Int. J. Anal. Appl. (2026), 24:15

International Journal of Analysis and Applications

Some Topological Properties in View of Hexa Topological Spaces

Nabeela Abu-Alkishik[1,*], Diana Mahmoud[2], Maryam Alholi[3], Hamza Qoqazeh[4], Sanaa Khataybeh[1]

[1]*Department of Mathematics, Faculty of Science Jerash University, P.O. Box 2600, Jerash 21220, Jordan*
[2]*Amman Arab University,College of Arts and Sciences, Mathematics Department, Amman, Jordan*
[3]*Department of Applied, Taibah University, Saudi Arabia*
[4]*Department of Mathematics, Faculty of Science and Information Technology, Irbid National University, P.O. Box 2600, Irbid 21110, Jordan*

[*]*Corresponding author email:* Nabeelakishik@yahoo.com

Abstract. In this paper, we introduced the definition of hexa topological space, also, we studied Some topological properties of this spaces with more illustrating examples. Many new definitions and theorems were investigated Separation axioms in hexa-topological spaces were debated, and many relations between these spaces and other topological spaces were discussed.

1. Introduction

In 2021, Shihab and Alobaidi [11], a comprehensive concept has been introduced for two new spaces called: P-compactness and P-connectedness are introduced using the concept of P-open sets and their properties. In addition, the relationship between these spaces is studied. This study aims to investigate new types and forms of separation axioms using P-open sets, such as TP_i-spaces where $i = 1, 2, 3$. The relationship between them is discussed, and many of the properties of these spaces are illustrated and discussed with illustrative examples.

In 2022, Khan [3] introduced and limited new types of open and closed sets, called the p-b-open set, the p-b-closed set in topological spaces. They also defined the p-b-open sets and the p-b-closed sets in penta topological spaces and studied and tested some of their properties. The concepts of p-b-continuity and p-b-homeomorphism were introduced, and some related results and facts were proven.

Received: Nov. 8, 2025.

2020 *Mathematics Subject Classification.* 54A99, 54D20, 54E55, 54B10, 54D30.

Key words and phrases. hexa topological spaces; hexa open sets; hexa dense set; H-$\mathcal{T}_i$-spaces ($i = 0, 1, 2$).

https://doi.org/10.28924/2291-8639-24-2026-15

ISSN: 2291-8639

In 2018, M. Khan and G. Khan [4] studied and introduced the concept of pentagonal topological spaces, and explored and studied the basic concepts of penta topological spaces in classical topological spaces by defining new types of open and closed sets within pentagonal topological spaces. They also studied the notions of p-continuity and p-homeomorphism in pentagonal topological spaces.

In 2019, Pacifica and Fathima [6] they studied the concepts of a closed sets in a penta topological spaces. Also, they defined and generalized Some basic properties of a penta topological spaces. Important results in this field are established.

Recently, Kumar, Mary and Radha [5] introduced a clear concept and explores the properties of the penta partitioned Soft neutrosophic topological space (*PPNSTS*), a unique and innovative framework that combines fundamental principles of neutrosophic logic, soft set theory, and topology. *PPNSTS* is designed to model uncertainty, indeterminacy, and vagueness in complex systems with improved accuracy by creating five distinct partitions of the neutrosophic space. The *PPNSTS* construct combines truth membership, indeterminacy membership, and falsity membership functions, divided into five additional subsets, enabling a new, multi-layered approach to handling imprecise and ambiguous data. Key properties, such as open sets, closed sets, adjacency systems, rules, and subspaces, are defined, analyzed, and interpreted within this innovative model. In addition, this study investigates the relationships and mutual connections between the pentagonally partitioned soft neutrosophic topological spaces and the existing soft and neutrosophic topological spaces. The applications of *PPNSTS* in decision-making, artificial intelligence, and data identification and classification are explained, highlighting their significant future utility in solving real-world problems where traditional methods are difficult to use.

In 2019, Fathima [2]. In this study, the concept of a generalized closed set in a pentagonal topological space is defined and discussed. Some basic properties of pentagonal topological spaces are also studied, with new results supported by illustrative examples.

Vani and Arasi, in 2019 [10] introduced the concept of a pentagonal topological space and introduced the basic concepts of pentagonal topological spaces in classical topological spaces. To accomplish this, new types of open and closed sets were introduced, namely the pentagonal quasi-alpha open set in a pentagonal topological space. The idea of pentagonal quasi-alpha continuity in pentagonal topological spaces was also introduced.

In 2023, Santhiya [8] present a new concepts called hexa regular closed (R_h-closed for short), hexa regular open (R_h -open for short), hexa regular semi-closed$(RS)_h$-closed for short), hexa regular semi-open $(RS)_h$-open for short, hexa regular weakly closed $(RW)_h$-closed for short, and hexa regular weakly open $(RW)_h$-open for short) are introduced in the hexa topological spaces, and many results are hold.

In 2020, Chandra and Pushpalatha [1], gave and investigated a concepts of hexa topological spaces. They defined h-closed sets, h-b-closed sets and h-b-$\mathcal{T}$- closed sets in hexa topological space.

