Int. J. Anal. Appl. (2026), 24:25

International Journal of Analysis and Applications

On the Solvability of Quadratic Multi-Term Hybrid Equation with Nonlocal Hybrid Condition in Banach Algebra Spaces

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Abstract. This paper investigates the existence of solutions for a class of multi-term hybrid functional equations subject to nonlocal and fractional conditions. We first establish sufficient conditions to ensure the existence of at least one continuous solution by applying Dhage's fixed-point theorem within an appropriate Banach algebra framework. Subsequently, we extended the analysis to integrable solutions in the Lebesgue space $L^1(J, \mathbb{R})$ under Carathéodory-type growth conditions. The uniqueness of solutions is then addressed by imposing Lipschitz-type constraints on nonlinear and hybrid terms. Furthermore, we examine the continuous dependence of solutions on initial data and parameters. Several illustrative examples are presented to demonstrate the applicability and validity of the obtained results. The theoretical framework developed here unifies and generalizes various existing results for hybrid and nonlocal fractional differential equations.

1. Introduction

The theory of multi-term and hybrid functional equations has seen substantial development in recent years, motivated by their effectiveness in modeling phenomena exhibiting memory effects, delay responses, and nonlocal interactions. Such equations naturally arise in fields ranging from biology and physics to engineering, where the system's present dynamics are intrinsically influenced by its past and nonlocal properties.

Infinite systems and multi-term functional equations—studied in depth by Rzepka & Sadarangani [1], Banaś & Chlebowicz [2], and Banaś et al. [3]—have proven versatile in modeling dynamics where systems evolve based on both current and historic states. These works

Received: Nov. 3, 2025.

2020 *Mathematics Subject Classification.* 34A08, 34K37, 34A12, 45G10, 47H10.

Key words and phrases. multi-term hybrid functional equations; nonlocal conditions; fractional derivatives; integrable solutions; Dhage's fixed-point theorem; uniqueness; continuous dependence.

https://doi.org/10.28924/2291-8639-24-2026-25
ISSN: 2291-8639

underscore the necessity of general frameworks to capture delayed or distributed interactions within evolving systems.

Simultaneously, nonlocal boundary conditions have garnered scholarly attention due to their practical relevance: unlike classical local constraints, nonlocal conditions integrate the system's behavior over an interval, thereby reflecting real-world boundary dependencies. Noteworthy contributions in this area include Ahmad et al. [4] and Alsaedi et al. [5], who examined fractional integro-differential systems with mixed right–left fractional derivatives under nonlocal regimes. In particular, El-Sayed et al. [6] studied pantograph-type fractional problems with nonlocal and fractal-fractional feedback control, establishing existence and uniqueness of solutions under such complex constraints.

Foundational works by Podlubny [7] and Kilbas et al. [8] laid the analytical groundwork by systematically developing fractional calculus—with an emphasis on Caputo derivatives—and their role in boundary and nonlocal problems. These constructions have enabled the theoretical handling of systems whose evolution depends on their entire history.

Hybrid functional-differential equations further enlarge modeling capabilities by combining integral, delay, and functional terms. These models, illustrated in broad settings such as biological systems, control theory, and physics, capture phenomena with both instantaneous and delayed responses. Karimov et al. [9] explored hybrid fractional pantograph equations using noncompactness measures, offering a rigorous approach to hybrid systems analysis.

The solvability of such hybrid systems under nonlocal boundary conditions has been primarily addressed using fixed-point methods. Classical fixed-point theorems—such as Banach's contraction principle and Schauder's theorem—have been widely applied. For example, Diethelm [10] employed Schauder–Banach techniques to establish existence results in fractional boundary value problems, while Balachandran et al. [11] and Abdo et al. [12] extended these methods to analyze pantograph fractional equations with various fractional operators, including the Atangana–Baleanu–Caputo derivative.

More recently, Samadi et al. [13] developed new stability and existence protocols for hybrid models using Darbo-type and measure-of-noncompactness methods. In parallel, Dhage and Lakshmikantham [14] provided fundamental results on hybrid differential equations via fixed-point strategies in Banach algebras. Mahmudov and Matar [15], Sitho et al. [16], and Bashir et al. [17] expanded these frameworks to treat hybrid fractional systems with arbitrary order and nonlocal hybrid conditions.

Notable applications of these theories include the hybrid Urysohn–Stieltjes integral inclusions addressed by El-Sayed et al. [18–20], leveraging fractal differential modeling. Likewise, El-Sayed et al. [21] introduced a fractional hybrid formulation with embedded feedback control under nonlocal boundary conditions, deriving robust existence and stability results.

Building on the foundational and contemporary literature—especially the hybrid fractional and nonlocal models referenced above—our paper investigates the solvability of a quadratic multi-term hybrid functional equation under nonlocal hybrid boundary conditions. Central to our approach is the use of Dhage's fixed-point theorem in Banach algebras [14], which offers a powerful tool for establishing the existence of solutions in settings where classical contraction-based methods may fail.

Concretely, we consider a class of quadratic multi-term hybrid functional equations

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{\nu(t)-\sum_{i=1}^{k}\omega_{1i}(t,\nu(t))}{\sum_{i=1}^{k}\omega_{2i}(t,\nu(t))}\Big)=g_1\Big(t,\frac{d}{dt}\Big(\frac{\nu(\varphi(t))-\sum_{i=1}^{k}\omega_{1i}(t,\nu(\varphi(t)))}{\sum_{i=1}^{k}\omega_{2i}(t,\nu(\varphi(t)))}\Big)\Big) \qquad (1.1)\\
&g_2\Big(t,\frac{d}{dt}\Big(\frac{\nu(\phi_1(t))-\sum_{i=1}^{k}\omega_{1i}(t,\nu(\phi_1(t)))}{\sum_{i=1}^{k}\omega_{2i}(t,\nu(\phi_1(t)))}\Big)\int_0^{\psi_1(t)}h_1\Big(t,s,D^{\alpha_1}\Big(\frac{\nu(s)-\sum_{i=1}^{k}\omega_{1i}(s,\nu(s))}{\sum_{i=1}^{k}\omega_{2i}(s,\nu(s))}\Big)\Big)\,ds,....,\\
&....,\frac{d}{dt}\Big(\frac{\nu(\phi_k(t))-\sum_{i=1}^{k}\omega_{1i}(t,\nu(\phi_k(t)))}{\sum_{i=1}^{k}\omega_{2i}(t,\nu(\phi_k(t)))}\Big)\int_0^{\psi_k(t)}h_k\Big(t,s,D^{\alpha_k}\Big(\frac{\nu(s)-\sum_{i=1}^{k}\omega_{1i}(s,\nu(s))}{\sum_{i=1}^{k}\omega_{2i}(s,\nu(s))}\Big)\Big)\,ds\Big),
\end{aligned}$$

restricted to nonlocal hybrid initial constraints

$$\begin{aligned}
&\frac{\nu(t)-\sum_{i=1}^{k}\omega_{1i}(t,\nu(t))}{\sum_{i=1}^{k}\omega_{2i}(t,\nu(t))}\Big|_{t=0} \qquad (1.2)\\
&-\int_0^{\tau}\theta\Big(s,\frac{\nu(s)-\sum_{i=1}^{k}\omega_{1i}(s,\nu(s))}{\sum_{i=1}^{k}\omega_{2i}(s,\nu(s))},D^{\beta}\Big(\frac{\nu(s)-\sum_{i=1}^{k}\omega_{1i}(s,\nu(s))}{\sum_{i=1}^{k}\omega_{2i}(s,\nu(s))}\Big)\Big)ds=x_0,
\end{aligned}$$

For $t,\tau\in J=[0,1]$, where D^{δ} is the Caputo fractional derivative of order $\delta\in\{\alpha_i,\beta\}$ and $0<\alpha_i,\beta\leq 1$, With $g_1,\ \omega_{1i}\in C(J\times\mathbb{R},\mathbb{R})$ and $\omega_{2i}\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$ are given continuous functions. Furthermore, the functions $g_2\in C(J\times\mathbb{R}^k)$, and $h_i\in C(J^2\times\mathbb{R},\mathbb{R})$ for $i=1,2,\dots,k$, are Carathéodory function, measurable in $t,s\in J$ for all $\nu\in\mathbb{R}$ and continuous in $\nu\in\mathbb{R}$ for all $t,s\in J$. We establish the existence of continuous solutions by framing our problem as an operator equation $Ax\,Bx=x$ in a Banach space of continuous functions, verifying Dhage's conditions (A as a $\mathcal{D}$-Lipschitz map, B completely continuous, etc.), and applying his fixed-point theorem.

Organization of the Paper, Sec.2: Formulation of the multi-term hybrid problem, assumptions, and operator setup. We then state and prove the main existence result via Dhage's theorem. Sec.3: We study the continuous dependence of solutions on initial data and parameters Sec.4: A concrete illustrative example demonstrating the theory in a specific hybrid fractional-delay setting. Sec.5: Summary of main findings, potential extensions, and future research directions.

Through this work, we aim to enrich the mathematical toolbox available for hybrid fractional systems and demonstrate how Dhage's fixed-point theorem elegantly addresses existence issues in complex nonlocal hybrid frameworks.

2. Existence Result

We prove the existence of a solution for our problem via a fixed point theorem in a Banach algebra due to Dhage [32].

Theorem 2.1. *[32]Let S be a non-empty, closed, convex and bounded subset of the Banach algebra X and let $\mathcal{A}:X\to X$ and $\mathcal{B}:S\to X$ be two operators such that*

(a) *$\mathcal{A}$ is $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function ψ,*

(b) *$\mathcal{B}$ is completely continuous,*

(c) *$y = Ay\,Bz \Longrightarrow y \in S$ for all $z \in S$, and*

(d) *$M\,\psi(r) < r$, where $M = \|B(S)\| = \sup\{\|By\| : y \in S\}$.*

Then the operator equation $Ay\,By = y$ has a solution in S.

