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Codimension Two Bifurcation Analysis of a Discrete Coupled Competition Duopoly Game

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Abstract. This paper analytically examines the coupled competition duopoly game model. This study examines the codimension-two bifurcations of different types of the model through bifurcation theory and numerical continuation methods. The model undergoes codimension-two bifurcation, heteroclinic bifurcation near the 1:2 point, a homoclinic structure near the 1:3 resonance point, and an invariant cycle bifurcated by a period 4 orbit near the 1:4 resonance point. Subsequently, numerical simulations are performed to validate the theoretical study.

1. Introduction

In recent years, there has been an increase in interest in discrete-time dynamical systems. One of the primary reasons for this scenario is that they may exhibit more complicated and diverse dynamic behaviors than their counterparts who operate in continuous time, such as [1–5]. Using bifurcation theory and center manifold theory, it has been demonstrated that various discrete-time species, discrete economic models and epidemic models can exhibit a variety of codimension one bifurcations. This applies to multiple different models. Fold, flip, transcritical, and Neimark-Sacker bifurcations are a few examples [6–10]. Recent studies [11–14] have shown that some discrete-time models can exhibit codimension-two bifurcations. We'll focus on resonance bifurcations at 1:2, 1:3, and 1:4. To be more clear, this is the primary purpose of this study for coupled competition duopoly game model [15]. Recent research [16–19] have demonstrated that some discrete-time models can display codimension-two bifurcations. We will focus on codimension-two bifurcations

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of 1:2, 1:3, and 1:4 resonances. That is the primary objective of this research. The proposed coupled competition maps have been numerically studied in [15].

$$\begin{cases} x_{n+1} = x_n + \alpha_1 x_n (A_1 - 2x_n - By_n), \\ y_{n+1} = y_n + \alpha_2 y_n (A_2 - 2y_n + Bx_n), \end{cases} \tag{1.1}$$

The stability and the codimension bifurcations of model (1.1) have been investigated, as concluded in [20]. This model has been shown to exhibit chaotic behavior and undergo bifurcations of the flip, pitchfork, and Neimark-Sacker types through numerical and analytical experiments. Numerous instances have demonstrated this. The present study aims to examine a specific competition model (1.1) for the existence of codimension-two bifurcation sets and codimension-two bifurcations at equilibrium points associated with 1:2, 1:3, and 1:4 strong resonances. The investigation will specifically assess the feasibility of these occurrences. The findings of our theoretical analyses are verified by numerical simulations.

The structure of the paper is as follows: Section 2 identifies the presence of codimension-two bifurcation sets at the equilibrium points of a discrete competition model (1.1), whereas Section 3 examines the codimension-two bifurcation of the same competition model (1.1). To validate theoretical findings, simulations are provided in Section 4, while the conclusion is presented in Section 5.

2. Codimension-two Bifurcation Analysis

This section provides an extensive investigation to show that the map defined in equation (1.1) exhibits a codimension-two bifurcation. This result is derived from the methodical implementation of normal form theory and bifurcation theory, as elucidated in references [21–24]. We can find the important combinations of parameters that cause qualitative changes in the dynamics by transforming the original map into its simplified normal form. The research shows that there is a codimension-two bifurcation point, where two different bifurcation scenarios come simultaneously or cooperate at the same time. This makes the local dynamical behavior around this critical point more interesting. We select α_1 and α_2 as bifurcation parameters to conduct a bifurcation analysis at $E^*(x^*, y^*)$. We define three bifurcation sets $F_{1j} = \{(\alpha_1, \alpha_2, A_1, A_2, B) : \alpha_1\alpha_2 H = 6 - j, G = j - 6, j = 2, 3, 4\}$, corresponding to the occurrences of $1:2$, $1:3$ and $1:4$ resonances, respectively.

2.1. **1:2 Resonance.**

Considering the parameters $(\tilde{\alpha}_1, \tilde{\alpha}_2, A_1, A_2, B)$ arbitrarily from F_{12}. We examine map (1.1) characterized by the parameters $(\tilde{\alpha}_1, \tilde{\alpha}_2, A_1, A_2, B)$ as follows:

$$\begin{cases} x \to x + \tilde{\alpha}_1 x (A_1 - 2x - By), \\ y \to y + \tilde{\alpha}_2 y (A_2 - 2y + Bx), \end{cases} \tag{2.1}$$

The map (2.1) possesses a unique interior fixed point denoted as $E^*(x^*, y^*)$, with corresponding eigenvalues $\lambda_1 = \lambda_2 = -1$. Let $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be bifurcation parameters. We examine a perturbation of map (2.1) as follows:

$$\begin{cases} x \to x + (\tilde{\alpha}_1 + \alpha_1^*)x(A_1 - 2x - By), \\ y \to y + (\tilde{\alpha}_2 + \alpha_2^*)y(A_2 - 2y + Bx), \end{cases} \tag{2.2}$$

the perturbation parameters $|\alpha_1^*|, |\alpha_2^*| \ll 1$ are small.

Let $u = x - x^*$, $v = y - y^*$, $\alpha_1 = \tilde{\alpha}_1 + \alpha_1^*$ and $\alpha_2 = \tilde{\alpha}_2 + \alpha_2^*$. Then, map (2.2) becomes

$$\begin{cases} u \to (1 - 2\alpha_1 x^*)u - \alpha_1 B x^* v - 2\alpha_1 u^2 - \alpha_1 B uv, \\ v \to \alpha_2 B y^* u + (1 - 2\alpha_2 y^*)v + \alpha_2 B uv - 2\alpha_2 v^2. \end{cases} \tag{2.3}$$

Let

$$T = \begin{pmatrix} \alpha_1 B x^* & \frac{\alpha_1 B x^*}{2 - 2\alpha_1 x^*} \\ 2 - 2\alpha_1 x^* & 0 \end{pmatrix},$$

and utilize the transform

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix},$$

Then, map (2.3) is changed into

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \to \begin{pmatrix} -1 + a(\alpha_1, \alpha_2) & 1 + b(\alpha_1, \alpha_2) \\ c(\alpha_1, \alpha_2) & -1 + d(\alpha_1, \alpha_2) \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} \hat{g}(\hat{x}, \hat{y}, \alpha_1, \alpha_2) \\ \hat{h}(\hat{x}, \hat{y}, \alpha_1, \alpha_2) \end{pmatrix} \tag{2.4}$$

where

$$\hat{\hat{g}}(\hat{x}, \hat{y}, \alpha_1, \alpha_2) = \sum_{2 \le j+k \le 3} \hat{g}_{jk} \hat{x}^j \hat{y}^k,$$
$$\hat{h}(\hat{x}, \hat{y}, \alpha_1, \alpha_2) = \sum_{2 \le j+k \le 3} \hat{h}_{jk} \hat{x}^j \hat{y}^k,$$

and

$$a(\alpha_1, \alpha_2) = \frac{4 - 4\alpha_1 x^* - 4\alpha_2 y^* + \alpha_1 \alpha_2 (B^2 + 4)x^* y^*}{2 - 2\alpha_1 x^*},$$
$$b(\alpha_1, \alpha_2) = \frac{\alpha_1 \alpha_2 B^2 x^* y^* - (2 - 2\alpha_1 x^*)^2}{(2 - 2\alpha_1 x^*)^2},$$
$$c(\alpha_1, \alpha_2) = -4 + 4\alpha_1 x^* + 4\alpha_2 y^* - \alpha_1 \alpha_2 (B^2 + 4)x^* y^*,$$
$$d(\alpha_1, \alpha_2) = \frac{(2 - 2\alpha_1 x^*)^2 - \alpha_1 \alpha_2 B^2 x^* y^*}{2 - 2\alpha_1 x^*},$$
$$\hat{g}_{20} = \alpha_1 \alpha_2 (B^2 + 4)x^* - 4\alpha_2, \ \hat{g}_{11} = \frac{\alpha_1 \alpha_2 B^2 x^*}{2 - 2\alpha_1 x^*},$$
$$\hat{h}_{20} = 2(1 - \alpha_1 x^*)\Big(4\alpha_2 - 2\alpha_1 B - \alpha_1 \alpha_2 (B^2 + 4)x^*\Big),$$
$$\hat{h}_{11} = -\alpha_1 B\Big(2 + (2\alpha_1 + \alpha_2 B)x^*\Big), \ \hat{h}_{02} = -\frac{\alpha_1^2 B x^*}{1 - \alpha_1 x^*},$$
$$\hat{g}_{30} = \hat{g}_{02} = \hat{g}_{21} = \hat{g}_{12} = \hat{g}_{03} = \hat{h}_{30} = \hat{h}_{21} = \hat{h}_{12} = \hat{h}_{03} = 0.$$

Propose the linear coordinate transformation that is not singular.

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 1+b(\alpha_1,\alpha_2) & 0 \\ -a(\alpha_1,\alpha_2) & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix},$$

the map (2.3) becomes

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ \theta_1(\alpha_1,\alpha_2) & -1+\theta_2(\alpha_1,\alpha_2) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} \tilde{g}(\tilde{x},\tilde{y},\alpha_1,\alpha_2) \\ \tilde{h}(\tilde{x},\tilde{y},\alpha_1,\alpha_2) \end{pmatrix} \tag{2.5}$$

where

$$\begin{aligned}
&\theta_1(\alpha_1,\alpha_2) = -4+4\alpha_1 x^* + 4\alpha_2 y^* - \alpha_1\alpha_2(B^2+4)x^*y^*,\\
&\theta_2(\alpha_1,\alpha_2) = 3-2\alpha_1 x^* - 2\alpha_2 y^*,\\
&\tilde{g}_{20} = -\frac{\alpha_1\alpha_2 B^2 x^*}{1-\alpha_1 x^*},\ \tilde{g}_{11} = \frac{\alpha_1\alpha_2 B^2 x^*}{2-2\alpha_1 x^*},\ \hat{g}_{02} = 0\\
&\tilde{h}_{20} = \frac{\alpha_1 B x^*(1-\alpha_2 y^*)\Big(-4\alpha_1 - 2\alpha_2 B + \alpha_1\alpha_2(B^2+4)y^*\Big)}{1-\alpha_1 x^*},\\
&\tilde{h}_{11} = \frac{\alpha_1 B x^*\Big(8\alpha_1 + 2\alpha_2 B - 2\alpha_2^2 B y^* - \alpha_1\alpha_2(B^2+8)y^*\Big)}{2-2\alpha_1 x^*},\ \hat{h}_{02} = -\frac{\alpha_1^2 B x^*}{1-\alpha_1 x^*},\\
&\hat{g}_{30} = \hat{g}_{21} = \hat{g}_{12} = \hat{g}_{03} = \hat{h}_{30} = \hat{h}_{21} = \hat{h}_{12} = \hat{h}_{03} = 0.
\end{aligned}$$