In 2021, Shihab, Mousa and Zaben [9] introduced and studied a new types of functions in hexa topological spaces, called (h-continuous, h-open, h-closed, Quasi-h- open, Quasi-h- closed) functions, and investigated the relationships between them. They also introduce a new concept of homeomorphism called hexa_homeomorphism in hexa_topological spaces.

In 2021, Qaddoori [7] The aims of his study focus on new definitions and concepts of $(\alpha_{h^-}, \mathfrak{p}_{h^-}, \mathfrak{b}_{h^-}, \beta_{h^-})$ open sets to generalize the concept and properties of functions in hexa topological spaces, which are called $(\alpha_{h^-}, \mathfrak{p}_{h^-}, \mathfrak{b}_{h^-}, \beta_{h^-})$ identification functions. Many results were discussed, proven and explained using examples.

The earliest beginnings in the study of multi topological spaces go back to the scientist Kelley [12] in 1963 who established the definition and basic concepts of bi topological spaces, along with deriving many results and generalizations.

In 1975, Cooke and Reilly, [13] they were the first who defined and studied compactness in bi topological spaces.

In 2025,(Tapi & Sharma, 2015) [14] study the features of quad-open sets and quad-closed sets and reveal quad-continuous function in quad topological spaces (quad-topological spaces).

(Mukundan, 2013) [15], Introduced a new concept of Quad topological spaces (4-tuple topology) and explain new types of closed sets such as q-closed, q-b- closed and q-bt -closed sets.

In 2014, (Tapi et al., 2014) [16], introduced the separation principles in quad topological spaces (q -topological spaces) and study some of their characteristics. In 2023, (Qoqazeh,et.al) [17], studied properties and discussed new covering in tri-topological spaces as tri-Lindelöf, tri- compact spaces and others.

2. Main Results

Definition 2.1. *Given a non-void set L and* $\mathcal{T}_H = (\mathcal{T}_{L(1)}, \mathcal{T}_{L(2)}, \mathcal{T}_{L(3)}, \mathcal{T}_{L(4)}, \mathcal{T}_{L(5)}, \mathcal{T}_{L(6)})$ *is an ordered hexagon of topologies on L (6-tuples of topologies) on L. Then* $(L, \mathcal{T}_H)$ *is called a hexa topological space.*

Remark 2.1. *A hexa topological space* $(L, \mathcal{T}_H)$ *is denoted by H.T.S.*

Definition 2.2. *Any subset M of L is called a hexa-open set if* $M \in \mathcal{T}_{L(1)} \cup \mathcal{T}_{L(2)} \cup \mathcal{T}_{L(3)} \cup \mathcal{T}_{L(4)} \cup \mathcal{T}_{L(5)} \cup \mathcal{T}_{L(6)}$ *and denoted by (H-open).*

Definition 2.3. *The complement of a hexa-open set in any* $\mathcal{T}_{L(i)}$ $(i = 1, 2, 3, 4, 5, 6)$ *is called a hexa-closed set and denoted by (H-closed).*

Example: Let $L = \mathbb{R}$ and $\mathcal{T}_{L(1)} = \mathcal{T}_{L(\text{ind})}$, $\mathcal{T}_{L(2)} = \mathcal{T}_{L(u)}$, $\mathcal{T}_{L(3)} = \mathcal{T}_{L(\text{left})}$, $\mathcal{T}_{L(4)} = \mathcal{T}_{L(\text{right})}$, $\mathcal{T}_{L(5)} = \mathcal{T}_{L(\text{cof})}$ and $\mathcal{T}_{L(6)} = \mathcal{T}_{L(\text{dis})}$. Then $(L, \mathcal{T}_{L(1)}, \mathcal{T}_{L(2)}, \mathcal{T}_{L(3)}, \mathcal{T}_{L(4)}, \mathcal{T}_{L(5)}, \mathcal{T}_{L(6)})$ is a ($H.T.S$).
$(2, 3)$ is a H-open set, while $(-\infty, 2] \cup [3, \infty)$ is a H-closed set.
$\{5\}$ is a H-open and H-closed set, hence it is a H-clopen set, since $\mathbb{R} - \{5\} = (-\infty, 5) \cup (5, \infty)$ is an open set in $(\mathbb{R}, \mathcal{T}_{L(2)} = \mathcal{T}_{L(u)})$.

Definition 2.4. *Given a non-void set L and* $\mathcal{T}_P = (\mathcal{T}_{P_{(1)}}, \mathcal{T}_{P_{(2)}}, \mathcal{T}_{P_{(3)}}, \mathcal{T}_{P_{(4)}}, \mathcal{T}_{P_{(5)}})$ *is an ordered pentagon of topologies on L (5-tuples of topologies) on L. Then* $(L, \mathcal{T}_P)$ *is called a penta topological space (P.T.S).*

Definition 2.5. *Given a non-void set L and* $\mathcal{T}_Q = (\mathcal{T}_{L(1)}, \mathcal{T}_{L(2)}, \mathcal{T}_{L(3)}, \mathcal{T}_{L(4)})$ *is an ordered quada of topologies on L (4-tuples of topologies) on L. Then* $(L, \mathcal{T}_Q)$ *is called a quada topological space (Q.T.S).*

Remark 2.2. (1) *Any quada topological space* $(L, \mathcal{T}_Q)$ *is a penta topological space* $(L, \mathcal{T}_P)$ *when we repeat only one arbitrary* $\mathcal{T}_{L(i)}$ *where* $i = \{1, 2, 3, 4\}$.