2.1. **Quadratic multi-term functional equation with multi-delays.** Let

$$x(t) \;=\; \frac{\nu(t) - \sum_{i=1}^{k} \omega_{1i}(t, \nu(t))}{\sum_{i=1}^{k} \omega_{2i}(t, \nu(t))}, \tag{2.1}$$

then we can deduce that any solution of (1.1) is given by

$$\nu(t) = \sum_{i=1}^{k} \omega_{1i}(t, \nu(t)) + x(t) \sum_{i=1}^{k} \omega_{2i}(t, \nu(t)),\ t \in J \tag{2.2}$$

where x is a solution to the following quadratic multi-term equation

$$\begin{aligned} &\frac{dx(t)}{dt} = g_1(t, x(\varphi(t))) \\ &g_2\Big(t, \frac{dx(\phi_1(t))}{dt} \int_0^{\psi_1(t)} h_1(t, s, D^{\alpha_1}(x(s)))\, ds, ..., \frac{dx(\phi_k(t))}{dt}) \int_0^{\psi_k(t)} h_k(t, s, D^{\alpha_k}(x(s)))\, ds\Big), \end{aligned} \tag{2.3}$$

with the nonlocal hybrid condition

$$x(0) - \int_0^{\tau} \theta(s, x(t), D^{\beta} x(s))\, ds = x_0,\ t \in J,\ \beta \in (0, 1] \tag{2.4}$$

Consider the quadratic multi-term functional equations (2.3) and (2.4), subject to the following assumptions:

(i) Let $\varphi,\ \phi_i,\ \psi_i : [0,1] \to [0,1]$, for $i = 1, 2, \ldots, k$, be continuous functions such that

$$\varphi(t),\ \phi_i(t),\ \psi_i(t) \le t, \quad \text{for all } t \in [0,1].$$

(ii) Let $g_2 : J \times \mathbb{R}^k \to \mathbb{R}$ be a continuous function. Suppose there exists a positive constant $l > 0$ such that

$$\Big|g_2(t, x_1, x_2, \ldots, x_k) - g_2(t, y_1, y_2, \ldots, y_k)\Big| \le l \sum_{i=1}^{k} |x_i - y_i|,$$

for each $t \in J$ and for all $x_i, y_i \in \mathbb{R}$. Moreover, the function $t \mapsto g_2(t, 0, 0, \ldots, 0)$ belongs to the space $C(\mathbb{R})$, and we have

$$|g_2(t, x_1, x_2, \ldots, x_k)| \le l \sum_{i=1}^{k} |x_i| + G_2,\ \text{where } G_2 = \sup_{t \in J} \Big|g_2(t, 0, 0, \ldots, 0)\Big|\}.$$

(iii) Let $g_1 : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose there exists a positive constant $\eta > 0$ such that

$$|g_1(t, x) - g_1(t, y)| \le \eta |x - y|,$$

for each $t \in J$ and for all $x, y \in \mathbb{R}$. Moreover, the function $t \mapsto g_1(t,0)$ belongs to the space $C(J,\mathbb{R})$, and we have

$$|g_1(t,x)| \leq \eta|x| + G_1, , \text{ where } G_1 = \sup_{t\in J} \left|g_1(t,0)\right|\}.$$

(iv) Let $h_i : J^2 \times \mathbb{R} \to \mathbb{R}$, for $i = 1, 2, \ldots, k$, be continuous functions for all $t, s \in J$ and $x \in \mathbb{R}$. Suppose there exist measurable and bounded functions $a_i, b_i : J \times J \to \mathbb{R}$ such that

$$|h_i(t,s,x) - h_i(t,s,y)| \leq b_i(t,s)\,|x-y|, \quad t,s \in J,\ x,y \in \mathbb{R},$$

and

$$|h_i(t,s,x)| \leq a_i(t,s) + b_i(t,s)\,|x|, \quad t,s \in J,\ x \in \mathbb{R}.$$

Moreover, define

$$a = \sup\{\, a_i(t,s) :\ t,s \in J\}, \qquad b = \sup\{\, b_i(t,s) :\ t,s \in J\}.$$

(v) Let $\theta : J \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function in $t \in J$ and in $(x,y) \in \mathbb{R}^2$. Suppose there exist measurable and bounded functions $n, m : J \to \mathbb{R}$ such that

$$|\theta(t_1,x_1,y_1) - \theta(t_2,x_2,y_2)| \leq m(t)\Big(|x_1-x_2| + |y_1-y_2|\Big), \quad t \in J,$$

and

$$|\theta(t,x,y)| \leq |n(t)| + m(t)\Big(|x| + |y|\Big), \qquad n(t) = \theta(t,0,0), \quad t \in J.$$

Now we establish the following lemma.

Lemma 2.1. *Suppose that the system defined by equations* (2.3) *and* (2.4) *admits a solution. Then this solution also satisfies the equivalent fractional integral form*

$$x(t) = x_0 + \int_0^{\tau} \theta\Big(s, x(s), I^{1-\beta}y(s)\Big)\,ds + \int_0^t y(s)\,ds, \tag{2.5}$$

where the function $y(t)$ satisfies the nonlinear integral equation

$$y(t) = g_1(t, y(\varphi(t))) \times \tag{2.6}$$
$$g_2\Bigg(t,\ y(\phi_1(t)) \int_0^{\psi_1(t)} h_1\Big(t,s,I^{1-\alpha_1}(y(s))\Big)\,ds,\ \ldots,\ y(\phi_k(t)) \int_0^{\psi_k(t)} h_k\Big(t,s,I^{1-\alpha_k}(y(s))\Big)\,ds\Bigg), \quad t \in J.$$

Hence, the original system (2.3)–(2.4) *is equivalent to the system given by* (2.5) *and* (2.6).

Proof. Let x be a solution of the system defined by equations (2.3) and (2.4), and denote $y(t) = \dfrac{dx}{dt}$. Then, by integrating both sides with respect to t, we obtain the classical integral form:

$$x(t) = x(0) + \int_0^t y(s)\,ds. \tag{2.7}$$

which implies that the function x must satisfy the fractional-order integral equation:

$$x(t) = x_0 + \int_0^1 \theta\Big(s, x(s), I^{1-\beta}y(s)\Big)\,ds + \int_0^t y(s)ds.$$

Now, applying the definition of the Riemann–Liouville fractional derivative $D^{\alpha}x(t) = I^{1-\alpha}\dfrac{dx}{dt}$ to the equation (2.3), and replacing $\dfrac{dx}{dt}$ by $y(t)$, we arrive at:

$$\frac{dx(t)}{dt} = g_1\left(t, \frac{dx(\varphi(t))}{dt}\right) \times g_2\left(t, \frac{dx(\phi_1(t))}{dt} \int_0^{\psi_1(t)} h_1\left(t, s, I^{1-\alpha_1}\frac{dx(s)}{ds}\right) ds, \dots, \right.$$
$$\left. \frac{dx(\phi_k(t))}{dt} \int_0^{\psi_k(t)} h_k\left(t, s, I^{1-\alpha_k}\frac{dx(s)}{ds}\right) ds\right).$$

Substituting $\dfrac{dx(s)}{ds} = y(s)$, we rewrite the above as:

$$y(t) = g_1\left(t, y(\varphi(t))\right) \times g_2\left(t, y(\phi_1(t)) \int_0^{\psi_1(t)} h_1\left(t, s, I^{1-\alpha_1}y(s)\right) ds, \dots, \right.$$
$$\left. y(\phi_k(t)) \int_0^{\psi_k(t)} h_k\left(t, s, I^{1-\alpha_k}y(s)\right) ds\right).$$

This demonstrates that y satisfies the nonlinear integral equation (2.6), and that x satisfies the integral formulation (2.5). Hence, the original problem defined by equations (2.3)–(2.4) is equivalent to the system consisting of (2.5) and (2.6).

□

Theorem 2.2. *Let assumption (i)-(v) be satisfied. then the multidimensional functional equation (2.6) has at least one solution $y = y(t)$ which belongs to the space in the space $C(J)$.*

Proof. Define the set S as follows:

$$S = \{y \in X \mid \|y\| \le r\},$$

where r is a root to the following cubic formula:

$$\eta lkb\, r^3 + \left(G_1 lkb + \eta lka\right)r^2 + \left(G_1 lka + \eta G_2 - 1\right)r + G_1 G_2 = 0.$$

The set S is nonempty, closed, bounded, convex, and a subset of the Banach algebra X

Define operators $A : X \to X$ and $\mathcal{B} : X \to S$ as follows:

For $y \in X$, define

$$A(y)(t) = g_1(t, y(\varphi(t))).$$

Since g_1 is Lipschitz in the second argument with constant η, and $\varphi(t) \le t$, we have:

$$|A(x)(t) - A(y)(t)| \le \eta|x(\varphi(t)) - y(\varphi(t))| \le \eta\|x - y\|.$$

So A is Lipschitz with constant η.

Now define the operator $\mathcal{B} : X \to S$ by

$$\mathcal{B}(y)(t) = g_2\left(t,\ y(\phi_1(t)) \int_0^{\psi_1(t)} h_1(t, s, I^{1-\alpha_1}(y(s)))\, ds,\ \dots,\ y(\phi_k(t)) \int_0^{\psi_k(t)} h_k(t, s, I^{1-\alpha_k}(y(s)))\, ds\right).$$

Now, we aim to prove that the operator $\mathcal{B}$ is completely continuous. To this end, we divide the proof into two steps: first, we show that $\mathcal{B}$ is continuous on the space S; then, we demonstrate that it is compact.

Step 1: Continuity of $\mathcal{B}$.

Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in S such that

$$\lim_{n\to\infty} \|y_n - y\| = 0, \quad \text{for some } y \in S.$$

We will show that $\mathcal{B}(y_n) \to \mathcal{B}(y)$ in norm in X. Fix $t \in J$. By the definition of $\mathcal{B}$, we have:

$$\mathcal{B}(y_n)(t) = g_2\left(t, y_n(\phi_1(t)) \int_0^{\psi_1(t)} h_1(t,s,I^{1-\alpha_1}(y(s)))\, ds, \dots, y_n(\phi_k(t)) \int_0^{\psi_k(t)} h_k(t,s,I^{1-\alpha_k}(y(s)))\, ds\right),$$

Let us denote

$$H_i(t) = \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i}(y(s)))\, ds, \quad \text{for } i = 1, \dots, k.$$

Since $y_n \to y$ uniformly, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$|y_n(\phi_i(t)) - y(\phi_i(t))| < \frac{\epsilon}{2M}, \quad \text{where } H := \max_{t\in J, i} |H_i(t)| < \infty,$$

by the continuity and boundedness of h_i and the compactness of J.

Now consider the convergence of the arguments of g_2. For each $i = 1, \dots, k$

$$y_n(\phi_i(t))H_i(t) \to y(\phi_i(t))H_i(t) \quad \text{as } n \to \infty.$$

Because each component converges and g_2 is continuous by assumption, then by continuity of g_2, we have:

$$\lim_{n\to\infty} \mathcal{B}(y_n)(t) = \mathcal{B}(y)(t), \quad \forall t \in J.$$

Hence, $\mathcal{B}$ is continuous from $C(J,\mathbb{R}) \to C(J,\mathbb{R})$.

Step 2: Compactness of $\mathcal{B}$.

To prove that $\mathcal{B}$ is compact on S, it suffices to show that the image $\mathcal{B}(S)$ is relatively compact in $X = C([0,1])$. By the Arzelà-Ascoli theorem, this is guaranteed if $\mathcal{B}(S)$ is uniformly bounded and equicontinuous.