Finally, we present the subsequent transformation:

$$\begin{aligned}
\tilde{x} &= \xi + \sum_{2\le j+k\le 3} \varphi_{jk}(\alpha_1,\alpha_2)\xi^j\eta^k,\\
\tilde{y} &= \eta + \sum_{2\le j+k\le 3} \psi_{jk}(\alpha_1,\alpha_2)\xi^j\eta^k,
\end{aligned} \tag{2.6}$$

The inverse transformation of the variable from transformation (2.6) is given by the expression:

$$\begin{aligned}
\xi &= \tilde{x} - \sum_{j+k=2} \varphi_{jk}(\alpha_1,\alpha_2)\tilde{x}^j\tilde{y}^k - \sum_{j+k=3} \delta_{jk}(\alpha_1,\alpha_2)\tilde{x}^j\tilde{y}^k + O\Big((|\tilde{x}|+|\tilde{y}|)^4\Big),\\
\eta &= \tilde{y} - \sum_{j+k=2} \psi_{jk}(\alpha_1,\alpha_2)\tilde{x}^j\tilde{y}^k - \sum_{j+k=3} \beta_{jk}(\alpha_1,\alpha_2)\tilde{x}^j\tilde{y}^k + O\Big((|\tilde{x}|+|\tilde{y}|)^4\Big),
\end{aligned} \tag{2.7}$$

By employing transformation (2.6) and its inverse transformation (2.7) within map (2.5), we obtain

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ \theta_1(\alpha_1,\alpha_2) & -1+\theta_2(\alpha_1,\alpha_2) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} \bar{g}(\xi,\eta,\alpha_1,\alpha_2) \\ \bar{h}(\xi,\eta,\alpha_1,\alpha_2) \end{pmatrix} \tag{2.8}$$

where

$$\begin{aligned}
\bar{g}(\xi,\eta,\alpha_1,\alpha_2) &= \sum_{2\le j+k\le 3} \gamma_{jk}(\alpha_1,\alpha_2)\xi^j\eta^k + O\Big((|\xi|+|\eta|)^4\Big),\\
\bar{h}(\xi,\eta,\alpha_1,\alpha_2) &= \sum_{2\le j+k\le 3} \sigma_{jk}(\alpha_1,\alpha_2)\xi^j\eta^k + O\Big((|\xi|+|\eta|)^4\Big),
\end{aligned}$$

$$
\begin{aligned}
\gamma_{20}(\alpha_1,\alpha_2) &= \tilde{g}_{20} + \psi_{20} - 2\varphi_{20} - \varphi_{02}\theta_1^2 + \varphi_{11}\theta_1,\\
\gamma_{11}(\alpha_1,\alpha_2) &= \tilde{g}_{11} + \psi_{11} - 2\varphi_{02}(\theta_2 - 1)\theta_1 + (\theta_2 - \theta_1 - 2)\varphi_{11} + 2\varphi_{20},\\
\gamma_{02}(\alpha_1,\alpha_2) &= \psi_{02} + (1-\theta_2)\varphi_{11} + (\theta_2^2 - 2\theta_2 + 2)\varphi_{02} - \varphi_{20},\\
\gamma_{30}(\alpha_1,\alpha_2) &= \tilde{g}_{11}\psi_{20} + 2\theta_1(1-\theta_2)\varphi_{02}\psi_{20} - (1+\theta_1-\theta_2)\varphi_{11}\psi_{20} + 2\varphi_{20}\psi_{20}\\
&\quad + \psi_{30} - 2\theta_1^2\varphi_{02}\varphi_{20} - 2\theta_1\tilde{h}_{20}\varphi_{02} + 2\theta_1\varphi_{11}\varphi_{20} + (\tilde{h}_{20} - \tilde{g}_{20}\theta_1)\varphi_{11}\\
&\quad - 2\varphi_{20}^2 + 4\tilde{g}_{20}\varphi_{20} + \delta_{30} + \delta_{12}\theta_1^2 - \varphi_{30} - \delta_{21}\theta_1 - \delta_{03}\theta_1^3,\\
\gamma_{21}(\alpha_1,\alpha_2) &= \psi_{21} + 2\varphi_{20}^2 - \varphi_{21} + 2\tilde{g}_{02}\psi_{20} + \tilde{g}_{11}\psi_{11} - (1+\theta_1-\theta_2)\varphi_{11}\psi_{11}\\
&\quad + 2(1-\theta_2)\varphi_{11}\psi_{20} - 2\theta_1^2\varphi_{02}\varphi_{11} + 2\Big((1-\theta_2)\tilde{h}_{20} - \theta_1\tilde{h}_{11}\varphi_{02}\Big)\\
&\quad - \Big(\tilde{h}_{20} - \tilde{h}_{11} + \tilde{g}_{11}\theta_1 + (\theta_2 - 3)\tilde{g}_{20}\Big) + (3\tilde{g}_{11} - 2\tilde{g}_{20})\varphi_{20} + 2\varphi_{20}\psi_{11}\\
&\quad + 2\theta_1\varphi_{11}^2 - 2\varphi_{20}\psi_{20} + (\theta_2 - \theta_1 - 3)\varphi_{11}\varphi_{20} + 2(1-\theta_2)\theta_1\varphi_{02}\psi_{11}\\
&\quad - 2(1-\theta_2)^2\varphi_{02}\psi_{20} + 2(1-\theta_2)\theta_1\varphi_{02}\varphi_{20} + 3\delta_{03}(1-\theta_2)\theta_1^2\\
&\quad + \delta_{21}(1 + 2\theta_1 - \theta_2) - \delta_{12}(2 - 2\theta_2 + \theta_1)\theta_1 - 3\delta_{30},\\
\gamma_{12}(\alpha_1,\alpha_2) &= \psi_{12} - \varphi_{12} + 2\varphi_{20}\psi_{02} - 2\theta_1^2\varphi_{20}^2 + 2(\tilde{g}_{02} - \tilde{g}_{11})\varphi_{20} + 2\tilde{g}_{11}\varphi_{11}\\
&\quad - 2(\varphi_{02} - \varphi_{11})\varphi_{20} + 2\tilde{g}_{02}\psi_{11} + (\theta_2 - \theta_1 - 1)\varphi_{11}\psi_{02} - 2(1-\theta_2)^2\varphi_{02}\psi_{11}\\
&\quad + 2(1-\theta_2)\varphi_{11}\psi_{11} - 2\Big(\tilde{h}_{02}\theta_1 - \tilde{g}_{20} - (1-\theta_2)\tilde{h}_{11}\Big)\varphi_{02} + (\theta_2 - \theta_1 - 1)\varphi_{11}^2\\
&\quad + 2\delta_{12}(1-\theta_2)\theta_1 - (\tilde{g}_{11}\theta_2 + \tilde{g}_{02}\theta_1 - \tilde{h}_{02} + \tilde{h}_{11})\varphi_{11} + \tilde{g}_{11}\psi_{02} - 2\varphi_{20}\psi_{11}\\
&\quad - 2(\theta_2 - 2)\theta_1\varphi_{11}\varphi_{02} + 2\theta_1(1-\theta_2)\varphi_{02}\psi_{02} - 3\delta_{03}(1-\theta_2)^2\theta_1 - 2\delta_{21}(1-\theta_2)\\
&\quad + \delta_{12}(1-\theta_2)^2 - \delta_{21}\theta_1 + 3\delta_{30},\\
\gamma_{03}(\alpha_1,\alpha_2) &= 2\tilde{g}_{02}\psi_{02} - 2(1-\theta_2)^2\varphi_{02}\psi_{02} + 2(1-\theta_2)\varphi_{11}\psi_{02} - 2\varphi_{20}\psi_{02} + \psi_{03}\\
&\quad + 2(1-\theta_2)\theta_1^2\varphi_{02}^2 + \Big(\tilde{g}_{11} + 2(1-\theta_2)\tilde{h}_{02}\Big)\varphi_{02} - (1+\theta_1-\theta_2)\varphi_{11}\varphi_{02}\\
&\quad + 2\varphi_{02}\varphi_{20} - \delta_{30} - \delta_{12}(1-\theta_2)^2 + \delta_{21}(1-\theta_2) + \delta_{03}(1-\theta_2)^3 - \varphi_{03}\\
&\quad - \Big(\tilde{h}_{02} - (1-\theta_2)\tilde{g}_{02}\Big)\varphi_{11} - 2\tilde{g}_{02}\varphi_{20},\\
\sigma_{20}(\alpha_1,\alpha_2) &= \tilde{h}_{20} - \psi_{02}\theta_1^2 + (\psi_{11} + \varphi_{20})\theta_1 - (2-\theta_2)\psi_{20},\\
\sigma_{11}(\alpha_1,\alpha_2) &= \tilde{h}_{11} + 2\psi_{02}(1-\theta_2)\theta_1 - (2 - 2\theta_2 + \theta_1)\psi_{11} + 2\psi_{20} + \varphi_{11}\theta_1,\\
\sigma_{02}(\alpha_1,\alpha_2) &= \tilde{h}_{02} - \Big(2 - (3-\theta_2)\theta_2\Big)\psi_{02} + (1-\theta_2)\psi_{11} - \psi_{20} + \theta_1\varphi_{02},\\
\sigma_{30}(\alpha_1,\alpha_2) &= 2\theta_1(1-\theta_2)\psi_{20}\psi_{02} - 2\theta_1^2\varphi_{20}\psi_{02} - 2\theta_1\tilde{h}_{20}\psi_{02} - (1+\theta_1-\theta_2)\psi_{11}\psi_{20}\\
&\quad + 2\theta_1\varphi_{20}\psi_{11} - (\tilde{g}_{20}\theta_1 - \tilde{h}_{20})\psi_{11} + 2\psi_{20}^2 + (\tilde{h}_{11} + 2\tilde{g}_{20})\psi_{20} - 2\varphi_{20}\psi_{20}\\
&\quad - (1-\theta_2)\psi_{30} + 2\tilde{h}_{20}\varphi_{20} + \theta_1\varphi_{30} - \beta_{03}\theta_1^3 - \beta_{21}\theta_1 + \beta_{30} + \beta_{12}\theta_1^2,
\end{aligned}
$$

