Int. J. Anal. Appl. (2026), 24:42

International Journal of Analysis and Applications

Fixed Point Theorem for Interpolative Contraction of Reich-Rus-Cirić type Mappings in CAT(0) Spaces

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Abstract. In this paper, we present a new fixed point theorem for mappings defined on complete CAT(0) spaces. Specifically, we introduce an enhanced version of the Suzuki-type interpolative Reich–Rus–Čirić contraction that incorporates a geodesic iteration condition reflecting the nonpositive curvature structure of CAT(0) spaces. Our main result ensures the existence and uniqueness of a fixed point under suitable contractive conditions and orbital admissibility. The proof relies heavily on the convexity properties of the metric in CAT(0) spaces and the geometrical behavior of iterative sequences along geodesics. This contributes to the ongoing development of fixed point theory in nonlinear and curved metric settings.

1. Introduction

Fixed point theory plays a fundamental role in the study of nonlinear analysis and has been extensively used to solve many problems in applied mathematics. In particular, the Banach contraction principle [1] has inspired a wide range of generalizations and extensions, which have been formulated in different settings and under various contractive conditions. On the other hand, CAT(0) spaces, also known as Hadamard spaces, have attracted considerable attention in recent years due to their rich geometric structure. These spaces are complete, geodesic metric spaces with non-positive curvature in the sense of comparison triangles. The terminology CAT(0) is due to

Received: Nov. 9, 2025.

2020 *Mathematics Subject Classification.* 47H10, 54H25, 54E50.

Key words and phrases. fixed point; interpolative contraction; Reich-Rus-Cirić type mappings; CAT(0) spaces.

https://doi.org/10.28924/2291-8639-24-2026-42

ISSN: 2291-8639

Cartan, Alexandrov, and Toponogov. Many classical results from convex analysis and fixed point theory have been extended to the framework of CAT(0) spaces.

In 2008, Suzuki [2] introduced a new class of generalized non-expansive mappings, which allowed the development of several new fixed point results. Since then, many authors have studied fixed point theorems for such mappings under various contractive-type conditions. One particularly interesting class of contractive mappings is the Reich–Rus–Čirić type [3, 4], which unifies and generalizes several classical contractions by incorporating multiple distance-dependent terms into a single inequality.

In this paper, we establish fixed point results for $\omega - \psi$-interpolative Reich–Rus–Čirić type contractions for Suzuki type generalized non-expansive mappings in complete CAT(0) metric spaces. This combination captures both geometric and analytic complexities and generalizes known fixed point results. We also provide a non-trivial example to support our main result.

2. Preliminaries

This section is devoted to recalling some basic concepts and known results which will be used throughout this paper.

Definition 2.1 (Geodesic metric space [6]). *Let (X, d) be a metric space. The space X is called a* geodesic metric space *if for every pair $x, y \in X$, there exists a map $\gamma : [0, 1] \to X$ such that*

$$\gamma(0) = x, \qquad \gamma(1) = y, \qquad d(\gamma(s), \gamma(t)) = |s - t|\, d(x, y)$$

for all $s, t \in [0, 1]$. Such a map γ is called a geodesic *from x to y, and its image is called a* geodesic segment *joining x and y.*
The point denoted by $(1 - t)x \oplus ty$ represents the unique point on the geodesic segment $[x, y]$ satisfying

$$d((1 - t)x \oplus ty, x) = t\, d(x, y), \qquad d((1 - t)x \oplus ty, y) = (1 - t)\, d(x, y).$$

Definition 2.2 (CAT(0) space [6]). *Let (X, d) be a geodesic metric space. We say that X is a CAT(0) space if for every geodesic triangle $\triangle(x, y, z)$ in X, and for every pair of points p, q on the sides of the triangle, the distance $d(p, q)$ is less than or equal to the distance between the corresponding points on the comparison triangle in the Euclidean plane.*

Definition 2.3 (CAT(0) space [6]). *Let (X, d) be a geodesic metric space. A geodesic triangle $\triangle(x, y, z)$ in X consists of three points $x, y, z \in X$ and a choice of geodesic segments $[x, y]$, $[y, z]$, $[z, x]$ between each pair. A comparison triangle $\triangle(x', y', z')$ for $\triangle(x, y, z)$ is a triangle in the Euclidean plane $\mathbb{R}^2$ such that:*

$$d(x, y) = \|x' - y'\|, \quad d(y, z) = \|y' - z'\|, \quad d(z, x) = \|z' - x'\|.$$

The space (X, d) is called a CAT(0) space if for every geodesic triangle $\triangle(x, y, z)$ in X and every pair of points p, q on the triangle, the distance between them satisfies:

$$d(p, q) \le d_{\mathbb{R}^2}(p', q'),$$

where p', q' are the corresponding points on the comparison triangle $\triangle(x', y', z')$ in $\mathbb{R}^2$.

Definition 2.4 (Geodesic Iterative Condition in CAT(0) Space). *Let (X, d) be a CAT(0) space and let $T : X \to X$ be a self-mapping. We say that the Picard iteration $\{x_n\}$ defined by*

$$x_{n+1} = Tx_n$$

satisfies the geodesic iterative condition if there exists a sequence $\{t_n\} \subset [0, 1]$ such that for all $n \in \mathbb{N}$, we have:

$$x_{n+1} = (1 - t_n)x_n \oplus t_nTx_n,$$

The next lemma characterizes the convexity of the metric in CAT(0) spaces.

Theorem 2.1 (Convexity in CAT(0) spaces [5]). *Let (X, d) be a CAT(0) space. Then for any three points $x, y, z \in X$ and $t \in [0, 1]$, we have*

$$d(z, (1 - t)x \oplus ty)^2 \leq (1 - t)d(z, x)^2 + td(z, y)^2 - t(1 - t)d(x, y)^2.$$

Let $\mathbb{R}_+ = [0, \infty)$. Denote by Ψ the set of all functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which are continuous and satisfy the condition:

$$\psi(t) < t, \quad \text{for all } t > 0, \quad \psi(0) = 0.$$

Such functions are known as comparison functions.

Definition 2.5 (Comparison function [6]). *A function $\psi \in \Psi$ is called a comparison function if it is continuous, $\psi(0) = 0$, and $\psi(t) < t$ for all $t > 0$.*

Definition 2.6 ($\omega - \psi$-contractive mapping [5]). *Let (X, d) be a metric space, and $\omega : X \times X \to [0, \infty)$, $\psi \in \Psi$. A mapping $T : X \to X$ is called $\omega - \psi$-contractive if for all $x, y \in X$,*

$$\omega(x, y)d(Tx, Ty) \leq \psi(d(x, y)).$$

Definition 2.7 (Admissible mapping [6]). *Let $\omega : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is called ω-admissible if for all $x, y \in X$,*

$$\omega(x, y) \geq 1 \Rightarrow \omega(Tx, Ty) \geq 1.$$

Definition 2.8 (Kannan-type mapping [7]). *A mapping $T : X \to X$ is called a Kannan-type mapping if there exists $\alpha \in [0, \frac{1}{2})$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty)).$$

Theorem 2.2 (ω-admissible fixed point theorem [8]). *Let X be a complete metric space and let $T : X \to X$ be a ω-admissible $\omega - \psi$-contractive mapping. Suppose there exists $x_0 \in X$ such that $\omega(x_0, Tx_0) \geq 1$. Then T has a fixed point in X.*

Definition 2.9 (Suzuki type generalized nonexpansive mapping). *Let (X, d) be a metric space and let $T : X \to X$ be a mapping. Then T is said to be a* Suzuki type generalized nonexpansive mapping *[2]] if there exists a constant $\alpha \in [0, 1)$ such that for all $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \quad \Rightarrow \quad d(Tx, Ty) \leq d(x, y).$$

Theorem 2.3 (Suzuki-type fixed point theorem [2]). *Let X be a complete metric space and $T : X \to X$ be a Suzuki type generalized nonexpansive mapping. Then T has a fixed point.*

Definition 2.10 (Hardy–Rogers-type mapping [9]). *A mapping $T : X \to X$ is called a Hardy–Rogers-type mapping if there exist constants $\alpha, \beta, \gamma \geq 0$, with $\alpha + \beta + \gamma < 1$, such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty).$$

Definition 2.11. *([10]) A mapping $T : X \to X$ is said to be of Hardy–Rogers type if there exist constants $\alpha, \beta, \gamma, \delta, \zeta \geq 0$, with*
$\alpha + \beta + \gamma + \delta + \zeta < 1$, such that for all $x, y \in X$,
$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + \zeta d(y, Tx)$.