To prove the uniform boundedness, Now, fix $t \in J$, and define

$$A_i(t) = y(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i}(y(s)))\, ds.$$

Using the growth condition of h_i, we estimate:

$$\begin{aligned}
\left|\int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i}(y(s)))\, ds\right| &\leq \int_0^{\psi_i(t)} \left|h_i(t,s,I^{1-\alpha_i}(y(s)))\right| ds \\
&\leq \int_0^{\psi_i(t)} a_i(t,s)\, ds + \int_0^{\psi_i(t)} b_i(t,s)|I^{1-\alpha_i}(y(s))|\, ds \\
&\leq a \cdot \psi_i(t) + b \cdot \psi_i(t) \cdot \|I^{1-\alpha_i}(y)\| \leq a + bM_y,
\end{aligned}$$

where $M_y := \sup_{t\in J} |I^{1-\alpha_i}(y(t))|$ is finite by the assumption that $y \in C(J)$.

Thus,

$$|A_i(t)| \leq \|y\| \cdot (a + bM_y) \leq r(a + bM_y).$$

Applying the growth bound of g_2, we get:

$$|\mathcal{B}(y)(t)| = \left|g_2(t, A_1(t), \ldots, A_k(t))\right| \leq l \sum_{i=1}^{k} |A_i(t)| + G_2 \leq lkr(a + bM_y) + G_2.$$

Therefore, for all $x \in S$, and all $t \in J$,

$$|\mathcal{B}(y)(t)| \leq M_1 := lkr(a + bM_y) + G_2.$$

Hence, we conclude:

$$\|\mathcal{B}(y)\| := \sup_{t \in J} |\mathcal{B}(y)(t)| \leq M_1, \quad \forall y \in S.$$

So, The operator $\mathcal{B}$ maps the set $S \subset C(J)$ into a uniformly bounded subset of $C(J)$, with uniform bound

$$M_1 = lkr(a + bM_y) + G_2.$$

Now, to prove the equicontinuity of $\mathcal{B}(S)$, we aim to show that the set $\mathcal{B}(S) \subset C([0, 1])$ is equicontinuous. That is, we need to prove the following:

$$\forall \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } |t_1 - t_2| < \delta \Rightarrow |\mathcal{B}(y)(t_1) - \mathcal{B}(y)(t_2)| < \varepsilon,\ \forall y \in S.$$

Let $y \in S$ be arbitrary.

We now estimate the difference $|\mathcal{B}(y)(t_1) - \mathcal{B}(y)(t_2)|$, for $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2|$ small. For

$$A_i(t) := y(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t, s, I^{1-\alpha_i}(y(s)))\, ds, \quad i = 1, \ldots, k.$$

Then, $\mathcal{B}(y)(t) = g_2(t, A_1(t), \ldots, A_k(t))$. Hence,

$$\begin{aligned}
|\mathcal{B}(y)(t_1) - \mathcal{B}(y)(t_2)| &= |g_2(t_1, A_1(t_1), \ldots, A_k(t_1)) - g_2(t_2, A_1(t_2), \ldots, A_k(t_2))| \\
&\leq l\left(|t_1 - t_2| + \sum_{i=1}^{k} |A_i(t_1) - A_i(t_2)|\right),
\end{aligned}$$

So we need to estimate, $|A_i(t_1) - A_i(t_2)|$, we have:

$$\begin{aligned}
|A_i(t_1) - A_i(t_2)| &= \left|y(\phi_i(t_1)) \int_0^{\psi_i(t_1)} h_i(t_1, s, I^{1-\alpha_i}(y(s)))\, ds\ - y(\phi_i(t_2)) \int_0^{\psi_i(t_2)} h_i(t_2, s, I^{1-\alpha_i}(y(s)))\, ds\right| \\
&\leq \left|y(\phi_i(t_1)) - y(\phi_i(t_2))\right| \cdot \left|\int_0^{\psi_i(t_1)} h_i(t_1, s, I^{1-\alpha_i}(y(s)))\, ds\right| \\
&\quad + |y(\phi_i(t_2))| \cdot \left|\int_0^{\psi_i(t_1)} h_i(t_1, s, I^{1-\alpha_i}(y(s)))\, ds - \int_0^{\psi_i(t_2)} h_i(t_2, s, I^{1-\alpha_i}(y(s)))\, ds\right| \\
&:= Z_1 + Z_2.
\end{aligned}$$

We begin by bounding Z_1.
Since $y \in S \subset C([0,1])$ and ϕ_i is continuous, then $y(\phi_i(t))$ is uniformly continuous in t on $[0,1]$. Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|t_1 - t_2| < \delta \Rightarrow |y(\phi_i(t_1)) - y(\phi_i(t_2))| < \frac{\varepsilon}{4Hk},$$

where $H = \sup_{t,s,y} |h_i(t,s,y)| < \infty$ and k is the number of terms. Also,

$$\left| \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i}(y(s)))\, ds \right| \leq \int_0^1 (a + b\|I^{1-\alpha_i}(y)\|)\, ds \leq a + b\|I^{1-\alpha_i}(y)\| := H$$

Thus,

$$Z_1 < \frac{\varepsilon}{4Mk} \cdot H = \frac{\varepsilon}{4k}.$$

We now proceed to bound Z_2. We denote:

$$H_i(t) = \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i}(y(s)))\, ds.$$

Then:

$$Z_2 = |y(\phi_i(t_2))| \cdot |H_i(t_1) - H_i(t_2)|.$$

we have $|y(\phi_i(t_2))| \leq r$. It remains to estimate

$$\begin{aligned}
|H_i(t_1) - H_i(t_2)| &\leq \left| \int_0^{\psi_i(t_1)} h_i(t_1,s,I^{1-\alpha_i}(y(s)))\, ds - \int_0^{\psi_i(t_1)} h_i(t_2,s,I^{1-\alpha_i}(y(s)))\, ds \right| \\
&\quad + \left| \int_{\psi_i(t_1)}^{\psi_i(t_2)} h_i(t_2,s,I^{1-\alpha_i}(y(s)))\, ds \right| \\
&\leq \int_0^1 |h_i(t_1,s,I^{1-\alpha_i}(y(s))) - h_i(t_2,s,I^{1-\alpha_i}(y(s)))|\, ds \\
&\quad + \int_{\min(\psi_i(t_1),\psi_i(t_2))}^{\max(\psi_i(t_1),\psi_i(t_2))} |h_i(t_2,s,I^{1-\alpha_i}(y(s)))|\, ds.
\end{aligned}$$

Since h_i is continuous and bounded, and ψ_i is continuous on $[0,1]$, both terms can be made less than $\frac{\varepsilon}{4kr}$ for $|t_1 - t_2| < \delta$ (by uniform continuity and boundedness).

Hence,

$$Z_2 < r \cdot \frac{\varepsilon}{4kr} = \frac{\varepsilon}{4k}.$$

Combining both substeps, we get:

$$|A_i(t_1) - A_i(t_2)| < \frac{\varepsilon}{2k}.$$

Then, summing over $i = 1$ to k, we get:

$$|\mathcal{B}(y)(t_1) - \mathcal{B}(y)(t_2)| \leq l\left(|t_1 - t_2| + \sum_{i=1}^{k} |A_i(t_1) - A_i(t_2)|\right) < l\left(\delta + \frac{\varepsilon}{2}\right) < \varepsilon.$$

Therefore, $\mathcal{B}(S)$ is equicontinuous on $[0,1]$.

Next, we show that hypothesis (c) of Theorem 2.1 is satisfied. Let $y \in X$ and $z \in S$ be arbitrary such that $y = \mathcal{A}y\mathcal{B}z$. Then, we have

$$\begin{aligned}
& |y(t)| \\
= \ & \left| g_1(t, y(\varphi(t))) \cdot g_2\left(t,\, z(\phi_1(t)) \int_0^{\psi_1(t)} h_1(t,s,I^{1-\alpha_1}z(s))\,ds,\, \ldots,\, z(\phi_k(t)) \int_0^{\psi_k(t)} h_k(t,s,I^{1-\alpha_k}z(s))\,ds\right)\right| \\
\le \ & |g_1(t, y(\varphi(t)))| \cdot \left| g_2\left(t,\, z(\phi_1(t)) \int_0^{\psi_1(t)} h_1(t,s,I^{1-\alpha_1}z(s))\,ds,\, \ldots,\, z(\phi_k(t)) \int_0^{\psi_k(t)} h_k(t,s,I^{1-\alpha_k}z(s))\,ds\right)\right| \\
\le \ & [|g_1(t,0)| + \eta|y(\varphi(t))|] \cdot \left[|g_2(t,0,\ldots,0)| + l \sum_{i=1}^{k} |z(\phi_i(t))| \int_0^{\psi_i(t)} |h_i(t,s,I^{1-\alpha_i}z(s))|\,ds \right] \\
\le \ & [|g_1(t,0)| + \eta|y(\varphi(t))|] \cdot \left[G_2 + l \sum_{i=1}^{k} |z(\phi_i(t))| \int_0^{\psi_i(t)} \left(a_i(t,s) + b_i(t,s)|I^{1-\alpha_i}z(s)|\right) ds \right] \\
\le \ & [|g_1(t,0)| + \eta\|y\|] \cdot \left[G_2 + l \sum_{i=1}^{k} \|z\| \left(\int_0^t a_i(t,s)\,ds + \int_0^t b_i(t,s) \left(\int_0^s \frac{(s-\sigma)^{-\alpha_i}}{\Gamma(1-\alpha_i)} |z(\sigma)|\,d\sigma \right) ds \right) \right] \\
\le \ & [G_1 + \eta\|y\|] \cdot \left[G_2 + l \sum_{i=1}^{k} \|z\| \left(a_i + b_i\|z\|\right) \right] \\
\le \ & [G_1 + \eta r] \cdot \left[G_2 + lr \sum_{i=1}^{k} (a + br) \right] \\
= \ & [G_1 + \eta r] \cdot [G_2 + lrk(a + br)] = r.
\end{aligned}$$

Let r be the positive root of the cubic equation:

$$G_1 lkb\, r^3 + (\eta lkb + G_1 lka) r^2 + (\eta lka + G_1 G_2 - 1) r + \eta G_2 = 0$$

This shows that hypothesis (c) of Theorem 2.1 is satisfied. Finally, we have estimating the product norm

Define $M = \sup_{y \in S} \|\mathcal{B}(y)\| \le lkr(a + bM_y) + G_2$. Then,

$$\Psi(r) \cdot M = [G_1 + \eta r] \cdot [G_2 + lrk(a + br)] < r,$$

where $\Psi(r) = G_1 + \eta r$ by assumption.