$$\begin{aligned}
\sigma_{21}(\alpha_1,\alpha_2) =& -\psi_{21}(1-\theta_2)+2\tilde{h}_{20}\varphi_{11}+\varphi_{21}\theta_1+2\varphi_{20}\psi_{20}+2(1-\theta_2)\theta_1\psi_{02}\psi_{11}\\
&-2(1-\theta_2)^2\psi_{02}\psi_{20}+2(1-\theta_2)\theta_1\varphi_{20}\psi_{02}-(1+\theta_1-\theta_2)\varphi_{20}\psi_{11}+\tilde{h}_{11}\varphi_{20}\\
&-(1+\theta_1-\theta_2)\psi_{11}^2+2(2-\theta_2)\psi_{11}\psi_{20}+2\theta_1\varphi_{11}\psi_{11}-2\varphi_{11}\psi_{20}-2\psi_{20}^2\\
&-2\theta_1^2\varphi_{11}\psi_{02}+\Big((1-\theta_2)\tilde{g}_{20}-\tilde{h}_{20}+2\tilde{h}_{11}-\tilde{g}_{11}\theta_1\Big)\psi_{11}+2(\tilde{h}_{02}-\tilde{g}_{20}+\tilde{g}_{11})\psi_{20}\\
&-\beta_{12}\theta_1^2-2\beta_{12}(1-\theta_2)\theta_1+2\Big((1-\theta_2)\tilde{h}_{20}-\theta_1\tilde{h}_{11}\Big)\psi_{02}+3\beta_{03}(1-\theta_2)\theta_1^2\\
&+\beta_{21}(1-\theta_2)+2\beta_{21}\theta_1-3\beta_{30},\\
\sigma_{12}(\alpha_1,\alpha_2) =& \tilde{h}_{11}\varphi_{11}-(1+\theta_1-\theta_2)\varphi_{11}\psi_{11}+\Big((1-\theta_2)(3-2\theta_2)-\theta_1\Big)\psi_{02}\psi_{11}+2\varphi_{11}\psi_{20}\\
&+2\theta_1(1-\theta_2)\varphi_{11}\psi_{02}+\theta_1\varphi_{12}-(1-\theta_2)\psi_{12}+2\tilde{h}_{20}\varphi_{02}-2\varphi_{02}\psi_{20}-2\psi_{11}\psi_{20}\\
&+2(\tilde{g}_{02}-\tilde{g}_{11})\psi_{20}+2(1-\theta_2)\psi_{11}^2+2\theta_1\varphi_{02}\psi_{11}+2\theta_1(1-\theta_2)\psi_{02}^2-2\theta_1^2\varphi_{02}\psi_{02}\\
&-\Big(\tilde{g}_{02}\theta_1-3\tilde{h}_{02}+\tilde{h}_{11}-\tilde{g}_{11}(1-\theta_2)\Big)\psi_{11}+\Big(2(1-\theta_2)\tilde{h}_{11}-2\tilde{h}_{02}\theta_1+\tilde{h}_{11}\Big)\psi_{02}\\
&+2\psi_{02}\psi_{20}-3\beta_{03}(1-\theta_2)^2\theta_1-2\beta_{21}(1-\theta_2)+\beta_{12}(1-\theta_2)^2-\beta_{21}\theta_1+3\beta_{30}\\
&+2\beta_{12}(1-\theta_2)\theta_1,\\
\sigma_{03}(\alpha_1,\alpha_2) =& -2(1-\theta_2)^2\psi_{02}^2+2(1-\theta_2)\psi_{02}\psi_{11}-2\psi_{02}\psi_{20}+2\theta_1(1-\theta_2)\varphi_{02}\psi_{02}\\
&+2(2-\theta_2)\tilde{h}_{02}\psi_{02}-(1-\theta_2)\psi_{03}-(1+\theta_1-\theta_2)\varphi_{02}\psi_{11}-2\tilde{g}_{02}\psi_{20}+2\varphi_{02}\psi_{20}\\
&-\Big(\tilde{h}_{02}-\tilde{g}_{02}(1-\theta_2)\Big)\psi_{11}+\beta_{21}(1-\theta_2)-\beta_{12}(1-\theta_2)^2+\tilde{h}_{11}\varphi_{02}+\theta_1\varphi_{03}\\
&+\beta_{03}(1-\theta_2)^3-\beta_{30}.
\end{aligned}$$

We remove quadratic terms by taking

$$\begin{aligned}
\gamma_{20}(\alpha_1,\alpha_2)&=\gamma_{11}(\alpha_1,\alpha_2)=\gamma_{02}(\alpha_1,\alpha_2)=\sigma_{20}(\alpha_1,\alpha_2)=\sigma_{11}(\alpha_1,\alpha_2)\\
&=\sigma_{02}(\alpha_1,\alpha_2)=0
\end{aligned}$$

and can obtain $\varphi_{jk}(\alpha_1,\alpha_2)$ and $\psi_{jk}(\alpha_1,\alpha_2)$ for $j+k=2$. Further, in order to annihilate all cubic but those resonant terms, the vanishing conditions

$$\begin{aligned}
\gamma_{30}(\alpha_1,\alpha_2)&=\gamma_{21}(\alpha_1,\alpha_2)=\gamma_{12}(\alpha_1,\alpha_2)=\gamma_{03}(\alpha_1,\alpha_2)=\sigma_{12}(\alpha_1,\alpha_2)\\
&=\sigma_{03}(\alpha_1,\alpha_2)=0
\end{aligned}$$

yield the system for $\varphi_{jk}(\alpha_1,\alpha_2)$ and $\psi_{jk}(\alpha_1,\alpha_2)$, from which we can obtain $\varphi_{jk}(\alpha_1,\alpha_2)$ and $\psi_{jk}(\alpha_1,\alpha_2)$ for $j+k=3$. After these transformations, map (2.8) can finally be transformed into the following normal form for 1 : 2 resonance at the critical condition

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\to\begin{pmatrix}-\xi+\eta\\ \theta_1(\alpha_1,\alpha_2)\xi+[-1+\theta_2(\alpha_1,\alpha_2)]\eta+C(\alpha_1,\alpha_2)\xi^3+D(\alpha_1,\alpha_2)\xi^2\eta+O\Big((|\xi|+|\eta|)^4\Big)\end{pmatrix}$$

where $C(\alpha_1,\alpha_2)=\sigma_{30}(\alpha_1,\alpha_2)$ and $D(\alpha_1,\alpha_2)=\sigma_{21}(\alpha_1,\alpha_2)$. When $(\alpha_1,\alpha_2)=(\tilde{\alpha}_1,\tilde{\alpha}_2)$, we have $\theta_1(\tilde{\alpha}_1,\tilde{\alpha}_2)=\theta_2(\tilde{\alpha}_1,\tilde{\alpha}_2)=0$, and

$$C(\tilde{\alpha}_1,\tilde{\alpha}_2)=\tilde{h}_{30}(\tilde{\alpha}_1,\tilde{\alpha}_2)+\tilde{g}_{20}\tilde{h}_{20}(\tilde{\alpha}_1,\tilde{\alpha}_2)+\frac{1}{2}\tilde{h}_{20}^2(\tilde{\alpha}_1,\tilde{\alpha}_2)+\frac{1}{2}\tilde{h}_{20}\tilde{h}_{11}(\tilde{\alpha}_1,\tilde{\alpha}_2),$$

$$\begin{aligned} D(\tilde{\alpha}_1,\tilde{\alpha}_2) &= \tilde{h}_{21}(\tilde{\alpha}_1,\tilde{\alpha}_2) + 3\tilde{g}_{20}\tilde{h}_{11}(\tilde{\alpha}_1,\tilde{\alpha}_2) + \frac{5}{4}\tilde{h}_{20}(\tilde{\alpha}_1,\tilde{\alpha}_2)\tilde{h}_{11}(\tilde{\alpha}_1,\tilde{\alpha}_2) \\ &\quad + \tilde{h}_{20}(\tilde{\alpha}_1,\tilde{\alpha}_2)\tilde{h}_{02}(\tilde{\alpha}_1,\tilde{\alpha}_2) + 3\tilde{g}_{20}^2(\tilde{\alpha}_1,\tilde{\alpha}_2) + \frac{5}{2}\tilde{g}_{20}(\tilde{\alpha}_1,\tilde{\alpha}_2)\tilde{h}_{20}(\tilde{\alpha}_1,\tilde{\alpha}_2) \\ &\quad + \frac{5}{2}\tilde{g}_{11}(\tilde{\alpha}_1,\tilde{\alpha}_2)\tilde{h}_{20}(\tilde{\alpha}_1,\tilde{\alpha}_2) + \tilde{h}_{20}^2(\tilde{\alpha}_1,\tilde{\alpha}_2) + \frac{1}{2}\tilde{h}_{11}^2(\tilde{\alpha}_1,\tilde{\alpha}_2). \end{aligned}$$

We get the following results by using the 1 : 2 resonance theorems from [25,26].

Theorem 2.1. *The following bifurcation behaviors are admitted by map (1.1) if* $C(\tilde{\alpha}_1,\tilde{\alpha}_2) \neq 0$ *and* $D(\tilde{\alpha}_1,\tilde{\alpha}_2) + 3C(\tilde{\alpha}_1,\tilde{\alpha}_2) \neq 0$

(i) *The critical point is elliptic if* $C(\tilde{\alpha}_1,\tilde{\alpha}_2) > 0$*, and saddle if* $C(\tilde{\alpha}_1,\tilde{\alpha}_2) < 0$*. The bifurcation scenario close to the* 1 : 2 *points is determined by* $D(\tilde{\alpha}_1,\tilde{\alpha}_2) + 3C(\tilde{\alpha}_1,\tilde{\alpha}_2) \neq 0$.

(ii) *A pitchfork bifurcation curve is defined as* $PF = \{(\theta_1,\theta_2) : \theta_1 = 0\}$*, and non-trivial fixed points exist for* $\theta_1 < 0$.

(iii) *A non-degenerate Neimark-Sacker bifurcation curve exists* $H = \{(\theta_1,\theta_2) : \theta_1 = -\theta_2 + O\Big((|\theta_1| + |\theta_2|)^2\Big), \theta_1 < 0\}$;

(iv) *A heteroclinic bifurcation curve exists* $HL = \{(\theta_1,\theta_2) : \theta_1 = -\frac{5}{3}\theta_2 + O\Big((|\theta_1| + |\theta_2|)^2\Big), \theta_1 < 0\}$.