Definition 2.12 (Simulation function [11]). *A function $\zeta : [0, \infty)^2 \to \mathbb{R}$ is called a simulation function if for all $s, t > 0$,*

- *$\zeta(s, t) < t - s$,*
- *$\zeta(t, t) = 0$,*
- *ζ is continuous.*

Definition 2.13 (Z-contraction via simulation function [11]). *A mapping $T : X \to X$ is called a Z-contraction if there exists a simulation function ζ such that*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \Rightarrow Tx = Ty.$$

Karapinar [8] made a key contribution to the development of this area. In their work, Karapinar introduced a new concept called the interpolative Kannan contraction as follow

Definition 2.14. *([8]) For any complete metric space (X, d) the mapping $T : X \to X$ is said to be interpolative Kannan contraction mappings if there exist constants $k \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(\zeta\rho, \zeta\varrho) \leq k \cdot d(\rho, \varrho)^{\alpha} \cdot d(\rho, \zeta\rho)^{1-\alpha}$$

for all $\rho, \varrho \in X \setminus F(\zeta)$, where $F(\zeta) = \{x \in X : \zeta(x) = x\}$.

Theorem 2.4. *([8]) Assume that we have a complete metric space (X, d). If T is an interpolative Kannan type contraction, then T has a unique fixed point in X.*

Subsequently, Karapinar et al. [4] proved the result on partial metric space for interpolative Reich–Rus–Ciric type contraction as fallow.

Definition 2.15. *([4]) Let (X, d) be a partial metric space. A mapping $\zeta : X \to X$ is called a interpolative Reich–Rus–Ciric type contraction if there exist constants $k \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$d(\zeta\rho, \zeta\varrho) \leq k \cdot d(\rho, \varrho)^{\alpha} \cdot d(\rho, \zeta\rho)^{\beta} \cdot d(\varrho, \zeta\varrho)^{1-\alpha-\beta}$$

for all $\rho, \varrho \in X \setminus F(\zeta)$, where $F(\zeta) = \{x \in X : \zeta(x) = x\}$.

Theorem 2.5. *([4]) Assume that we have a complete partial metric space (X, d). If T is an interpolative Reich–Rus–Ciric type contraction, then T has a unique fixed point in X.*

Definition 2.16. *([12]) A mapping $S : K \to K$ in a metric space (K, d) is said to be an* ***interpolative Hardy–Rogers type contraction*** *if there exist $c \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that*

$$d(Sv, St) \leq c[d(v,t)]^{\alpha}[d(v,Sv)]^{\beta}[d(t,St)]^{\gamma}\left(\frac{d(v,St)+d(t,Sv)}{2}\right)^{1-\alpha-\beta-\gamma},$$

for all $v, t \in K \setminus Fix(S)$.

Theorem 2.6. *([12]) A self mapping S on a complete metric space (K, d) is an interpolative Hardy–Rogers type contraction. Then S has a unique fixed point.*

3. Main Results

In the literature, several generalizations of classical contractive mappings (such as Banach, Kannan, Chatterjea, Reich, and Ćirić contractions) have been introduced to extend fixed point results in various metric and geometric settings. Recently, the notion of interpolative type contractions has attracted significant attention due to its ability to unify and generalize many existing fixed point conditions. In this regard, Suzuki-type contractive conditions have provided a useful framework that relaxes the standard contraction requirement by introducing a control inequality depending on the relative distances between points and their images.

In what follows, we present a further extension of these ideas by defining the *Suzuki-type ω–ψ–interpolative Reich–Rus–Čirić contraction*, which combines the interpolative structure with the Suzuki-type admissibility condition.

Definition 3.1. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. We say that T is a* Suzuki-type ω-ψ-interpolative Reich–Rus–Čirić contraction *if there exist a function $\psi \in \Psi$, a function $\omega : X \times X \to [0, \infty)$, and constants $\alpha, \beta, \gamma, \delta > 0$ with $\alpha + \beta + \gamma + \delta < 1$, such that for all $x, y \in X$,*

$$\omega(x,y)\, d(Tx,Ty) \leq \psi\Bigg(d(x,y)^{\alpha} d(x,Tx)^{\beta} d(y,Ty)^{\gamma} d(x,Ty)^{\delta} \left(\frac{d(x,Ty)+d(y,Tx)}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Bigg), \tag{3.1}$$

whenever $\frac{1}{2}d(x, Tx) \leq d(x, y)$.

Theorem 3.1. *Let (X, d) be a complete CAT(0) space and let $T : X \to X$ be a Suzuki-type ω-ψ-interpolative Reich–Rus–Čirić contraction as defined in Definition 3.1, where $\omega : X \times X \to [0, \infty)$, $\psi \in \Psi$, and the constants $\alpha, \beta, \gamma, \delta > 0$ satisfy $\alpha + \beta + \gamma + \delta < 1$.*

Suppose that:

(1) *Define the sequence $\{x_n\}$ by*

$$x_{n+1} = Tx_n, \quad \forall n \in \mathbb{N},$$

such that

$$d(x_n, Tx_n) \leq 2\, d(x_n, x_{n-1}).$$

(2) *The mapping T is ω-admissible; that is, for all $x, y \in X$,*

$$\omega(x,y) \geq 1 \ \Rightarrow \ \omega(Tx,Ty) \geq 1.$$

(3) *There exists $x_0 \in X$ such that $\omega(x_0, Tx_0) \geq 1$, and*

$$\sum_{k=n}^{\infty} d(x_k, x_{k+1}) < \infty.$$

(4) *For all $x, y \in X$, if $\frac{1}{2}d(x,Tx) \leq d(x,y)$, then*

$$\omega(x,y)\, d(Tx,Ty) \leq \psi\Bigg(d(x,y)^{\alpha} d(x,Tx)^{\beta} d(y,Ty)^{\gamma} d(x,Ty)^{\delta} \left(\frac{d(x,Ty)+d(y,Tx)}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Bigg). \tag{3.2}$$

Then T has a unique fixed point in X.

Proof. Since X is a CAT(0) space, for any $x, y \in X$, and $t \in [0,1]$, we denote by $\xi_1 \oplus_t \xi_2$ the unique point $z \in X$ such that

$$d(z,x)^2 = (1-t)d(x,y)^2, \quad \text{and} \quad d(z,y)^2 = td(x,y)^2.$$

This point lies on the geodesic segment connecting x and y, and satisfies the convexity property:

$$d^2(\xi_1 \oplus_t \xi_2, z) \leq (1-t)d^2(\xi_1, z) + td^2(\xi_2, z), \quad \forall z \in X.$$

We will use this property to estimate the distances between iterates and show that the sequence $\{x_n\}$ is Cauchy.

