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An Application of Neutrix Calculus to Modified Degenerate Gamma Function

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Abstract. The modified degenerate Gamma function $\Gamma^*_\lambda(x)$ is defined for positive values of x; however, it is not defined for zero or negative values of x. In this study, the concepts of neutrix and neutrix limit are employed to extend the definition of the modified degenerate Gamma function $\Gamma^*_\lambda(x)$ for all real values of x. The results demonstrate that the established definitions and findings recover the classical results for Euler's Gamma function $\Gamma(x)$ as $\lambda \to 0$ for all real values of x. Additionally, explicit equations for $\Gamma^*_\lambda(0)$ and $\Gamma^*_\lambda(-n)$, where n is a positive integer, are given.

1. Introduction and Preliminaries

The concept of neutrix and its limit is a comprehensive approach to make sense of divergent expressions, and was developed by van der Corput [3]. A neutrix is a set that contains pieces that are discarded during this elimination process. Neutrix limit is a generalized limit concept that attempts to make meaningful the classically undefined or divergent limits by ignoring certain meaningless parts. This concept was developed to interpret some divergent integrals or series encountered, especially in physics and engineering, in a meaningful way. Some expressions, such as $\Gamma(0)$, where Γ is Euler's Gamma function, do not give a finite limit in the classical sense. But in such expressions, neutrix helps us to eliminate the meaningless parts and define the remaining part as finite. In [13], the authors obtained finite renormalizations by applying neutrix calculus to quantum field theory. They showed that while none of the physically measurable results in renormalizable quantum field theory are changed, quantum gravity is rendered more manageable with the neutrix concept.

Neutrices are commutative additive groups of negligible functions that do not contain any constants except zero. Their calculus was developed in connection with asymptotic series and divergent integrals. Encountered, especially in physics and engineering, in a meaningful way. Negligible functions $f(\epsilon)$ of a neutrix N are defined on a domain N' with values in an additive

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group $N^{''}$. Let $N^{'}$ be a set contained in a topological space with a limit point a which does not belong to $N^{'}$. If $f(\epsilon)$ is a function defined on $N^{'}$ with values in $N^{''}$ and it is possible to find a limit point c such that $f(\epsilon)-c\in N$, then c is called the neutrix limit of f as tends to a and we write $\text{N}-\lim_{\epsilon\to 0} f(\epsilon)=c$. Also note that, if $f(\epsilon)$ tends to c in the normal sense as ϵ tends to zero, it converges to c in the neutrix sense, where the neutrix contains all functions that converge to zero as $\epsilon\to 0$. The reader may find the general concept of neutrix in [3].

There are several special functions that have particular significance, and they have been used widely in many applications such as mathematics, physics, and statistics. One of the most nonelementary special functions is the Euler's Gamma function, and its integral representation is defined for $x>0$ by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt, \tag{1.1}$$

[1]. Using the regularization technique given in [8], the Gamma function is defined for $-n<x<-n+1$, $n=1,2,\ldots$ with the equation

$$\Gamma(x)=\int_0^\infty t^{x-1}\left[e^{-t}-\sum_{i=1}^{n-1}\frac{(-t)^i}{i!}\right]dt. \tag{1.2}$$

In recent years, the concept of neutrix for dealing with special functions has attracted much attention, [6,7,14–17]. For example, in the classical sense, the Gamma function is not defined for the negative integers. Using the neutrix limit, it was proved in [5] that the Gamma function is defined for all real values of x as

$$\Gamma(x)=\underset{\epsilon\to 0}{\text{N}-\lim}\int_\epsilon^\infty t^{x-1}e^{-t}\,dt. \tag{1.3}$$

Also, it was also shown in [5] that

$$\Gamma(0)=\Gamma^{'}(1), \tag{1.4}$$

$$\Gamma(-n)=\int_1^\infty t^{-n-1}e^{-t}\,dt+\int_0^1 t^{-n-1}\left[e^{-t}-\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}t^i\right]dt+\sum_{i=0}^{n-1}\frac{(-1)^i}{i!(i-n)}, \tag{1.5}$$

and

$$\Gamma(-n)+\frac{1}{n}\Gamma(-n+1)=\frac{(-1)^n}{n!} \tag{1.6}$$

for $n=1,2,\ldots$.