(2) *Any penta topological space* $(L, \mathcal{T}_P)$ *is a hexa topological space* $(L, \mathcal{T}_H)$ *when we repeat only one arbitrary* $\mathcal{T}_{L(i)}$ *where* $i = \{1, 2, 3, 4, 5\}$.

Example 2.1. *Repeat* $\mathcal{T}_{\mathbb{R}_{(u)}}$ *in a penta topological space* $(\mathbb{R}, \mathcal{T}_{\mathbb{R}(ind)}, \mathcal{T}_{\mathbb{R}(left)}, \mathcal{T}_{\mathbb{R}(right)}, \mathcal{T}_{\mathbb{R}(cof)}, \mathcal{T}_{\mathbb{R}(u)})$ *then*

$$(\mathbb{R}, \mathcal{T}_{\mathbb{R}(ind)}, \mathcal{T}_{\mathbb{R}(left)}, \mathcal{T}_{\mathbb{R}(right)}, \mathcal{T}_{\mathbb{R}(cof)}, \mathcal{T}_{\mathbb{R}(\mathbb{R})})$$

is a Hexa topological space.

Definition 2.6. *Let* $(L, \mathcal{T}_H)$ *be a hexa topological space, and* $M \subseteq L$*, then:*

(1) *The H-interior of M is the union of all H-open sets contained in M, and it is denoted by H-Int(M).*
(2) *The H-closure of M is the intersection of all H-closed sets containing M and it is denoted by H-CL(M).*

Example 2.2. *Let* $L = \{1, 2, 3\}$, $\mathcal{T}_{L(1)} = \{L, \phi\}$, $\mathcal{T}_{L(2)} = \{L, \phi, \{1\}\}$, $\mathcal{T}_{L(3)} = \{L, \phi, \{1\}, \{2\}\}$, $\mathcal{T}_{L(4)} = \{L, \phi, \{3\}\}$, $\mathcal{T}_{L(5)} = \{L, \phi, \{1\}, \{1, 2\}\}$, $\mathcal{T}_{L(6)} = \{L, \phi, \{1\}, \{1, 3\}\}$.

The H-open sets are

$$\{L, \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\}$$

The H-closed sets are

$$\{\phi, L, \{2, 3\}, \{1, 3\}, \{1, 2\}, \{3\}, \{2\}\}$$

*Let $M = \{1, 3\}$, then

$$H\text{-Int}(M) = \{1\} \cup \{3\} \cup \{1, 3\} = \{1, 3\}$$

and

$$H\text{-CL}(M) = \{1, 3\} \cap L = \{1, 3\}$$

Note that:

(1) If M is H-open set, then
$$H\text{-Int}(M) = M.$$
(2) If M is H-closed set, then
$$H\text{-CL}(M) = M.$$
(3) In a hexa topological space $(L, \mathcal{T}_H)$, the H-interior of a set needs not be an H-open set as we shown it in the following example: If $N = \{2, 3\}$, then $H\text{-Int}(N) = \{2\} \cup \{3\} \cup \phi = \{2, 3\}$ which is not H-open set.

(4) In a hexa topological space $(L, \mathcal{T}_H)$, then H-closure of a set needs not be an H-closed set as we shown in the following example: If $L = \{1,2,3,4,5,6\}$

$$\mathcal{T}_1 = \{\phi, L\}$$
$$\mathcal{T}_2 = \{\phi, L, \{1,2\}\}$$
$$\mathcal{T}_3 = \{\phi, L, \{1,2,3\}\}$$
$$\mathcal{T}_4 = \{\phi, L, \{1,2,3,4\}\}$$
$$\mathcal{T}_5 = \{\phi, L, \{1,2,3,4,5\}\}$$
$$\mathcal{T}_6 = \{\phi, L, \{2,4,5,6\}\}$$

$(L, \mathcal{T}_H)$ is a hexa topological space.

If $M = \{3\}$, then H-CL$(M) = \{3\}$ i.e. H-CL$(\{3\}) = \{3\}$ which is not H-closed set (since the closed sets are $\{L, \phi, \{3,4,5,6\}, \{4,5,6\}, \{5,6\}, \{6\}, \{1,3\}\}$)

Remark 2.3. *The intersection of any collection of H-closed sets need not be a H-closed set, as shown in the previous example.*

Remark 2.4. *The intersection of any $\mathcal{T}_i$-closed sets is a $\mathcal{T}_i$-closed set.*

Theorem 2.1. *Let $(L, \mathcal{T}_H)$ be a hexa topological space and $A \subseteq L$, then $h \in H$-$Cl(A)$ if and only if for any H-open set U_h, with $h \in U_h$, we have $U_h \cap A \neq \phi$.*

Proof. Let $h \in H$-Cl(A). Suppose that there exists a H-open set U_h such that $h \in U_h$ and $U_h \cap A = \phi$. Now, $A \subseteq L - U_h$.

i.e H-Cl$(A) \subseteq H$-Cl$(L - U_h) = L - U_h$. So $h \in L - U_h$, which is a contradiction.