Therefore, all conditions of the fixed point Theorem 2.1 are satisfied and hence the operator equation $y = A(y)\mathcal{B}(y)$ has a solution in S. As a result, the the nonlinear integral equation (2.6) has a solution defined on J. □

Theorem 2.3. *Assume that the conditions of Theorem 2.2 are fulfilled and if*

$$lL(\eta r + G_1) + \eta lr + \eta G_2 \le 1.$$

Then the solution of the equation (2.6) is unique.

Proof. Assume that $y_1, y_2 \in C(J)$ are two solutions of the nonlinear integral equation (2.6):

$$y_i(t) = g_1\Big(t, y_i(\varphi(t))\Big) \cdot g_2\Bigg(t,\ y_i(\phi_1(t)) \int_0^{\psi_1(t)} h_1(t,s,I^{1-\alpha_1} y_i(s))\, ds,$$

$$\ldots,\ y_i(\phi_k(t)) \int_0^{\psi_k(t)} h_k(t,s,I^{1-\alpha_k} y_i(s))\, ds\Bigg), \quad i = 1, 2.$$

For each $t \in J$, consider

$$\begin{aligned} |y_1(t) - y_2(t)| &= \Big|g_1(t, y_1(\varphi(t)))g_2\Big(t, \ldots\Big) - g_1(t, y_2(\varphi(t)))g_2\Big(t, \ldots\Big)\Big| \\ &= \Big|g_1(t, y_1(\varphi(t)))\Big(g_2(\cdot, y_1) - g_2(\cdot, y_2)\Big) + \Big(g_1(t, y_1(\varphi(t))) - g_1(t, y_2(\varphi(t)))\Big)g_2(t, y_2)\Big|, \end{aligned}$$

where for brevity, $g_2(\cdot, y_i)$ denotes

$$g_2\Bigg(t, y_i(\phi_1(t)) \int_0^{\psi_1(t)} h_1(t,s,I^{1-\alpha_1} y_i(s))ds, \ldots, y_i(\phi_k(t)) \int_0^{\psi_k(t)} h_k(t,s,I^{1-\alpha_k} y_i(s))ds\Bigg).$$

Using the triangle inequality, we get

$$|y_1(t) - y_2(t)| \le |g_1(t, y_1(\varphi(t)))| \cdot |g_2(\cdot, y_1) - g_2(\cdot, y_2)| + |g_1(t, y_1(\varphi(t))) - g_1(t, y_2(\varphi(t)))| \cdot |g_2(\cdot, y_2)|.$$

Thus,

$$\begin{aligned} &|y_1(t) - y_2(t)| \\ &\le \Big(\eta|y_1(\varphi(t))| + G_1\Big) \cdot l \sum_{i=1}^{k} \left|y_1(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i} y_1(s))ds - y_2(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i} y_2(s))ds\right| \\ &\quad + \eta|y_1(\varphi(t)) - y_2(\varphi(t))| \cdot \left(l \sum_{i=1}^{k} \left|y_2(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i} y_2(s))ds\right| + G_2\right). \end{aligned}$$

For each $i = 1, \ldots, k$, we estimate the integrals difference

$$\begin{aligned} &\left|y_1(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i} y_1(s))ds - y_2(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i} y_2(s))ds\right| \\ &\quad \le |y_1(\phi_i(t)) - y_2(\phi_i(t))| \int_0^{\psi_i(t)} |h_i(t,s,I^{1-\alpha_i} y_1(s))|ds \\ &\qquad + |y_2(\phi_i(t))| \int_0^{\psi_i(t)} |h_i(t,s,I^{1-\alpha_i} y_1(s)) - h_i(t,s,I^{1-\alpha_i} y_2(s))|ds. \end{aligned}$$

Using assumption (iv) on h_i, we have

$$\begin{aligned} &\left|y_1(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i} y_1(s))ds - y_2(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i} y_2(s))ds\right| \\ &\le |y_1(\phi_i(t)) - y_2(\phi_i(t))| \int_0^{\psi_i(t)} \Big(a_i(t,s) + b_i(t,s)|I^{1-\alpha_i} y_1(s)|\Big)ds \\ &\quad + |y_2(\phi_i(t))| \int_0^{\psi_i(t)} b_i(t,s)|I^{1-\alpha_i} (y_1 - y_2)(s)|ds. \end{aligned}$$

using the boundedness of the kernels and fractional integrals, we obtain constants

$$a = \max_i \sup_{t,s\in J} a_i(t,s), \quad b = \max_i \sup_{t,s\in J} b_i(t,s).$$

Also, since the fractional integral operator $I^{1-\alpha_i}$ is bounded on $C(J)$, say with norm C_{α_i}, we have

$$|I^{1-\alpha_i}(y_1 - y_2)(s)| \leq C_{\alpha_i}\|y_1 - y_2\|.$$

Therefore,

$$\left| y_1(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i} y_1(s))ds - y_2(\phi_i(t)) \int_0^{\psi_i(t)} h_i(t,s,I^{1-\alpha_i} y_2(s))ds \right|$$
$$\leq \|y_1 - y_2\| \left[\int_0^{\psi_i(t)} \Big(a_i(t,s) + b_i(t,s)C_{\alpha_i}\|y_1\| \Big) ds \right] + \|y_2\| b\psi_i(t) C_{\alpha_i}\|y_1 - y_2\|$$
$$\leq \|y_1 - y_2\| \left[a_i + b_i C_{\alpha_i}\|y_1\| + b\psi_i(t) C_{\alpha_i}\|y_2\| \right].$$

Summing over $i = 1, \ldots, k$, and using boundedness of $\|y_i\|$, we get $\sum_{i=1}^k \Big| \ldots \Big| \leq L\|y_1 - y_2\|$, We define a constant $L > 0$, depending on $a_i, b_i, C_{\alpha_i}, \|y_i\|$, and ψ_i, by introducing the auxiliary sup-constants for each $i = 1, \ldots, k$:

$$A_i := \sup_{t\in J} \int_0^{\psi_i(t)} a_i(t,s)\, ds, \qquad B_i := \sup_{t\in J} \int_0^{\psi_i(t)} b_i(t,s)\, ds,$$

$$b_i^* := \sup_{(t,s)\in J\times[0,\sup \psi_i]} b_i(t,s), \qquad \psi_i^* := \sup_{t\in J} \psi_i(t).$$

Accordingly, L can be expressed as $L = \sum_{i=1}^k \Big(A_i + C_{\alpha_i} B_i \,\|y_1\| + C_{\alpha_i}\, b_i^*\, \psi_i^* \,\|y_2\| \Big)$.

Returning to the original difference estimate, we have

$$|y_1(t) - y_2(t)| \leq \Big(\eta\|y_1\| + G_1 \Big) lL\|y_1 - y_2\| + \eta l\|y_2\|\|y_1 - y_2\| + \eta G_2\|y_1 - y_2\|.$$

Taking supremum over t,

$$\|y_1 - y_2\|[1 - (1 \leq lL(\eta\|y_1\| + G_1) + \eta l\|y_2\| + \eta G_2)] \leq 0$$

By assumption $lL(\eta\|y_1\| + G_1) + \eta l\|y_2\| + \eta G_2 \leq 1$ and the boundedness of $\|y_i\|$, the coefficient on the right side is strictly grater than 1. Hence,

$$\|y_1 - y_2\| = 0,$$

which implies $y_1 = y_2$ on J.

□

Theorem 2.4. *Let the assumption (v) hold, then there is at least one continuous solution, $x \in C(J, \mathbb{R})$ of (2.5).*

Proof. Define the operator $\mathcal{F} : C(J) \to C(J)$ by

$$(\mathcal{F}x)(t) := x_0 + \int_0^\tau \theta\Big(s, x(s), I^{1-\beta}y(s)\Big)ds + \int_0^t y(s)ds,$$

Consider a closed bounded convex ball $B_\rho = \{x \in C(J) : \|x\| \leq \rho\}$ where, for $\|m\|_{L^1} < 1$), we have

$$\rho \geq \frac{|x_0| + \|n\|_{L^1} + r_y\Big(1 + \|m\|_{L^1}C_\beta\Big)}{1 - \|m\|_{L^1}}.$$

For $x \in B_\rho$, and fixed $y \in C(J)$ with $\|y\| \leq r_y$, we have

$$\begin{aligned} |(\mathcal{F}x)(t)| &\leq |x_0| + \int_0^\tau |\theta(s, x(s), I^{1-\beta}y(s))|ds + \int_0^t |y(s)|ds \\ &\leq |x_0| + \int_0^\tau \Big(|n(s)| + m(s)(|x(s)| + |I^{1-\beta}y(s)|)\Big)ds + r_y. \end{aligned}$$

Using the boundedness of y and continuity of the fractional integral,

$$|I^{1-\beta}y(s)| \leq C_\beta\|y\| \leq C_\beta r_y,$$

where $C_\beta = \frac{1}{\Gamma(1-\beta)} \max_{s\in J} \int_0^s (s-\sigma)^{-\beta}d\sigma$ is finite.

Thus,

$$|(\mathcal{F}x)(t)| \leq |x_0| + \|n\|_{L^1} + \|m\|_{L^1}(\rho + C_\beta r_y) + r_y,$$

which is finite and independent of t. Choosing

$$\rho \geq |x_0| + \|n\|_{L^1} + \|m\|_{L^1}(\rho + C_\beta r_y) + r_y,$$

so that $\mathcal{F}(B_\rho) \subset B_\rho$.

We now prove that the image $\mathcal{F}(B_\rho)$ is equicontinuous.

Let $x \in B_\rho$ and consider for $t_1, t_2 \in J$, with $t_1 < t_2$,

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| = \left|\int_0^{t_2} y(s)\,ds - \int_0^{t_1} y(s)\,ds\right| = \left|\int_{t_1}^{t_2} y(s)\,ds\right| \leq \int_{t_1}^{t_2} |y(s)|\,ds.$$

From the assumptions on g_1, g_2, and h_i, and using the boundedness of y for all $s \in J$. Hence,

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \leq r_y|t_2 - t_1|.$$

This estimate is independent of the particular function $x \in B_\rho$. Therefore, the family $\mathcal{F}(B_\rho)$ is equicontinuous.

Since θ is continuous and bounded, $\mathcal{F}$ is continuous. By Arzelà–Ascoli theorem, the image $\mathcal{F}(B_\rho)$ is equicontinuous and uniformly bounded, thus relatively compact in $C(J)$.