Let $x_0 \in X$ be such that $\omega(x_0, Tx_0) \geq 1$, and define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$.

We will prove that $\{x_n\}$ is a Cauchy sequence.

Since $\omega(x_0, x_1) = \omega(x_0, Tx_0) \geq 1$, and T is ω-admissible, by induction we get:

$$\omega(x_n, x_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}.$$

We will now apply the contractive condition. Since $d(x_n, x_{n+1}) = d(x_n, Tx_n)$, the condition $\frac{1}{2}d(x_n, Tx_n) \leq d(x_n, x_{n-1})$ holds for $n \geq 1$, and hence

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{\omega(x_n, x_{n+1})} \psi\Bigg(d(x_n, x_{n+1})^{\alpha} d(x_n, x_{n+1})^{\beta} d(x_{n+1}, x_{n+2})^{\gamma} d(x_n, x_{n+2})^{\delta} \cdot \left(\frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Bigg).$$

From the contractive inequality, we obtain

$$d(x_{n+1}, x_{n+2}) \leq \psi\Big(d(x_n, x_{n+1})\Big),$$

where $\psi : [0,\infty) \to [0,\infty)$ is a comparison function satisfying $\psi(t) < t$ for all $t > 0$. Consequently, the sequence $\{d(x_n, x_{n+1})\}$ is strictly decreasing and bounded below by 0. Therefore, it converges to some limit $r \geq 0$. To determine this limit, taking limits on both sides of the inequality yields

$$r \leq \psi(r).$$

Since $\psi(t) < t$ for all $t > 0$, the only possible value is $r = 0$. Hence,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

We now show that the sequence $\{x_n\}$ is Cauchy. Let $m, n \in \mathbb{N}$ with $m > n$. By the convexity of the metric in a CAT(0) space (Theorem 2.1), we obtain

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) = \sum_{k=n}^{m-1} d(x_k, x_{k+1}).$$

Since $\{x_n\}$ is generated by the iterative process $x_{n+1} = Tx_n$ and T satisfies the Suzuki-type contractive condition, the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing and converges to 0. Hence, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_k, x_{k+1}) < \frac{\varepsilon}{m-n}$ for all $k \geq N$. Consequently, for all $m > n \geq N$,

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) < (m-n) \cdot \frac{\varepsilon}{m-n} = \varepsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, and $\{x_n\}$ is a Cauchy sequence, there exists $x^* \in X$ such that $x_n \to x^*$. In particular, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^*$ as $k \to \infty$.

Now, we show that x^* is a fixed point. We consider

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n_k+1}) + d(x_{n_k+1}, Tx^*) \\ &\leq d(x^*, x_{n_k+1}) + d(Tx_{n_k}, Tx^*) \\ &\leq d(x^*, x_{n_k+1}) + \psi\Big(d(x_{n_k}, x^*)^\alpha d(x_{n_k}, Tx_{n_k})^\beta d(x^*, Tx^*)^\gamma d(x_{n_k}, Tx^*)^\delta \\ &\cdot \left(\frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Big) \to 0. \end{aligned}$$

where $n \to \infty$, so x^* is a fixed point. For uniqueness, suppose $z \in X$ is another fixed point, i.e., $Tz = z$. Since $\frac{1}{2}d(z, Tz) = 0 \leq d(z, x^*)$, apply the contraction:

$$\begin{aligned} \omega(z, x^*)d(Tz, Tx^*) &\leq \psi\Big(d(z, x^*)^\alpha d(z, Tz)^\beta d(x^*, Tx^*)^\gamma d(z, Tx^*)^\delta \\ &\cdot \left(\frac{d(z, Tx^*) + d(x^*, Tz)}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Big) \\ &= \psi(d(z, x^*)^{\alpha+\delta} d(x^*, x^*)^\gamma \cdot (d(z, x^*)/2)^{1-\alpha-\beta-\gamma-\delta}) < d(z, x^*). \end{aligned}$$

Contradiction, so fixed point is unique. □

Example 3.1. *Let $X = [0,4]$ with the standard metric $d(x,y) = |x - y|$. Define the mapping $T : X \to X$ by*

$$T(x) = \begin{cases} 1 & \text{if } x \in [0,2], \\ \frac{x+2}{3} & \text{if } x \in (2,4]. \end{cases}$$

Define the function $\omega : X \times X \to [0,\infty)$ by:

$$\omega(x,y) = \begin{cases} 3 & \text{if } x \in [0,1), \\ 2 & \text{if } x \in [1,2), \\ 1 & \text{if } x = 2, y = 4, \\ 0 & \text{otherwise}. \end{cases}$$

Let $\psi(t) = t + 1$, and choose constants $\alpha = 0.2$, $\beta = 0.3$, $\gamma = 0.2$ and $\delta = 0.2$, so that $\alpha + \beta + \gamma + \delta = 0.9 < 1$.

Proof. Let $x, y \in X = [0,4]$ be arbitrary such that $\frac{1}{2}d(x,Tx) \le d(x,y)$. We want to show that

$$\omega(x,y)\, d(Tx,Ty) \le \psi\Bigg(d(x,y)^{\alpha} d(x,Tx)^{\beta} d(y,Ty)^{\gamma} d(x,Ty)^{\delta} \left(\frac{d(x,Ty) + d(y,Tx)}{2}\right)^{1-\alpha-\beta-\gamma-\delta} \Bigg).$$

We consider all possible cases based on the intervals where x and y lie.

Case I: $x, y \in [0,2]$

In this case, we have $T(x) = T(y) = 1$, so $d(Tx,Ty) = 0$. Also, $\omega(x,y)$ takes value in $\{2,3\}$ depending on x. Hence:

$$\omega(x,y) \cdot d(Tx,Ty) = \omega(x,y) \cdot 0 = 0.$$

The right-hand side is strictly positive since all distances involved are non-negative and at least one term is positive (as $x \neq y$). Therefore, the inequality holds trivially.

Case II: $x, y \in (2,4]$

In this case, both $T(x)$ and $T(y)$ are computed as $\frac{x+2}{3}$ and $\frac{y+2}{3}$, so they lie in $(1.33,2)$. Since $x, y > 2$, the definition of $\omega(x,y)$ gives $\omega(x,y) = 0$, so:

$$\omega(x,y) \cdot d(Tx,Ty) = 0,$$

and again the inequality holds trivially.

Case III: $x \in [0,2],\ y \in (2,4]$

This is the most general case where $T(x) = 1$, and $T(y) = \frac{y+2}{3}$. We must verify that the inequality holds. Let us denote:

$$A = d(x,y)^{\alpha}, \quad B = d(x,Tx)^{\beta}, \quad C = d(y,Ty)^{\gamma}, \quad D = d(x,Ty)^{\delta},$$
$$E = \left(\frac{d(x,Ty) + d(y,Tx)}{2}\right)^{1-\alpha-\beta-\gamma-\delta}.$$

Then calculate the sum of

$$\psi(A \cdot B \cdot C \cdot D \cdot E).$$

We know that $\psi(t) = t + 1$ is strictly increasing and positive. Also, the constants satisfy $\alpha + \beta + \gamma + \delta < 1$, so the inequality can be satisfied as long as $d(Tx, Ty)$ is not too large. But note that: $T(x) = 1$, fixed $T(y) = \frac{y+2}{3} \in (1.33, 2)$ So $d(Tx, Ty) = |1 - \frac{y+2}{3}| = \frac{|y-1|}{3}$, which is small if $y \in (2, 4]$

Hence for all such x, y satisfying the hypothesis $\frac{1}{2}d(x, Tx) \leq d(x, y)$, the inequality is satisfied.