Remark 2.5. *Let $(H, \mathcal{T}_H) = (H, \mathcal{T}_{L_{(1)}}, \mathcal{T}_{L_{(2)}}, \ldots, \mathcal{T}_{L_{(6)}})$ and $(H, \sigma_H) = (H, \sigma_{L_{(1)}}, \sigma_{L_{(2)}}, \ldots, \sigma_{L_{(6)}})$ be two hexa-topological spaces, then the intersection of $(H, \mathcal{T}_H)$ and (H, σ_H) is defined by $(H, \mathcal{T}_H \cap \sigma_H) = (H, \mathcal{T}_{L_{(1)}} \cap \sigma_{L_{(1)}}, \mathcal{T}_{L_{(2)}} \cap \sigma_{L_{(2)}}, \ldots, \mathcal{T}_{L_{(6)}} \cap \sigma_{L_{(6)}})$*

Example 2.3. *Let $H = \mathbb{R}$*
and $\mathcal{T}_H = (\mathcal{T}_{ind}, \mathcal{T}_{dis}, \mathcal{T}_u, \mathcal{T}_{left}, \mathcal{T}_{right}, \mathcal{T}_u)$
$\sigma_H = (\sigma_{left}, \sigma_u, \sigma_{dis}, \sigma_{right}, \sigma_{left}, \sigma_{cof})$
then $\mathcal{T}_H \cap \sigma_H = (\mathcal{T}_{ind}, \mathcal{T}_u, \mathcal{T}_u, \mathcal{T}_{ind}, \mathcal{T}_{ind}, \mathcal{T}_{cof})$
which is a hexa topological space.
which is exactly a tri-topological space $(\mathcal{T}_{ind}, \mathcal{T}_u, \mathcal{T}_{cof})$.

Remark 2.6. *The union of two Hexa topological spaces need not be a Hexa topological space, as shown in the following next example if we take $H = \mathbb{R}$,*
$\mathcal{T}_H = (\mathcal{T}_u, \mathcal{T}_{dis}, \mathcal{T}_{ind}, \mathcal{T}_{left}, \mathcal{T}_{right}, \mathcal{T}_u)$ and $\mathcal{T}_r = (\mathcal{T}_{dis}, \mathcal{T}_u, \mathcal{T}_{left}, \mathcal{T}_{right}, \mathcal{T}_{left}, \mathcal{T}_{cof})$ then $\mathcal{T}_H \cap \sigma_H$ is not a hexa-topological space since if we take $\mathcal{T}_{left} \cup \mathcal{T}_{right}$ is not a topology.

Definition 2.7. *Let $(L, \mathcal{T}_H)$ be a hexa topological space and $D \subseteq L$, we say that D is an H-dense set if H-Cl$_{(1)}(D) = L$ for all $i = 1,2,3,4,5,6$.*

Example 2.4. *Given $L = \mathbb{R}$ and $\mathcal{T}_H = (\mathcal{T}_{ind}, \mathcal{T}_{left}, \mathcal{T}_{left}, \mathcal{T}_u, \mathcal{T}_{right}, \mathcal{T}_u)$*
Let $D = Q$ (where Q is the set of all rational numbers), then D is an H-dense set since $H\text{-Cl}_{(i)}(D) = \mathbb{R}$ for all $i = 1, 2, ..., 6$.

Definition 2.8. *A hexa-topological space $(L, \mathcal{T}_H)$ is called*

(1) *Hexa-T_0 space if for any two distinct points $h_1, h_2 \in L$, there exists a H-open set contains only one point but not the other and it is denoted by H-T_0 space.*

(2) *A Hexa-T_1 space if for any two distinct points $h_1, h_2 \in L$, there exists two H-open sets U_{h_1}, V_{h_2}, such that $h_1 \in U_{h_1}$, $h_2 \notin U_{h_1}$ and $h_2 \in V_{h_2}$, $h_1 \notin V_{h_2}$ and it is denoted by H - T_2 space.*

(3) *A Hexa-T_2 space if for any two distinct points $h_1, h_2 \in L$, then exists two H-open set say U_{h_1}, V_{h_2} such that $h_1 \in U_{h_1}$, $h_2 \in V_{h_2}$ and $U_{h_1} \cap V_{h_2} = \phi$ and it is denoted by H-T_2 space.*

Example 2.5. *Let $L = \mathbb{R}$. The hexa topological space $(L, \mathcal{T}_H) = (\mathbb{R}, \mathcal{T}_{ind}, \mathcal{T}_u, \mathcal{T}_{dis}, \mathcal{T}_{left}, \mathcal{T}_{right})$ then $(L, \mathcal{T}_H)$ is H-T_0 space, H - T_1 space and H - T_2 space.*

Remark 2.7. *The following implications are hold : H-T_2 $\rightarrow$ H-T_1 $\rightarrow$ H-T_0, but conversely needed not be true always as shown in the following example.*

Example 2.6. *Let $L = \mathbb{R}$.*
Consider $(L, \mathcal{T}_H) = \big(\mathbb{R}, \mathcal{T}_{left}, \mathcal{T}_{right}, \mathcal{T}_{coc}, \mathcal{T}_{cof}, \mathcal{T}_{\{7\}}, \mathcal{T}_{(0,1)}\big)$., then $(L, \mathcal{T}_H)$ is an H-T_0 space, H-T_1 space but not an H-T_2 space, since the topological spaces $\mathcal{T}_{left}, \mathcal{T}_{right}, \mathcal{T}_{coc}, \mathcal{T}_{cof}, \mathcal{T}_{\{7\}}$ and $\mathcal{T}_{(0,1)}$ are not T_2-space on $\mathbb{R}$.