The operator $\mathcal{F}$ is continuous, maps a convex closed bounded set into a relatively compact subset of itself. By Schauder's fixed point theorem,there is a continuous solution of the integral equation (2.5). □

Corollary 2.1. *Under the assumptions of Theorem 2.4, for each solution $y \in C(J, \mathbb{R})$ of the functional integral equation* (2.6)*, there exists a unique solution $x \in C(J, \mathbb{R})$ for equation* (2.5)*, provided that*

$$\mathcal{M}T < 1,$$

where $\mathcal{M} := \sup_{t\in J} m(t)$, and $m(t)$ is as in assumption (v)*.*

Proof. Let $y \in C(J, \mathbb{R})$ be a given solution of equation (2.6). Define the operator $\mathcal{F} : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$(\mathcal{F}x)(t) := x_0 + \int_0^\tau \theta\left(s, x(s), I^{1-\beta}y(s)\right) ds + \int_0^t y(s)\, ds.$$

We show that $\mathcal{F}$ is a contraction. Let $x_1, x_2 \in C(J, \mathbb{R})$. Then for any $t \in J$, we have:

$$\begin{aligned}
|(\mathcal{F}x_1)(t) - (\mathcal{F}x_2)(t)| &= \left|\int_0^\tau \left[\theta\left(s, x_1(s), I^{1-\beta}y(s)\right) - \theta\left(s, x_2(s), I^{1-\beta}y(s)\right)\right] ds\right| \\
&\le \int_0^\tau m(s)|x_1(s) - x_2(s)|\, ds \\
&\le \left(\sup_{s\in J} m(s)\right) \int_0^\tau |x_1(s) - x_2(s)|\, ds \le \mathcal{M}T\|x_1 - x_2\|.
\end{aligned}$$

Taking the supremum over $t \in J$, we obtain:

$$\|\mathcal{F}x_1 - \mathcal{F}x_2\| \le \mathcal{M}T\|x_1 - x_2\|.$$

Hence, $\mathcal{F}$ is a contraction provided that $\mathcal{M}T < 1$. By the Banach fixed point theorem, the operator $\mathcal{F}$ admits a unique fixed point $x \in C(J, \mathbb{R})$, which is the unique solution of equation (2.5) corresponding to the given $y \in C(J, \mathbb{R})$. □

3. Continuous Dependence

Definition 3.1. *The unique solution $x \in C(J, \mathbb{R})$ of equation* (2.5) *is said to depend continuously on the parameter x_0 if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|x_0 - x_0^*| \le \delta \quad \Rightarrow \quad \|x - x^*\| \le \epsilon,$$

where x^* is the solution corresponding to the perturbed initial condition x_0^*, satisfying

$$x^*(t) = x_0^* + \int_0^\tau \theta(s, x^*(s), I^{1-\beta}y(s))\, ds + \int_0^t y(s)\, ds.$$

Theorem 3.1. *Assume that the conditions of Theorem 2.4 and corollary 2.1 hold. Then the unique solution $x \in C(J, \mathbb{R})$ of equation* (2.5) *depends continuously on the initial condition x_0.*

Proof. Let x and x^* be the solutions of equation (2.5) corresponding to the initial values x_0 and x_0^*, respectively. Then, for all $t \in J$, we have

$$|x(t) - x^*(t)| \le |x_0 - x_0^*| + \int_0^\tau \left|\theta(s, x(s), I^{1-\beta}y(s)) - \theta(s, x^*(s), I^{1-\beta}y(s))\right| ds.$$

Using assumption (v), which implies that θ is Lipschitz continuous with respect to the second and third arguments, we have

$$\left|\theta(s,x(s),I^{1-\beta}y(s))-\theta(s,x^*(s),I^{1-\beta}y(s))\right|\leq m(s)|x(s)-x^*(s)|,$$

Thus,

$$|x(t)-x^*(t)|\leq|x_0-x_0^*|+\int_0^\tau m(s)|x(s)-x^*(s)|\,ds.$$

Taking the supremum over $t\in J$, we get

$$\|x-x^*\|\leq|x_0-x_0^*|+\mathcal{M}\int_0^t\|x-x^*\|\,ds.$$

Taking the supremum over $t\in[0,\tau]$, we obtain:

$$\|x-x^*\|=\sup_{t\in[0,\tau]}|x(t)-x^*(t)|\leq|x_0-x_0^*|+\mathcal{M}\tau\|x-x^*\|.$$

If $\mathcal{M}\tau<1$, then:

$$\|x-x^*\|\leq\frac{1}{1-\mathcal{M}\tau}|x_0-x_0^*|.$$

Therefore, for every $\varepsilon>0$, choosing $\delta=\varepsilon(1-\mathcal{M}\tau)$, ensures that

$$|x_0-x_0^*|\leq\delta\quad\Rightarrow\quad\|x-x^*\|\leq\varepsilon.$$

Hence, the unique solution x depends continuously on the initial condition x_0. □

Definition 3.2. *The unique solution $x\in C(J,\mathbb{R})$ of the equation (2.5) depends continuously on y, if $\epsilon>0$, $\exists\,\delta>0$ such that*

$$|y-y^*|\leq\delta\quad\Rightarrow\quad\|x-x^*\|\leq\epsilon,$$

where x^* is the unique solution of the integral equation

$$x^*(t)=x_0+\int_0^\tau\theta(s,x^*(s),I^{1-\beta}y^*(s))ds+\int_0^t y^*(s)ds.$$

Theorem 3.2. *The unique solution of the integral equation (2.5) depends continuously on y.*

Proof. Let $x(t)$ and $x^*(t)$ be the unique solutions of the equation (2.5) corresponding to $y(t)$ and $y^*(t)$, respectively. Then, we have:

$$x(t)-x^*(t)=\int_0^\tau\left[\theta(s,x(s),I^{1-\beta}y(s))-\theta(s,x^*(s),I^{1-\beta}y^*(s))\right]ds+\int_0^t(y(s)-y^*(s))\,ds.$$

Taking absolute values and applying the triangle inequality, we obtain:

$$|x(t)-x^*(t)|\leq\int_0^\tau\left|\theta(s,x(s),I^{1-\beta}y(s))-\theta(s,x^*(s),I^{1-\beta}y^*(s))\right|ds+\int_0^t|y(s)-y^*(s)|ds.$$

By assumption (v), the function θ is Lipschitz continuous in both arguments. In particular, for all $s\in J$, we have:

$$\left|\theta(s,x(s),I^{1-\beta}y(s))-\theta(s,x^*(s),I^{1-\beta}y^*(s))\right|\leq m(s)|x(s)-x^*(s)|+m(s)\left|I^{1-\beta}y(s)-I^{1-\beta}y^*(s)\right|.$$

Substituting this into the inequality, we get:

$$|x(t) - x^*(t)| \leq \int_0^\tau m(s)|x(s) - x^*(s)|\, ds + \int_0^\tau m(s) \left|I^{1-\beta}y(s) - I^{1-\beta}y^*(s)\right| ds + \int_0^t |y(s) - y^*(s)|\, ds.$$

Now define

$$\delta(t) := \sup_{r\in[0,t]} |x(r) - x^*(r)|.$$

Then,

$$\delta(t) \leq \mathcal{M} \int_0^\tau \delta(s)\, ds + \mathcal{M} \int_0^\tau \left|I^{1-\beta}y(s) - I^{1-\beta}y^*(s)\right| ds + \int_0^t |y(s) - y^*(s)|\, ds.$$

Using the boundedness of the fractional integral operator $I^{1-\beta}\colon L^1(J) \to C(J)$, we know there exists a constant $C_\beta > 0$ such that:

$$\left\|I^{1-\beta}y - I^{1-\beta}y^*\right\| \leq C_\beta \|y - y^*\|_{L^1(J)}.$$

Hence,

$$\delta(t) \leq \mathcal{M}\tau\delta(t) + \mathcal{M}\tau C_\beta \|y - y^*\|_{L^1(J)} + \|y - y^*\|_{L^1(J)}.$$

So we obtain:

$$\delta(t)\,(1 - \mathcal{M}\tau) \leq \Big(\mathcal{M}\tau C_\beta + 1\Big) \|y - y^*\|_{L^1(J)}.$$

If $\mathcal{M}\tau < 1$, then:

$$\delta(t) \leq \frac{\mathcal{M}\tau C_\beta + 1}{1 - \mathcal{M}\tau} \|y - y^*\|_{L^1(J)}.$$

Thus, we conclude:

$$\|x - x^*\| \leq C\|y - y^*\|_{L^1(J)}, \quad \text{where } C := \frac{\mathcal{M}\tau C_\beta + 1}{1 - \mathcal{M}\tau}.$$

Therefore, for every $\varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{C}$ ensures that

$$\|y - y^*\|_{L^1(J)} \leq \delta \quad \Rightarrow \quad \|x - x^*\| \leq \varepsilon.$$

This proves the continuous dependence of the unique solution $x \in C(J, \mathbb{R})$ on the function y. □

3.1. **Continuous Dependence on Nonlinear Perturbation θ.**

Definition 3.3. *The unique solution $x \in C(J, \mathbb{R})$ of the integral equation* (2.5) *is said to depend continuously on the function θ if for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever*

$$\sup_{(t,z,u)\in J\times\mathbb{R}\times\mathbb{R}} |\theta(t, z, u) - \theta^*(t, z, u)| \leq \delta,$$

it follows that

$$\|x - x^*\| \leq \epsilon,$$

where x^* is the unique solutions of the integral equations

$$x^*(t) = x_0 + \int_0^\tau \theta^*(s, x^*(s), I^{1-\beta}y(s))\, ds + \int_0^t y(s)\, ds,$$

Theorem 3.3. *Assume that all the assumptions of Theorem 2.4 hold, particularly that the function θ and its perturbation θ^* satisfy a uniform Lipschitz condition in the second and third arguments with a Lipschitz constant $m(t)$ such that $\mathcal{M}\tau \leq 1$. Then the unique solution $x \in C(J, \mathbb{R})$ of equation* (2.5) *depends continuously on the function θ.*

Proof. Let x and x^* be the unique solutions of the integral equations corresponding to θ and θ^*, respectively. Then, we have

$$x(t) - x^*(t) = \int_0^\tau \Big[\theta(s, x(s), I^{1-\beta}y(s)) - \theta^*(s, x^*(s), I^{1-\beta}y(s))\Big] ds.$$

Adding and subtracting $\theta(s, x^*(s), I^{1-\beta}y(s))$ inside the integrand, we obtain

$$\begin{aligned}|x(t) - x^*(t)| \leq &\int_0^\tau \Big|\theta(s, x(s), I^{1-\beta}y(s)) - \theta(s, x^*(s), I^{1-\beta}y(s))\Big| ds \\ &+ \int_0^\tau \Big|\theta(s, x^*(s), I^{1-\beta}y(s)) - \theta^*(s, x^*(s), I^{1-\beta}y(s))\Big| ds.\end{aligned}$$

Using the Lipschitz continuity of θ with respect to its second argument, we get:

$$|x(t) - x^*(t)| \leq \int_0^\tau m(s)|x(s) - x^*(s)| ds + \tau\delta.$$

Taking the supremum over $t \in J$, we have

$$\|x - x^*\| \leq \int_0^\tau m(s)\|x - x^*\| ds + \tau\delta \leq \mathcal{M}\tau\|x - x^*\| + \tau\delta.$$

Since $\mathcal{M}\tau < 1$, we conclude $\|x - x^*\| \leq \frac{\tau\delta}{1-\mathcal{M}\tau}$.
Hence, for any $\epsilon > 0$, we can choose $\delta = \frac{(1-\mathcal{M}\tau)}{\tau}\epsilon$ to ensure that $\|x - x^*\| \leq \epsilon$, completing the proof. □

3.2. **Applications.** we obtain some particular cases which are useful for the theory of qualitative analysis of some functional integral equations and important for some modeles and real problems.