3. 1 : 3 Resonance

This section aims to elucidate the dynamic behavior of map (1.1) at the 1 : 3 resonance point. We can choose the parameters $(\hat{\alpha}_1, \hat{\alpha}_2, A_1, A_2, B)$ from F_{13} and analyze the mapping (1.1) using these parameters:

$$\begin{cases} x \to x + \hat{\alpha}_1 x(A_1 - 2x - By), \\ y \to y + \hat{\alpha}_2 y(A_2 - 2y + Bx), \end{cases} \tag{3.1}$$

Now let's examine this map in this way:

$$\begin{cases} x \to x + \alpha_1 x(A_1 - 2x - By), \\ y \to y + \alpha_2 y(A_2 - 2y + Bx), \end{cases} \tag{3.2}$$

where $|\alpha_1 - \hat{\alpha}_1|, |\alpha_2 - \hat{\alpha}_2| \ll 1$. Bifurcation theory [22,24,25] will be used to show the normal form at the 1 : 3 resonance point next. Consider $\hat{u} = x - x^*$, $\hat{v} = y - y^*$, if we move the fixed point $E^*(x^*, y^*)$ to the starting point:

$$\begin{cases} \hat{u} \to (1 - 2\alpha_1 x^*)\hat{u} - \alpha_1 B x^* \hat{v} - 2\alpha_1 \hat{u}^2 - \alpha_1 B \hat{u}\hat{v}, \\ \hat{v} \to \alpha_2 B y^* \hat{u} + (1 - 2\alpha_2 y^*)\hat{v} + \alpha_2 B \hat{u}\hat{v} - 2\alpha_2 \hat{v}^2. \end{cases} \tag{3.3}$$

The matrix of the Jacobian at E^* for the map (3.3) is

$$A(\alpha_1,\alpha_2) = \begin{pmatrix} 1 - 2\alpha_1 x^* & -\alpha_1 B x^* \\ \alpha_2 B y^* & 1 - 2\alpha_2 y^* \end{pmatrix},$$

The eigenvalues that correspond to the map (3.3) are $\lambda_{1,2} = (\pm\sqrt{3}i - 1)/2$. The adjoint eigenvector $p(\alpha_1,\alpha_2) \in \mathbb{C}$ and the corresponding eigenvector $q(\alpha_1,\alpha_2) \in \mathbb{C}$ may be easily derived in the same way:

$$\begin{aligned} A(\hat{\alpha}_1,\hat{\alpha}_2)q(\hat{\alpha}_1,\hat{\alpha}_2) &= \frac{\sqrt{3}i-1}{2}q(\hat{\alpha}_1,\hat{\alpha}_2), \\ A^T(\hat{\alpha}_1,\hat{\alpha}_2)p(\hat{\alpha}_1,\hat{\alpha}_2) &= -\frac{\sqrt{3}i+1}{2}p(\hat{\alpha}_1,\hat{\alpha}_2),\ \langle p(\hat{\alpha}_1,\hat{\alpha}_2), q(\hat{\alpha}_1,\hat{\alpha}_2)\rangle = 1, \end{aligned}$$

the standard scalar product in $\mathbb{C}$, $\langle p,q\rangle = \bar{p}_1q_1+\bar{p}_2q_2$, where $\langle .,.\rangle$ denotes this. Take

$$q(\hat{\alpha}_1,\hat{\alpha}_2) = \begin{pmatrix} \hat{\alpha}_1Bx^* \\ \frac{3-\sqrt{3}i}{2} - 2\hat{\alpha}_1x^* \end{pmatrix},$$

$$p(\hat{\alpha}_1,\hat{\alpha}_2) = \begin{pmatrix} \frac{3-\sqrt{3}i(3-4\hat{\alpha}_2y^*)}{6\hat{\alpha}_1Bx^*} \\ -\frac{\sqrt{3}i}{3} \end{pmatrix},$$

Any vector $U = (\hat{u},\hat{v})^T \in \mathbb{R}^2$ can be expressed in the following manner:

$$U = zq(\hat{\alpha}_1,\hat{\alpha}_2) + \bar{z}\bar{q}(\hat{\alpha}_1,\hat{\alpha}_2), z \in \mathbb{C}.$$

Consequently, (3.3) can be change into

$$z \mapsto \frac{\sqrt{3}i-1}{2}z + \sum_{2\le k+l\le 3}\frac{1}{k!l!}\, g_{kl}z^k\bar{z}^l, \tag{3.4}$$

where

$$\begin{aligned} g_{20} &= \frac{1}{2}\Bigg\{4\hat{\alpha}_2\left(3+i\sqrt{3}-4\hat{\alpha}_2y^*\right) + \hat{\alpha}_2B^2\left(1-\sqrt{3}i\right)(3-2\hat{\alpha}_2y^*) \\ &\quad - 2\hat{\alpha}_1\hat{\alpha}_2B\left(3+i\sqrt{3}-\left(2+2i\sqrt{3}\right)y^*\right)\Bigg\}, \\ g_{11} &= \frac{B\sqrt{3}i}{2}\Bigg\{\hat{\alpha}_2B(-3+2\hat{\alpha}_2y^*) - \hat{\alpha}_1\left(3-\sqrt{3}i-4\hat{\alpha}_2y^*\right)\Bigg\}, \\ g_{02} &= \frac{1}{2}\Bigg\{\hat{\alpha}_2B^2\left(1+i\sqrt{3}\right)(-3+2\hat{\alpha}_2y^*) + 4\hat{\alpha}_2\left(-3+i\sqrt{3}+4\hat{\alpha}_2y^*\right) \\ &\quad - 4\hat{\alpha}_1B\left(\sqrt{3}i+\left(1-\sqrt{3}i\right)\hat{\alpha}_2y^*\right)\Bigg\}, \end{aligned}$$

$$g_{30} = g_{21} = g_{12} = g_{03} = 0.$$

We now present the subsequent transformation to exclude certain second-order terms:

$$z = \omega + \frac{1}{2}h_{20}\omega^2 + h_{11}\omega\bar{\omega} + \frac{1}{2}h_{02}\bar{\omega}^2 \tag{3.5}$$

The coefficients h_{kl}, where $k+l=2$, will be provided subsequently. Consequently, we can acquire:

$$
\begin{aligned}
\omega = {} & z - \frac{1}{2}h_{20}z^2 - h_{11}z\bar{z} - \frac{1}{2}h_{02}\bar{z}^2 + \frac{1}{2}(h_{20}^2 + h_{11}\bar{h}_{02})z^3 \\
& + \left(\frac{3}{2}h_{20}h_{11} + |h_{11}|^2 + \frac{1}{2}|h_{02}|^2\right)z^2\bar{z} + \left(\frac{1}{2}h_{11}\bar{h}_{20} + h_{11}^2 + h_{02}\bar{h}_{11}\right. \\
& \left. + \frac{1}{2}h_{02}h_{20}\right)z\bar{z}^2 + \frac{1}{2}\left(h_{02}\bar{h}_{20} + h_{11}h_{02}\right)\bar{z}^3 + O\left(|z|^4\right),
\end{aligned}
\tag{3.6}
$$

By applying transformation (3.5) and its inverse transformation (3.6), model (3.4) is reformulated as

$$
\omega \mapsto \frac{\sqrt{3}i - 1}{2}\omega + \sum_{2 \le k+l \le 3} \frac{1}{k!l!}\, Q_{kl}\omega^k\bar{\omega}^l + O\left(|\omega|^4\right),
\tag{3.7}
$$

where

$$
\begin{aligned}
Q_{20} = {} & g_{20} + \sqrt{3}h_{20}i,\ Q_{11} = g_{11} - \frac{1}{2}(3 - \sqrt{3}i)h_{11},\ Q_{02} = g_{02} \\
Q_{30} = {} & \frac{3}{2}\left(3 - \sqrt{3}i\right)h_{20}g_{20} + 3g_{11}\bar{h}_{02} - \frac{3}{2}\left(\sqrt{3}i - 1\right)h_{11}\bar{g}_{02} \\
& + \frac{3}{2}\left(3 + \sqrt{3}i\right)h_{20}^2 + g_{30}, \\
Q_{21} = {} & \frac{1}{2}\left(5 + \sqrt{3}i\right)h_{11}g_{20} + \left(2 - \sqrt{3}i\right)h_{20}g_{11} + 2\bar{h}_{11}g_{11} \\
& + \left(1 - \sqrt{3}i\right)h_{11}\bar{g}_{11} + g_{02}\bar{h}_{02} + \frac{1}{2}\left(1 + \sqrt{3}i\right)h_{02}\bar{g}_{02} \\
& + g_{21} - \frac{1}{2}\left(3 - 5\sqrt{3}i\right)h_{11}h_{20} - \left(3 - \sqrt{3}i\right)\bar{h}_{11}h_{11}, \\
Q_{12} = {} & g_{20}h_{02} + \frac{1}{2}\left(1 - \sqrt{3}i\right)h_{11}\bar{g}_{20} + \left(\bar{h}_{20} + \left(3 + \sqrt{3}i\right)h_{11}\right)g_{11} \\
& + \left(1 + \sqrt{3}i\right)h_{02}\bar{g}_{11} + g_{12} + \left(\frac{1}{2}\left(1 - \sqrt{3}i\right)h_{20} + 2\bar{h}_{11}\right)g_{02} \\
& - \frac{1}{2}\left(3 + \sqrt{3}i\right)h_{11}\bar{h}_{20} - \left(3 + \sqrt{3}i\right)h_{11}^2 - 2\sqrt{3}ih_{02}\bar{h}_{11}, \\
Q_{30} = {} & \frac{3}{2}\left(1 + \sqrt{3}i\right)h_{02}\bar{g}_{20} + 3g_{11}\bar{h}_{02} + 3\bar{h}_{20}g_{02} \\
& + \frac{3}{2}\left(1 + \sqrt{3}i\right)h_{11}g_{02} + \frac{3}{2}\left(3 - \sqrt{3}i\right)h_{02}\bar{h}_{20} + g_{03}.
\end{aligned}
$$

By setting

$$
h_{20} = \frac{\sqrt{3}i}{3}g_{20},\ h_{11} = \frac{3 + \sqrt{3}i}{6}g_{11},\ h_{02} = 0.
\tag{3.8}
$$