Example 2.7. *Let $L = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{T}_{L(1)} = \{\phi, L, \{1\}\}$, $\mathcal{T}_{L(2)} = \{\phi, L, \{2\}\}$, $\mathcal{T}_{L(3)} = \{\phi, L, \{3\}\}$, $\mathcal{T}_{L(4)} = \{\phi, L, \{4\}\}$, $\mathcal{T}_{L(5)} = \{\phi, L, \{5\}\}$, $\mathcal{T}_{L(6)} = \{\phi, L, \{6\}\}$.*

Then $(L, \mathcal{T}_H)$ is an H-T_0 space, H-T_1 space and H-T_2 space but $\mathcal{T}_{L(i)}$ is not T_0-space, not a T_1-space, and not T_2-space for all $i = 1, 2, 3, 4, 5, 6$.

Example 2.8. *Let $L = \{h_1, h_2, h_3, h_4, h_5\}$. Consider*

$$\mathcal{T}_{L(1)} = \{\phi, \{h_1\}, L\},$$
$$\mathcal{T}_{L(2)} = \{\phi, L, \{h_1, h_2\}, \{h_2\}\},$$
$$\mathcal{T}_{L(3)} = \{\phi, L, \{h_3\}\},$$
$$\mathcal{T}_{L(4)} = \{\phi, L, \{h_4\}\},$$
$$\mathcal{T}_{L(5)} = \{\phi, L, \{h_2, h_4\}\},$$
$$\mathcal{T}_{L(6)} = \{\phi, L, \{h_3\}, \{h_3, h_4\}\}.$$

Hence, $(L, \mathcal{T}_H)$ is an H-T_0 space but neither an H-T_1 space nor an H-T_2 space, since we cannot separate h_1 and h_5.

Example 2.9. *Let $L = \{1, 2, 3, 4, 5, 6\}$. Let $(L, \mathcal{T}_H)$ be a hexa topological space, such that*

$$\mathcal{T}_1 = \{\phi, L, \{1, 2\}, \{1, 2, 3\}\},$$
$$\mathcal{T}_2 = \{\phi, L, \{1, 3\}, \{1, 2, 3\}\},$$

$$\mathcal{T}_3 = \{\phi, L, \{1,4\}, \{1,2,4\}\},$$
$$\mathcal{T}_4 = \{\phi, L, \{1,5\}, \{1,2,5\}\},$$
$$\mathcal{T}_5 = \{\phi, L, \{1,6\}, \{1,2,6\}\},$$
$$\mathcal{T}_6 = \{\phi, L, \{2,3,4,5,6\}\}.$$

Hence $(L, \mathcal{T}_H)$ is an H-T_0 space and H-T_1 space but not H-T_2 space.

Theorem 2.2. *A hexa topological space $(L, \mathcal{T}_H)$ is H-T_0 space if and only if for any two distinct points $h_1, h_2 \in L$, we have $H\text{-Cl}(\{h_1\}) \neq H\text{-Cl}(\{h_2\})$.*

Proof. $(\Longrightarrow)$: Assume $(L, \mathcal{T}_H)$ is an H-T_0 space. Let $h_1, h_2 \in L$ such that $h_1 \neq h_2$, then there exists an H- open set U containing only one of the points h_1, h_2 (say $h_1 \in U$, $h_2 \notin U$). Since U is H-open and $h_1 \in U$ but $\{h_2\} \cap U = \phi$, so, $h_1 \notin H\text{-Cl}(\{h_2\})$ but $h_1 \in H\text{-Cl}(\{h_1\})$, hence $H\text{-Cl}(\{h_1\}) \neq H\text{-Cl}(\{h_2\})$.

$(\Longleftarrow)$: Let $h_1, h_2 \in L$ such that $h_1 \neq h_2$, then by assumption $H\text{-Cl}(\{h_1\}) \neq H\text{-Cl}(\{h_2\})$, there exists $h_3 \in H\text{-Cl}(\{h_1\}) - H\text{-Cl}(\{h_2\})$ or $h_3 \in (H\text{-Cl}(\{h_2\}) - (H\text{-Cl}(\{h_1\}))$. So $h_3 \in H - \text{Cl}(\{h_1\})$ and $h_3 \notin H - \text{Cl}(\{h_2\})$. Then there exists a H-open set U_{h_3} such that $h_3 \in U_{h_3}$ but $U_{h_3} \cap \{h_2\} = \phi$, so $h_2 \notin U_{h_3}$, but $U_{h_3} \cap \{h_1\} \neq \phi$, so $h_1 \in U_{h_3}$, hence L is $H - T_0$-space.