(1) Let $g_1(t, y) = 1$, then the multi-term functional equation (2.6) takes the form

$$y(t) = g_2\bigg(t, y(\phi_1(t)) \int_0^{\psi_1(t)} h_1(t, s, y(s)\, ds, ..., y(\phi_k(t))) \int_0^{\psi_k(t)} h_k(t, s, y(s)\, ds\bigg), \tag{3.1}$$

then under the assumptions of Theorem 2.2, the multi-term functional equation (3.1) has at least one continuous solution $y \in C(J, \mathbb{R})$.

(2) Let $g_1(t, y) = 1$, $g_2(t, .) = g(t, y) = 1 + \sum_{i=1}^k y$, and $\phi_i(t) = t$ then cubic multi-term functional equation (2.6) takes the form

$$y(t) = 1 + \sum_{i=1}^{k} y^2(t) \int_0^{\psi(t)} h_i(t, s, y(s)) ds, \tag{3.2}$$

and under the assumptions of Theorem 2.2, then the cubic integral equation (3.2) has at least one continuous solution $y \in C(J, \mathbb{R})$.

4. Integrable Solutions of Equation (2.2)

We examine the existence of integrable solutions of equation (2.2) under the following modified assumption:

(i^*) Each function

$$\omega_{1i}, \omega_{2i} : J \times \mathbb{R} \to \mathbb{R}, \quad i = 1, 2, \dots, k$$

satisfies the Carathéodory conditions, and there exist constants $k_1, k_2, N_1, N_2 > 0$ such that:

$$|\omega_{1i}(t, x)| \le N_1 + k_1|x|, \quad |\omega_{2i}(t, x)| \le N_2 + k_2|x|, \quad \forall i = 1, \dots, k.$$

Theorem 4.1. *Assume that $x \in C(J, \mathbb{R})$ and that the assumptions of Theorem 2.2 hold, with assumption (i^*). Then equation* (2.2) *has at least one solution $v \in L_1(J, \mathbb{R})$.*

Proof. Define the operator $A^* : L_1(J, \mathbb{R}) \to L_1(J, \mathbb{R})$ by

$$(A^* v)(t) \ = \ \sum_{i=1}^{k} \omega_{1i}(t, v(t)) \ + \ x(t) \sum_{i=1}^{k} \omega_{2i}(t, v(t)).$$

Put $N = \sup_{t \in J} |x(t)|$ and

$$r^* = \frac{kN_1 + NkN_2}{1 - k_1 k - N k_2 k}, \qquad B^*_{r^*} = \{v \in L_1(J, \mathbb{R}) : \|v\|_{L_1} \le r^*\}.$$

Let $v \in B^*_{r^*}$. Using the growth bounds from assumption (iii^*), we estimate:

$$\begin{aligned} |(A^* v)(t)| &\le \sum_{i=1}^{k} |\omega_{1i}(t, v(t))| + |x(t)| \sum_{i=1}^{k} |\omega_{2i}(t, v(t))| \\ &\le \sum_{i=1}^{k} (N_1 + k_1|v(t)|) + |x(t)| \sum_{i=1}^{k} (N_2 + k_2|v(t)|) \\ &= kN_1 + k_1 k|v(t)| + |x(t)|(kN_2 + k_2 k|v(t)|). \end{aligned}$$

Integrating both sides:

$$\begin{aligned} \|A^* v\|_{L_1} &\le kN_1 + NkN_2 + (k_1 k + N k_2 k)\|v\|_1 \\ &\le kN_1 + NkN_2 + (k_1 k + N k_2 k) r^* = r^*. \end{aligned}$$

Thus, A^* maps $B^*_{r^*}$ into itself.

Compactness. We now show A^* is compact. Let $\psi \subset B^*_{r^*}$ be arbitrary and fix $v \in \psi$. For $h > 0$ define the Steklov average of $A^* v$:

$$(A^* v)_h(t) = \frac{1}{h} \int_t^{t+h} A^* v(s)\, ds.$$

Then

$$(A^* v)_h(t) - A^* v(t) = \frac{1}{h} \int_t^{t+h} \Big(A^* v(s) - A^* v(t) \Big)\, ds,$$

so, integrating over J,

$$\|(A^*\nu)_h - A^*\nu\|_{L_1} \leq \int_J \frac{1}{h}\int_t^{t+h} |A^*\nu(s) - A^*\nu(t)|\, ds\, dt.$$

Split the inner difference using the definition of A^*. After elementary manipulations we get (writing the domain of integration as J and summing over $i = 1, \dots, k$):

$$\begin{aligned}
\|(A^*\nu)_h - A^*\nu\|_{L_1} &\leq \sum_{i=1}^{k} \int_J \frac{1}{h}\int_t^{t+h} \Big|\omega_{1i}(s,\nu(s)) - \omega_{1i}(t,\nu(t))\Big|\, ds\, dt \\
&\quad + \sum_{i=1}^{k} \int_J \frac{1}{h}\int_t^{t+h} |x(s) - x(t)|\,|\omega_{2i}(s,\nu(s))|\, ds\, dt \\
&\quad + \sum_{i=1}^{k} \int_J |x(t)| \frac{1}{h}\int_t^{t+h} \Big|\omega_{2i}(s,\nu(s)) - \omega_{2i}(t,\nu(t))\Big|\, ds\, dt \\
&=: I_1(h,\nu) + I_2(h,\nu) + I_3(h,\nu).
\end{aligned}$$

We show that each term $I_j(h,\nu)$ tends to 0 as $h \to 0$, uniformly for $\nu \in \psi$.

(i) *The term $I_2(h,\nu)$.* Since $x \in C(J)$ we have $\sup_{|s-t|\leq h} |x(s) - x(t)| \to 0$ as $h \to 0$. Moreover, for every $\nu \in \psi$ the growth condition in (iii^*) gives

$$|\omega_{2i}(s,\nu(s))| \leq N_2 + k_2|\nu(s)|,$$

and hence $\omega_{2i}(\cdot,\nu(\cdot)) \in L_1(J)$ with a uniform L_1-bound on ψ. Therefore by Fubini/Tonelli and the uniform smallness of $|x(s) - x(t)|$ on $|s - t| \leq h$,

$$\sup_{\nu\in\psi} I_2(h,\nu) \;\longrightarrow\; 0 \quad (h \to 0).$$

(ii) *The terms $I_1(h,\nu)$ and $I_3(h,\nu)$.* Fix $\nu \in \psi$. For a.e. t the map $s \mapsto \omega_{ji}(s,\nu(s))$ belongs to $L_1(J)$ (this follows from the Carathéodory and growth assumptions). It is a standard property of Steklov averages that for any $f \in L_1(J)$,

$$\frac{1}{h}\int_t^{t+h} |f(s) - f(t)|\, ds \xrightarrow[h\to 0]{} 0 \qquad \text{in } L^1(J),$$

and hence

$$\int_J \frac{1}{h}\int_t^{t+h} |f(s) - f(t)|\, ds\, dt \xrightarrow[h\to 0]{} 0.$$

Apply this fact with $f(\cdot) = \omega_{1i}(\cdot,\nu(\cdot))$ to see $I_1(h,\nu) \to 0$. For $I_3(h,\nu)$ observe $|x(t)| \leq N$ and apply the same Steklov-average argument to $f(\cdot) = \omega_{2i}(\cdot,\nu(\cdot))$ to get $I_3(h,\nu) \to 0$. The uniform growth bounds from (iii^*) supply an integrable dominating function which allows one to upgrade pointwise (in ν) convergence to uniform convergence over the bounded family ψ.

Combining the three estimates we obtain

$$\sup_{\nu\in\psi} \|(A^*\nu)_h - A^*\nu\|_{L_1} \xrightarrow[h\to 0]{} 0.$$

By Kolmogorov's compactness criterion [23], this implies the set $A^*(\psi)$ is relatively compact in $L_1(J)$. Since $\psi \subset B^*_{r^*}$ was arbitrary, A^* maps bounded sets into relatively compact sets; therefore A^* is compact.

Finally, A^* is continuous (by the Carathéodory conditions and the linear growth) and maps the closed, bounded, convex set $B^*_{r^*}$ into itself. Schauder's fixed point theorem [22] yields at least one fixed point $\nu \in B^*_{r^*}$, which is an L_1-solution of (2.2). □

Theorem 4.2. *For the function $y \in C(J, \mathbb{R})$. Let the assumptions of Theorem 2.2 and Theorem 4.1 be satisfied. Then the multi-term functional problem (1.1)-(1.2) has at least one solution $\nu \in L_1(J, \mathbb{R})$ which is the solution of the problem (2.2), where y is the solution of (2.5).*

4.1. **Uniqueness of the solution of equation (2.2).** In this subsection, we demonstrate that the solution of the functional equation (2.2) is unique under the following assumptions:

(ii^*) The functions $\omega_{1i} : J \times \mathbb{R} \to \mathbb{R}$ and $\omega_{2i} : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\} \quad i = 1, 2, \ldots, k$, are continuous and satisfy the following Lipschitz conditions:

$$\begin{aligned} \left|\omega_{1i}(t, x_1) - \omega_{1i}(t, x_2)\right| &\leq k_1(t)|x_1 - x_2|, \\ |\omega_{2i}(t, x_1) - \omega_{2i}(t, x_2)| &\leq k_2(t)|x_1 - x_2|, \end{aligned} \tag{4.1}$$

for all $t \in J$, $x_1, x_2 \in \mathbb{R}$, where $k_1, k_2 \in L_1(J, \mathbb{R}_+)$ are nonnegative functions.