Subsequently, we possess $Q_{20} = Q_{11} = 0, Q_{02} = g_{02}$ and $Q_{30}, Q_{21}, Q_{12}, Q_{03}$ can be simplified as follows. Consequently, the transformation (3.5) is established and

$$
\begin{aligned}
Q_{30} &= \frac{3 - \sqrt{3}i}{2}g_{11}\bar{g}_{02} + \sqrt{3}ig_{20}^2 + g_{30} \\
Q_{21} &= \frac{3 + 2\sqrt{3}i}{3}g_{11}g_{20} + \frac{3 - \sqrt{3}i}{3}|g_{11}|^2 + g_{21}
\end{aligned}
$$

$$\begin{aligned} Q_{12} &= \frac{3+\sqrt{3}i}{6}g_{20}g_{02}+\frac{3-\sqrt{3}i}{3}\bar{g}_{11}g_{02}+\frac{3+\sqrt{3}i}{3}g_{11}^2-\frac{\sqrt{3}i}{3}g_{11}\bar{g}_{20}+g_{12} \\ Q_{03} &= \sqrt{3}ig_{11}g_{02}-\sqrt{3}i\bar{g}_{20}g_{02}+g_{03}. \end{aligned}$$

To further eliminate some cubic terms, we take

$$\omega=\zeta+\frac{1}{6}h_{30}\zeta^3+\frac{1}{2}h_{21}\zeta^2\bar{\zeta}+\frac{1}{2}h_{12}\zeta\bar{\zeta}^2+\frac{1}{6}h_{03}\bar{\zeta}^3 \tag{3.9}$$

Utilizing (3.9) and its inverse transformation, the map (3.6) is transformed into the subsequent form:

$$\zeta\mapsto\frac{\sqrt{3}i-1}{2}\zeta+\frac{g_{02}}{2}\bar{\zeta}^2+\sum_{k+l=3}\frac{1}{k!l!}\tilde{Q}_{kl}\zeta^k\bar{\zeta}^l+O\left(|\zeta|^4\right), \tag{3.10}$$

where

$$\begin{aligned} &\tilde{Q}_{30}=Q_{30}-\frac{3-\sqrt{3}i}{2}h_{30},\ \tilde{Q}_{21}=Q_{21},\ \tilde{Q}_{12}=Q_{12}+\sqrt{3}ih_{12}, \\ &\tilde{Q}_{03}=Q_{03}-\frac{3-\sqrt{3}i}{2}h_{03}. \end{aligned}$$

By setting

$$h_{30}=\frac{3+\sqrt{3}i}{6}Q_{30},\ h_{21}=0,\ h_{12}=\frac{\sqrt{3}i}{3}Q_{12},\ h_{03}=\frac{3+\sqrt{3}i}{6}Q_{03},$$

Consequently, we have $\tilde{Q}_{30}=\tilde{Q}_{21}=\tilde{Q}_{03}=0$. Therefore, the transformation (3.9) is established. By employing transformation (3.9), map (3.10) ultimately assumes the subsequent normal form of the bifurcation exhibiting 1 : 3 resonance:

$$\zeta\mapsto\frac{\sqrt{3}i-1}{2}\zeta+\hat{B}(\hat{\alpha}_1,\hat{\alpha}_2)\bar{\zeta}^2+\hat{C}(\hat{\alpha}_1,\hat{\alpha}_2)\zeta|\zeta|^2+O\left(|\zeta|^4\right), \tag{3.11}$$

where

$$\begin{aligned} \hat{B}(\hat{\alpha}_1,\hat{\alpha}_2) &= \frac{g_{02}}{2}, \\ \hat{C}(\hat{\alpha}_1,\hat{\alpha}_2) &= \frac{\left(3+2\sqrt{3}i\right)g_{20}g_{11}}{6}+\frac{\left(3-\sqrt{3}i\right)g_{11}^2}{6}+\frac{g_{21}}{2}. \end{aligned}$$

Let

$$\begin{aligned} B_1(\hat{\alpha}_1,\hat{\alpha}_2) &= -\frac{3}{2}\left(1+\sqrt{3}i\right)\hat{B}(\hat{\alpha}_1,\hat{\alpha}_2), \\ C_1(\hat{\alpha}_1,\hat{\alpha}_2) &= -3|\hat{B}(\hat{\alpha}_1,\hat{\alpha}_2)|^2-\frac{3}{2}\left(1+\sqrt{3}i\right)\hat{C}(\hat{\alpha}_1,\hat{\alpha}_2). \end{aligned}$$

Theorem 9.11 in Kuznetsov [22] and Theorem 2 in Li and He [25] allow us to get the following theorem:

Theorem 3.1. *Let $(\hat{\alpha}_1,\hat{\alpha}_2)\in F_{13}$. Assume $B_1(\hat{\alpha}_1,\hat{\alpha}_2)\neq 0$ and $Re\,(C_1(\hat{\alpha}_1,\hat{\alpha}_2))\neq 0$, then (1.1) has the following dynamical behaviors at the fixed point E^**

(i) The invariant closed curve near the 1 : 3 resonance point is unstable if $Re\,(C_1(\hat{\alpha}_1,\hat{\alpha}_2))>0$; conversely, if $Re\,(C_1(\hat{\alpha}_1,\hat{\alpha}_2))<0$, the invariant closed curve is stable;

(ii) The map's trivial fixed point E_0 has a non-degenerate Neimark-Sacker bifurcation (3.11).;

(iii) The saddle fixed $E_k (k = 1, 2, 3)$ of (3.11) corresponds to a saddle cycle of period three;

(iv) There is a homoclinic structure formed by the stable and unstable invariant manifolds of the period three cycle intersecting transversally in an exponentially narrow parameter region.

In Sect. 6, we will give some values of parameters such that 1 : 3 resonance occurs as $\hat{\alpha}_1$ and $\hat{\alpha}_2$ varying with $B_1(\hat{\alpha}_1, \hat{\alpha}_2) \neq 0$ and $\Re\left(C_1(\hat{\alpha}_1, \hat{\alpha}_2)\right) \neq 0$.

4. 1:4 Resonance of Map (1.1) at E^*

Subsequently, we examine the 1 : 4 resonance of map (1.1) at E^*, with parameters (α_1, α_2) fluctuating within a minor vicinity of F_{14}. We examine system (1.1) utilizing parameters $(\bar{\alpha}_1, \bar{\alpha}_2, A_1, A_2, B)$ from F_{14} as follows:

$$\begin{cases} x \to x + \bar{\alpha}_1 x(A_1 - 2x - By), \\ y \to y + \bar{\alpha}_2 y(A_2 - 2y + Bx), \end{cases} \tag{4.1}$$

We now examine this map in the following manner:

$$\begin{cases} x \to x + \alpha_1 x(A_1 - 2x - By), \\ y \to y + \alpha_2 y(A_2 - 2y + Bx), \end{cases} \tag{4.2}$$

where $|\alpha_1 - \bar{\alpha}_1|, |\alpha_2 - \bar{\alpha}_2| \ll 1$. Let $\bar{u} = x - x^*$, $\bar{v} = y - y^*$, Transform the fixed point $E^*(x^*, y^*)$ to the origin, we obtain

$$\begin{cases} \bar{u} \to (1 - 2\alpha_1 x^*)\bar{u} - \alpha_1 B x^* \bar{v} - 2\alpha_1 \bar{u}^2 - \alpha_1 B \bar{u}\bar{v}, \\ \bar{v} \to \alpha_2 B y^* \bar{u} + (1 - 2\alpha_2 y^*)\bar{v} + \alpha_2 B \bar{u}\bar{v} - 2\alpha_2 \bar{v}^2. \end{cases} \tag{4.3}$$

A map (4.3) at E^* has a Jacobian matrix that is

$$A(\alpha_1, \alpha_2) = \begin{pmatrix} 1 - 2\alpha_1 x^* & -\alpha_1 B x^* \\ \alpha_2 B y^* & 1 - 2\alpha_2 y^* \end{pmatrix},$$

The corresponding eigenvalues of map (4.3) are $\lambda_{1,2} = \pm i$. We choose

$$q(\bar{\alpha}_1, \bar{\alpha}_2) = \begin{pmatrix} \bar{\alpha}_1 B x^* \\ 1 - 2\bar{\alpha}_1 x^* - i \end{pmatrix},$$

$$p(\bar{\alpha}_1, \bar{\alpha}_2) = \begin{pmatrix} \frac{1 - i(1 - 2\bar{\alpha}_2 y^*)}{2\bar{\alpha}_1 B x^*} \\ -\frac{i}{2} \end{pmatrix}.$$

We can get any vector $U = (\bar{u}, \bar{v})^T \in \mathbb{R}^2$ is put as follows

$$U = zq(\bar{\alpha}_1, \bar{\alpha}_2) + \bar{z}\bar{q}(\bar{\alpha}_1, \bar{\alpha}_2), z \in \mathbb{C}.$$

In this way, (4.3) can be changed into the following shape:

$$z \mapsto iz + \sum_{2 \le k+l \le 3} \frac{1}{k!l!} g_{kl} z^k \bar{z}^l, \tag{4.4}$$

where

$$\begin{aligned}
g_{20} &= (4+4i)\alpha_2 - 8\alpha_2^2 y^* + \alpha_2 B^2(1-i)(1-\alpha_2 y^*) \\
&\quad - 2\alpha_1 B\Big(1-(1+i)\alpha_2 y^*\Big), \\
g_{11} &= iB\left\{\alpha_2 B(-1+\alpha_2 y^*) + \alpha_1\Big((-1+i)+2\alpha_2 y^*\Big)\right\}, \\
g_{02} &= (1+i)\left\{\alpha_2 B^2(-1+\alpha_2 y^*) + 4\alpha_2\Big(i+(1-i)\alpha_2 y^*\Big)\right. \\
&\quad \left. + \alpha_1 B(1+i)\Big(-1+(1+i)\alpha_2 y^*\Big)\right\}, \\
g_{30} &= g_{21} = g_{12} = g_{03} = 0.
\end{aligned}$$

We apply a transformation in order to eliminate certain quadratic terms

$$z = \omega + \frac{1}{2}h_{20}\omega^2 + h_{11}\omega\bar{\omega} + \frac{1}{2}h_{02}\bar{\omega}^2 \tag{4.5}$$

the following will give the coefficients h_{kl} where $k+l=2$. Model (4.4) is transformed into by utilizing transformation (4.5) and its inverse transformation.