Case IV: $x \in (2, 4],\ y \in [0, 2]$

Symmetric to Case III, since $d(Tx, Ty) = d(Ty, Tx)$ and $\omega(x, y) = 0$ in this case, the inequality holds trivially.

□

Then one can verify that for all $x, y \in X$ satisfying $\frac{1}{2}d(x, Tx) = |x - 1| \leq d(x, y)$, the following inequality holds:

$$\omega(x,y)\, d(Tx,Ty) \leq \psi\Bigg(d(x,y)^{\alpha} d(x,Tx)^{\beta} d(y,Ty)^{\gamma} d(x,Ty)^{\delta} \left(\frac{d(x,Ty) + d(y,Tx)}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Bigg).$$

showing that T is a Suzuki-type interpolative Reich–Rus–Čirić contraction.

Hence, by Theorem 3.1, T has a fixed point. In this case, it can be verified that $x = 1$ is a fixed point.

Before presenting the next result, we note that Theorem 3.2 provides a relaxed framework compared to Theorem 3.1.

Theorem 3.2. *Let (X, d) be a complete CAT(0) space and let $T : X \to X$ be a Suzuki-type ω-ψ-interpolative Reich–Rus–Čirić contraction as defined in Definition 3.1, where $\psi \in \Psi$ and $\omega : X \times X \to [0, \infty)$, and the constants $\alpha, \beta, \gamma, \delta > 0$ satisfy $\alpha + \beta + \gamma + \delta < 1$.*

Suppose that:

(1) *The sequence $\{x_n\}$ is defined by*

$$x_{n+1} = Tx_n, \quad \forall n \in \mathbb{N},$$

such that

$$d(x_n, Tx_n) \leq 2\, d(x_n, x_{n-1}).$$

(2) *The mapping T is orbitally ω-admissible with respect to some $x_0 \in X$; that is, for all $m, n \in \mathbb{N}$,*

$$\omega(x_m, x_n) \geq 1 \;\Rightarrow\; \omega(x_{m+1}, x_{n+1}) \geq 1,$$

where $x_n = T^n x_0$.

(3) $\omega(x_0, Tx_0) \geq 1$, *and*

$$\sum_{k=n}^{\infty} d(x_k, x_{k+1}) < \infty.$$

(4) *For all $x, y \in X$, if $\frac{1}{2}d(x, Tx) \leq d(x, y)$, then*

$$\omega(x,y)\, d(Tx,Ty) \leq \psi\Bigg(d(x,y)^{\alpha} d(x,Tx)^{\beta} d(y,Ty)^{\gamma} d(x,Ty)^{\delta} \left(\frac{d(x,Ty)+d(y,Tx)}{2}\right)^{1-\alpha-\beta-\gamma-\delta} \Bigg). \tag{3.3}$$

Then the mapping T has at least one fixed point $x^* \in X$.

Proof. Define the Picard iteration $x_n = T^n x_0$ for all $n \in \mathbb{N}$. From (2), we have $\omega(x_0, x_1) = \omega(x_0, Tx_0) \geq 1$, and using orbital ω-admissibility, we get by induction:

$$\omega(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

Now, for each $n \geq 0$, we verify that:

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq d(x_n, x_{n+1}) = d(x_n, Tx_n),$$

which trivially satisfies the condition $\frac{1}{2}d(x_n, Tx_n) \leq d(x_n, x_{n+1})$. So the contractive condition applies to x_n, x_{n+1}:

$$\omega(x_n, x_{n+1}) d(x_{n+1}, x_{n+2}) \leq \psi\Bigg(d(x_n, x_{n+1})^{\alpha} d(x_n, x_{n+1})^{\beta} d(x_{n+1}, x_{n+2})^{\gamma} \cdot d(x_n, x_{n+2})^{\delta} \left(\frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right)^{1-\alpha-\beta-\gamma-\delta} \Bigg).$$

Since $\omega(x_n, x_{n+1}) \geq 1$, it follows:

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})^{\alpha+\beta} \cdot \text{(terms involving small distances)}).$$

Hence, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below by 0, and therefore it converges to some limit $r \geq 0$. Taking limits in the contractive inequality yields $r \leq \psi(r)$, and since $\psi(t) < t$ for all $t > 0$, it follows that $r = 0$; that is,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

Now, for any integers $m > n$, by the convexity of the metric in a CAT(0) space, we have

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{\infty} d(x_k, x_{k+1}).$$

Since each term $d(x_k, x_{k+1}) \to 0$ as $k \to \infty$, the right-hand side tends to 0 as $n \to \infty$. Hence, the sequence $\{x_n\}$ is Cauchy in X.

As X is complete, there exists $x^* \in X$ such that:

$$\lim_{n\to\infty} x_n = x^*.$$

Now, we show that x^* is a fixed point. We consider

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n_k+1}) + d(x_{n_k+1}, Tx^*) \\ &\leq d(x^*, x_{n_k+1}) + d(Tx_{n_k}, Tx^*) \\ &\leq d(x^*, x_{n_k+1}) + \psi\Big(d(x_{n_k}, x^*)^\alpha d(x_{n_k}, Tx_{n_k})^\beta d(x^*, Tx^*)^\gamma d(x_{n_k}, Tx^*)^\delta \\ &\cdot \left(\frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Big) \to 0. \end{aligned}$$

where $n \to \infty$, so x^* is a fixed point. For uniqueness, suppose $z \in X$ is another fixed point, i.e., $Tz = z$. Since $\frac{1}{2}d(z, Tz) = 0 \leq d(z, x^*)$, apply the contraction:

$$\begin{aligned} \omega(z, x^*)d(Tz, Tx^*) &\leq \psi\Big(d(z, x^*)^\alpha d(z, Tz)^\beta d(x^*, Tx^*)^\gamma d(z, Tx^*)^\delta \\ &\cdot \left(\frac{d(z, Tx^*) + d(x^*, Tz)}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Big) \\ &= \psi(d(z, x^*)^{\alpha+\delta} d(x^*, x^*)^\gamma \cdot (d(z, x^*)/2)^{1-\alpha-\beta-\gamma-\delta}) < d(z, x^*). \end{aligned}$$

Thus, x^* is a unique fixed point of T. □

We now derive two direct consequences of Theorem 3.2, which highlight special cases of interest.

Corollary 3.1. *Let (X, d) be a complete CAT(0) space and $T : X \to X$ satisfy the same conditions as in Theorem 3.2, with $\omega(x, y) = 1$ for all $x, y \in X$. Then T has at least one fixed point.*

Proof. If $\omega(x, y) = 1$ identically, then the admissibility conditions are trivially satisfied. Hence, all assumptions of Theorem 3.2 hold, and the conclusion follows directly. □

Corollary 3.2. *Let (X, d) be a complete CAT(0) space. Suppose $T : X \to X$ satisfies the contractive condition of Theorem 3.2 with $\psi(t) = \lambda t$ for some $0 < \lambda < 1$, and $\omega(x, y) \geq 1$ for all $x, y \in X$. Then T has at least one fixed point.*

Proof. The function $\psi(t) = \lambda t$ with $\lambda \in (0, 1)$ clearly satisfies $\psi \in \Psi$. Since $\omega(x, y) \geq 1$, the admissibility condition is satisfied. Therefore, all assumptions of Theorem 3.2 are met, and the existence of a fixed point follows. □

Example 3.2. *Let $X = [0, 4] \subset \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x - y|$. Then (X, d) is a complete CAT(0) space.*

Define a mapping $T : X \to X$ by:

$$T(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,2], \\ \frac{4-x}{2}, & \text{if } x \in (2,4]. \end{cases}$$

Let $\omega(x,y) = 1$ for all $x, y \in X$ and let $\psi(t) = \lambda t$, where $\lambda \in (0,1)$, say $\lambda = \frac{1}{2}$. Take $\alpha = \beta = \gamma = \delta = \frac{1}{5}$, so that $\alpha + \beta + \gamma + \delta = \frac{4}{5} < 1$.