Theorem 4.3. *The solution to equation (2.2) is unique for any $x \in C(J, \mathbb{R})$, provided that the conditions of Corollary 2.1 and assumption (ii^*) hold, with the additional constraint that $k_1 + N\, k_2 < 1$.*

Proof. From assumption (ii^*) and Corollary 2.1, we know that the solution $\nu(t)$ of (2.2) is continuous.

Let ν_1 and ν_2 be two solutions of (2.2). For each $t \in J$, we compute:

$$\begin{aligned} |\nu_1(t) - \nu_2(t)| &= \left|\sum_{i=1}^{k} \omega_{1i}(t, \nu_1(t)) + x(t) \sum_{i=1}^{k} \omega_{2i}(t, \nu_1(t)) - \sum_{i=1}^{k} \omega_{1i}(t, \nu_2(t)) - x(t) \sum_{i=1}^{k} \omega_{2i}(t, \nu_2(t))\right| \\ &\leq \left|\sum_{i=1}^{k} [\omega_{1i}(t, \nu_1(t)) - \omega_{1i}(t, \nu_2(t))]\right| + |x(t)| \left|\sum_{i=1}^{k} [\omega_{2i}(t, \nu_1(t)) - \omega_{2i}(t, \nu_2(t))]\right| \\ &\leq \sum_{i=1}^{k} k_{1i}(t)|\nu_1(t) - \nu_2(t)| + |x(t)| \sum_{i=1}^{k} k_{2i}(t)|\nu_1(t) - \nu_2(t)| \\ &= \left(\sum_{i=1}^{k} k_{1i}(t) + |x(t)| \sum_{i=1}^{k} k_{2i}(t)\right)|\nu_1(t) - \nu_2(t)|. \end{aligned}$$

Let $k_1(t) := \sum_{i=1}^{k} k_{1i}(t)$, $k_2(t) := \sum_{i=1}^{k} k_{2i}(t)$, and set $N := \sup_{t \in I} |x(t)|$. Then:

$$|\nu_1(t) - \nu_2(t)| \leq (k_1(t) + |x(t)| k_2(t))\, |\nu_1(t) - \nu_2(t)|.$$

Integrating over $J = [0,1]$, we obtain:

$$\begin{aligned}\|v_1 - v_2\|_1 &\leq \int_0^1 k_1(t)|v_1(t) - v_2(t)|\,dt + \int_0^1 |x(t)|k_2(t)|v_1(t) - v_2(t)|\,dt \\ &\leq k_1\|v_1 - v_2\|_1 + Nk_2\|v_1 - v_2\|_1 \\ &= (k_1 + Nk_2)\|v_1 - v_2\|_1.\end{aligned}$$

Hence,

$$[1 - (k_1 + Nk_2)]\,\|v_1 - v_2\|_1 \leq 0.$$

Since $k_1 + Nk_2 < 1$, we conclude that $\|v_1 - v_2\|_1 = 0$, which implies $v_1(t) = v_2(t)$ almost everywhere in J.

Therefore, the solution to the problem (2.2) is unique in $L_1(J, \mathbb{R})$. □

With this, we can deduce a unique result for the solutions of the multi-term functional problem (1.1)-(1.2):

Theorem 4.4. *Let the assumptions of Corollary 2.1 and Theorem 4.3 be satisfied. Then the solution of multi-term functional problem (1.1)-(1.2) is unique.*

4.2. **Continuous Dependence of the Solution of Equation (2.2).** Consider the operator defined by:

$$v(t) = \sum_{i=1}^{k} \omega_{1i}(t, v(t)) + x(t)\sum_{i=1}^{k} \omega_{2i}(t, v(t)).$$

Definition 4.1. *The solution v of equation (2.2) continuously depends on the function x if for every $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\|x - x^*\| \leq \delta \quad \Rightarrow \quad \|v - v^*\| \leq \epsilon.$$

Theorem 4.5. *Suppose that the conditions of Theorem 4.3 and assumption (vi) are satisfied. Then the solution of equation (2.2) depends continuously on the function x.*

Proof. Let v and v^* be the solutions corresponding to x and x^*, respectively. Then for all $t \in J$, we have:

$$\begin{aligned}|v(t) - v^*(t)| &= \left|\sum_{i=1}^{k} \omega_{1i}(t, v(t)) + x(t)\sum_{i=1}^{k} \omega_{2i}(t, v(t)) - \sum_{i=1}^{k} \omega_{1i}(t, v^*(t)) - x^*(t)\sum_{i=1}^{k} \omega_{2i}(t, v^*(t))\right| \\ &\leq \sum_{i=1}^{k} \left|\omega_{1i}(t, v(t)) - \omega_{1i}(t, v^*(t))\right| + \sum_{i=1}^{k} \left|\omega_{2i}(t, v(t))\right| \cdot |x(t) - x^*(t)| \\ &\quad + \sum_{i=1}^{k} |x^*(t)| \cdot \left|\omega_{2i}(t, v(t)) - \omega_{2i}(t, v^*(t))\right|.\end{aligned}$$

Using the Lipschitz conditions, we obtain:

$$\begin{aligned}|\nu(t)-\nu^*(t)| &\le k\cdot k_1(t)|\nu(t)-\nu^*(t)|+\delta\cdot\sum_{i=1}^{k}|\omega_{2i}(t,\nu(t))|+k_2(t)|\nu(t)-\nu^*(t)|\cdot\sum_{i=1}^{k}|x^*(t)|\\ &=\left[k\cdot k_1(t)+k_2(t)\sum_{i=1}^{k}|x^*(t)|\right]\cdot|\nu(t)-\nu^*(t)|+\delta\cdot\sum_{i=1}^{k}|\omega_{2i}(t,\nu(t))|.\end{aligned}$$

From the continuity of ω_{2i} and boundedness of ν, there exists a constant $G>0$ such that $|\omega_{2i}(t,\nu(t))|\le G$ for all $t\in J$ and $i=1,\dots,k$. Then:

$$\|\nu-\nu^*\|\le\sup_{t\in J}\left\{[kk_1(t)+k_2(t)kN]\,\|\nu-\nu^*\|+k\delta G\right\}.$$

Since $k_1,k_2\in L_1(J)$, we can integrate both sides over $J=[0,1]$ and obtain:

$$\|\nu-\nu^*\|\le\left(\int_0^1[kk_1(t)+kk_2(t)N]\,dt\right)\|\nu-\nu^*\|+kG\delta.$$

Set:

$$\Lambda:=\int_0^1[kk_1(t)+kk_2(t)N]\,dt<1,$$

which is possible due to the integrability of k_1 and k_2, and the smallness of N or appropriate normalization.

Then,

$$(1-\Lambda)\|\nu-\nu^*\|\le kG\delta\quad\Rightarrow\quad\|\nu-\nu^*\|\le\frac{kG\delta}{1-\Lambda}.$$

Given any $\epsilon>0$, choose

$$\delta=\frac{(1-\Lambda)\epsilon}{kG},$$

which ensures that $\|\nu-\nu^*\|\le\epsilon$.

Hence, the solution ν depends continuously on x. □

Theorem 4.6. *Suppose that the conditions of Theorem 4.3 are satisfied, and in particular, that assumption* (ii^*) *holds. Then the solution of equation* (2.2) *depends continuously on the function x.*

Proof. Let ν and ν^* be two solutions of equation (2.2) corresponding to functions x and x^*, respectively. Then for each $t\in J$, we write:

$$\nu(t)=\sum_{i=1}^{k}\omega_{1i}(t,\nu(t))+x(t)\sum_{i=1}^{k}\omega_{2i}(t,\nu(t)),$$

$$\nu^*(t)=\sum_{i=1}^{k}\omega_{1i}(t,\nu^*(t))+x^*(t)\sum_{i=1}^{k}\omega_{2i}(t,\nu^*(t)).$$

Subtracting these, we have:

$$
\begin{aligned}
|\nu(t)-\nu^*(t)| &\leq \left|\sum_{i=1}^{k}\left[\omega_{1i}(t,\nu(t))-\omega_{1i}(t,\nu^*(t))\right]\right| \\
&\quad + \left|\sum_{i=1}^{k}\left[x(t)\omega_{2i}(t,\nu(t))-x^*(t)\omega_{2i}(t,\nu^*(t))\right]\right| \\
&\leq \sum_{i=1}^{k}|\omega_{1i}(t,\nu(t))-\omega_{1i}(t,\nu^*(t))| \\
&\quad + \sum_{i=1}^{k}\left|x(t)\omega_{2i}(t,\nu(t))-x^*(t)\omega_{2i}(t,\nu^*(t))\right|.
\end{aligned}
$$

Now apply the triangle inequality and the Lipschitz condition (ii^*):

$$
\begin{aligned}
|\nu(t)-\nu^*(t)| &\leq \sum_{i=1}^{k}k_1(t)|\nu(t)-\nu^*(t)| \\
&\quad + \sum_{i=1}^{k}\left[|\omega_{2i}(t,\nu(t))||x(t)-x^*(t)|+|x^*(t)||\omega_{2i}(t,\nu(t))-\omega_{2i}(t,\nu^*(t))|\right] \\
&\leq kk_1(t)|\nu(t)-\nu^*(t)|+kk_2(t)|\nu(t)-\nu^*(t)||x^*(t)|+kk_2(t)|\nu(t)||x(t)-x^*(t)|.
\end{aligned}
$$

Take the supremum norm over J:

$$\|\nu-\nu^*\| \leq \|k_1+k_2N\|_1\cdot\|\nu-\nu^*\|+\|k_2\|_1\cdot\|\nu\|\cdot\|x-x^*\|.$$

Assuming $\|x-x^*\|\leq\delta$, and provided that $\|k_1+k_2N\|_1<1$, it follows that:

$$\|\nu-\nu^*\|\leq\frac{\|k_2\|_1\cdot\|\nu\|}{1-\|k_1+k_2N\|_1}\cdot\delta.$$

This shows that for any $\epsilon>0$, choosing $\delta>0$ small enough ensures $\|\nu-\nu^*\|\leq\epsilon$. Hence, the solution depends continuously on the function x.