$$\omega \mapsto i\omega + \sum_{2\le k+l\le 3} \frac{1}{k!l!}\, Q_{kl}\omega^k\bar{\omega}^l + O\left(|\omega|^4\right), \tag{4.6}$$

where

$$\begin{aligned}
Q_{20} &= g_{20} + (1+i)h_{20},\ Q_{11} = g_{11} - (1-i)h_{11},\ Q_{02} = g_{02} + (1+i)h_{02}, \\
Q_{30} &= 3\,(1-i)\,h_{20}g_{20} - 3\left(i\bar{g}_{02} + (1+i)\bar{h}_{02}\right)h11 + 3(1-i)h_{20}^2 + 3g_{11}\bar{h}_{02} + g_{30} \\
Q_{21} &= \left(i\bar{g}_{02} + (1+i)\bar{h}_{02}\right)h_{02} + (1-2i)g_{11}h_{20} + 2g_{11}\bar{h}_{11} + g_{02}\bar{h}_{02} + g_{21} \\
&\quad + \left((1+3i)h_{20} - 2i\bar{g}_{11} - 2(1-i)\bar{h}_{11} + (2+i)g_{20}\right)h_{11}, \\
Q_{12} &= \left(2(1+i)h_{11} + \bar{h}_{20}\right)g_{11} + g_{20}h_{02} + \left((1-i)h_{20} + 2i\bar{g}_{11} + 2(1-i)\bar{h}_{11}\right)h_{02}, \\
&\quad - 2(1+i)h_{11}^2 - \left(i\bar{g}_{20} + (1+i)\bar{h}_{20}\right)h_{11} + g_{12} - ih_{20}g_{02} + 2g_{02}\bar{h}_{11}, \\
Q_{30} &= g_{03} + 3g_{11}h_{02} + 3\left((i-1)h_{11} + i\bar{g}_{20} + (1+i)\bar{h}_{20}\right)h_{02} \\
&\quad 3g_{02}\bar{h}_{20} + 3ih_{11}g_{02}.
\end{aligned}$$

By setting

$$h_{20} = \frac{1}{2}g_{20}(i-1),\ h_{11} = \frac{1}{2}g_{11}(i+1),\ h_{02} = \frac{1}{2}g_{02}(i+1), \tag{4.7}$$

then we have $Q_{20} = Q_{11} = Q_{02} = 0$ and $Q_{30}, Q_{21}, Q_{12}, Q_{03}$ can be simplified. This leads to the definition of the transformation (4.5). The following transformation is introduced to better simplify the system:

$$\omega = \zeta + \frac{1}{6}h_{30}\zeta^3 + \frac{1}{2}h_{21}\zeta^2\bar{\zeta} + \frac{1}{2}h_{12}\zeta\bar{\zeta}^2 + \frac{1}{6}h_{03}\bar{\zeta}^3 \tag{4.8}$$

Map (4.6) takes on the following shape after employing (4.8) and its inverse transformation:

$$\zeta \mapsto i\zeta + \sum_{k+l=3} \frac{1}{k!l!} \tilde{Q}_{kl}\zeta^k\bar{\zeta}^l + O\left(|\zeta|^4\right), \tag{4.9}$$

where

$$\tilde{Q}_{30} = Q_{30} + 2ih_{30},\ \tilde{Q}_{21} = Q_{21},\ \tilde{Q}_{12} = Q_{12} + 2ih_{12},$$
$$\tilde{Q}_{03} = Q_{03}.$$

By setting

$$h_{30} = \frac{1}{2}Q_{30}i,\ h_{21} = 0,\ h_{12} = \frac{1}{2}Q_{12}i,\ h_{03} = 0,$$

then we have $\tilde{Q}_{30} = \tilde{Q}_{12} = 0$. Hence, transformation (4.8) is defined. Using transformation (4.8), map (4.9) finally becomes the following normal form of the bifurcation with 1 : 4 strong resonance

$$\zeta \mapsto i\zeta + \tilde{Q}_{21}(\bar{\alpha}_1, \bar{\alpha}_2)\zeta^2\bar{\zeta} + \tilde{Q}_{03}(\bar{\alpha}_1, \bar{\alpha}_2)\zeta^3 + O\left(|\zeta|^4\right), \tag{4.10}$$

where

$$\tilde{Q}_{21} = \frac{1+3i}{4}g_{20}g_{11} + \frac{1-i}{2}|g_{11}|^2 - \frac{1+i}{4}|g_{02}|^2 + \frac{1}{2}g_{21},$$
$$\tilde{Q}_{03} = \frac{i-1}{4}g_{02}g_{11} + \frac{1+i}{4}g_{11}g_{20} + \frac{1}{6}g_{03}.$$

Let

$$C_1(\bar{\alpha}_1, \bar{\alpha}_2) = -4i\tilde{Q}_{21}(\bar{\alpha}_1, \bar{\alpha}_2),\ D_1(\bar{\alpha}_1, \bar{\alpha}_2) = -4i\tilde{Q}_{03}(\bar{\alpha}_1, \bar{\alpha}_2).$$

If $D_1(\bar{\alpha}_1, \bar{\alpha}_2) \neq 0$, we denote $A(\bar{\alpha}_1, \bar{\alpha}_2) = \frac{C_1(\bar{\alpha}_1, \bar{\alpha}_2)}{|D_1(\bar{\alpha}_1, \bar{\alpha}_2)|}$. By employing a comparable reasoning found in Lemma 9.15 of [22], we can get the subsequent results.

Theorem 4.1. *The set F_{14} contains $(\bar{\alpha}_1, \bar{\alpha}_2)$. When both $ReA(\bar{\alpha}_1, \bar{\alpha}_2) \neq 0$ and $ImA(\bar{\alpha}_1, \bar{\alpha}_2) \neq 0$, a number of complex bifurcation curves are admissible in map (1.1):*

- *(i) A Neimark-Sacker bifurcation curve exists at the trivial fixed point E_0 of map (4.10). Moreover, when $\lambda = -i$, a stable invariant circle exists; conversely, when $\lambda = i$, the invariant circle vanishes.*
- *(ii) A fold bifurcation curve is present at eight nontrivial fixed points $S_k, E_k (k = 1, 2, 3, 4)$ of the map (4.10). Eight nontrivial fixed points appear or vanish via the fold bifurcation curve at the corresponding parameter values.*
- *(iii) There are Neimark-Sacker bifurcations at S_k and E_k (k = 1, 2, 3, 4). Furthermore, it is worth noting that four "small" invariant circles bifurcate from S_k and E_k (k = 1, 2, 3, 4).*
- *(iv) A "small" homoclinic loop bifurcation curve is shown. Neimark-Sacker bifurcation curves generate the "small" invariant circles from $S_k, E_k (k = 1, 2, 3, 4)$, and they vanish at homoclinic loop bifurcation curves.*
- *(v) There is a "square" heteroclinic cycle located around E_0.*

(vi) All fixed points $S_k, E_k (k = 1, 2, 3, 4)$ are surrounded by a "clover" heteroclinic cycle. It generates either a stable or an unstable "big" invariant circle, depending on the sign of the saddle quantity $\sigma = tr\Lambda(S_k)$.

(vii) A fold bifurcation curve exists at two significant invariant circles. The outer invariant circle is stable, while two invariant circles converge and disappear.

5. Numerical Results

5.1. **Numerical continuation.** The purpose of this section is to explore the local bifurcation structure around the fixed pointsE_1 and E^*. To do this, we carry out a series of numerical continuation operations by utilizing the MATLAB-based toolbox MATCONT [22, 25]. To verify the analytical results, we used MatCont for numerical bifurcation analysis. This section shows a variety of behaviors, starting with a 1:2 resonance bifurcation. A pair of complex conjugate eigenvalues traverses the unit circle with a modulus ratio of 1:2. Higher-order resonances like 1:3 and 1:4 interact with period-doubling bifurcations and quasi-periodic oscillations. We also detect phase-locking, invariant tori, and chaotic dynamics, as described in the following parts.

Example 1: Fix $A_1 = 5.9$, $A_2 = 5.8$, $B = 0.5$, $\alpha_2 = 0.25$, and vary α_1. The report of MATCONTM is as follows.

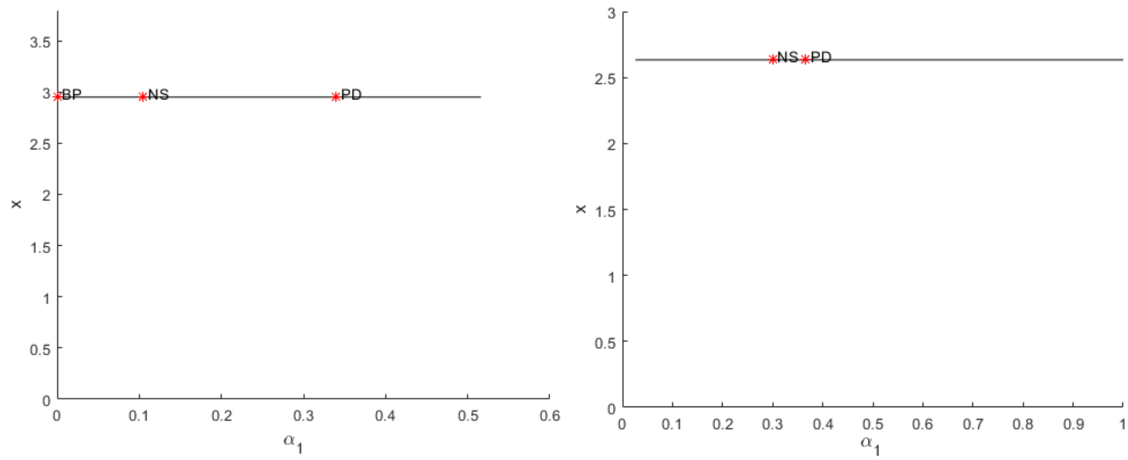
```
label PD, x= (2.950000 0.000000 0.338983),
Normal form coefficient for PD = 4.296380e-01 ,
label NS, x= (2.950000 0.000000 0.104240),
Neutral Saddle
label BP, x= (2.950000 0.000000 -0.000000).
```

By Theorem 1, E_1 has a flip bifurcation detected ad PD in Fig.(1)(a) and there is also a Neimark-sacker bifurcation, which has not been shown theoretically, detected as NS.