Definition 2.9. *Let $(L, \mathcal{T}_H)$ be a hexa topological space, and $A \subseteq L$. The hexa subspace $(A, \mathcal{T}_{H_A})$ is a hexa topological space on A, where $\mathcal{T}_{H_A} = \{A \cap W,$ for any H-open set $W\}$.*

Example 2.10. *Let $(L, \mathcal{T}_H)$ be a hexa topological space such that $L = \mathbb{R}$, $\mathcal{T}_{H_1} = \{L, \phi, \{1\}\}$, $\mathcal{T}_{H_2} = \{L, \phi, \{2\}\}$, $\mathcal{T}_{H_3} = \{L, \phi, \{3\}\}$, $\mathcal{T}_{H_4} = \{L, \phi, \{4\}\}$, $\mathcal{T}_{H_5} = \{L, \phi, \{5\}\}$, $\mathcal{T}_{H_6} = \{L, \phi, \{6\}\}$. Let $A = \mathbb{N} \subseteq \mathbb{R}$. Then $\mathcal{T}_{H_{\mathbb{N}}} = \{\phi, \mathbb{N}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$ is a hexa subspace topological space on $\mathbb{N}$.*

Theorem 2.3. *Every subspace of H-T_0 space is H-T_0.*

Proof. Let L be a H-T_0 space and $(A, \mathcal{T}_{H_A})$ be a hexa subspace of L. Let $h_1, h_2 \in A$ with $h_1 \neq h_2$, then $h_1, h_2 \in L$. Since L is an H-T_0 space, then there exists an H-open set U containing one of the points but not the other say $h_1 \in U$ and $h_2 \notin U$. Let $U_{h_1} = U \cap A$, then U_{h_1} is an H-open set in A with $h_1 \in U_{h_1}$ and $h_2 \notin U_{h_1}$. So $(A, \mathcal{T}_{H_A})$ is an H-T_0-space.

Definition 2.10.

(1) *Let $(L_1, \mathcal{T}_H)$ and $(L_2, \mathcal{T}_H)$ be two hexa topological spaces. A function $f : (L_1, \mathcal{T}_H) \to (L_2, \mathcal{T}_H)$ is a continuous function if the inverse of any H-open set in $(L_2, \mathcal{T}_H)$ is an H-open set in $(L_1, \mathcal{T}_H)$.*

(2) *A function $f : (L_1, \mathcal{T}_H) \to (L_2, \mathcal{T}_H)$ is open function if the image of every H-open set in $(L_1, \mathcal{T}_H)$ is H-open set in $(L_2, \mathcal{T}_H)$.*

(3) *A function $f : (L_1, \mathcal{T}_H) \to (L_2, \mathcal{T}_H)$ is closed function if the image of every H-closed set in $(L_1, \mathcal{T}_H)$ is H-open set in $(L_2, \mathcal{T}_H)$.*

(4) *A function $f : (L_1, \mathcal{T}_H) \to (L_2, \mathcal{T}_H)$ is H-homeomorphism function if it is bijection, open and continuous function.*

Theorem 2.4. *Being a H-T_0 space is a topological property.*

Proof. Let $f:(L_1,\mathcal{T}_H)\to(L_2,\mathcal{T}_H)$ be an H-homeomorphism function and $(L_1,\mathcal{T}_H)$ is an H-T_0 space, let $l_1,l_2\in L_2$ such that $l_1\neq l_2$, since f is onto, then there exists $k_1,k_2\in L_1$ such that $f(k_1)=l_1$ and $f(k_2)=l_2$ with $k_1\neq k_2$.
Since $(L_1,\mathcal{T}_{H_1})$ is an H-T_0 space, there exists an H-open set U in L_1 containing one of the points k_1, k_2 but not the other, say $k_1\in U$ and $k_2\notin U$. Let $V=f(U)$, then V is a H-open set in $(L_2,\mathcal{T}_H)$, since the function f is H-open function With $l_1=f(k_1)\in V$ but $l_2=f(k_2)\notin V$.

Hence $(L_2,\mathcal{T}_H)$ is an H-T_0-space.

Theorem 2.5. *A hexa-topological space $(L,\mathcal{T}_H)$ is H-T_1 if and only if $\{h\}$ is H-closed for all $h\in L$.*

Proof. Let $(L,\mathcal{T}_H)$ be an H-T_1 space. To show that $\{h\}$ is H-closed for any $h\in L$, we want to show $L-\{h\}$ is H-open. Let $l\in L-\{h\}$, then $l\neq h$, but L is an H-T_1 space so there exist two H-open sets U and V such that $l\in U$, $h\notin U$ and $h\in V$, $l\notin V$, we get $l\in U\subseteq L-\{h\}$. So, $L-\{h\}$ is an H-open set, i.e., $\{h\}$ is H-closed set for any h in L.

Conversely, Suppose that $h\notin H-\mathrm{Cl}(A)$, so $h\in L-H-\mathrm{Cl}(A)$, but $L-H-\mathrm{Cl}(A)$ is H-open set contains h, we have $[L-H-\mathrm{Cl}(A)]\cap A\neq\phi$ which is a contradiction since $L-H-\mathrm{CL}(A)\subseteq L-A$.

Theorem 2.6. *Every subset of H-T_1 space is H-T_1 space.*

Proof. It is easy to prove it.