□

5. Example

Consider the multi-term hybrid functional problem in the hybrid form. Set $J=[0,1]$. Let $\nu: J\to\mathbb{R}$ and define

$$x(t)=\frac{\nu(t)-\sum_{i=1}^{2}\omega_{1i}(t,\nu(t))}{\sum_{i=1}^{2}\omega_{2i}(t,\nu(t))},\qquad t\in J,$$

and suppose x satisfies the quadratic multi-term hybrid equation

$$\frac{dx(t)}{dt}=g_1\Big(t,x(\phi(t))\Big)\,g_2\Big(t,\ \frac{dx(\varphi_1(t))}{dt}\int_0^{\psi_1(t)}h_1(t,s,D^{\alpha_1}x(s))\,ds,\ \frac{dx(\varphi_2(t))}{dt}\int_0^{\psi_2(t)}h_2(t,s,D^{\alpha_2}x(s))\,ds\Big),$$

with the nonlocal hybrid condition

$$x(0)-\int_0^{\tau}\theta(s,x(s),D^{\beta}x(s))\,ds=x_0,\qquad \beta\in(0,1].$$

denote $y(t) = \dfrac{dx}{dt}$ and which implies that the function x must satisfy the fractional-order integral equation:

$$x(t) = x_0 + \int_0^1 \theta\Big(s, x(s), I^{1-\beta}y(s)\Big)\,ds + \int_0^t y(s)ds.$$

We take the concrete functions and delays

$$\phi(t) = \tfrac{t}{2}, \quad \varphi_1(t) = \tfrac{t}{3}, \quad \varphi_2(t) = \tfrac{t}{4}, \quad \psi_1(t) = \tfrac{t}{2}, \quad \psi_2(t) = \tfrac{t}{3}, \quad t \in J.$$

Choose the nonlinearities and kernels

$$g_1(t,x) = \tfrac{1}{10} + \tfrac{x}{5}, \qquad g_2(t,z_1,z_2) = \tfrac{1}{10} + \tfrac{|z_1|+|z_2|}{6},$$
$$h_1(t,s,x) = 1 + \tfrac{|x|}{4}, \qquad h_2(t,s,x) = 2 + \tfrac{|x|}{5},$$

and two-term coefficients

$$\omega_{11}(t,x) = 1 + \tfrac{x}{4}, \quad \omega_{12}(t,x) = 2 + \tfrac{x}{5}, \quad \omega_{21}(t,x) = 1 + \tfrac{x}{6}, \quad \omega_{22}(t,x) = 3 + \tfrac{x}{7}.$$

Finally take $\alpha_1 = \alpha_2 = \frac{1}{2}$

We now verify, that this example satisfies the existence of a continuous solution x and the existence of an integrable ν solving the original hybrid problem.

The maps $\phi, \varphi_i, \psi_i : J \to J$ are continuous and satisfy $\phi(t), \varphi_i(t), \psi_i(t) \leq t$ for all $t \in J$ because

$$\phi(t) = \tfrac{t}{2} \leq t, \quad \varphi_1(t) = \tfrac{t}{3} \leq t, \quad \varphi_2(t) = \tfrac{t}{4} \leq t, \quad \psi_i(t) \leq t.$$

Estimates for g_1, g_2 and choice of constants.

Compute Lipschitz / growth constants explicitly.

For any $x, y \in \mathbb{R}$

$$|g_1(t,x) - g_1(t,y)| = \frac{1}{5}|x-y|.$$

So we may take $\eta = \frac{1}{5}$ and $G_1 = \sup_{t\in J}|g_1(t,0)| = \frac{1}{10}$.

For any $(z_1,z_2), (w_1,w_2) \in \mathbb{R}^2$,

$$|g_2(t,z_1,z_2) - g_2(t,w_1,w_2)| \leq \frac{1}{6}\Big(|z_1 - w_1| + |z_2 - w_2|\Big).$$

So we may take $l = \frac{1}{6}$ and $G_2 = \sup_{t\in J}|g_2(t,0,0)| = \frac{1}{10}$.

Estimates for kernels h_i.

For $i = 1, 2$ we have

$$|h_1(t,s,x) - h_1(t,s,y)| \leq \tfrac{1}{4}|x-y|, \qquad |h_2(t,s,x) - h_2(t,s,y)| \leq \tfrac{1}{5}|x-y|.$$

Thus set $b_1 = \frac{1}{4},\ b_2 = \frac{1}{5}$, and choose $b = \max\{b_1, b_2\} = \frac{1}{4}$. Also growth bounds

$$|h_1(t,s,x)| \leq 1 + \tfrac{|x|}{4}, \qquad |h_2(t,s,x)| \leq 2 + \tfrac{|x|}{5},$$

so take $a_1 = 1,\ a_2 = 2,\ a = \max\{a_1, a_2\} = 2$.

With the above constants,

$$\eta = \tfrac{1}{5}, \quad G_1 = \tfrac{1}{10}, \quad l = \tfrac{1}{6}, \quad G_2 = \tfrac{1}{10}, \quad a = 2, \quad b = \tfrac{1}{4}, \quad k = 2,$$

Now choose $r > 0$ large enough so that

$$(G_1 + \eta r)\,(G_2 + l\,r\,k\,(a + b\,r)) < r$$

Substituting the above constants gives

$$\left(\tfrac{1}{10} + \tfrac{r}{5}\right)\left(\tfrac{1}{10} + \tfrac{2}{3}r + \tfrac{1}{12}r^2\right) < r.$$

Expanding and moving r to the left-hand side,

$$p(r) = \frac{1}{60}r^3 + \frac{17}{120}r^2 - \frac{137}{150}r + \frac{1}{100} < 0.$$

The cubic $p(r)$ has three real zeros, numerically

$$r_1 \approx -12.788701096, \qquad r_2 \approx 0.010967587, \qquad r_3 \approx 4.277733509.$$

Since the leading coefficient is positive, we have

$$p(r) < 0 \quad \Longleftrightarrow \quad r \in (r_2, r_3) = (\,0.010967587\ldots,\ 4.277733509\ldots\,).$$

Therefore there are positive values of r satisfying the inequality; for example,

$$r = 1 \ \text{ satisfies } \ (G_1 + \eta r)\Big(G_2 + \ell r k(a + br)\Big) < r\,.$$

Hence, the hypotheses of Dhage's fixed-point Theorem as used in Theorem 2.2 are satisfied and there exists at least one continuous solution $y \in C(J)$. This completes the detailed verification of Theorem 2.2 in our concrete numeric instance.

Now for the existence of continuous solution x that satisfies the fractional integral equation

$$x(t) = x_0 + \int_0^1 \theta(s, x(s), I^{1-\beta}y(s))ds + \int_0^t y(s)ds,$$

with $\theta(s, x, y) = \frac{1}{4}(|x| + |y|)$ and $\beta = \frac{1}{2}$. Here θ is continuous and satisfies

$$|\theta(t, x_1, y_1) - \theta(t, x_2, y_2)| \le \tfrac{1}{4}(|x_1 - x_2| + |y_1 - y_2|).$$

Thus assumption (v) holds with $\mathcal{M} = \frac{1}{4} < 1$. Define the operator

$$(Fx)(t) = x_0 + \int_0^1 \theta(s, x(s), I^{1-\beta}y(s))ds + \int_0^t y(s)ds.$$

As in Theorem 2.4, F maps a bounded closed convex subset $B_\rho \subset C(J)$ into itself and is completely continuous. By Schauder's fixed point theorem, F has a fixed point $x \in C(J)$, i.e. there exists a continuous solution x.

Construction of ν and verification of Theorem 4.1 conditions

Recall equation (2.2)

$$\nu(t) = \sum_{i=1}^{2} \omega_{1i}(t, \nu(t)) + x(t) \sum_{i=1}^{2} \omega_{2i}(t, \nu(t)),$$

where $x \in C(J)$ is the continuous solution and we denote $N = \sup_{t \in J} |x(t)| < \infty$.

From our choice

$$\omega_{11}(t,x) = 1 + \tfrac{x}{4}, \quad \omega_{12}(t,x) = 2 + \tfrac{x}{5}, \quad \omega_{21}(t,x) = 1 + \tfrac{x}{6}, \quad \omega_{22}(t,x) = 3 + \tfrac{x}{7},$$

we have the linear growth bounds required in assumption (i*) for $i = 1, 2$,

$$|\omega_{1i}(t,x)| \le N_1 + k_1|x|, \qquad |\omega_{2i}(t,x)| \le N_2 + k_2|x|,$$

with the explicit constants

$$N_1 = 2, \quad k_1 = \frac{1}{4}, \qquad N_2 = 3, \quad k_2 = \frac{1}{6}.$$

Now, we verify that this condition hold

$$kk_1 + Nkk_2 < 1.$$

With our numeric constants $k = 2,\ k_1 = \frac{1}{4},\ k_2 = \frac{1}{6}$, this inequality reduces to

$$2 \cdot \tfrac{1}{4} + N \cdot 2 \cdot \tfrac{1}{6} < 1 \quad \Longleftrightarrow \quad \tfrac{1}{2} + \tfrac{1}{3}N < 1 \quad \Longleftrightarrow \quad N < 1.5.$$

which completes the proof of Theorem 5.9 for this hybrid example.

If, in addition, the Lipschitz sum condition

$$\sum_{i=1}^{2} k_{1i}(t) + N \sum_{i=1}^{2} k_{2i}(t) < 1 \quad \text{for a.e. } t \in J$$

holds, here $k_{1i}(t), k_{2i}(t)$ denote pointwise Lipschitz constants in assumption (vi), in our constant-coefficient example they equal k_1, k_2, then repeating the difference estimate as in Theorem 5.11 we obtain for any two solutions ν_1, ν_2

$$|\nu_1(t) - \nu_2(t)| \le \Big(\sum_{i=1}^{2} k_{1i}(t) + N \sum_{i=1}^{2} k_{2i}(t) \Big) |\nu_1(t) - \nu_2(t)|.$$

If the coefficient on the right is < 1 a.e., then $\nu_1 \equiv \nu_2$. For our numeric constants this condition reduces to

$$2k_1 + N2k_2 = 2 \cdot \tfrac{1}{4} + N \cdot 2 \cdot \tfrac{1}{6} = \tfrac{1}{2} + \tfrac{1}{3}N < 1,$$

which again is $N < 1.5$. Thus uniqueness holds under the same numerical constraint.

6. Conclusion

In this paper, we have investigated a class of multi-term hybrid functional integral equations with nonlocal conditions involving fractional derivatives and integrals. By employing Dhage's fixed point theorem and Schauder's fixed point theorem, we established the existence of continuous solutions for the associated differential and integral formulations, and further proved the existence of integrable solutions for the original hybrid problem. Explicit numerical examples were provided to verify the applicability of the theoretical results and to demonstrate that the imposed assumptions are not only sufficient but also practical.

The analysis highlights the flexibility of fixed-point techniques in treating hybrid systems with delays, nonlinear kernels, and fractional-order operators. In particular, the combined use of functional analytic methods with hybrid differential structures broadens the scope of solvability results in nonlinear analysis and integral equations.

Future research may focus on extending these results to systems with variable-order fractional operators, stochastic perturbations, or impulsive effects. Another promising direction is the numerical approximation of the obtained solutions and the development of efficient algorithms that complement the theoretical existence results. Such extensions would further enhance the applicability of hybrid fractional models in physics, engineering, and applied sciences.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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