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Analysis of Best Proximity Points in $\mathcal{F}$-Metric Spaces with Applications

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Abstract. The primary objective of this study is to introduce the concept of α-interpolative proximal contractions within the framework of $\mathcal{F}$-metric spaces and to derive corresponding best proximity point results for these newly defined contractions. As a direct implication of the main theorems, best proximity results for single-valued mappings are also established. Moreover, we extend our investigation to partially ordered $\mathcal{F}$-metric spaces, examining how graph structures influence best proximity point theory in this context. In addition, several fixed point theorems are obtained as immediate consequences of the proposed results. To illustrate the validity of our theoretical developments, non-trivial examples are provided.

1. Introduction

The idea of a metric space (MS), first formulated by the French mathematician Fréchet [1], served as a foundation for the study of fixed point (FP) theory. In 1989, I. A. Bakhtin expanded this concept by introducing b-metric spaces (b-MSs), which relaxed some of the rigid constraints of classical metric spaces. Subsequently, in 1993, Czerwik [3] provided a detailed investigation of b-MSs, emphasizing their flexibility through the replacement of the standard triangle inequality with the b-metric inequality. This extension allowed for a more general notion of distance and enabled the study of a wider variety of mathematical structures. More recently, Jleli et al. [4] proposed $\mathcal{F}$-metric spaces ($\mathcal{F}$-MSs), which further generalize both classical and b-MSs, offering a versatile framework that has significantly contributed to the advancement of FP theory and has opened new directions in functional analysis and topology.

By formalizing the concept of distance between points, metric spaces provide a natural setting for fixed point (FP) theorems to establish conditions under which certain mappings admit fixed points. The foundational result in this area is the renowned Banach contraction principle (BCP) [5],

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introduced by Stefan Banach in 1922. Later, Kannan [6] relaxed the continuity assumption which was necessary in BCP and established a FP theorem under a weaker contractive framework. This study of obtaining FPs has undergone significant generalizations beyond classical metric spaces (MSs) to accommodate more complex structures and relationships among elements. One such direction was initiated by Jachymski [7], who in 2008 introduced the notion of graphic contractions in MSs endowed with a directed graph. This approach arose from the observation that many mappings arising in applied sciences, economics, and computer science naturally preserve certain relational structures, which cannot be captured by the ordinary metric framework alone. By embedding a graph structure on the underlying set, Jachymski [7] unified and extended several existing FP theorems, particularly the BCP to cases where the contractive condition is required only along edges of the graph rather than globally. Subsequently, Bojor [8] advanced the incorporation of graph structures into FP theory and established a result that generalizes Kannan's FP theorem. Additional insights can be found in [9, 10].

Building on this framework, Samet et al. [11] later introduced the notion of α-admissibility within MSs and derived fixed point results for mappings satisfying this property. Motivated by this trend, Karapınar [12] introduced the concept of interpolative Kannan-type contractions, offering a deeper understanding of the structural relationships between classical and modern contraction principles, and paving the way for further exploration in generalized metric frameworks.

Since FP results usually require self-mappings, a natural generalization arises through best proximity point (BPP) theory, which extends the FP framework to non-self mappings $\aleph : \Omega \to \Lambda$, where (Ω, Λ) denotes a pair of subsets of a MS (Φ, d). In situations where a genuine FP may not exist, BPPs provide the optimal approximate solution, ensuring that the distance between a point and its image attains the minimum possible separation between the two sets. Instead, the aim is to locate points $\upsilon^* \in \Omega$ whenever

$$d(\upsilon^*, \aleph\upsilon^*) = dist(\Omega, \Lambda),$$

where

$$dist(\Omega, \Lambda) = \inf\{d(\upsilon, \varrho) : \upsilon \in \Omega, \varrho \in \Lambda\}.$$

These points are known as BPPs of the mapping $\aleph$. The foundation for this area was laid by Fan [13] in 1969 through a seminal theorem on best approximations. Later, Sadiq Basha [14] provided necessary and sufficient conditions ensuring the attainment of the minimum distance between two subsets for specific classes of mappings. More recently, Lateef [15] advanced this line of study in the framework of $\mathcal{F}$-MS, proving BPP results for generalized contractions and coupled BPPs under arbitrary binary relations. For comprehensive treatments and further developments in this direction, we refer the reader to [16-23].

The present work is devoted to introduce the concept of α-interpolative proximal contraction within the setting of $\mathcal{F}$-MSs, thereby extending and generalizing several existing results in BPP theory. The study establishes a series of BPP theorems corresponding to this new class of contractions, highlighting their structural properties and the conditions under which such points exist and are unique.

As a natural consequence of our primary results, we derive BPP results for single-valued self-mappings, demonstrating that these outcomes serve as special cases of the general framework. Furthermore, we explore partially ordered $\mathcal{F}$-MSs, where the ordering relation enables the derivation of more refined and generalized results. The role of graph structures in enhancing the analytical framework of BPP theory within $\mathcal{F}$-MSs is also examined, providing deeper insight into the interplay between order, topology, and contraction mappings.

In addition, the paper presents FP theorems that emerge as direct corollaries of the established BPP results, thereby bridging the gap between proximity theory and classical FP theory. To validate the theoretical findings, we construct non-trivial and illustrative examples that demonstrate the applicability and non-vacuity of the developed results.

2. Preliminaries

Frechet [1] introduced the notion of MS in this way.

Definition 2.1. *Let $\Phi \neq \emptyset$ and $d : \Phi\times\Phi \to [0, +\infty)$ be a function satisfying the following conditions*

(D_1) $d(\upsilon, \varrho) = 0 \Longleftrightarrow \upsilon = \varrho$,
(D_2) $d(\upsilon, \varrho) = d(\varrho, \upsilon)$,
(D_3) $d(\upsilon, \varrho) \leq d(\upsilon, v) + d(v, \varrho)$,

for all $\upsilon, v, \varrho \in \Phi$, then (Φ, d) is said to be a MS.

In the setting of MS, Stefan Banach [5] proved the following result which is considered as a pioneer theorem in this theory.

Theorem 2.1. *([5]) Let (Φ, d) be a complete MS and $\aleph : \Phi \to \Phi$ be a self mapping.If there exists some constant $\lambda \in [0, 1)$ such that*

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda d(\upsilon, \varrho)$$

for all $\upsilon, \varrho \in \Phi$, then $\aleph$ has a unique FP.

From the contractive condition in the BCP, it follows directly that the mapping $\Im$ is necessarily continuous. This observation naturally leads to the question: is it possible for a mapping to have a FP without being continuous? In response, Kannan [6] (1968) proved a FP theorem for self-mappings in CMSs, showing that even in the absence of continuity, certain contractive-type conditions are sufficient to ensure the existence of a unique FP.

Theorem 2.2. *[6]Let $\aleph : \Phi \to \Phi$. Suppose that there is $\lambda \in [0, \frac{1}{2})$ such that*

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda \left(d(\upsilon, \aleph\upsilon) + d(\varrho, \aleph\varrho)\right),$$

$\forall \upsilon, \varrho \in \Phi$, then $\aleph$ has a unique FP in Φ.