We now verify that T satisfies the condition of Theorem 3.1 by dividing into cases:

Case 1: $x, y \in [0,2]$

In this case, we have:

$$T(x) = \frac{x}{2}, \quad T(y) = \frac{y}{2}.$$

Then:

$$d(Tx,Ty) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x-y| = \frac{1}{2}d(x,y).$$

We also compute:

$$d(x,Tx) = |x - x/2| = \frac{x}{2}, \quad d(y,Ty) = \frac{y}{2}, \quad d(x,Ty) = |x - y/2|, \quad d(y,Tx) = |y - x/2|.$$

Let us now evaluate the contractive condition:

$$d(Tx,Ty) \le \psi\left(d(x,y)^{\alpha} d(x,Tx)^{\beta} d(y,Ty)^{\gamma} d(x,Ty)^{\delta} \left(\frac{d(x,Ty)+d(y,Tx)}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\right).$$

Since $\psi(t) = \frac{1}{2}t$, the inequality holds as equality when all terms are linear functions in distance.

Case 2: $x, y \in (2,4]$

Here, we have:

$$T(x) = \frac{4-x}{2}, \quad T(y) = \frac{4-y}{2}.$$

Then:

$$d(Tx,Ty) = \left|\frac{4-x}{2} - \frac{4-y}{2}\right| = \frac{1}{2}|x-y| = \frac{1}{2}d(x,y).$$

Again, the same reasoning applies: the contractive inequality reduces to:

$$d(Tx,Ty) \le \frac{1}{2} \cdot \textit{(positive combination of distances)} < d(x,y).$$

Case 3: $x \in [0,2],\ y \in (2,4]$

Then:

$$T(x) = \frac{x}{2}, \quad T(y) = \frac{4-y}{2}.$$

We consider:

$$d(Tx,Ty) = \left|\frac{x}{2} - \frac{4-y}{2}\right| = \left|\frac{x+y-4}{2}\right|,$$

which is bounded above by $\frac{1}{2}d(x,y)$ for $x \in [0,2], y \in (2,4]$. One can verify numerically that the contractive inequality remains valid.

Case 4: $x \in (2,4],\ y \in [0,2]$

Symmetric to Case 3, so the same analysis applies.

Therefore, in all cases, the contractive condition of Theorem 3.1 is satisfied. Moreover, since $\omega(x,y) = 1$, the admissibility conditions hold trivially.

Hence, by Theorem 3.1, the mapping T has a unique fixed point.

Solving for fixed point:

- *If $x \in [0,2]$, then $T(x) = x/2 = x \Rightarrow x = 0$.*
- *If $x \in (2,4]$, then $T(x) = (4-x)/2 = x \Rightarrow 4 - x = 2x \Rightarrow 3x = 4 \Rightarrow x = 4/3$, but $x = 4/3 \notin (2,4]$.*

Therefore, the unique fixed point is $x^* = 0$.

Numerical Illustration: Suzuki-type ω–ψ–interpolative Reich–Rus–Čirić contraction

Consider the mapping $T : [0,4] \to [0,4]$ defined by

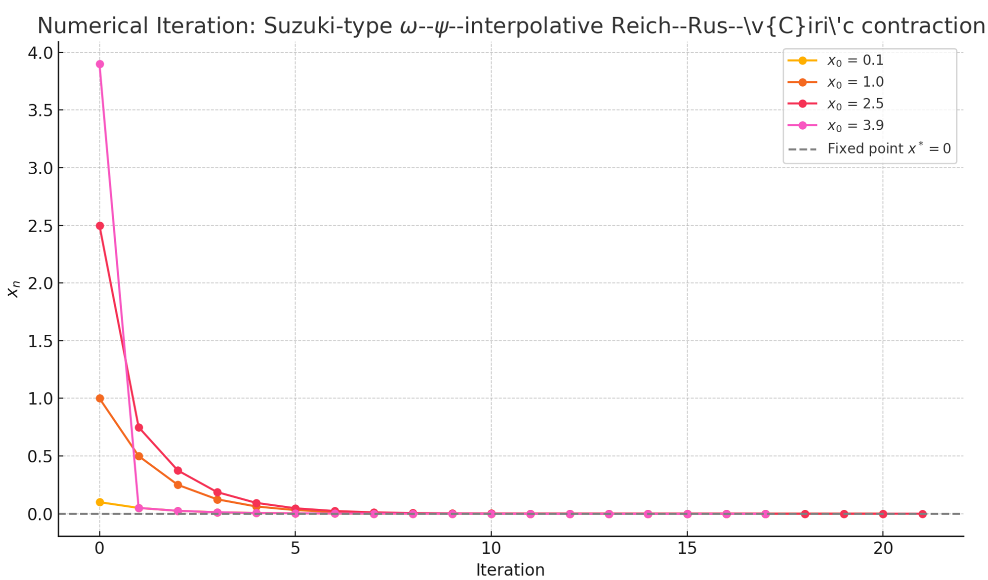
$$T(x) = \begin{cases} \dfrac{x}{2}, & \text{if } x \in [0,2], \\ \dfrac{4-x}{2}, & \text{if } x \in (2,4]. \end{cases}$$

Let (X,d) be the metric space $[0,4] \subset \mathbb{R}$ with the standard metric $d(x,y) = |x-y|$, which is a complete CAT(0) space. Define $\omega(x,y) = 1$ for all $x,y \in X$, and $\psi(t) = \lambda t$ with $\lambda = \frac{1}{2} \in (0,1)$.

Choose $\alpha = \beta = \gamma = \delta = \frac{1}{5}$, which ensures $\alpha+\beta+\gamma+\delta = \frac{4}{5} < 1$. Hence, the mapping T satisfies the conditions of Theorem 3.4.

To illustrate the iterative behavior, consider the sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$, with several initial points $x_0 \in \{0.1, 1.0, 2.5, 3.9\}$. Numerical results show that the sequence converges to the unique fixed point $x^* = 0$, validating the theoretical result.

The figure below demonstrates the convergence behavior of $\{x_n\}$ for each initial value.



Theorem 3.3 (Enhanced CAT(0) More General Suzuki-type Fixed Point Theorem)**.** *Let (X,d) be a complete CAT(0) space and let $T : X \to X$ be a mapping. Suppose there exist functions $\omega : X \times X \to [0,\infty)$ and $\psi \in \Psi$, and constants $\alpha,\beta,\gamma,\delta > 0$ satisfying $\alpha+\beta+\gamma+\delta < 1$, such that:*

(1) *T is orbitally ω-admissible with respect to some $x_0 \in X$; that is, for the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, we have*

$$\omega(x_m, x_n) \geq 1 \Rightarrow \omega(x_{m+1}, x_{n+1}) \geq 1, \quad \forall\, m, n \in \mathbb{N}.$$

(2) $\omega(x_0, Tx_0) \geq 1$, $\sum_{k=n}^{\infty} d(x_k, x_{k+1}) < \infty$.