It is important to consider which neutrix makes an expression meaningful. Different sets of neutrix can lead to different limits for the same expression. Therefore, when calculating a neutrix limit, it is essential to clearly specify which neutrix is being used. In [5], N is the neutrix having domain $N^{'}=(0,\infty)$, $N^{''}=\mathbb{R}$ and negligible functions finite linear sums of the functions $\epsilon^\lambda\ln^{r-1}\epsilon$, $\ln^r\epsilon$ where $\lambda<0$, $r=1,2,\ldots$ and all functions $o(\epsilon)$ which converge to zero in the normal sence as ϵ tends to zero. In this study, we use the neutrix N_λ, again having domain $N^{'}=(0,\infty)$, $N^{''}=\mathbb{R}$ and negligible functions finite linear sums of the functions $\epsilon^\lambda\ln^{r-1}(f(\lambda)\epsilon)$, $\ln^r(f(\lambda)\epsilon)$ where $\lambda<0$, $r=1,2,\ldots$, $\lim_{\lambda\to 0}f(\lambda)=1$ and all functions $o(\epsilon)$ which converge to zero in the normal sence as

ϵ tends to zero. Note that since we have the relation $N \subset N_\lambda$ for the given neutrix sets N and N_λ, if $\mathrm{N}-\lim_{\epsilon\to 0} f(\epsilon) = c$, then we have $\mathrm{N}_\lambda-\lim_{\epsilon\to 0} f(\epsilon) = c$.

In recent years, the degenerate versions of the Gamma function, namely the modified Gamma and the modified degenerate Gamma functions, were introduced and investigated, [9–12], and the related degenerate Laplace transforms were studied, [2,4,9,10].

The modified degenerate Gamma function is denoted as

$$\Gamma^*_\lambda(z) = \int_0^\infty t^{z-1}(1+\lambda)^{-\frac{t}{\lambda}}\, dt, \quad \lambda \in (0,1) \text{ and } Re(z) > 0, \tag{1.7}$$

[10], and it satisfies the properties

$$\Gamma^*_\lambda(1) = \frac{\lambda}{\ln(1+\lambda)}, \tag{1.8}$$

$$\Gamma^*_\lambda(z+1) = \frac{\lambda z}{\ln(1+\lambda)}\Gamma^*_\lambda(z), \tag{1.9}$$

$$\Gamma^*_\lambda(n+1) = \frac{\lambda^{n+1} n!}{(\ln(1+\lambda))^{n+1}}, \;\; n = 1, 2, \ldots, \tag{1.10}$$

and

$$\Gamma^*_\lambda(z+1) = \frac{\lambda^{n+1} z(z-1)\ldots(z-n)}{(\ln(1+\lambda))^{n+1}}\Gamma^*_\lambda(z-n), \;\; n \geq 0. \tag{1.11}$$

Additionally, the relationship between Euler's classical Gamma function Γ and Γ^*_λ is given by

$$\Gamma^*_\lambda(z) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^z \Gamma(z), \;\; z \in \mathbb{C}, Re(z) > 0. \tag{1.12}$$

In this paper, we extend the equation

$$\Gamma^*_\lambda(x) = \int_0^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\, dt, \quad \lambda \in (0,1) \text{ and } x > 0,$$

to all real values of x using the neutrix and the neutrix limit. We also demonstrate that this extended definition still adheres to the relationship outlined in the equation (1.12). Additionally, we present several equations for $\Gamma^*_\lambda(-n)$, $n = 0, 1, 2, \ldots$.

2. Main Results

In this section, we utilize the regularization approach developed by Gel'fand and Shilov in [8] specifically, as well as the concept of neutrix to make sense of divergent expressions. Using this technique, we subtract enough terms of the Taylor series of the function $(1+\lambda)^{-\frac{t}{\lambda}}$ to extend the domain of the convergence of some integrals.