To extend the classical notion of BCP to settings involving graph structures, Jachymski [7] introduced the concept of a graphic contraction, which incorporates the edge-preserving property of a directed graph.

A directed graph G is an ordered pair $G = (V(G), E(G))$, where

- $V(G)$ is a nonempty set whose elements are called vertices, and
- $E(G) \subseteq V(G) \times V(G)$ is a set of directed edges (or arcs) between the vertices.

The graph G is said to be loop-inclusive if every vertex is connected to itself, that is,

$$\Delta = \{(\upsilon, \upsilon) : \upsilon \in V(G)\} \subseteq E(G).$$

Furthermore, G is assumed to contain no multiple edges, meaning that between any ordered pair of vertices there exists at most one directed edge.

A path from υ to ϱ in G of length N (a natural number) is a sequence $\{\upsilon_i\}_{i=0}^N$ of $N+1$ vertices such that $\upsilon_0 = \upsilon$, $\upsilon_N = \varrho$ and $(\upsilon_{i-1}, \upsilon_i) \in E(G)$, $\forall\ i = 1, \cdots, N$. Let $\widetilde{G}$ denote the symmetric graph obtained from G by ignoring the direction of edges, i.e.,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}),$$

where G^{-1} is the converse graph of G, defined by

$$E(G^{-1}) = \{(\upsilon, \varrho) \in \Phi \times \Phi : (\varrho, \upsilon) \in E(G)\}.$$

The graph G is said to be weakly connected if the symmetric graph $\widetilde{G}$ is connected, i.e., if for any two distinct vertices $\upsilon, \varrho \in V(G)$, there exists a path from υ to ϱ in $\widetilde{G}$.

Jachymski [7] proved the following foundational result.

Theorem 2.3. *([7]) Let (Φ, d) be a MS associated with a directed graph $G = (\Phi, E(G))$ and $\aleph : \Phi \to \Phi$. Suppose that*

Definition 2.2. *(a) $\aleph$ is edge-preserving, that is, for all $\upsilon, \varrho \in \Phi$ with $(\upsilon, \varrho) \in E(G)$, we have $(\aleph\upsilon, \aleph\varrho) \in E(G)$,*

(b) there exists $\lambda \in (0,1)$ such that, $\forall\ \upsilon, \varrho \in \Phi$ with $(\upsilon, \upsilon) \in E(G)$, we have

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda d(\upsilon, \varrho). \tag{2.1}$$

Then $\aleph$ is called a graphic contraction on (Φ, d, G). Moreover, if (Φ, d) is complete and there exists $\upsilon_0 \in \Phi$ with $(\upsilon_0, \aleph\upsilon_0) \in E(G)$, then $\aleph$ has a FP in Φ. Moreover, if G is weakly connetced, then this FP is unique.

In [12], Karapınar examined Theorem 2.2 in the framework of interpolation theory, where the main result was derived using an interpolative Kannan-type contraction.

Theorem 2.4. *[12] Let $\aleph : \Phi \to \Phi$. Assume that there is $\lambda \in [0,1)$ and $\beta \in (0,1)$ such that*

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda\, [d(\upsilon, \aleph\upsilon)]^{\beta}\, [d(\varrho, \aleph\varrho)]^{1-\beta},$$

for all $\upsilon, \varrho \in \Phi$ with $\upsilon, \varrho \in \Phi \backslash Fix(\aleph)$, with $Fix(\aleph) = \{\upsilon \in \Phi : \aleph\upsilon = \upsilon\}$, then $\aleph$ has a unique FP in Φ.

Sadiq Basha [14] introduced the concept of best proximity point in this way.

Definition 2.3. *([14]) Let (Φ, d) be a MS and let $\Omega, \Lambda \subseteq \Phi$ be two nonempty subsets. For a mapping $\aleph : \Omega \to \Lambda$, a point $\upsilon^* \in \Omega$ is said to be a best proximity point of $\aleph$ if*

$$d(\upsilon^*, \aleph\upsilon^*) = d(\Omega, \Lambda),$$

where

$$d(\Omega, \Lambda) = \inf\{d(\upsilon, \varrho) : \upsilon \in \Omega, \varrho \in \Lambda\}.$$

Consistent with the approach of Sadiq Basha [14], let $\Omega, \Lambda \subseteq \Phi$ be two nonempty subsets of the MS Φ and

$$\begin{aligned} \Omega_0 &= \{\upsilon \in \Omega : \text{there exists } \varrho \in \Lambda \text{ such that } d(\upsilon, \varrho) = dist(\Omega, \Lambda)\} \\ \Lambda_0 &= \{\varrho \in \Lambda : \text{there exists } \upsilon \in \Omega \text{ such that } d(\upsilon, \varrho) = dist(\Omega, \Lambda)\}. \end{aligned}$$

Definition 2.4. *([14]) Let Ω and Λ are non-empty subsets of MS Φ. Then, (Ω, Λ) is said to satisfy the weak P-property if for all $\upsilon_1, \upsilon_2 \in \Omega$ and $\varrho_1, \varrho_2 \in \Lambda$, the following holds*

$$\left.\begin{array}{l} d(\upsilon_1, \varrho_1) = dist(\Omega, \Lambda) \\ d(\upsilon_2, \varrho_2) = dist(\Omega, \Lambda) \end{array}\right\} \Rightarrow d(\upsilon_1, \upsilon_2) \le d(\varrho_1, \varrho_2).$$

Czerwik [3] introduced the idea of b-MSs by altering the triangle inequality inherent in standard MSs.

for all $\upsilon, v, \varrho \in \Phi$ and for some $b \ge 1$,

$$d(\upsilon, \varrho) \le b\left[d(\upsilon, v) + d(v, \varrho)\right].$$

Later on, Jleli et al. [4] unveiled a captivating generalization of MS and b-MS, known as $\mathcal{F}$-MS.

Let Ξ be the collection of functions $\hbar : (0, +\infty) \to \mathbb{R}$ such that

($\mathfrak{F}_1$) for all $t_1, t_2 \in (0, +\infty)$ such that $t_1 < t_2 \Longrightarrow \hbar(t_1) < \hbar(t_2)$,

($\mathfrak{F}_2$) for $\{t_n\} \subseteq (0, +\infty)$, $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} \hbar(t_n) = -\infty$ are equivalent.

Definition 2.5. *([4]) Let $\Phi \ne \emptyset$ and $d : \Phi \times \Phi \to [0, +\infty)$ be a distance function such that*

(D_1) $d(\upsilon, \varrho) = 0 \Longleftrightarrow \upsilon = \varrho$,

(D_2) $d(\upsilon, \varrho) = d(\varrho, \upsilon)$,

(D_3) for every $(u_\imath)_{\imath=1}^{p} \subset \Phi$ with $(u_1, u_p) = (\upsilon, \varrho)$, we have

$$d(\upsilon, \varrho) > 0 \Rightarrow \hbar(d(\upsilon, \varrho)) \le \hbar\left(\sum_{\imath=1}^{p-1} d(u_\imath, u_{\imath+1})\right) + \eth,$$

for all $(\upsilon, \varrho) \in \Phi \times \Phi$ and for $p \in \mathbb{N}$ with $p \ge 2$. If there exists a combination $(\hbar, \eth) \in \Xi \times [0, +\infty)$ satisfying the aforementioned properties, then d is referred to as an $\mathcal{F}$- metric on Φ, and the combination (Φ, d) is termed an $\mathcal{F}$-MS.