Example 2: Fix $A_1 = 5.9$, $A_2 = 5.8$, $B = 0.2$, $\alpha_2 = 0.42$, and vary α_1. The report of MATCONTM is as follows.

```
label NS, x= (2.633663 3.163366 0.299604),
Neutral Saddle
label PD, x= (2.633663 3.163366 0.364944),
 Normal form coefficient for PD = 5.583321e-01.
```

According to Theorems 2 and 3 the fixed point E^* has flip and Neimark-sacker bifurcations, respectively. It is depicted in Fig. (1).

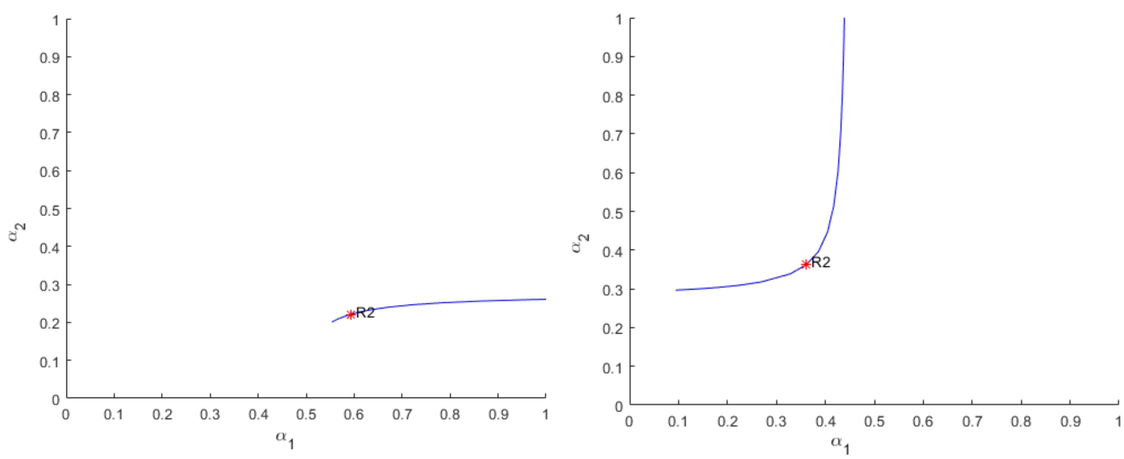


(a) **(b)**

Figure 1. The continuation of E_1 and E^* in (α_1, x)-plane. The branch point (BP), Neimark-Sacker point (NS), and period doubling point (PD) are found in Theorems 1 (a) and 2 and 3 (b).

Example 3: Fix $A_1 = 5.9$, $A_2 = 5.8, B = 0.5$, and vary α_1 and α_2. We have $E^* = (2.0941176, 3.4235294)$ is a $1:2$ resonance at $(\alpha_1, \alpha_2)_1 = (0.3617, 0.3629)$ and $(\alpha_1, \alpha_2)_2 = (0.5933, 0.2213)$ according to Theorem 4, where $|\lambda_1| = 1$ and $|\lambda_2| = 1.0786946 \simeq 1$. This is depicted in Fig. (2). The MATCONTM report is as follows:

```
label R2, x= (2.094118 3.423529 0.593346 0.24252),
Normal form coefficient for R2: [c,d]= -5.576197e-01, -3.205910e+00,
label R2, x= (2.094118 3.423529 0.361711 0.362940),
Normal form coefficient for R2: [c,d]= 2.626999e-01, -3.872191e+00.
```



(a) **(b)**

Figure 2. Continuation of E^* in (α_1, α_2)-plane where the $1:2$ resonance point is labeled as R_2.

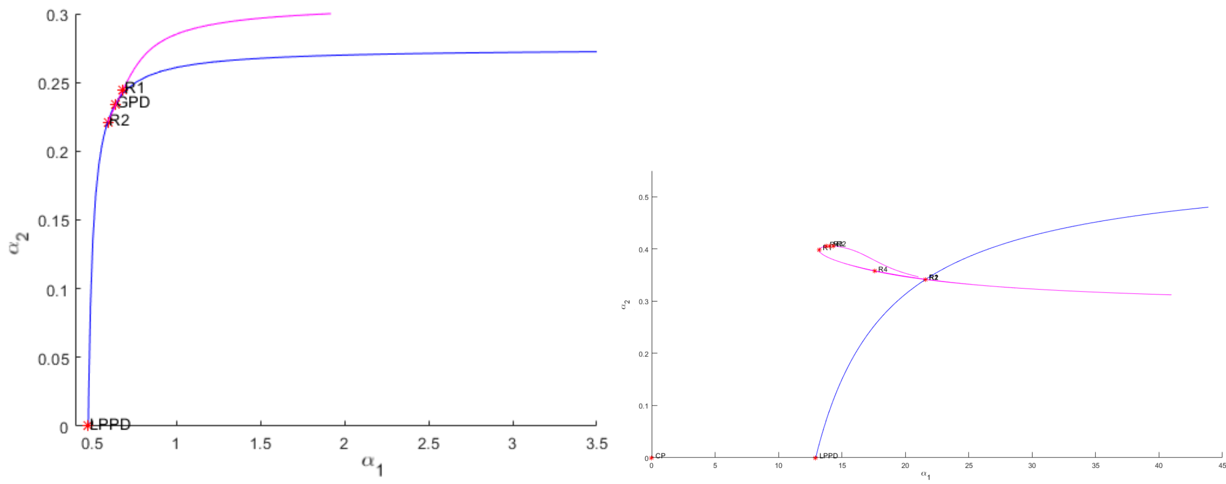
Further more, continuation of the bifurcation at E^* is reported as follows:

```
label R2, x= (2.094118 3.423529 0.593346 0.24252),
Normal form coefficient for R2: [c,d]= -5.576197e-01, -3.205910e+00,
label GPD, x= (2.094118 3.423529 0.636330 0.233594),
Normal form coefficient for GPD: [a/e,be]= -5.176538e-01, -1.135171e-09.
```

This is shown in Fig.(3)(a).

Example 4: Fix $A_1 = 1.9, A_2 = 1.8, B = 1.8$, and vary α_1 and α_2. We have $E^* = (0.0773, 0.9696)$ is a $1 : 3$ resonance at $(\alpha_1, \alpha_2)_1 = (14.6724, 0.3766)$ according to Theorem 5 where $|\lambda_{1,2}| = 1$ $\theta_{1,2} = \pm 120.3334$. This is depicted in Fig. (3)(b). The MATCONTM report is as follows

```
 label = R1, x= (0.064618 0.955773 13.204553 0.398235 1.000000),
Normal form coefficient for R1: s=1,
label = R4, x= (0.037827 0.995080 13.749222 0.404998 -0.000000),
Normal form coefficient for R4: A= 3.153868e-02-3.273009e-01i,
label= R3, x= (0.034550 0.996337 14.033526 0.405489 -0.500000),
Normal form coefficient for R3: Re(c1)= 1.081499e-02,
label =R2, x= (0.032050 0.997137 14.302355 0.405404 -1.000000),
Normal form coefficient for R2: [c,d]= -9.971172e+04, 1.7836789e+3,
label =R2, x= (0.077348 0.969613 21.577330 0.341410),
Normal form coefficient for R2: [c,d]= -1.911158e+03, -6.495115e+01.
 label= LPPD, x= (0.077348 0.969613 12.928571  -0.000000),
Normal form coefficient for LPPD: [a/e,be]= 5.439640e-10, 1.836239e-06,
First Lyapunov coefficient for second iterate = 1.836239e-06,
label= R4, x= (0.077348 0.969613 17.589093 0.357487 0.000000),
Normal form coefficient for R4: A= -4.587149e+06+3.683014e+08i,
label = CP, x=(0.077962 0.970653 -0.000000 0.0000000),
 Normal form coefficient for CP, s=-7.713883e-15.
```



(a) **(b)**

Figure 3. Continuation of E^* in (α_1, α_2)-plane where (a)generalized period doubling bifurcation is labeled GPD and limit point- period doubling is labeled LPPD, and (b)1:2, 1:3, and 1:4 resonance are labeled as R2,R3, and R4, respectively.

A magnification of Fig.(3)(b) is illustrated in Fig.(4) below

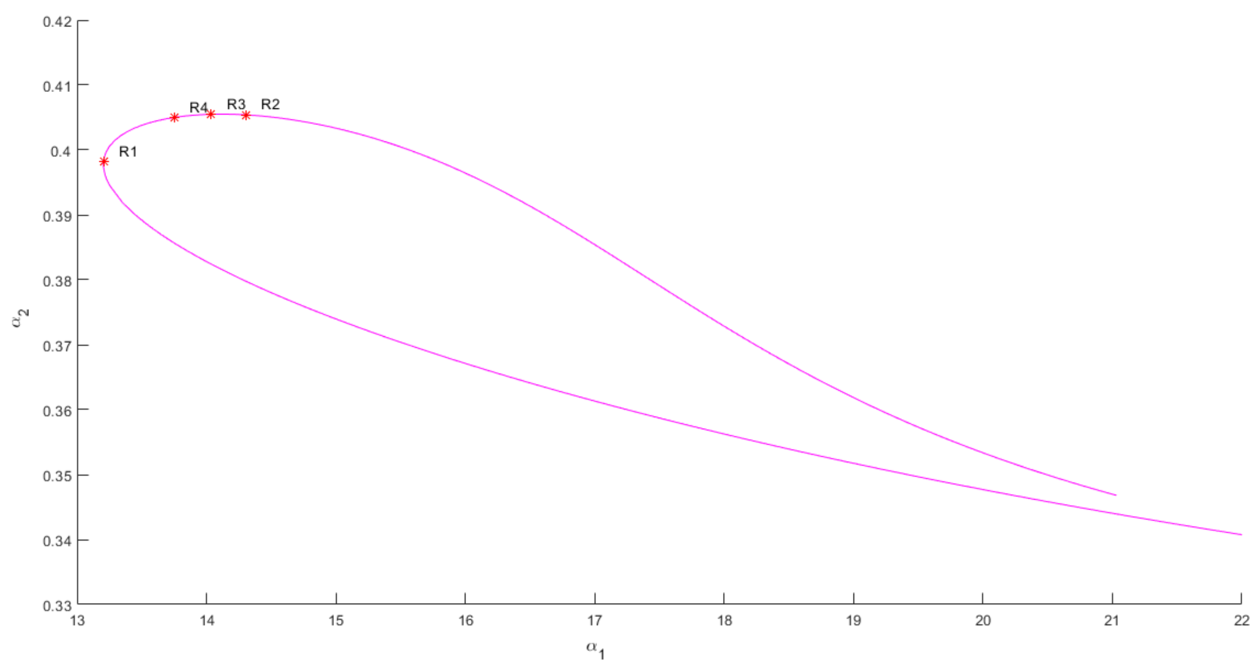


Figure 4. A magnification of Fig.(3)(b).