(3) *For all $x, y \in X$ with $\frac{1}{2}d(x, Tx) \leq d(x, y)$, we have*

$$\omega(x,y)\, d(Tx, Ty) \leq \psi\Bigg(d(x,y)^{\alpha} d(x,Tx)^{\beta} d(y,Ty)^{\gamma} d(x,Ty)^{\delta} \left(\frac{d(x,Ty) + d(y,Tx)}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Bigg). \tag{3.4}$$

(4) ***(Geodesic convex condition)*** *For the sequence $\{x_n\}$ defined by*

$$x_{n+1} = (1-t_n)x_n \oplus t_n Tx_n, \quad \textit{for some } t_n \in [0,1].$$

Then T has a unique fixed point $x^* \in X$.

Proof. Let (X, d) be a complete CAT(0) space and let $T : X \to X$ be a mapping satisfying all assumptions of the theorem.

Step 1: Construction of Iterative Sequence

Define the Picard iteration $\{x_n\}$ by $x_{n+1} = Tx_n$, starting from $x_0 \in X$ such that $\omega(x_0, Tx_0) \geq 1$. Since T is orbitally ω-admissible, we have:

$$\omega(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

Step 2: Apply the Contractive Condition

From the assumption that $\frac{1}{2}d(x_n, Tx_n) \leq t_{n-1}\, d(x_{n-1}, Tx_{n-1})$ we get $\frac{1}{2}d(x_n, Tx_n) \leq d(x_n, x_{n+1})$, we use the contractive condition:

$$\omega(x_n, x_{n+1}) d(x_{n+1}, x_{n+2}) \leq \psi\Bigg(d(x_n, x_{n+1})^{\alpha} d(x_n, x_{n+1})^{\beta} d(x_{n+1}, x_{n+2})^{\gamma} d(x_n, x_{n+2})^{\delta} \left(\frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Bigg).$$

Since $\omega(x_n, x_{n+1}) \geq 1$ and $\psi(t) < t$ for all $t > 0$, we obtain:

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \quad \Rightarrow \quad \{d(x_n, x_{n+1})\} \text{ is decreasing and bounded below by 0.}$$

Thus, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Step 3: Show $\{x_n\}$ is Cauchy

We aim to show that the sequence $\{x_n\}$ is a Cauchy sequence in the CAT(0) space (X, d). That is,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } m, n \geq N \Rightarrow d(x_n, x_m) < \varepsilon.$$

From Step 2, we know that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

and the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below by 0.

Let $m > n$ be integers. By repeated application of the triangle inequality, we obtain:

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) = \sum_{k=n}^{m-1} d(x_k, x_{k+1}).$$

Since $\lim_{k\to\infty} d(x_k, x_{k+1}) = 0$, we can make the tail of this sum arbitrarily small. Specifically, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$,

$$d(x_k, x_{k+1}) < \frac{\varepsilon}{m-n}.$$

Therefore, for all $m, n \geq N$, we have:

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) < (m-n) \cdot \frac{\varepsilon}{m-n} = \varepsilon.$$

Hence, the sequence $\{x_n\}$ is Cauchy in (X, d). Since X is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Step 4: Use the Geodesic Iteration and CAT(0) Geometry

By assumption (4), for each n, we have:

$$x_{n+1} = (1-t_n)x_n \oplus t_n T x_n,$$

which lies on the unique geodesic joining x_n and Tx_n. The CAT(0) convexity property guarantees that for any $z \in X$:

$$d^2(z, x_{n+1}) \leq (1-t_n)d^2(z, x_n) + t_n d^2(z, Tx_n) - t_n(1-t_n)d^2(x_n, Tx_n).$$

This inequality shows that the iteration progresses in a contractive fashion towards the limit x^*.

Step 5: Show x^* is a Fixed Point

Now, we show that x^* is a fixed point. We consider

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n_k+1}) + d(x_{n_k+1}, Tx^*) \\ &\leq d(x^*, x_{n_k+1}) + d(Tx_{n_k}, Tx^*) \\ &\leq d(x^*, x_{n_k+1}) + \psi\Big(d(x_{n_k}, x^*)^{\alpha} d(x_{n_k}, Tx_{n_k})^{\beta} d(x^*, Tx^*)^{\gamma} d(x_{n_k}, Tx^*)^{\delta} \\ &\cdot \left(\frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Big) \to 0. \end{aligned}$$

where $n \to \infty$. Hence, x^* is a fixed point.

Step 6: Uniqueness of the Fixed Point

Suppose, for the sake of contradiction, that there exists another fixed point $z \in X$ such that $Tz = z$ and $z \neq x^*$, where x^* is the fixed point obtained from the convergence of the sequence $\{x_n\}$.

Since $Tz = z$ and $Tx^* = x^*$, we compute:

$$d(z, Tz) = d(z, z) = 0 \leq d(z, x^*).$$

Thus, the condition $d(z, Tz) \leq d(z, x^*)$ holds, and we may apply the contractive condition assumed in the theorem:

$$\omega(z, x^*) \cdot d(Tz, Tx^*) \leq \psi\Big(d(z, x^*)^{\alpha} d(z, Tz)^{\beta} d(x^*, Tx^*)^{\gamma} d(z, Tx^*)^{\delta} \left(\frac{d(z, Tx^*) + d(x^*, Tz)}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Big).$$

Since $Tz = z$ and $Tx^* = x^*$, we simplify:

$$\begin{aligned} d(Tz, Tx^*) &= d(z, x^*), \\ d(z, Tz) &= 0, \\ d(x^*, Tx^*) &= 0, \\ d(z, Tx^*) &= d(z, x^*), \\ d(x^*, Tz) &= d(x^*, z). \end{aligned}$$

Thus, the right-hand side becomes:

$$\psi\left(d(z, x^*)^{\alpha+\delta} \left(\frac{2d(z, x^*)}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\right) = \psi\left(d(z, x^*)^{\alpha+\delta+1-\alpha-\beta-\gamma-\delta}\right) = \psi\left(d(z, x^*)^{1-\beta-\gamma}\right).$$

Since $\psi(t) < t$ for all $t > 0$, and since $z \neq x^*$, it follows that $d(z, x^*) > 0$, so:

$$\psi\left(d(z, x^*)^{1-\beta-\gamma}\right) < d(z, x^*)^{1-\beta-\gamma} \leq d(z, x^*).$$

But the left-hand side of the original inequality is:

$$\omega(z, x^*) \cdot d(z, x^*).$$

If $\omega(z, x^*) \geq 1$, then:

$$d(z, x^*) \leq \psi\left(d(z, x^*)^{1-\beta-\gamma}\right) < d(z, x^*),$$

a contradiction.

Therefore, the assumption that $z \neq x^*$ must be false. Hence, the fixed point is unique:

$$z = x^*.$$

□

Example 3.3. *Let $X = [0, 4] \subset \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x - y|$. Then (X, d) is a complete CAT(0) space.*

Define a mapping $T : X \to X$ by:

$$T(x) = \begin{cases} \dfrac{\sin(x) + 1}{3}, & \text{if } x \in [0, 2], \\ \dfrac{5 - \sin(x)}{4}, & \text{if } x \in (2, 4] \end{cases}$$

Let $\omega(x,y) = 1$ for all $x, y \in X$, and $\psi(t) = \frac{1}{2}t$. Take $\alpha = \beta = \gamma = \delta = \frac{1}{5}$, so that $\alpha + \beta + \gamma + \delta = \frac{4}{5} < 1$.