Jleli et al. [4] proved the following Theorem in

Theorem 2.5. *([4]) Let (Φ, d) be an $\mathcal{F}$-metric space and $\aleph : \Phi \to \Phi$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) (Φ, d) is $\mathcal{F}$-complete,
(ii) there exists $\lambda \in (0, 1)$ such that

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda d(\upsilon, \varrho)$$

Then $\mathcal{H}$ has a unique fixed point $\upsilon^* \in \Phi$. Moreover, for any $\upsilon_0 \in \Phi$, the sequence $\{\upsilon_n\} \subset \Phi$ defined by

$$\upsilon_{n+1} = \aleph\upsilon_n, \;\; n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to υ^*.

In the context of $\mathcal{F}$-MSs, Lateef [15] established BPP result for contraction mappings.

Theorem 2.6. *([15]) Given two non-empty, closed subsets Ω and Λ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Φ, d) such that $\Omega_0 \neq \emptyset$ and the combination (Ω, Λ) satisfies P-property. Suppose $\aleph : \Omega \to \Lambda$ be a contraction mapping such that $\aleph(\Omega_0) \subseteq \Lambda_0$. Then $\aleph$ has a BPP.*

Here we give the basic concepts in $\mathcal{F}$-MS which are crucial in our main result.

Definition 2.6. *([4]) Let (Φ, d) be an $\mathcal{F}$-MS.*

(i) Let $\{\upsilon_n\}$ be a sequence in Φ. We say that $\{\upsilon_n\}$ is $\mathcal{F}$-convergent to $\upsilon \in \Phi$ if $\{\upsilon_n\}$ is convergent to υ with respect to the $\mathcal{F}$-metric d.

(ii) A sequence $\{\upsilon_n\}$ is $\mathcal{F}$-Cauchy, if

$$\lim_{n,m\to\infty} d(\upsilon_n, \upsilon_m) = 0.$$

(iii) We say that (Φ, d) is $\mathcal{F}$-complete, if every $\mathcal{F}$-Cauchy sequence in Φ is $\mathcal{F}$-convergent to a certain element in Φ.

3. Main Results

In this section, we introduce the concept of α-interpolative proximal contraction in the framework of $\mathcal{F}$-MS and establish corresponding BPP results. Initially, we prove the existence of a BPP by assuming that the mapping $\aleph$ is continuous. Later, we replace the continuity condition with a more general assumption, thereby extending and improving the obtained results.

Definition 3.1. *Let (Φ, d) be an $\mathcal{F}$-MS and let Ω and Λ be the nonempty subsets of Φ. A mapping $\aleph : \Omega \to \Lambda$ is called an α-interpolative proximal contraction, if there exist a constant $\lambda \in (0, 1)$, a control function $\alpha : \Omega \times \Omega \to [0, \infty)$ and positive real constants γ, β with $\gamma + \beta < 1$ such that $\alpha(\upsilon, \varrho) \geq 1$ implies*

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda (d(\upsilon, \varrho))^{\gamma} \left(d(\upsilon, \aleph\upsilon) - d(\Omega, \Lambda)\right)^{\beta} \left(d(\varrho, \aleph\varrho) - d(\Omega, \Lambda)\right)^{1-\gamma-\beta}, \tag{3.1}$$

for all $\upsilon, \varrho \in \Omega \backslash B_{est}(\aleph)$, where $B_{est}(\aleph) = \{\upsilon \in \Omega : d(\upsilon, \aleph\upsilon) = d(\Omega, \Lambda)\}$.

Example 3.1. *Let $\Phi = [0,3]$ be equipped with the $\mathcal{F}$-metric $d : \Phi \times \Phi \to \mathbb{R} \cup \{0\}$ defined by*

$$d(\upsilon, \varrho) = \left|\upsilon - \varrho\right|$$

for all $\upsilon, \varrho \in \Phi$, then (Φ, d) is an $\mathcal{F}$-MS with $\hbar(t) = \ln t$, for $t > 0$ and $\eth = 0$. Let $\Omega = [0,1]$ and $\Lambda = [2,3]$, then $d(\Omega, \Lambda) = 1$. Define the mapping $\aleph : \Omega \to \Lambda$ by

$$\aleph\upsilon = 2 + \frac{\upsilon}{2}.$$

Let the control function $\alpha : \Omega \times \Omega \to [0, \infty)$ be defined as follows

$$\alpha(\upsilon, \varrho) = 1 + \left|\upsilon - \varrho\right|.$$

Choose the constants

$$\lambda = 0.80, \quad \gamma = 0.40, \quad \beta = 0.50,$$

so that $\gamma + \beta < 1$. Here

$$\begin{aligned} B_{est}(\aleph) &= \{\upsilon \in \Omega : d(\upsilon, \aleph\upsilon) = d(\Omega, \Lambda)\} \\ &= \left\{\upsilon \in [0,1] : 2 - \frac{\upsilon}{2} = 1\right\}, \end{aligned}$$

but the equation $2 - \frac{\upsilon}{2} = 1$ gives $\upsilon = 2$. So in fact

$$B_{est}(\aleph) = \emptyset.$$

Hence $\Omega \backslash B_{est}(\aleph) = \Omega$. Now, for any $\upsilon, \varrho \in \Omega \backslash B_{est}(\aleph)$, we have

$$d(\aleph\upsilon, \aleph\varrho) = \frac{1}{2}\left|\upsilon - \varrho\right| \leq 0.80\left|\upsilon - \varrho\right|^{0.40} (d(\upsilon, \aleph\upsilon) - 1)^{0.50} (d(\varrho, \aleph\varrho) - 1)^{0.10}.$$

Thus, $\aleph$ satisfies the inequality (3.1) and is an α-interpolative proximal contraction.

Theorem 3.1. *Let (Φ, d) be an $\mathcal{F}$-complete $\mathcal{F}$-MS and let Ω and Λ be the nonempty closed subsets of Φ satisfying the weak P-property and $\Omega_0 \neq \emptyset$. Let $\aleph : \Omega \to \Lambda$ is an α-interpolative proximal contraction and $\aleph(\Omega_0) \subseteq \Lambda_0$. Assume that the following conditions holds*

(i) $\aleph$ is an α-proximal admissible;
(ii) there exist $\upsilon_0, \upsilon_1 \in \Omega_0$ such that $d(\upsilon_1, \aleph\upsilon_0) = d(\Omega, \Lambda)$ and $\alpha(\upsilon_0, \upsilon_1) \geq 1$;
(iii) $\aleph$ is continuous.
Then $\aleph$ has a unique BPP in Ω.

