be ij–S^*_α–continuous and $\mathcal{M}$ ij–S_α–Lindelöf.
Let $\{W_\lambda\} \subseteq ij\text{-}S_\alpha O(C)$ cover $\Gamma(\mathcal{M})$. Then $\{\Gamma^{-1}(W_\lambda)\} \subseteq ij\text{-}S_\alpha O(\mathcal{M})$ covers $\mathcal{M}$. Choose a countable subfamily $\{\Gamma^{-1}(W_{\lambda_k})\}_{k\in\mathbb{N}} covering \mathcal{M}$. Then $\{W_{\lambda_k}\}_{k=1}^m$ covers $\Gamma(\mathcal{M})$. □

Corollary 3.3. *If $\Gamma : \mathcal{M} \to C$ is onto and ij–S^*_α–continuous, then*

$$\mathcal{M}\ ij\text{–}S_\alpha\text{–Lindelöf} \Rightarrow C\ ij\text{–}S_\alpha\text{–Lindelöf}.$$

Definition 3.9. *$\Gamma : (\mathcal{M}, \gamma_1, \gamma_2) \to (C, \rho_1, \rho_2)$ is ij–S_α–perfect if it is onto, ij–S_α–closed, and every fiber $\Gamma^{-1}(c)$ is ij–S_α–Lindelöf.*

Example 3.12. *Let $C = \mathbb{R}$ with $\rho_1 = \gamma_{\text{Euc}}$ and arbitrary ρ_2, let $K = [0, 1]$ with $\kappa_1 = \gamma_{\text{Euc}}$ and arbitrary κ_2. Put $\mathcal{M} = C \times K$ with $\gamma_1 = \rho_1 \times \kappa_1$ and $\gamma_2 = \rho_2 \times \kappa_2$. Fix $(i, j) = (1, 2)$ and let $\Gamma = \pi_C : \mathcal{M} \to C$ be the first projection. Γ is onto ,12–S_α–closed: if $F \in 12\text{-}S_\alpha C(\mathcal{M})$ then F is γ_1–closed, and $\pi_C(F)$ is ρ_1–closed because K is κ_1–compact; hence $\pi_C(F) \in 12\text{-}S_\alpha C(C)$. $\Gamma^{-1}(c) = \{c\} \times K$ are κ_1–compact, thus 12–S_α–Lindelöf.*
Therefore Γ is 12–S_α–perfect.

Theorem 3.12. *If $\Gamma : (\mathcal{M}, \gamma_1, \gamma_2) \to (\mathcal{C}, \rho_1, \rho_2)$ is ij–S_α–perfect and $\mathcal{M}$ is i–Lindelöf, then $\mathcal{C}$ is ij–S_α–Lindelöf and*

$$L_{ij}^{S_\alpha}(\mathcal{C}) \ \le\ L_i(\mathcal{M}) \cdot \sup_{c \in \mathcal{C}} L_{ij}^{S_\alpha}\Big(\Gamma^{-1}(c)\Big).$$

Proof. Let $\mathcal{W} \subseteq ij\text{-}S_\alpha O(\mathcal{C})$ cover $\mathcal{C}$. For each $c \in \mathcal{C}$, the subfamily

$$\mathcal{W}(c) = \{W \in \mathcal{W} :\ c \in W\}$$

induces a cover

$$\{\Gamma^{-1}(W) \cap \Gamma^{-1}(c) : W \in \mathcal{W}(c)\}$$

of $F_c = \Gamma^{-1}(c)$ by ij–S_α–open sets in the subspace F_c. By ij–S_α–Lindelöfness of F_c there exists a subfamily $\mathcal{W}_c \subseteq \mathcal{W}(c)$ with

$$|\mathcal{W}_c| \le \kappa := \sup_{z \in \mathcal{C}} L_{ij}^{S_\alpha}(\Gamma^{-1}(z))$$

and

$$F_c \ \subseteq \ \bigcup_{W \in \mathcal{W}_c} \Big(\Gamma^{-1}(W) \cap F_c\Big) \ \subseteq \ \bigcup_{W \in \mathcal{W}_c} \Gamma^{-1}(W).$$

Define the closed set

$$E_c := \mathcal{M} \setminus \bigcup_{W \in \mathcal{W}_c} \Gamma^{-1}(W).$$

Then $\Gamma(E_c)$ is ij–S_α–closed in $\mathcal{C}$ because Γ is ij–S_α–closed, and $c \notin \Gamma(E_c)$.
Set $V_c := \mathcal{C} \setminus \Gamma(E_c)$. Each V_c is ij–S_α–open, contains c, and satisfies

$$\Gamma^{-1}(V_c) \ \subseteq \ \bigcup_{W \in \mathcal{W}_c} \Gamma^{-1}(W). \tag{$*$}$$

Hence $\{V_c : c \in \mathcal{C}\}$ is an ij–S_α–open cover of $\mathcal{C}$ refining $\mathcal{W}$ *with countable-κ control on preimages* via $(*)$.