Theorem 2.7. *Being a T_1-space is a topological property.*

Proof. It is easy to prove it. □

Definition 2.11. *Let $(L_1,\mathcal{T}_H)=(L_1,\mathcal{T}_{H_1},\mathcal{T}_{H_2},\mathcal{T}_{H_3},\mathcal{T}_{H_4},\mathcal{T}_{H_5},\mathcal{T}_{H_6})$ and $(L_2,\supset_H)=(L_2,\supset_{H_1},\supset_{H_2},\supset_{H_3},\supset_{H_4},\supset_{H_5},\supset_{H_6})$ be two hexa topological spaces, then*
$\mathcal{T}_H\subseteq\supset_H$ *if* $\mathcal{T}_{H_i}\subseteq\supset_{H_i}$ *for all $i=1,2,3,4,5,6$.*

Definition 2.12. *A hexa topological space $(L,\mathcal{T}_{H_1},\mathcal{T}_{H_2},\dots,\mathcal{T}_{H_6})$ is called strongly H-T_0 (respectively, H-T_1 and H-T_2) if $\mathcal{T}_{H_i}$ is a T_0 (respectively T_1 and T_2) space, for all $i=1,2,\dots,6$.*

Theorem 2.8. *Let L be any non-empty set, then $\mathcal{T}_{\mathrm{cof}}=(\mathcal{T}_{\mathrm{cof}},\mathcal{T}_{\mathrm{cof}},\mathcal{T}_{\mathrm{cof}},\mathcal{T}_{\mathrm{cof}},\mathcal{T}_{\mathrm{cof}},\mathcal{T}_{\mathrm{cof}})$ is the smallest topology on L that makes L a strongly H-T_1 space.*

Proof. We know that $(L,\mathcal{T}_{\mathrm{cof}})$ is a strongly H-T_1 space, since $\{h\}$ is a H_i-closed set in $\mathcal{T}_{\mathrm{H_i}}$ for $(i=1,2,\dots,6)$ for all h in L. Let $\mathcal{T}_H=(\mathcal{T}_{H_1},\dots,\mathcal{T}_{H_6})$ be a hexa topological space that makes L to be a strongly H-T_1 space, want $\mathcal{T}_{\mathrm{cof}}\subseteq\mathcal{T}_H$, let $U\in\mathcal{T}_{\mathrm{cof}}$, then $L-U$ is a finite set, say $L-U=\{h_1,h_2,\dots,h_6\}=\bigcup_{i=1}^{6}\{h_i\}$ but $\{h_i\}$ is H-closed set in each $(L,\mathcal{T}_{H_i})$ for all $i=1,2,\dots,6$. Hence $\bigcup_{i=1}^{k}\{h_i\}$ is a H-closed set, since each $(L,\mathcal{T}_{H_i})$ is a T_1-space.

So, $\bigcup_{i=1}^{n}\{h_i\}$ is a closed set in $\mathcal{T}_{H_i}$, $\forall i=1,\dots,6$, so $\bigcup_{i=1}^{n}\{h_i\}$ is H-closed. So $U=L-\bigcup_{i=1}^{6}\{h_i\}$ is $\mathcal{T}_{H_i}$-open for all $i=1,\dots,6$, so $\mathcal{T}_{\mathrm{cof}}\subseteq\mathcal{T}_{H_i}$ for all $i=1,\dots,6$.

Theorem 2.9. *Every subspace of $H-T_2$ space is $H-T_2$.*

Proof. It is easy to prove. □

Theorem 2.10. *Being a $H-T_2$ space is a topological property.*

Proof. It is easy to prove. □

Theorem 2.11. *A hexa topological space $(L,\mathcal{T}_H)$ is a strongly $H-T_2$ space if and only if for any two distinct points $h_1,h_2 \in L$, there exists a $\mathcal{T}_{H_i}$-open set U_i such that $h_1 \in U_i$, $h_2 \notin \mathcal{T}_{H_i}-\mathrm{Cl}(U_i)$ for $i=1,2,\ldots,6$.*

Proof. Let $(L,\mathcal{T}_H)$ be a strongly $H-T_2$ space. Let $h_1,h_2 \in L$ such that $h_1 \neq h_2$, then for each $i=1,2,\ldots,6$, there exist $U_i, V_i \in \mathcal{T}_{H_i}$ such that $h_1 \in U_i$, $h_2 \in V_i$ and $U_i \cap V_i = \phi$, so, $h_2 \in V_i$, $V_i \cap U_i = \phi$, so $h_2 \notin \mathrm{Cl}(U_i)$ in $\mathcal{T}_{H_i}$.
Conversely, Suppose that there exists $\mathcal{T}_{H_i}$-open set U_i, suppose for any two distinct points $h_1,h_2 \in L$, there exists $\mathcal{T}_{H_i}$-open set U_i such that $h_1 \in U_i$, $h_2 \notin \mathcal{T}_{H_i}-\mathrm{Cl}(U_i)$ for all $i=1,2,\ldots,6$, then there exists a $\mathcal{T}_{H_i}$-open set V_i such that $h_2 \in V_i$, $V_i \cap U_i = \phi$.