Proof. Let $(\hbar, \eth) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow \hbar(t) < \hbar(\epsilon) - \eth. \tag{3.2}$$

In view of condition (ii), there exist elements $\upsilon_0, \upsilon_1 \in \Omega_0$ such that □

$$d\left(v_1, \aleph v_0\right) = d(\Omega, \Lambda) \text{ and } \alpha\left(v_0, v_1\right) \geq 1. \tag{3.3}$$

Since $\aleph\left(\Omega_0\right) \subseteq \Lambda_0$, it follows that $\aleph v_1$ belongs to $\aleph\left(\Omega_0\right)$, which is contained in Λ_0. Hence, there exists some $v_2 \in \Omega_0$ satisfying

$$d\left(v_2, \aleph v_1\right) = d(\Omega, \Lambda). \tag{3.4}$$

Now, by combining Equations (3.3) and (3.4) together with the definition of α-proximal admissibility, we obtain

$$\begin{aligned} \alpha\left(v_0, v_1\right) &\geq 1 \\ d\left(v_1, \aleph v_0\right) &= d(\Omega, \Lambda) \\ d\left(v_2, \aleph v_1\right) &= d(\Omega, \Lambda), \end{aligned}$$

which leads to $\alpha\left(v_1, v_2\right) \geq 1$. Hence,

$$d\left(v_2, \aleph v_1\right) = d(\Omega, \Lambda) \text{ and } \alpha\left(v_1, v_2\right) \geq 1.$$

Proceeding in a similar manner, since $\aleph v_2 \in \Lambda_0$, there exists an element $v_3 \in \Omega_0$ such that

$$d\left(v_3, \aleph v_2\right) = d(\Omega, \Lambda).$$

Because $\aleph$ is α-proximal admissible, it follows that $\alpha\left(v_2, v_3\right) \geq 1$. Therefore,

$$d\left(v_3, \aleph v_2\right) = d(\Omega, \Lambda) \text{ and } \alpha\left(v_2, v_3\right) \geq 1.$$

By continuing this process inductively, we can construct a sequence $\{v_n\}$ in Ω_0 satisfying

$$d\left(v_{n+1}, \aleph v_n\right) = d(\Omega, N) \text{ and } \alpha\left(v_n, v_{n+1}\right) \geq 1, \tag{3.5}$$

for every $n \in \mathbb{N} \cup \{0\}$. If, for some $n_0 \in \mathbb{N}$, we have $v_{n_0} = v_{n_0+1}$, then

$$d\left(v_{n_0}, \aleph v_{n_0}\right) = d\left(v_{n_0+1}, \aleph v_{n_0}\right) = d(\Omega, \Lambda).$$

which shows that v_{n_0} is a BPP of $\aleph$. Assuming instead that $v_n \neq v_{n+1}$ for all $n \in \mathbb{N}$. Then, $d\left(v_n, \aleph v_n\right) > 0$ and $d\left(v_{n+1}, \aleph v_{n+1}\right) > 0$, for every $n \in \mathbb{N} \cup \{0\}$. Applying (D_3) to the points v_n and $\aleph v_n$, we get

$$\hbar\left(d\left(v_n, \aleph v_n\right)\right) \leq \hbar\left(d\left(v_n, v_{n+1}\right) + d\left(v_{n+1}, \aleph v_n\right)\right) + \eth.$$

Since $d\left(v_{n+1}, \aleph v_n\right) = d(\Omega, \Lambda)$, so the above inequality becomes

$$\begin{aligned} \hbar\left(d\left(v_n, \aleph v_n\right)\right) &\leq \hbar\left(d\left(v_n, v_{n+1}\right) + d(\Omega, \Lambda)\right) + \eth \\ &< \hbar\left(d\left(v_n, v_{n+1}\right) + d(\Omega, \Lambda)\right), \end{aligned}$$

which implies by $(\mathcal{F}_1)$ that

$$d\left(v_n, \aleph v_n\right) < d\left(v_n, v_{n+1}\right) + d(\Omega, \Lambda). \tag{3.6}$$

Similarly

$$d\left(\upsilon_{n+1},\aleph\upsilon_{n+1}\right)<d\left(\upsilon_{n+1},\upsilon_{n+2}\right)+d(\Omega,\Lambda). \tag{3.7}$$

By (3.1), we have

$$d\left(\aleph\upsilon_n,\aleph\upsilon_{n+1}\right)\leq\lambda\left(d\left(\upsilon_n,\upsilon_{n+1}\right)^{\gamma}\left(d\left(\upsilon_n,\aleph\upsilon_n\right)-d(\Omega,\Lambda)\right)^{\beta}\left(d\left(\upsilon_{n+1},\aleph\upsilon_{n+1}\right)-d(\Omega,\Lambda)\right)^{1-\gamma-\beta}\right).$$

Using (3.6) and (3.7) in above inequaliy, we get

$$d\left(\aleph\upsilon_n,\aleph\upsilon_{n+1}\right)\leq\lambda\left(\left(d\left(\upsilon_n,\upsilon_{n+1}\right)\right)^{\gamma}\left(d\left(\upsilon_n,\upsilon_{n+1}\right)\right)^{\beta}\left(d\left(\upsilon_{n+1},\upsilon_{n+2}\right)\right)^{1-\gamma-\beta}\right). \tag{3.8}$$

Since the pair (Ω,Λ) satisfies the weak P-property, we have $d\left(\upsilon_{n+1},\upsilon_{n+2}\right)\leq d\left(\aleph\upsilon_n,\aleph\upsilon_{n+1}\right)$. Hence, from (3.8), it follows that

$$d\left(\aleph\upsilon_n,\aleph\upsilon_{n+1}\right)\leq\lambda\left(\left(d\left(\upsilon_n,\upsilon_{n+1}\right)\right)^{\gamma+\beta}\left(d\left(\upsilon_{n+1},\upsilon_{n+2}\right)\right)^{1-(\gamma+\beta)}\right)$$

and therefore,

$$\left(d\left(\upsilon_{n+1},\upsilon_{n+2}\right)\right)^{\gamma+\beta}\leq\lambda\left(d\left(\upsilon_n,\upsilon_{n+1}\right)\right)^{\gamma+\beta}. \tag{3.9}$$

And so,

$$d\left(\upsilon_{n+1},\upsilon_{n+2}\right)\leq\lambda^{\frac{1}{\gamma+\beta}}d\left(\upsilon_n,\upsilon_{n+1}\right), \tag{3.10}$$

for every $n\in\mathbb{N}\cup\{0\}$. Set $\mu=\lambda^{\frac{1}{\gamma+\beta}}\in(0,1)$. Then we get

$$d\left(\upsilon_{n+1},\upsilon_{n+2}\right)\leq\mu d\left(\upsilon_n,\upsilon_{n+1}\right),$$

for every $n\in\mathbb{N}\cup\{0\}$. By repeating this argument, we deduce that

$$d\left(\upsilon_{n+1},\upsilon_{n+2}\right)\leq\mu d\left(\upsilon_n,\upsilon_{n+1}\right)\leq\mu^2 d\left(\upsilon_{n-1},\upsilon_n\right)\leq\cdots\leq\mu^{n+1}d\left(\upsilon_0,\upsilon_1\right), \tag{3.11}$$

which yields

$$\sum_{i=n+1}^{m-1}d\left(\upsilon_i,\upsilon_{i+1}\right)\leq\frac{\mu^n}{1-\mu}d\left(\upsilon_0,\upsilon_1\right),\ \text{ for } m>n.$$