Consider the family $\{\Gamma^{-1}(V_c) : c \in \mathcal{C}\}$; it covers $\mathcal{M}$. Since every γ_i–open set is ij–S_α–open, for each $m \in \mathcal{M}$ choose a γ_i–open O_m with $m \in O_m \subseteq \Gamma^{-1}(V_{\Gamma(m)})$. Then $\{O_m : m \in \mathcal{M}\}$ is a γ_i–open cover of $\mathcal{M}$. By i–Lindelöfness, there exists a subfamily indexed by $I \subseteq \mathcal{M}$ with $|I| \le L_i(\mathcal{M})$ such that $\{O_m : m \in I\}$ covers $\mathcal{M}$. Let

$$J := \{\Gamma(m) : m \in I\} \subseteq \mathcal{C};$$

then $|J| \le |I| \le L_i(\mathcal{M})$ and $\{V_c : c \in J\}$ covers $\mathcal{C}$ by surjectivity of Γ: for any $c_0 \in \mathcal{C}$ pick $m_0 \in \mathcal{M}$ with $\Gamma(m_0) = c_0$; some $m \in I$ has $m_0 \in O_m \subseteq \Gamma^{-1}(V_{\Gamma(m)})$, hence $c_0 \in V_{\Gamma(m)}$.

Finally, for each $c \in J$, $V_c \subseteq \bigcup \mathcal{W}_c$ and $|\mathcal{W}_c| \le \kappa$. Therefore $\bigcup_{c \in J} \mathcal{W}_c$ is a subfamily of $\mathcal{W}$ of cardinal $\le |J| \cdot \kappa \le L_i(\mathcal{M}) \cdot \kappa$ that covers $\mathcal{C}$. This proves $\mathcal{C}$ is ij–S_α–Lindelöf and yields the stated bound. □

Lemma 3.3. *The following are equivalent for $\Gamma : (\mathcal{M}, \gamma_1, \gamma_2) \to (\mathcal{C}, \rho_1, \rho_2)$:*

(1) *Γ is ij–S_α^*–continuous;*
(2) *for every $F \in ij\text{-}S_\alpha C(\mathcal{C})$, $\Gamma^{-1}(F) \in ij\text{-}S_\alpha C(\mathcal{M})$.*

Proposition 3.9. *For every $A \subseteq \mathcal{M}$ and ij–S_α^*–continuous Γ, $L_{ij}^{S_\alpha}\Big(\Gamma(A)\Big) \ \le\ L_{ij}^{S_\alpha}(A)$.*

Proposition 3.10. *Suppose $\Gamma : \mathcal{M} \to C$ is onto and ij–S_α–*quotient *in the sense that $W \in ij\text{-}S_\alpha O(C) \iff \Gamma^{-1}(W) \in ij\text{-}S_\alpha O(\mathcal{M})$. Then C is ij–S_α–Lindelöf iff $\mathcal{M}$ is so.*

Theorem 3.13. *If Γ is ij–semi–α'–continuous and $\mathcal{M}$ is i–Lindelöf, then $\Gamma(\mathcal{M})$ is ij–S_α–Lindelöf. Equivalently, $L_{ij}^{S_\alpha}\big(\Gamma(\mathcal{M})\big) \le L_i(\mathcal{M})$.*

Proof. Let $\{W_\lambda\} \subseteq ij\text{-}S_\alpha O(C)$ cover $\Gamma(\mathcal{M})$. By ij–α'–continuity, $\{\Gamma^{-1}(W_\lambda)\} \subseteq \gamma_i$ covers $\mathcal{M}$. Use i–Lindelöfness to extract a countable subfamily; push forward to cover $\Gamma(\mathcal{M})$. □

Example 3.13. *Fix $(i, j) = (1, 2)$. Let $\mathcal{M} = \mathbb{R}$ with γ_1 Euclidean (so 1–Lindelöf) and arbitrary γ_2. Let $C = \mathbb{R}^2$ with ρ_1 the product Euclidean topology and ρ_2 discrete, so again 12–$S_\alpha O(C) = \rho_1$. Define $\Gamma : \mathcal{M} \to Y$ by $f(m) = (m, \sin m)$; then Γ is 1–continuous and hence 12–α'–continuous. By Theorem 3.13, $\Gamma(\mathcal{M}) = \{(m, \sin m) : m \in \mathbb{R}\}$ is 12–S_α–Lindelöf in C; here this is just Euclidean Lindelöfness of a continuous curve in $\mathbb{R}^2$.*