Theorem 2.1. *The neutrix limits as ϵ tends to zero of the integrals*

$$\int_\epsilon^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\, dt \;\; \text{and} \;\; \int_{\lambda\epsilon/\ln(1+\lambda)}^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\, dt$$

exist, and the equality

$$\operatorname{N\check{}-lim}_{\epsilon\to 0}\int_{\epsilon}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \operatorname{N\check{}-lim}_{\epsilon\to 0}\int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\Gamma(x) \tag{2.1}$$

is valid for $-n < x < -n+1$, $n = 1, 2, \ldots$.

Proof. We have

$$\int_{\epsilon}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \int_{\epsilon}^{\infty} t^{x-1}e^{-t\ln(1+\lambda)/\lambda}\,dt = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\int_{\ln(1+\lambda)\epsilon/\lambda}^{\infty} u^{x-1}e^{-u}\,du.$$

Now, omitting of the first $n-1$ terms of the series of the exponential function e^{-u} we get

$$\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{x}\int_{\epsilon}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \int_{\ln(1+\lambda)\epsilon/\lambda}^{\infty} u^{x-1}\left[e^{-u}-\sum_{i=1}^{n-1}\frac{(-u)^{i}}{i!}\right]du$$
$$+\sum_{i=1}^{n-1}\frac{(-1)^{i}}{i!}\int_{\ln(1+\lambda)\epsilon/\lambda}^{\infty} u^{x+i-1}\,du.$$

Since $0 \le i < n-1$ and $-n < x < -n+1$, we have

$$\int_{\ln(1+\lambda)\epsilon/\lambda}^{\infty} u^{x+i-1}\,du = \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{x+i}\frac{\epsilon^{x+i}}{x+i} \in N \subset N_{\lambda}.$$

Then we get

$$\begin{aligned}
\operatorname{N\check{}-lim}_{\epsilon\to 0}\int_{\epsilon}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt &= \operatorname{N\check{}-lim}_{\epsilon\to 0}\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\int_{\ln(1+\lambda)\epsilon/\lambda}^{\infty} u^{x-1}\left[e^{-u}-\sum_{i=1}^{n-1}\frac{(-u)^{i}}{i!}\right]du\\
&= \lim_{\epsilon\to 0}\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\int_{\ln(1+\lambda)\epsilon/\lambda}^{\infty} u^{x-1}\left[e^{-u}-\sum_{i=1}^{n-1}\frac{(-u)^{i}}{i!}\right]du\\
&= \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\int_{0}^{\infty} u^{x-1}\left[e^{-u}-\sum_{i=1}^{n-1}\frac{(-u)^{i}}{i!}\right]du = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\Gamma(x)
\end{aligned}\tag{2.2}$$

by using the equation (1.2) for $-n < x < -n+1$. Now, we give the existence of

$$\operatorname{N\check{}-lim}_{\epsilon\to 0}\int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt.$$

Since

$$\int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{x-1}e^{-t\ln(1+\lambda)/\lambda}\,dt = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\int_{\epsilon}^{\infty} t^{x-1}e^{-t}\,dt,$$

$$\operatorname{N-lim}_{\epsilon\to 0}\int_{\epsilon}^{\infty} t^{x-1}e^{-t}\,dt = \Gamma(x)$$

for $-n < x < -n+1$, and

$$\operatorname{N-lim}_{\epsilon\to 0}\int_{\epsilon}^{\infty} t^{x-1}e^{-t}\,dt = \operatorname{N\check{}-lim}_{\epsilon\to 0}\int_{\epsilon}^{\infty} t^{x-1}e^{-t}\,dt = \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{x}\operatorname{N\check{}-lim}_{\epsilon\to 0}\int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt,$$

then

$$\underset{\epsilon\to 0}{\text{N-lim}} \int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\Gamma(x) \tag{2.3}$$

for $-n < x < -n+1$. Hence the equation (2.1) follows by using the equations (2.2) and (2.3). □

We are now able to provide the following definition.