Since

$$\lim_{n\to+\infty}\frac{\mu^n}{1-\mu}d\left(\upsilon_0,\upsilon_1\right)=0,$$

so there exists some $N\in\mathbb{N}$ such that

$$0<\frac{\mu^n}{1-\mu}d\left(\upsilon_0,\upsilon_1\right)<\delta,\ \text{ for } n\geq N.$$

Hence, by (3.2) and $(\mathcal{F}_1)$, we get

$$\hbar\left(\sum_{i=n+1}^{m-1}d\left(\upsilon_i,\upsilon_{i+1}\right)\right)\leq\hbar\left(\frac{\mu^n}{1-\mu}d\left(\upsilon_0,\upsilon_1\right)\right)<\hbar\left(\varepsilon\right)-\eth \tag{3.12}$$

for $m>n\geq N$. Using (D_3) and (3.12), we obtain

$$d\left(\upsilon_n,\upsilon_m\right)>0,\ m>n\geq N$$

implies

$$\hbar\left(d\left(\upsilon_n,\upsilon_m\right)\right)\leq\hbar\left(\sum_{i=n+1}^{m-1}d\left(\upsilon_i,\upsilon_{i+1}\right)\right)+\eth<\hbar\left(\varepsilon\right)$$

which implies by $(\mathcal{F}_1)$ that

$$d\left(\upsilon_n,\upsilon_m\right)<\varepsilon$$

for $m>n\geq N$. Thus, $\{\upsilon_n\}$ is a Cauchy sequence in Ω. Since (Φ,d) is an $\mathcal{F}$-complete $\mathcal{F}$-MS and Ω is a closed subset of Φ, so there exists $\upsilon\in\Omega$ such that $\upsilon_n\to\upsilon$ as $n\to\infty$, that is,

$$\lim_{n\to\infty}d\left(\upsilon_n,\upsilon\right)=0. \tag{3.13}$$

Since $\aleph$ is continuous, one writes

$$\lim_{n\to\infty}d\left(\aleph\upsilon_n,\aleph\upsilon\right)=0. \tag{3.14}$$

Combining (3.5), (3.13) and (3.14), we get

$$d(\Omega,\Lambda)=\lim_{n\to\infty}d\left(\upsilon_{n+1},\aleph\upsilon_n\right)=d(\upsilon,\aleph\upsilon).$$

Consequently, υ is a BPP of $\aleph$.

Now, to prove uniqueness of the BPP of mapping $\aleph$, on contrary, suppose that $\varrho\in\Omega_0$ is another BPP (different from υ) of the mapping $\aleph$ such that

$$\begin{aligned}\alpha\left(\upsilon,\varrho\right)&\geq 1\\ d\left(\upsilon,\aleph\upsilon\right)&=d(\Omega,\Lambda)\\ d\left(\varrho,\aleph\varrho\right)&=d(\Omega,\Lambda),\end{aligned}$$

Since the pair of subsets (Ω,Λ) satisfies the weak property, then we have $d(\upsilon,\varrho)\leq d(\aleph\upsilon,\aleph\varrho)$, and mapping $\aleph$ is α-interpolative proximal contraction, then we have

$$\begin{aligned}d\left(\upsilon,\varrho\right)&\leq d\left(\aleph\upsilon,\aleph\varrho\right)\leq\alpha\left(\upsilon,\varrho\right)d\left(\aleph\upsilon,\aleph\varrho\right)\\ &\leq\lambda(d(\upsilon,\varrho))^{\gamma}\left(d(\upsilon,\aleph\upsilon)-d(\Omega,\Lambda)\right)^{\beta}\left(d(\varrho,\aleph\varrho)-d(\Omega,\Lambda)\right)^{1-\gamma-\beta}\\ &=0,\end{aligned}$$

which is a contradiction, hence BPP of the mapping $\aleph$ is unique.

Next, we provide a result where the continuity of the mapping $\aleph$ is substituted with a more general assumption.

Theorem 3.2. *Let (Φ,d) be an $\mathcal{F}$-complete $\mathcal{F}$-MS and let Ω and Λ be the nonempty closed subsets of Φ satisfying the weak P-property and $\Omega_0\neq\emptyset$. Let $\aleph:\Omega\to\Lambda$ is an α-interpolative proximal contraction and $\aleph\left(\Omega_0\right)\subseteq\Lambda_0$. Assume that the following conditions holds*

(i) $\aleph$ is an α-proximal admissible;

(ii) there exist $v_0, v_1 \in \Omega_0$ such that $d(v_1, \aleph v_0) = d(\Omega, \Lambda)$ and $\alpha(v_0, v_1) \geq 1$;

(iii) if there exists a sequence $\{v_n\}$ in Ω such that $\alpha(v_n, v_{n+1}) \geq 1$ for all n, and $v_n \to v \in \Omega$ as $n \to \infty$, then $\alpha(v_n, v) \geq 1$, for every n.

Then, $\aleph$ possesses a unique BPP in Ω.

Proof. Pursuing the proof of Theorem 3.1, there exists a Cauchy sequence $\{v_n\} \subset \Omega$ satisfying

$$d(v_{n+1}, \aleph v_n) = d(\Omega, \Lambda) \text{ and } \alpha(v_n, v_{n+1}) \geq 1,$$

for every $n \in \mathbb{N} \cup \{0\}$. Also, there exists $v \in \Omega$ such that $v_n \to v$ as $n \to \infty$. Thus, from the assumption (iii), we infer that $\alpha(v_n, v) \geq 1$ for all $n \in \mathbb{N}$. As $\aleph(\Omega_0) \subseteq \Lambda_0$, there is $\varrho \in \Omega_0$ so that

$$d(\varrho, \aleph v) = d(\Omega, \Lambda).$$

Thus, we get

$$\begin{aligned} \alpha(v_n, v) &\geq 1 \\ d(v_{n+1}, \aleph v_n) &= d(\Omega, \Lambda) \\ d(\varrho, \aleph v) &= d(\Omega, \Lambda). \end{aligned}$$

By (3.1) and $(\mathcal{F}_1)$, we have

$$\begin{aligned} \hbar(d(\aleph v_n, \aleph v)) &\leq \hbar(d(\aleph v_n, \aleph v)) \\ &\leq \hbar\Big(\lambda(d(v_n, v))^{\gamma}(d(v_n, \aleph v_n) - d(\Omega, \Lambda))^{\beta}(d(v, \aleph v) - d(\Omega, \Lambda))^{1-\gamma-\beta}\Big). \end{aligned} \tag{3.15}$$

Since

$$d(v_n, \aleph v_n) < d(v_n, v_{n+1}) + d(\Omega, \Lambda),$$

by the inequlaity (3.6). Thus the above inequality (3.15) becomes

$$\hbar(d(\aleph v_n, \aleph v)) \leq \hbar\Big(\lambda(d(v_n, v))^{\gamma}(d(v_n, v_{n+1}))^{\beta}(d(v, \aleph v) - d(\Omega, \Lambda))^{1-\gamma-\beta}\Big). \tag{3.16}$$