Proposition 3.11. *For every $W \in ij\text{-}S_\alpha O(C)$ there exists $V \in \rho_i$ with $\Gamma^{-1}(W) = \Gamma^{-1}(V)$. Then i–Lindelöf of $\mathcal{M}$ implies ij–S_α–Lindelöf of $\Gamma(\mathcal{M})$.*

Definition 3.10. *Let $(\mathcal{M}, \gamma_1, \gamma_2)$ be bitopological and $i \neq j \in \{1, 2\}$.*

1. *$\mathcal{M}$ is ij–α–*Lindelöf *iff every ij–α–open cover of $\mathcal{M}$ has a countable subcover.*
2. *$\mathcal{M}$ is ij–*pre–Lindelöf *iff every ij–pre–open cover of $\mathcal{M}$ has a countable subcover.*

Define cardinals

$$L_{ij}^{\alpha}(\mathcal{M}) := \min\{\kappa : \text{ every } ij\text{–}\alpha\text{–open cover of } \mathcal{M} \text{ has a subcover of size } \le \kappa\},$$

$$L_{ij}^{\text{pre}}(\mathcal{M}) := \min\{\kappa : \text{ every } ij\text{–pre–open cover has a subcover } \le \kappa\}.$$

Remark 3.5. *ij–pre–open $\supseteq$ ij–α–open, hence ij–pre–Lindelöf $\Rightarrow$ ij–α–Lindelöf and $L_{ij}^{\alpha}(\mathcal{M}) \le L_{ij}^{\text{pre}}(\mathcal{M})$. If γ_j is discrete, then ij–α–open $=$ ij–pre–open $= \gamma_i$–open, so ij–α–Lindelöf $\Leftrightarrow$ ij–pre–Lindelöf $\Leftrightarrow$ i–Lindelöf.*

Example 3.14. *Let $\mathcal{M} = [0, \omega_1)$ with γ_1 the order topology and γ_2 indiscrete; take $(i, j) = (1, 2)$. Then 12–pre–open sets are all subsets (since $1\text{int}(2\text{cl}\, A) = 1\text{int}\, \mathcal{M} = \mathcal{M}$), so the cover by singletons $\{\{\alpha\} : \alpha < \omega_1\}$ has no countable subcover. Hence $\mathcal{M}$ is not 12–pre–Lindelöf and $L_{12}^{\text{pre}}(\mathcal{M}) > \aleph_0$.*

Moreover γ_1–open $\subseteq$ 12–α–open, so the classical open cover $\{[0, \alpha) : \alpha < \omega_1\}$ (no countable subcover in (X, γ_1)) is a 12–α–open cover with no countable subcover. Thus X is not 12–α–Lindelöf and $L_{12}^{\alpha}(X) > \aleph_0$.

Proposition 3.12. *Every ij–pre–Lindelöf space is ij–α–Lindelöf. Equivalently $L_{ij}^{\alpha}(\mathcal{M}) \le L_{ij}^{\text{pre}}(\mathcal{M})$.*

Proof. ij–α–open $\subseteq$ ij–pre–open. Any ij–α–open cover is a special ij–pre–open cover, hence has a countable subcover. □

Proposition 3.13. *If every ij–pre–open set is ij–semi–open, then ij–α–open $=$ ij–pre–open (by Theorem 2.9), hence*

$$\mathcal{M} \text{ is } ij\text{–}\alpha\text{–Lindelöf} \iff \mathcal{M} \text{ is } ij\text{–pre–Lindelöf},$$

and $L_{ij}^{\alpha}(\mathcal{M}) = L_{ij}^{\text{pre}}(\mathcal{M})$.

Proposition 3.14. *Every ij–pre–Lindelöf space is i–Lindelöf. Hence $L_i(\mathcal{M}) \le L_{ij}^{\mathrm{pre}}(\mathcal{M})$.*

Proof. Every γ_i–open set is ij–pre–open. Any γ_i–open cover is an ij–pre–open cover. □

Remark 3.6. *Every ij–semi–α–Lindelöf space is ij–α–Lindelöf; thus $L_{ij}^{\alpha}(\mathcal{M}) \le L_{ij}^{S_\alpha}(\mathcal{M})$.*

Proposition 3.15. *If every γ_i–open subset of $\mathcal{M}$ is γ_j–closed, then ij–α–open $=$ ij–semi–α–open (Remark 2.17(ii)). Hence $\mathcal{M}$ is ij–α–Lindelöf $\iff$ $\mathcal{M}$ is ij–semi–α–Lindelöf, and $L_{ij}^{\alpha}(\mathcal{M}) = L_{ij}^{S_\alpha}(\mathcal{M})$.*