Definition 2.1. *The modified degenerate Gamma function* Γ^*_λ *is defined by*

$$\Gamma^*_\lambda(x) = \underset{\epsilon\to 0}{\text{N-lim}} \int_{\epsilon}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \underset{\epsilon\to 0}{\text{N-lim}} \int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt$$

for $-n < x < -n+1$, $n = 1, 2, \ldots$.

Theorem 2.2. *We have*

$$\Gamma^*_\lambda(x) = \int_0^{\infty} t^{x-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \sum_{i=1}^{n-1}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i}\frac{(-t)^i}{i!}\right]dt \tag{2.4}$$

for $-n < x < -n+1$, $n = 1, 2, \ldots$.

Proof. Let $-n < x < -n+1$, $n = 1, 2, \ldots$. Then we have

$$\int_{\epsilon}^{\infty} t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \int_{\epsilon}^{\infty} t^{x-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \sum_{i=1}^{n-1}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i}\frac{(-t)^i}{i!}\right]dt + \sum_{i=1}^{n-1}\frac{(-1)^i}{i!}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i}\int_{\epsilon}^{\infty} t^{x+i-1}\,dt.$$

Now, by taking the neutrix limit of both sides of the last equation as ϵ tends to zero and using the Definition (2.1), the result follows. □

Note that by using the equation (2.1) and the Definition (2.1), we get that the the equality

$$\Gamma^*_\lambda(x) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\Gamma(x) \tag{2.5}$$

is valid for $-n < x < -n+1$.

The relation

$$\Gamma^*_\lambda(x+1) = \frac{\lambda x}{\ln(1+\lambda)}\Gamma^*_\lambda(x), \quad x > 0 \tag{2.6}$$

is used to define $\Gamma^*_\lambda(x)$ for negative, non-integer valus of x. Thus, if $-n < x < -n+1$, then

$$\Gamma^*_\lambda(x) = \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n+1}\frac{\Gamma^*_\lambda(x+n+1)}{x(x+1)\ldots(x+n)}. \tag{2.7}$$

Moreover, the equation (2.7) shows that the function Γ^*_λ has simple poles at $x = 0, -1, -2, \ldots$ with the residues

$$Res(\Gamma^*_\lambda(x), -n) = \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n+1}\frac{\Gamma^*_\lambda(1)}{(-n)(-n+1)\ldots(-1)} = \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n}\frac{(-1)^n}{n!}.$$

Now, we define $\Gamma^*_\lambda(-n)$, $n = 1, 2, \ldots$. The Theorem 2.1 suggest that we can define $\Gamma^*_\lambda(-n)$ as

$$\Gamma^*_\lambda(-n) = \operatorname*{N\check{}-lim}_{\epsilon\to 0}\int_\epsilon^\infty t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \operatorname*{N\check{}-lim}_{\epsilon\to 0}\int_{\lambda\epsilon/\ln(1+\lambda)}^\infty t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt \tag{2.8}$$

for $n = 0, 1, 2, \ldots$ provided that the neutrix limit of the last integrals exists. First of all, we prove the following theorem.

Theorem 2.3. *The neutrix limits as ϵ tends to zero of the integrals*

$$\int_\epsilon^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt \quad \textit{and} \quad \int_{\lambda\epsilon/\ln(1+\lambda)}^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt$$

exist, and

$$\operatorname*{N\check{}-lim}_{\epsilon\to 0}\int_\epsilon^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \operatorname*{N\check{}-lim}_{\epsilon\to 0}\int_{\lambda\epsilon/\ln(1+\lambda)}^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \Gamma(0). \tag{2.9}$$

Proof. We have

$$\int_\epsilon^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \int_\epsilon^\infty t^{-1}e^{-t\ln(1+\lambda)/\lambda}\,dt = \int_{\ln(1+\lambda)\epsilon/\lambda}^\infty u^{-1}e^{-u}\,du.$$

Then integrating by parts we get

$$\int_\epsilon^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = -e^{\ln(1+\lambda)\epsilon/\lambda}\ln\left(\ln(1+\lambda)\epsilon/\lambda\right) + \int_{\ln(1+\lambda)\epsilon/\lambda}^\infty e^{-u}\ln u\,du.$$