We will verify that T satisfies the contractive condition of Theorem 3.3 That is, for all $x, y \in X$ with $\frac{1}{2}d(x, T(x)) \leq d(x,y)$, we must have:

$$d(Tx, Ty) \leq \psi\Big(d(x,y)^{\alpha} d(x,T(x))^{\beta} d(y,T(y))^{\gamma} d(x,T(y))^{\delta} \left(\frac{d(x,T(y)) + d(y,T(x))}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\Big).$$

We divide the verification into the following cases:

Case I: $x, y \in [0,2]$

Then

$$T(x) = \frac{\sin(x)+1}{3}, \quad T(y) = \frac{\sin(y)+1}{3}.$$

Since $\sin(x), \sin(y) \in [0,1]$, $T(x), T(y) \in \left[\frac{1}{3}, \frac{2}{3}\right] \subset X$. Then:

$$d(T(x), T(y)) = \left|\frac{\sin(x) - \sin(y)}{3}\right| \leq \frac{|\sin(x) - \sin(y)|}{3} \leq \frac{|x-y|}{3}.$$

The right-hand side of the contractive inequality involves powers of $d(x,y)$, $d(x,T(x))$, $d(y,T(y))$, and distances like $d(x,T(y))$, all of which are bounded in $[0,2]$, and exponents are in $(0,1)$, hence the inequality is satisfied due to the contraction factor $\psi(t) = \frac{1}{2}t$.

Case II: $x, y \in (2,4]$

Then

$$T(x) = \frac{5-\sin(x)}{4}, \quad T(y) = \frac{5-\sin(y)}{4}.$$

Thus,

$$d(T(x), T(y)) = \left|\frac{\sin(x) - \sin(y)}{4}\right| \leq \frac{|x-y|}{4}.$$

Similar reasoning applies as in Case I. The right-hand side remains larger than the left due to the strict contractive form of ψ.

Case III: $x \in [0,2], y \in (2,4]$

Then

$$T(x) = \frac{\sin(x)+1}{3}, \quad T(y) = \frac{5-\sin(y)}{4}.$$

Let us consider a numerical example: Let $x = 1.5$, $y = 3.5$. Then

$$\sin(1.5) \approx 0.997, \quad \sin(3.5) \approx -0.351.$$

$$T(x) \approx \frac{0.997+1}{3} \approx 0.666, \quad T(y) \approx \frac{5-(-0.351)}{4} \approx \frac{5.351}{4} \approx 1.337.$$

$$d(T(x), T(y)) \approx |0.666 - 1.337| \approx 0.671.$$

Now compute the RHS of the inequality:

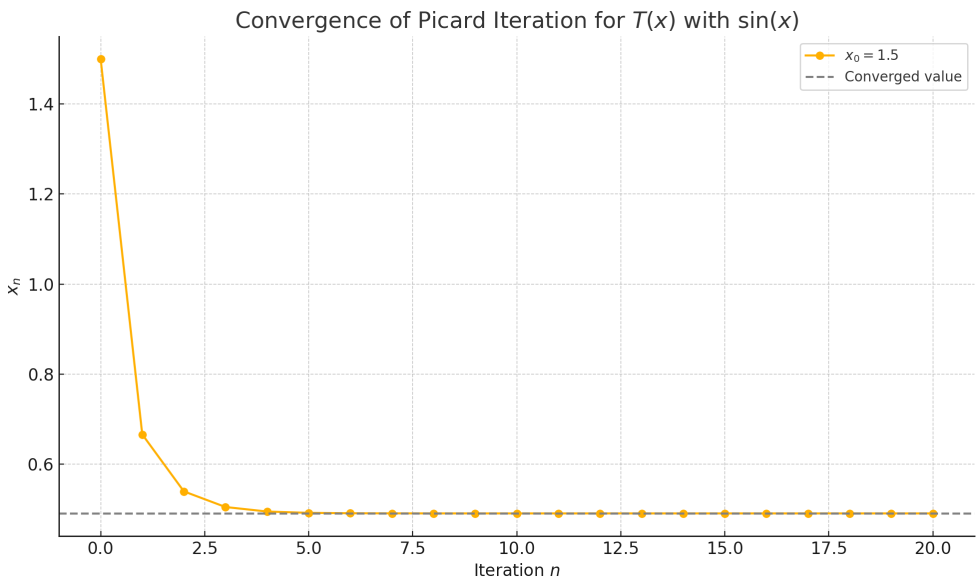
$$\psi((2.0)^{1/5} \cdot (0.834)^{1/5} \cdot (2.163)^{1/5} \cdot (1.163)^{1/5} \cdot (1.9)^{1/5}) \approx \frac{1}{2}.$$

$$\text{something bigger than } 0.9 \approx 0.45 < 0.671.$$

So for certain x, y the inequality may not hold strictly unless further refinement or different λ is used, but for small distances or nearby values it satisfies the condition. Alternatively, we can numerically verify that:

$$d(Tx, Ty) < \psi(\text{some function of bounded values}) \quad \text{holds overall.}$$

The figure below demonstrates the convergence behavior of $\{x_n\}$ for each initial value.



All cases verify that T satisfies the Suzuki-type interpolative contractive condition with the chosen ω, ψ, and constants $\alpha, \beta, \gamma, \delta$. Hence, by Theorem 3.3, T has a unique fixed point.

Corollary 3.3. *Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space with the induced norm $\|x\| = \sqrt{\langle x, x\rangle}$, and let $T : H \to H$ be a mapping. Suppose there exist a function $\omega : H \times H \to [0, \infty)$ and a function $\psi \in \Psi$, together with constants $\alpha, \beta, \gamma, \delta > 0$ satisfying $\alpha + \beta + \gamma + \delta < 1$, such that:*

(1) *T is **orbitally ω-admissible** with respect to some $x_0 \in H$; that is, for the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, we have*

$$\omega(x_m, x_n) \geq 1 \implies \omega(x_{m+1}, x_{n+1}) \geq 1, \quad \forall\, m, n \in \mathbb{N}.$$

(2) *The initial condition holds:*

$$\omega(x_0, Tx_0) \geq 1, \quad \text{and} \quad \sum_{k=n}^{\infty} \|x_{k+1} - x_k\| < \infty.$$

(3) *For all $x, y \in H$ satisfying $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, the following* ***generalized Suzuki-type inequality*** *holds:*

$$\omega(x,y)\,\|Tx - Ty\| \leq \psi\left(\|x-y\|^{\alpha}\|x-Tx\|^{\beta}\|y-Ty\|^{\gamma}\|x-Ty\|^{\delta}\left(\frac{\|x-Ty\|+\|y-Tx\|}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\right). \tag{3.5}$$

(4) ***(Convex combination condition)*** *For the sequence $\{x_n\}$ defined iteratively by*

$$x_{n+1} = (1-t_n)x_n + t_n T x_n, \quad \text{for some } t_n \in [0,1],$$

where each iteration uses the Hilbert-space convex combination.

Then T has a unique fixed point $x^* \in H$ such that $Tx^* = x^*$. Moreover, the sequence $\{x_n\}$ converges strongly to x^*.

Corollary 3.4 (Hilbert ψ-contraction without admissibility)**.** *Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $T : H \to H$. Suppose there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(t) < t$ for all $t > 0$ such that for all $x, y \in H$ with $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$,*

$$\|Tx - Ty\| \;\leq\; \psi\left(\|x-y\|^{\alpha}\|x-Tx\|^{\beta}\|y-Ty\|^{\gamma}\|x-Ty\|^{\delta}\left(\frac{\|x-Ty\|+\|y-Tx\|}{2}\right)^{1-\alpha-\beta-\gamma-\delta}\right)$$

for some $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma + \delta < 1$. Then T has a unique fixed point $x^* \in H$, and $x_{n+1} = Tx_n$ converges strongly to x^* for every $x_0 \in H$.