Since (Ω, Λ) satisfies the weak P-property, we deduce that $d(v_{n+1}, \varrho) \leq d(\aleph v_n, \aleph v)$. From (3.16), we have

$$\hbar(d(v_{n+1}, \varrho)) \leq \hbar\Big(\lambda(d(v_n, v))^{\gamma}(d(v_n, v_{n+1}))^{\beta}(d(v, \aleph v) - d(\Omega, \Lambda))^{1-\gamma-\beta}\Big).$$

Taking limit as $n \to \infty$ and using the fact that

$$\lim_{n\to\infty} \hbar\Big(\lambda(d(v_n, v))^{\gamma}(d(v_n, v_{n+1}))^{\beta}(d(v, \aleph v) - d(\Omega, \Lambda))^{1-\gamma-\beta}\Big) = -\infty,$$

we get that $\lim_{n\to\infty} d(v_{n+1}, \varrho) = 0$. By the uniqueness of limit, we obtain $\varrho = v$. Therefore, $d(v, \aleph v) = d(\Omega, \Lambda)$. The uniqueness of the BPP followed on the same lines as proved in Theorem 3.1. □

Example 3.2. *Let $\Phi = [0,3]$ be equipped with the $\mathcal{F}$-metric $d : \Phi \times \Phi \to \mathbb{R} \cup \{0\}$ defined by*

$$d(\upsilon, \varrho) = \left|\upsilon - \varrho\right|$$

for all $\upsilon, \varrho \in \Phi$, then (Φ, d) is an $\mathcal{F}$-MS with $\hbar(t) = \ln t$, for $t > 0$ and $\eth = 0$. Let $\Omega = [0,1]$ and $\Lambda = [2,3]$, then

$$d(\Omega, \Lambda) = \inf\left\{\left|\upsilon - \varrho\right| : \upsilon \in \Omega, \varrho \in \Lambda\right\} = 1,$$

and

$$\Omega_0 = \{\upsilon \in \Omega : d(\upsilon, \Lambda) = d(\Omega, \Lambda)\} = \{1\}, \quad \Lambda_0 = \{2\}.$$

Define the mapping $\aleph : \Omega \to \Lambda$ by

$$\aleph\upsilon = 2$$

for all $\upsilon \in \Omega$. Let the control function $\alpha : \Omega \times \Omega \to [0, \infty)$ be defined as follows

$$\alpha(\upsilon, \varrho) = 1, \text{ for all } \upsilon, \varrho \in \Omega$$

Choose the constants

$$\lambda = 0.80, \quad \gamma = 0.40, \quad \beta = 0.50,$$

so that $\gamma + \beta < 1$. Here

$$B_{est}(\aleph) = \{\upsilon \in \Omega : d(\upsilon, \aleph\upsilon) = d(\Omega, \Lambda)\} = \{1\}$$

Hence $\Omega \backslash B_{est}(\aleph) = [0, 1)$. Now, since Ω and Λ are closed in Φ and Ω_0 and Λ_0 are singletons, so Ω and Λ satisfy the weak P-property trivially. Now $\aleph(\Omega_0) = \{2\} \subseteq \Lambda_0$. It is simple to verify that $\aleph$ is α-interpolative proximal contraction and α-proximal admissible. Take

$$\upsilon_0 = \upsilon_1 = 1 \in \Omega_0.$$

Then

$$d(\upsilon_1, \aleph\upsilon_0) = 1 = d(\Omega, \Lambda)$$

and

$$\alpha(\upsilon_0, \upsilon_1) = \alpha(1,1) = 1 \geq 1.$$

Moreover, since $\aleph$ is constant, hence continuous. Hence all the conditions of Theorem 3.1 (respectively. Theorem 3.2) are satisfied and $\aleph$ has a unique BPP 1 in Ω.

Corollary 3.1. *Let (Φ, d) be an $\mathcal{F}$-complete $\mathcal{F}$-MS and let Ω and Λ be the nonempty closed subsets of Φ satisfying the weak P-property and $\Omega_0 \neq \emptyset$. Assume that $\aleph : \Omega \to \Lambda$ be a single-valued mapping such that $\aleph(\Omega_0) \subseteq \Lambda_0$ and there exist a constant $\lambda \in (0,1)$, a control function $\alpha : \Omega \times \Omega \to [0, \infty)$ and positive real constants γ, β with $\gamma + \beta < 1$ such that*

$$\alpha(\upsilon, \varrho) d(\aleph\upsilon, \aleph\varrho) \leq \lambda (d(\upsilon, \varrho))^{\gamma} (d(\upsilon, \aleph\upsilon) - d(\Omega, \Lambda))^{\beta} (d(\varrho, \aleph\varrho) - d(\Omega, \Lambda))^{1-\gamma-\beta}, \tag{3.17}$$

for all $\upsilon, \varrho \in \Omega \backslash B_{est}(\aleph)$, where $B_{est}(\aleph) = \{\upsilon \in \Omega : d(\upsilon, \aleph\upsilon) = d(\Omega, \Lambda)\}$. Assume that the following conditions holds

(i) $\aleph$ is an α-proximal admissible;

(ii) there exist $\upsilon_0, \upsilon_1 \in \Omega_0$ such that $d(\upsilon_1, \aleph\upsilon_0) = d(\Omega, \Lambda)$ and $\alpha(\upsilon_0, \upsilon_1) \geq 1$;

(iii) $\aleph$ is continuous.

Then $\aleph$ has a unique BPP in Ω.

Proof. Let $\upsilon, \varrho \in \Omega \backslash B_{\text{est}}(\aleph)$ such that $\alpha(\upsilon, \varrho) \geq 1$. By (3.17), we have

$$d(\aleph\upsilon, \aleph\varrho) \leq \alpha(\upsilon, \varrho) d(\aleph\upsilon, \aleph\varrho) \leq \lambda (d(\upsilon, \varrho))^{\gamma} (d(\upsilon, \aleph\upsilon) - d(\Omega, \Lambda))^{\beta} (d(\varrho, \aleph\varrho) - d(\Omega, \Lambda))^{1-\gamma-\beta}.$$

Thus

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda (d(\upsilon, \varrho))^{\gamma} (d(\upsilon, \aleph\upsilon) - d(\Omega, \Lambda))^{\beta} (d(\varrho, \aleph\varrho) - d(\Omega, \Lambda))^{1-\gamma-\beta},$$

for all $\upsilon, \varrho \in \Omega \backslash B_{\text{est}}(\aleph)$. Hence, the inequality (3.1) is satisfied, and consequently, the remaining part of the proof proceeds directly from Theorem 3.1 (respectively, Theorem 3.2). □

Corollary 3.2. *Let (Φ, d) be an $\mathcal{F}$-complete $\mathcal{F}$-MS and let Ω and Λ be the nonempty closed subsets of Φ satisfying the weak P-property and $\Omega_0 \neq \emptyset$. Let $\aleph : \Omega \to \Lambda$ be a single-valued continuous mapping such that $\aleph(\Omega_0) \subseteq \Lambda_0$. Assume that there exists a constant $\lambda \in (0,1)$ and positive real constants γ, β with $\gamma + \beta < 1$ such that*

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda (d(\upsilon, \varrho))^{\gamma} (d(\upsilon, \aleph\upsilon) - d(\Omega, \Lambda))^{\beta} (d(\varrho, \aleph\varrho) - d(\Omega, \Lambda))^{1-\gamma-\beta},$$

for all $\upsilon, \varrho \in \Omega \backslash B_{est}(\aleph)$, where $B_{est}(\aleph) = \{\upsilon \in \Omega : d(\upsilon, \aleph\upsilon) = d(\Omega, \Lambda)\}$. Then $\aleph$ has a unique BPP in Ω.