Since

$$e^{\ln(1+\lambda)\epsilon/\lambda}\ln\left(\ln(1+\lambda)\epsilon/\lambda\right) = \ln\left(\ln(1+\lambda)\epsilon/\lambda\right) + o(\epsilon) \in N_\lambda,$$

we have

$$\operatorname*{N\check{}-lim}_{\epsilon\to 0}\int_\epsilon^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \operatorname*{N\check{}-lim}_{\epsilon\to 0}\int_{\ln(1+\lambda)\epsilon/\lambda}^\infty e^{-u}\ln u\,du = \int_0^\infty e^{-u}\ln u\,du = \Gamma'(1) = \Gamma(0)$$

by using the equation (1.4). Now, we prove the existence of

$$\operatorname*{N\check{}-lim}_{\epsilon\to 0}\int_{\lambda\epsilon/\ln(1+\lambda)}^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt. \tag{2.10}$$

We write

$$\int_{\lambda\epsilon/\ln(1+\lambda)}^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \int_{\lambda\epsilon/\ln(1+\lambda)}^\infty t^{-1}e^{-t\ln(1+\lambda)/\lambda}\,dt = \int_\epsilon^\infty u^{-1}e^{-u}\,du.$$

Since

$$\operatorname*{N-lim}_{\epsilon\to 0}\int_\epsilon^\infty u^{-1}e^{-u}\,du = \Gamma(0),$$

and

$$\operatorname*{N-lim}_{\epsilon\to 0}\int_\epsilon^\infty u^{-1}e^{-u}\,du = \operatorname*{N\check{}-lim}_{\epsilon\to 0}\int_\epsilon^\infty u^{-1}e^{-u}\,du$$

we get the existence of (2.10) and the equality (2.9). □

Definition 2.2. *The modified degenerate Gamma function Γ^*_λ is defined at $x = 0$ by*

$$\Gamma^*_\lambda(0) = \mathrm{N}\text{-}\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \mathrm{N}\text{-}\lim_{\epsilon\to 0}\int_{\lambda\epsilon/\ln(1+\lambda)}^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt.$$

By using the equation (2.9) and the Definition 2.2, we get that the equality

$$\Gamma^*_\lambda(x) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^x \Gamma(x). \tag{2.11}$$

is valid for $x = 0$.

Corollary 2.1. *We have*

$$\Gamma^*_\lambda(0) = \int_0^1 t^{-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - 1\right] dt + \int_1^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt \tag{2.12}$$

Proof. Since

$$\int_\epsilon^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \int_\epsilon^1 t^{-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - 1\right] dt + \int_\epsilon^1 t^{-1}\,dt + \int_1^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt,$$

it follows that

$$\mathrm{N}\text{-}\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \int_0^1 t^{-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - 1\right] dt + \int_1^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt,$$

and the result follows. □

Corollary 2.2. *We have*

$$\Gamma^*_\lambda(0) = \frac{\ln(1+\lambda)}{\lambda}\left(\Gamma^*_\lambda\right)'(1). \tag{2.13}$$

Proof. By using the Corollary 2.1 we have

$$\begin{aligned}
\Gamma^*_\lambda(0) &= \int_0^1 t^{-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - 1\right] dt + \int_1^\infty t^{-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt \\
&= \left[(1+\lambda)^{-\frac{t}{\lambda}} - 1\right]\ln t\Big|_0^1 + (1+\lambda)^{-\frac{t}{\lambda}}\ln t\Big|_1^\infty + \frac{\ln(1+\lambda)}{\lambda}\int_0^\infty \ln t\,(1+\lambda)^{-\frac{t}{\lambda}}\,dt \\
&= \frac{\ln(1+\lambda)}{\lambda}\int_0^\infty \ln t\,(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \frac{\ln(1+\lambda)}{\lambda}\left[\frac{d}{dx}\Gamma^*_\lambda(x)\right]\Big|_{x=1} = \frac{\ln(1+\lambda)}{\lambda}\left(\Gamma^*_\lambda\right)'(1),
\end{aligned}$$

and the result follows. □

Theorem 2.3 proves that the equation (2.8) is valid for $n = 0$.
To illustrate our findings, let's assign meaning to the divergent integral $\int_0^\infty (1+\lambda)^{-\frac{tx}{\lambda}} t^{-1}\,dt$.