Example 5: Fix $A_1 = 1.9$, $A_2 = 1.8, B = 2.03$, and vary α_1 and α_2, the continuation of the PD curve is presented in Fig.(5). The MATCONTM report is as follows

```
label LPPD, x= (0.004352 0.901311 28.370567 1.250695),
Normal form coefficient for LPPD: [a/e,be]= -5.630804e-01, -1.884415e+07,
```

```
label LPPD, x= (0.005792 0.902582 22.468995 1.724068),
Normal form coefficient for LPPD: [a/e,be]= -5.148009e-01, -1.620579e+08,
label R1, x= (0.006249 0.902897 20.509344 1.747401),
Normal form coefficient for R1=-1,
label LPPD, x= (0.007987 0.934518 33.838462 0.716497),
Normal form coefficient for LPPD: [a/e,be]= 5.777744e-01, 1.437467e+05,
first Lyapunov coefficient for second iterate = 1.437467e+05.
```

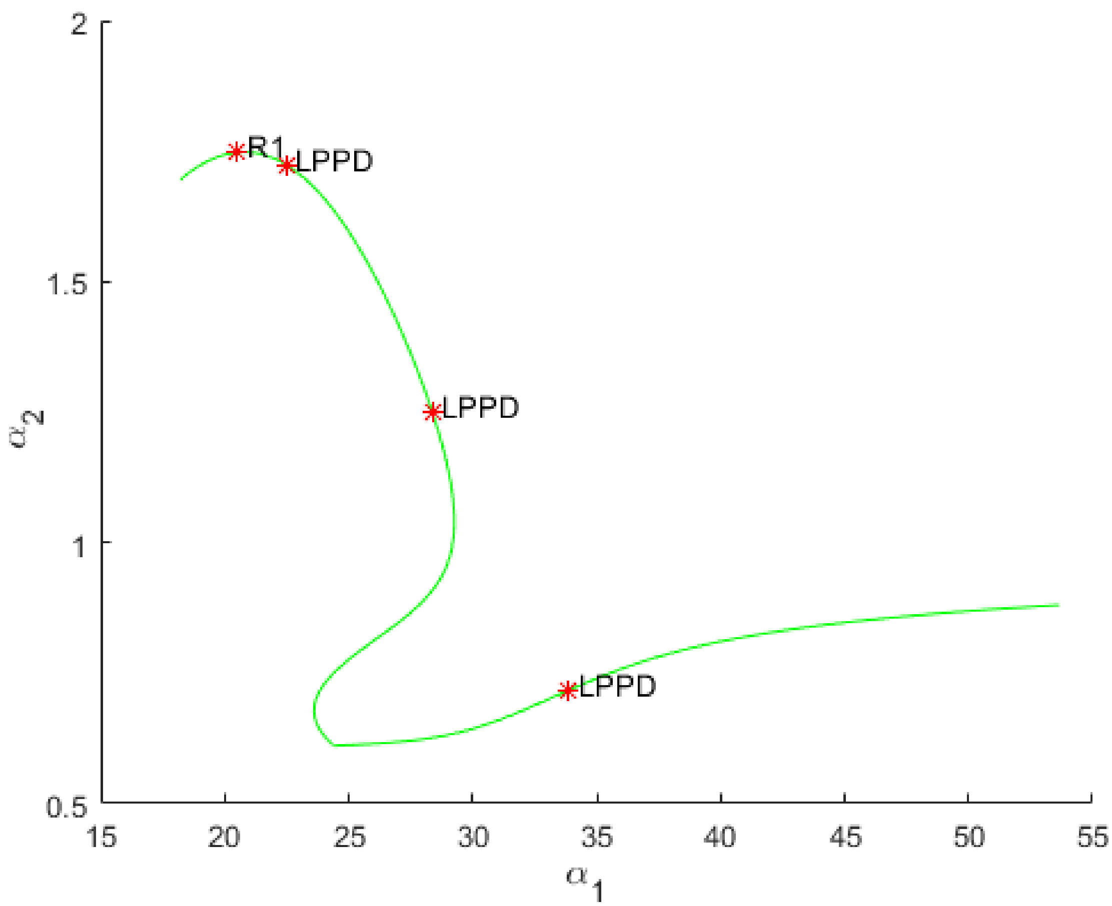


Figure 5. Period doubling curve of the second iterate.

Example 6: Fix $A_1 = 0.9$, $A_2 = 0.8, B = 2.05$, and vary α_1 and α_2, the continuation of $E^* = (0.0195, 0.4200)$ is presented in Fig.(6). The MATCONTM report is as follows

```
label LPPD, x= (0.034019 0.415123 47.594382 0.446535),
Normal form coefficient for LPPD: [a/e,be]= 4.229877e-01, 2.412199e+06,
first Lyapunov coefficient for the second iterate =  2.412199e+06,
label CP, x= (0.033655 0.414220 58.561701 -0.000000),
Normal form coefficient for CP s=4.129853e-03,
```

```
label R1, x= (0.020582 0.420100 29.645462 1.004500),
Normal form coefficient for R1=-1,
label R1, x= (0.030749 0.417998 29.968556 1.120381),
Normal form coefficient for R1=1,
label R1, x= (0.039532 0.405203 98.168742 1.239945),
Normal form coefficient for R1=-1,
label =R2, x= (0.019506 0.419994 87.960632 0.676719 -1.000000),
Normal form coefficient for R2: [c,d]= -2.919289e+05, 3.442778e+03,
label LPPD, x= (0.025208 0.414432 39.670716 0.000000),
Normal form coefficient for LPPD: [a/e,be]= -4.219238e-19, -2.293202e-14,
 label LPPD, x= (0.019471 0.420028 51.358679 0.000000),
Normal form coefficient for LPPD: [a/e,be]= -7.011869e-20, -8.381067e-15,
 label CP, x= (0.026656 0.413017 0.000000 0.000000),
Normal form coefficient for CP s= -5.617315e-19.
```

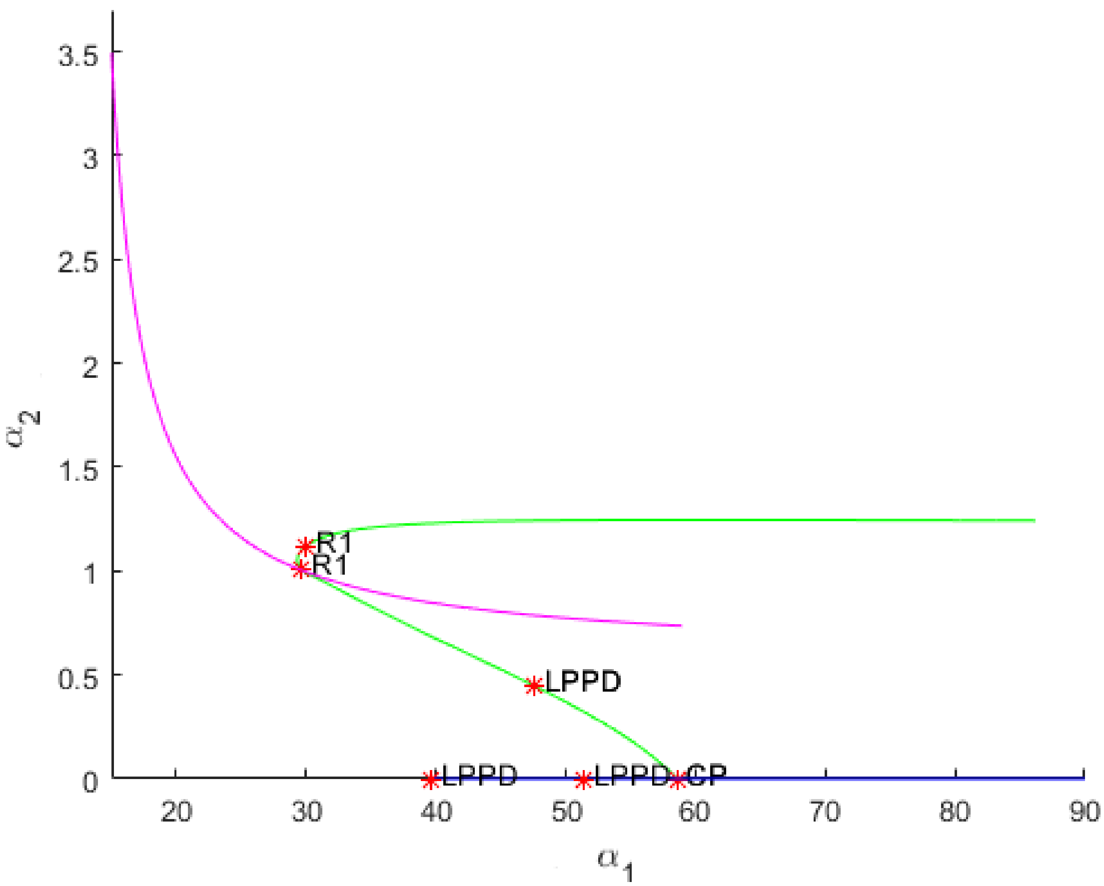


Figure 6. Period doubling curve of the second iterate.

6. Conclusion

We employed bifurcation theory and numerical continuation methods to study resonance-induced codimension-two bifurcations, specifically for 1:2, 1:3, and 1:4. We studied resonance events in depth to uncover and describe crucial dynamical system changes. A variety of numerical models supported our analytical results. These simulations confirmed our predictions and taught us how the system's structure moves. Mathematical theory and computer science demonstrate how nonlinear dynamical system approaches can handle multidisciplinary challenges. This combination of research and simulation improves our understanding of bifurcation and affords us new real-world applications. We found that resonance-induced bifurcations can alter the stability and longevity of economic models, adding to economic dynamics research. These findings affect market regulation, company competition, and economic instability. This study applies advanced mathematical methods to real-world systems. It highlights how versatile our method is and how vital cross-disciplinary research is when examining complex, changing processes.

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