Proof. Take $\alpha(\upsilon, \varrho) = 1$ in Theorem 3.1. □

4. Follow-up Results

In this segment, we are concerned with BPP deductions for ordered interpolative proximal contractions on a $\mathcal{F}$-MS endowed with a partial ordering/graph, with the aid of results presented in the preceding section. Define

$$\Delta = \{\upsilon, \varrho \in \Omega \text{ such that } \upsilon \leq \varrho \text{ or } \varrho \leq \upsilon\}$$

and

$$\alpha : \Omega \times \Omega \to [0, \infty), \quad \text{where } \alpha(\upsilon, \varrho) = \begin{cases} 1, & \text{if } (\upsilon, \varrho) \in \Delta, \\ 0, & \text{otherwise}. \end{cases}$$

Definition 4.1. *Let Φ be a nonempty set. The triple $(\Phi, d, \leq)$ is said to be a partially ordered $\mathcal{F}$-MS if the following conditions hold:*

(i) d is a $\mathcal{F}$-metric on Φ,

(ii) $\leq$ is a partial order on Φ.

Definition 4.2. *Consider $(\Phi, d, \leq)$ as a partially ordered $\mathcal{F}$-MS, and let Ω and Λ be two non-empty subsets of Φ. A mapping $\aleph : \Omega \to \Lambda$ is said to be proximally order-preserving if the following implication holds:*

$$\left.\begin{array}{l}\varrho_1 \leq \varrho_2 \\ d\left(\upsilon_1, \aleph\varrho_1\right)=d(\Omega, \Lambda) \\ d\left(\upsilon_2, \aleph\varrho_2\right)=d(\Omega, \Lambda)\end{array}\right\} \Longrightarrow \upsilon_1 \leq \upsilon_2,$$

for all $\upsilon_1, \upsilon_2, \varrho_1, \varrho_2 \in \Omega$.

Definition 4.3. *Let Ω and Λ be two non-empty subsets of partially ordered $\mathcal{F}$-MS $(\Phi, d, \leq)$. A mapping $\aleph : \Omega \to \Lambda$ is said to be ordered interpolative proximal contraction, if there exists a constant $\lambda \in (0,1)$ and the positive real numbers γ, β satisfying $\gamma+\beta<1$ such that*

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda\left((d(\upsilon, \varrho))^{\gamma}\left(d(\upsilon, \aleph\upsilon)-d(\Omega, \Lambda)\right)^{\beta}\left(d(\varrho, \aleph\varrho)-d(\Omega, \Lambda)\right)^{1-\gamma-\beta}\right),$$

for all $\upsilon, \varrho \in \Omega \backslash B_{\text{est}}(\aleph)$ with $(\upsilon, \varrho) \in \Delta$.

The next result follows directly from Theorem 3.1 (respectively, Theorem 3.2).

Theorem 4.1. *Let $(\Phi, d, \leq)$ be a $\mathcal{F}$-complete partially ordered $\mathcal{F}$-MS and let Ω and Λ be a non-emtpy closed subsets of Φ satisfying weak P-property and $\Omega_0 \neq \emptyset$. Let $\aleph : \Omega \to \Lambda$ be an ordered interpolative proximal contraction and $\aleph(\Omega_0) \subseteq \Lambda_0$. Assume that the following conditions hold:*

(i) $\aleph$ is proximally order-preserving,

(ii) there exist $\upsilon_0, \upsilon_1 \in \Omega_0$ such that $d\left(\upsilon_1, \aleph\upsilon_0\right)=d(\Omega, \Lambda)$ and $(\upsilon_0, \upsilon_1) \in \Delta$,

(iii) either $\aleph$ is continuous or if there exists a sequence $\{\upsilon_n\}$ in Ω satisfying $\upsilon_n \leq \upsilon_{n+1}$, for all n, and $\upsilon_n \to \upsilon \in \Omega$ as $n \to \infty$, then it follows that $\upsilon_n \leq \upsilon$, for every n.

Then, $\aleph$ possesses a unique BPP in Ω.

Definition 4.4. *Let (Φ, d) be an $\mathcal{F}$-MS endowed with a graph G, and let Ω and Λ be the non-empty subsets of Φ. A mapping $\aleph : \Omega \to \Lambda$ is called proximally G-preserving, if for all $\upsilon_1, \upsilon_2, \varrho_1, \varrho_2 \in \Omega$,*

$$\left.\begin{array}{l}(\varrho_1, \varrho_2) \in E(G) \\ d\left(\upsilon_1, \aleph\varrho_1\right)=d(\Omega, \Lambda) \\ d\left(\upsilon_2, \aleph\varrho_2\right)=d(\Omega, \Lambda)\end{array}\right\} \Rightarrow (\upsilon_1, \upsilon_2) \in E(G).$$

Definition 4.5. *Let (Φ, d) be an $\mathcal{F}$-MS endowed with a graph G, and let Ω and Λ be the non-empty subsets of Φ. A mapping $\aleph : \Omega \to \Lambda$ is said to be G-interpolative proximal contraction, if there exists $\lambda \in (0,1)$ and positive real numbers γ, β satisfying $\gamma+\beta<1$ such that*

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda(d(\upsilon, \varrho))^{\gamma}\left(d(\upsilon, \aleph\upsilon)-d(\Omega, \Lambda)\right)^{\beta}\left(d(\varrho, \aleph\varrho)-d(\Omega, \Lambda)\right)^{1-\gamma-\beta},$$

for all $\upsilon, \varrho \in \Omega \backslash B_{est}(\aleph)$ with $(\upsilon, \varrho) \in E(G)$.

Therefore, the next result is a direct consequence of Theorem 3.1 (respectively. Theorem 3.2).

Theorem 4.2. *Let (Φ, d) be an $\mathcal{F}$-complete $\mathcal{F}$-MS endowed with a graph G, and let Ω and Λ be non-empty closed subsets of Φ satisfying the weak P-property and $\Omega_0 \neq \emptyset$. Let $\aleph : \Omega \to \Lambda$ be G-interpolative proximal contraction and $\aleph(\Omega_0) \subseteq \Lambda_0$. Assume that the following conditions hold:*

(i) $\aleph$ is proximally G-preserving;

(ii) there exist $v_0, v_1 \in \Omega_0$ such that $d(v_1, \aleph v_0) = d(\Omega, \Lambda)$ and $(v_0, v_1) \in E(G)$;

(iii) either $\aleph$ is continuous or if $\{v_n\}$ be a sequence in Ω such that $(v_n, v_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$ and $v_n \to v \in \Omega$ as $n \to \infty$, then $(v_n, v) \in E(G)$ for each n.