Proposition 3.16. *If every γ_i–open set is γ_j–closed, then ij–pre–Lindelöf $\Rightarrow$ ij–semi–α–Lindelöf, and*

$$L_{ij}^{S_\alpha}(\mathcal{M}) \;\le\; L_{ij}^{\alpha}(\mathcal{M}) \;\le\; L_{ij}^{\mathrm{pre}}(\mathcal{M}).$$

Proof. ij–pre–Lindelöf $\Rightarrow$ ij–α–Lindelöf by 3.12. Under the hypothesis, ij–α–open $=$ ij–S_α–open (3.15). □

Proposition 3.17. *If every ij–pre–open is ij–semi–open, then ij–α–open $=$ ij–pre–open. Hence*

$$\mathcal{M}\; ij\text{–}S_\alpha\text{–Lindelöf} \;\Rightarrow\; \mathcal{M}\; ij\text{–}\alpha\text{–Lindelöf} \;\Rightarrow\; X\mathcal{M}\; ij\text{–pre–Lindelöf},$$

and $L_{ij}^{\mathrm{pre}}(\mathcal{M}) = L_{ij}^{\alpha}(\mathcal{M}) \le L_{ij}^{S_\alpha}(\mathcal{M})$.

Corollary 3.4. *If $\Gamma : (\mathcal{M}, \gamma_1, \gamma_2) \to (\mathcal{C}, \rho_1, \rho_2)$ is ij–α'–continuous and $\mathcal{M}$ is i–Lindelöf, then $\Gamma(\mathcal{M})$ is ij–S_α–Lindelöf. Thus $L_{ij}^{S_\alpha}\big(\Gamma(\mathcal{M})\big) \le L_i(\mathcal{M})$.*

Proof. Let $\{W_\lambda\} \subseteq ij\text{-}S_\alpha O(\mathcal{C})$ cover $\Gamma(\mathcal{M})$. Then $\{\Gamma^{-1}(W_\lambda)\} \subseteq \gamma_i$ covers $\mathcal{M}$; pick a countable subcover and push forward. □

Corollary 3.5. *If Γ is ij–S_α^*–continuous and $\mathcal{M}$ is ij–S_α–Lindelöf, then $\Gamma(\mathcal{M})$ is ij–S_α–Lindelöf. Hence $L_{ij}^{S_\alpha}\big(\Gamma(\mathcal{M})\big) \le L_{ij}^{S_\alpha}(\mathcal{M})$.*

4. Conclusion

Our goal was to comprehend the behavior of semi–α–openness at the level of covering properties in bitopological spaces. Two themes surfaced. First, the Čech–closure viewpoint via the hull $H_{ij} = j\mathrm{cl}\; i\mathrm{int}\; j\mathrm{cl}\; i\mathrm{int}$ is the proper mechanism to convey arguments that classically rely on closure operators. Second, once one works in the H_{ij}–framework, several familiar Lindelöf tools carry over with minimal friction. On the structural side we proved network and star criteria for ij–S_α–Lindelöfness, hereditary and dense-set transfer, ρ–glueing principles, and stable behavior under countable sums. A tube-type product theorem was established when the second topology is discrete and the i–side factor is compact. We introduced ij–S_α–perfect maps and showed they preserve ij–S_α–Lindelöfness with sharp cardinal bounds. At the level of invariants we compared $L_{ij}^{S_\alpha}$ with the α– and pre–Lindelöf numbers and analyzed the pairwise quantity $\widehat{L}_{\mathrm{pair}}^{S_\alpha}$. Examples drawn from ordinal, Sorgenfrey, Michael, discrete, and co–countable settings separate all notions and show that our hypotheses are close to optimal. Beyond these core results, the picture that emerges is that semi–α–Lindelöfness behaves like a robust shadow of i–Lindelöfness, the outermost operator separates the change and permits the writing of proofs only once, which can then be applied in di¤erent settings.

Directions for further work: Formulate ij–S_α analogues of the Menger/Hurewicz properties and corresponding games; compare with $L^{S_\alpha}_{ij}$. Study $C_i(\mathcal{M})$ with natural bitopologies and relate their tightness, network weight, and Baire category to $L^{S_\alpha}_{ij}(\mathcal{M})$. Characterize ij–S_α–perfect maps via closedness of graphs or inverse image operators; test stability under composition and pull-backs. When both $L^{S_\alpha}_{12}$ and $L^{S_\alpha}_{21}$ are finite, determine exact values of $\widehat{L}^{S_\alpha}_{\text{pair}}$ and identify extremal examples. Find necessary and sufficient conditions under which ij–S_α–star–Lindelöf implies ij–S_α–Lindelöf. The semi–α–Lindelöf framework is flexible enough to encompass classical phenomena and rigid enough to support clean cardinal bounds and preservation theorems. The tools developed here—especially the H_{ij}–operator and network methods—should be useful beyond the topic at hand, wherever mixed topological information must be organized across two interacting topologies.

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