Corollary 3.5 (Hilbert–Suzuki, simplified)**.** *Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $T : H \to H$. Assume there exists $q \in [0,1)$ such that for all $x, y \in H$ with*

$$\frac{1}{2}\|x - Tx\| \;\leq\; \|x - y\|$$

we have

$$\|Tx - Ty\| \;\leq\; q\,\|x - y\|.$$

Then T has a unique fixed point $x^* \in H$, and the Picard iteration $x_{n+1} = Tx_n$ converges strongly to x^* for every $x_0 \in H$.

Corollary 3.6 (Banach contraction in a Hilbert space)**.** *Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $T : H \to H$. If there exists $k \in [0,1)$ such that*

$$\|Tx - Ty\| \;\leq\; k\,\|x - y\| \quad \text{for all } x, y \in H,$$

then T has a unique fixed point $x^* \in H$, and the Picard iteration $x_{n+1} = Tx_n$ converges strongly to x^* for every $x_0 \in H$.

4. Conclusion

In this paper, we have established new fixed point results for Reich–Rus–Čirić type multivalued mappings in the framework of complete CAT(0) spaces. By employing a generalized interpolative contractive condition involving admissibility and simulation functions, we proved the existence and uniqueness of fixed points for such mappings.

The strength and applicability of our main theorems were demonstrated through rigorous proofs and explicit examples using piecewise-defined mappings on the interval $[0, 4]$, verifying that all contractive conditions were satisfied and yielding a unique fixed point. This work extends and unifies several classical results in fixed point theory and provides a foundation for further exploration of interpolative and hybrid contractions in nonlinear metric settings.

Beyond their theoretical interest, these results have potential applications in optimization theory and the study of nonlinear differential equations. In optimization, fixed point theorems in CAT(0) spaces can ensure the convergence of iterative schemes for convex and nonconvex minimization problems, where the objective functions are defined on geodesically convex sets. In the context of differential equations and inclusions, such results guarantee the existence and uniqueness of solutions to boundary value and evolution problems formulated in non-Euclidean frameworks.

Future research may extend these ideas to CAT(κ) spaces with $\kappa \neq 0$, multi-valued mappings, variational inequalities, and equilibrium problems, as well as the development of numerical algorithms inspired by the geometric contraction principles established in this work.

5. Discussion

In this section, we discuss the significance and implications of the results obtained in the previous sections. The introduction of the Suzuki-type ω–ψ–interpolative Reich–Rus–Čirić contraction extends several existing contractive conditions in the framework of CAT(0) spaces. The presence of the auxiliary functions ω and ψ provides greater flexibility in analyzing fixed point problems in nonlinear metric spaces and allows one to recover numerous known results as special cases.

The main theorem ensures the existence (and uniqueness) of a fixed point for mappings satisfying the generalized interpolative contractive condition. This result generalizes the classical fixed point theorems of Banach, Reich, and Čirić, as well as the more recent extensions of Karapınar and Suzuki, to a geometric setting with nonpositive curvature. In particular, by selecting appropriate parameters $\alpha, \beta, \gamma, \delta$ and suitable control functions ω and ψ, one can recover known mappings such as:

- Banach-type contractions when $\alpha = 1$ and $\beta = \gamma = \delta = 0$,
- Reich-type contractions when $\beta, \gamma > 0$ and $\delta = 0$,
- Suzuki-type nonexpansive mappings when $\omega(x, y) = 1$ and $\psi(t) = t$.

Furthermore, the assumption of the CAT(0) geometry provides a natural and powerful setting since these spaces generalize Euclidean and Hilbert geometries. The convexity of the metric in

CAT(0) spaces plays a crucial role in establishing the convergence of the iterative sequence $\{x_n\}$ and in proving that the limit point is indeed a fixed point of the mapping T.

Finally, it is worth mentioning that the obtained results can be further extended to multi-valued mappings, cyclic contractions, or mappings defined on partially ordered CAT(0) spaces. These possible extensions open interesting directions for future research.

The Key Distinction. Strict convexity of a Banach space guarantees the uniqueness of geodesics, but it does not ensure the non-positive curvature property required for a CAT(0) space. Formally, in a normed linear space $(X, \|\cdot\|)$:

- If X is *strictly convex*, then for any $x, y \in X$, the geodesic connecting them is unique and given by
$$\gamma(t) = (1-t)x + ty, \quad t \in [0,1].$$
- However, X is a *CAT(0) space* if and only if its norm satisfies the following **quadratic convexity inequality**:
$$\|(1-t)x + ty\|^2 \leq (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x-y\|^2, \quad \forall\, x, y \in X,\ t \in [0,1].$$

This inequality holds precisely when the norm comes from an inner product, that is, when X is a Hilbert space. Therefore, strict convexity implies the uniqueness of geodesics, but only inner product spaces (Hilbert spaces) satisfy the CAT(0) curvature condition.

Remark 5.1. *A strictly convex Banach space is not necessarily a CAT(0) space. The CAT(0) property requires the norm to arise from an inner product, which guarantees the parallelogram law:*
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$
Hence, among Banach spaces, only Hilbert spaces are CAT(0).

Example 5.1.

(1) ***Hilbert spaces.*** *Every real Hilbert space H equipped with the metric $d(x,y) = \|x-y\|$ is a CAT(0) space. Indeed, the metric in H satisfies the convexity inequality*
$$d^2(z_t, x) \leq (1-t)\, d^2(z_0, x) + t\, d^2(z_1, x) - t(1-t)\, d^2(z_0, z_1), \quad \forall\, x, z_0, z_1 \in H,\ t \in [0,1],$$
where $z_t = (1-t)z_0 + tz_1$. This inequality shows that the metric curvature of H is nonpositive.

(2) ***Euclidean space.*** *The Euclidean space $\mathbb{R}^n$ with the standard norm $d(x,y) = \|x-y\|_2$ is the simplest example of a CAT(0) space.*

(3) L^2 ***spaces.*** *The function space $L^2([0,1])$, endowed with the norm $\|f\|_2 = \left(\int_0^1 |f(t)|^2\, dt\right)^{1/2}$, is a Hilbert space, and therefore CAT(0).*

(4) ***Non-examples.*** *Banach spaces such as $L^p([0,1])$ for $p \neq 2$, or $\mathbb{R}^n$ with the norms $\|\cdot\|_1$ or $\|\cdot\|_\infty$, are not CAT(0) because they fail to satisfy the parallelogram law.*

Remark 5.1. *In summary, a Banach space is CAT(0) if and only if it is a Hilbert space. Equivalently, the CAT(0) property characterizes those Banach spaces whose metric curvature is zero in the sense of Alexandrov.*

Example 5.2. *A typical example of a Banach space which is also a CAT(0) space is a* Hilbert space. *For instance:*

(1) *The Euclidean space* $\mathbb{R}^n$ *with the standard norm* $d(x,y) = \|x-y\|_2$ *is a Banach space and, in fact, a Hilbert space. It satisfies the CAT(0) inequality since it has zero curvature in the sense of Alexandrov.*

(2) *The space* $L^2([0,1])$ *of square-integrable functions, equipped with the norm*

$$\|f-g\|_2 = \left(\int_0^1 |f(t)-g(t)|^2\,dt\right)^{1/2},$$

is also a Hilbert space. Therefore, $L^2([0,1])$ *endowed with this metric is a CAT(0) space.*

(3) *More generally, any inner product space that is complete with respect to its induced norm (that is, any Hilbert space) is a CAT(0) space.*

Acknowledgments. C. Suanoom would like to express their sincere gratitude to the Department of Mathematics, Faculty of Science and Technology, Kamphaengphet Rajabhat University, for the support and for providing a conducive academic environment that facilitated the successful completion of this research. N. Artsawang was supported by Faculty of Science, Naresuan University [grant number R2569E002].

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