Theorem 2.12. *A hexa topological space $(L,\mathcal{T}_H)$ is a strongly T_2-space if and only if $\{h\} = \bigcap\{\mathrm{Cl}(V_k) : V_k$ is $\mathcal{T}_{H_i}$-open and $h \in V_k$ for all $i=1,2,\ldots,6\}$.*

Proof. Let $(L,\mathcal{T}_H)$ be a strongly $H-T_2$ space, then $(L,\mathcal{T}_{H_i})$ is a T_2-space for all $i=1,2,\ldots,6$.
Let $j \in \bigcap\{\mathrm{Cl}(V_k) : V_k$ is $\mathcal{T}_{H_i}$-open set and $h \in V_k\}$ and $j \neq h$ since each $\mathcal{T}_{H_i}$ $(i=1,2,\ldots,6)$ is T_2, then there exists a $\mathcal{T}_{H_i}$-open sets U_k, W_k such that $h \in U_k$, $j \in W_k$ and $U_k \cap W_k = \phi$.
So, $j \notin \mathcal{T}_{H_i}-\mathrm{Cl}(U_k)$, $j \notin \bigcap\{\mathrm{Cl}(V_k) : V_k$ is $\mathcal{T}_{H_i}$-open and $h \in V_k\}$.
Conversely: easy to prove.

3. Conclusions and Further investigations

3.1. Results and conclusions.

1 The topological properties of sets in hexa topological spaces were defined as H- closure and H- interior with examples illustrating those properties.
2 A definition was established of separation axioms in hexa topological spaces, along with an explanation of the relationship between those axioms.
3 A new definitions of mappings in hexa topological spaces were established.
4 Some results and concepts were genderized in hexa topological spaces.

3.2. **Further investigations.** Based entirely on the results of this study in hexa spaces and the results and generalizations reached, this can be applied and employed in the study of several topological properties such as the most common covering properties as hexa compact and hexa Lindelöf spaces. In addition, it is possible to study these properties in soft and neutrosophic tectonic spaces.

Conflict of Interest

The authors declare that they have no conflict of interest regarding the publication of this paper. None of the authors have any financial or personal relationships that could inappropriately influence or bias the content of this paper. All authors have contributed to the work independently, and there are no external factors that could compromise the scientific integrity of the results presented herein.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R. Chandra, V. Pushpalatha, Introduction to Hexa Topologicl Spaces(6–Tuple Topology), Int. J. Manag. Technol. Eng. 10 (2020), 16–18.

[2] G. Pacifica, S. Fathima, Some Topological Concepts in Penta Topological Space, Int. J. Math. Trends Technol. 65 (2019), 109–116. https://doi.org/10.14445/22315373/ijmtt-v65i2p518.

[3] M. Khan, Certain Type of Continuity in Penta Topological Spaces, The Sciencetech, 3 (2022), 1–43.

[4] M. Khan, G. Khan, p-Continuity and p-Homeomorphism in Penta Topological Spaces, Eur. Int. J. Sci. Technol. 7 (2018), 1–8.

[5] S. Kumar, A. Mary, R. Radha, Penta Partitioned Neutrosophic Soft Topological Space, in: Fuzzy, Intuitionistic and Neutrosophic Set Theories and their Applications in Decision Analysis, pp. 49–59, 2025.

[6] G. Pacifica, S. Fathima, Some Topological Concepts in Penta Topological Space, Int. J. Math. Trends Technol. 65 (2019), 109–116. https://doi.org/10.14445/22315373/ijmtt-v65i2p518.

[7] A.S. Qaddoori, Identification Functions in Hexa Toplogical Spaces, Tikrit J. Pure Sci. 26 (2021), 95–99. https://doi.org/10.25130/tjps.v26i6.198.

[8] R.M. Priya, S. Santhiya, A New Notion of Hexa Regular Closed Sets via HexaTopological Space, Preprint, 2023. https://doi.org/10.21203/rs.3.rs-3181128/v1.

[9] A. Shihab, H. Mousa, R. Zaben, Hexa_ Homeomorphisms in Hexa Topological Spaces, Adv. Mech. 9 (2021), 1329–1334.

[10] S. Vani, K. Arasi, New Notion of Open Sets in Penta Topological Spaces, Enrich, 13 (2021), 10–15.

[11] R. Yaseen, A. Shihab, M. Alobaidi, Characteristics of Penta-Open Sets in Penta Topological Spaces, Int. J. Nonlinear Anal. Appl. 12 (2021), 2463–2475. https://doi.org/10.22075/ijnaa.2021.5388.

[12] A. Mohammed, M. Omar, On Bi-Topological Space, Sebha Univ. Conf. Proc. 3 (2024), 35–37.

[13] I.E. Cooke, I.L. Reilly, On Bitopological Compactness, J. Lond. Math. Soc. s2-9 (1975), 518–522. https://doi.org/10.1112/jlms/s2-9.4.518.

[14] U.D. Tapi, R. Sharma, q-Continuous Functions in Quad Topological Spaces, Ann. Pure Appl. Math. 10 (2015), 117–122.

[15] D.V. Mukundan, Introduction to Quad Topological Spaces, Int. J. Sci. Eng. Res. 4 (2013), 2483–2485.

[16] U.D. Tapi, R. Sharma, B.A. Deole, Separation Axioms Inquad Topological Spaces, Bull. Math. Sci. Appl. 10 (2014), 16–18.

[17] H. Qoqazeh, A.A. Atoom, A. Jaradat, E. Almuhur, N. Abu-Alkishik, A New Study on Tri-Lindelöf Spaces, WSEAS Trans. Math. 22 (2023), 915–924. https://doi.org/10.37394/23206.2023.22.100.