Then, $\aleph$ possesses a unique BPP in Ω.

Proof. It follows directly, if we define $\alpha : \Omega \times \Omega \to [0, \infty)$ in this way

$$\alpha(v, \varrho) = \begin{cases} 1, \text{ if } (v, \varrho) \in E(G), \\ 0, \text{ otherwise} \end{cases}$$

in Theorem 3.1 (respectively. Theorem 3.2). □

5. Fixed Point Consequences of Best Proximity Results

This section presents results that are closely connected to FP theory, particularly for α-interpolative contractions. When $\Omega = \Lambda = \Phi$, the corresponding contractive condition simplifies accordingly. In this case, since $d(v, \aleph v) = d(\Omega, \Lambda) = 0$ for a self mapping $\aleph$, it follows that $v = \aleph v$. Hence, the notion of a BPP naturally reduces to that of a FP. Consequently, an α-interpolative proximal contraction becomes an α-interpolative FP problem.

Definition 5.1. *Let (Φ, d) be an $\mathcal{F}$-MS. A mapping $\aleph : \Phi \to \Phi$ is said to be an α-interpolative contraction, if there exist $\lambda \in (0,1)$, a control function $\alpha : \Phi \times \Phi \to [0, \infty)$ and positive real numbers γ, β satisfying $\gamma + \beta < 1$ such that*

$$d(\aleph v, \aleph \varrho) \leq \lambda \left((d(v, \varrho))^{\gamma} (d(v, \aleph v))^{\beta} (d(\varrho, \aleph \varrho))^{1-\gamma-\beta} \right),$$

for all $v, \varrho \in \Phi \backslash Fiv(\aleph)$ with $\alpha(v, \varrho) \geq 1$.

The upcoming result is a consequence of Theorem 3.1 (resp. Theorem 3.2).

Theorem 5.1. *Let (Φ, d) be an $\mathcal{F}$-complete $\mathcal{F}$-MS and $\aleph : \Phi \to \Phi$ be an α-interpolative contraction. Assume that the following conditions hold:*

(i) $\aleph$ is α-admissible;

(ii) There exists $v_0 \in \Phi$ such that $\alpha(v_0, \aleph v_0) \geq 1$,

(iii) either $\aleph$ is continuous or if there exists a sequence $\{v_n\}$ in Φ such as $\alpha(v_n, v_{n+1}) \geq 1$ for each n and $v_n \to v \in \Phi$ as $n \to \infty$, then $\alpha(v_n, v) \geq 1$ for every n.

Then $\aleph$ has unique FP in Φ.

Definition 5.2. *Let ($\Phi, d, \leq$) as a partially ordered $\mathcal{F}$-MS. A mapping $\aleph : \Phi \to \Phi$ is called an ordered interpolative contraction if there exists a constant $\lambda \in (0,1)$ and positive real numbers γ, β satisfying $\gamma + \beta < 1$ in a way that*

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda \left((d(\upsilon, \varrho))^{\gamma} (d(\upsilon, \aleph\upsilon))^{\beta} (d(\varrho, \aleph\varrho))^{1-\gamma-\beta} \right),$$

for all $\upsilon, \varrho \in \Phi \backslash Fix(\aleph)$ with $(\upsilon, \varrho) \in \Delta$.

The following result is a direct consequence of Theorem 4.1.

Theorem 5.2. *Let $(\Phi, d, \leq)$ be an $\mathcal{F}$-complete partially ordered $\mathcal{F}$-MS and $\aleph : \Phi \to \Phi$ be an ordered interpolative contraction. Assume that the following conditions hold.*

(i) $\aleph$ is nondecreasing,

(ii) there exists $\upsilon_0 \in \Phi$ such that $\upsilon_0 \leq \aleph\upsilon_0$;

(iii) either $\aleph$ is continuous or If there exists a sequence $\{\upsilon_n\}$ in Φ such that $\upsilon_n \leq \upsilon_{n+1}$ for each n and $\upsilon_n \to \upsilon \in \Phi$ as $n \to \infty$, then $\upsilon_n \leq \upsilon$ for all n.

Then $\aleph$ admits unique FP in Φ.

Definition 5.3. *Let (Φ, d) be an $\mathcal{F}$-MS endowed with a graph G. A mapping $\aleph : \Phi \to \Phi$ is said to be G-interpolative contraction, if there exists a constant $\lambda \in (0,1)$ and positive real numbers γ, β satisfying $\gamma + \beta < 1$ such that*

$$d(\aleph\upsilon, \aleph\varrho) \leq \lambda \left((d(\upsilon, \varrho))^{\gamma} (d(\upsilon, \aleph\upsilon))^{\beta} (d(\varrho, \aleph\varrho))^{1-\gamma-\beta} \right)$$

for all $\upsilon, \varrho \in \Phi \backslash Fix(\aleph)$ with $(\upsilon, \varrho) \in E(G)$.

Definition 5.4. *Let (Φ, d) be an $\mathcal{F}$-MS endowed with a graph G. A mapping $\aleph : \Phi \to \Phi$ is called G-preserving, if for all $\upsilon_1, \upsilon_2 \in \Phi$,*

$$(\upsilon_1, \upsilon_2) \in E(G) \Longrightarrow (\aleph\upsilon_1, \aleph\upsilon_2) \in E(G).$$

From Theorem 4.2, we directly obtain the following result.

Theorem 5.3. *Let (Φ, d) be $\mathcal{F}$-complete $\mathcal{F}$-MS endowed with a graph G, and let $\aleph : \Phi \to \Phi$ be the G-interpolative contraction. Assume that the following conditions hold:*

(i) $\aleph$ is G-preserving;

(ii) there exists $\upsilon_0 \in \Phi$ such that $(\upsilon_0, \aleph\upsilon_0) \in E(G)$,

(iii) either $\aleph$ is continuous or if $\{\upsilon_n\}$ is a sequence in Φ such that $(\upsilon_n, \upsilon_{n+1}) \in E(G)$ for each n and $\upsilon_n \to \upsilon \in \Phi$ as $n \to \infty$, then $(\upsilon_n, \upsilon) \in E(G)$ for every n.

Then $\aleph$ has a unique FP.

6. Conclusion

In the framework of $\mathcal{F}$-MSs, the present study has introduced the novel concept of α-interpolative proximal contractions and established the existence of BPPs for this new class of mappings. By extending and refining previously known contractive conditions, the results achieved in this work unify and generalize several well-established findings in the existing literature. To demonstrate the applicability and validity of the theoretical results, explicit illustrative examples have been provided.

The conclusions of this research open promising directions for future investigation in the theory of BPPs for fuzzy and generalized mappings in $\mathcal{F}$-MSs. In particular, the potential applications of these results to fractional and classical differential equations offer a fertile area for continued study. Additionally, exploring the connections between the present findings and the emerging theory of orthogonal $\mathcal{F}$-MSs may yield further significant developments in this growing field.

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