Example 2.1. *We want to show that*

$$\mathrm{N}\text{-}\lim_{\epsilon\to 0}\int_\epsilon^\infty (1+\lambda)^{-\frac{tx}{\lambda}} t^{-1}\,dt = \Gamma^*_\lambda(0) - \ln x$$

for $\lambda \in (0,1)$ and $x > 0$.
Let $0 < \epsilon x < 1$. Then we have

$$\int_{\epsilon}^{\infty} (1+\lambda)^{-\frac{tx}{\lambda}} t^{-1}\, dt = \int_{\epsilon x}^{\infty} (1+\lambda)^{-\frac{u}{\lambda}} u^{-1}\, du$$
$$= \int_{\epsilon x}^{1} u^{-1}\left[(1+\lambda)^{-\frac{u}{\lambda}} - 1\right] du + \int_{\epsilon x}^{1} u^{-1}\, du + \int_{1}^{\infty} (1+\lambda)^{-\frac{u}{\lambda}} u^{-1}\, du.$$

Hence we get

$$\begin{aligned}
&\text{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} (1+\lambda)^{-\frac{tx}{\lambda}} t^{-1}\, dt \\
&= \text{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon x}^{\infty} (1+\lambda)^{-\frac{t}{\lambda}} t^{-1}\, dt \\
&= \lim_{\epsilon\to 0} \int_{\epsilon x}^{1} t^{-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - 1\right] dt + \text{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon x}^{1} t^{-1}\, dt + \int_{1}^{\infty} (1+\lambda)^{-\frac{t}{\lambda}} t^{-1}\, dt \\
&= \int_{0}^{1} t^{-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - 1\right] dt - \ln x + \int_{1}^{\infty} (1+\lambda)^{-\frac{t}{\lambda}} t^{-1}\, dt \\
&= \Gamma_{\lambda}^{*}(0) - \ln x,
\end{aligned}$$

by using the equation (2.12).

Theorem 2.4. *The neutrix limit as ϵ tends to zero of the integrals*

$$\int_{\epsilon}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\, dt \quad \textit{and} \quad \int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\, dt$$

exist, and the equality

$$\text{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\, dt = \text{N}\!-\!\lim_{\epsilon\to 0} \int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\, dt = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{-n} \Gamma(-n) \tag{2.14}$$

is valid for $n = 1, 2, \ldots$.

Proof. Firstly, we use the induction method. Let us assume that the neutrix limit

$$\text{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} t^{-n}(1+\lambda)^{-\frac{t}{\lambda}}\, dt$$

exists. On integrating by parts, we obtained

$$\int_{\epsilon}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\, dt = \frac{1}{n}(1+\lambda)^{-\frac{\epsilon}{\lambda}}\epsilon^{-n} - \frac{\ln(1+\lambda)}{\lambda n} \int_{\epsilon}^{\infty} t^{-n}(1+\lambda)^{-\frac{t}{\lambda}}\, dt$$
$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{1}{n} \epsilon^{i-n} \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i} - \frac{\ln(1+\lambda)}{\lambda n} \int_{\epsilon}^{\infty} t^{-n}(1+\lambda)^{-\frac{t}{\lambda}}\, dt.$$

Now taking the neutrix limit of both sides, we get

$$\begin{aligned}\underset{\epsilon\to 0}{\text{N}\check{}\text{-lim}}\int_{\epsilon}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt &= \underset{\epsilon\to 0}{\text{N}\check{}\text{-lim}}\left[\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{1}{n}\epsilon^{i-n}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^i\right]+\frac{(-1)^n}{n!}\frac{1}{n}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^n\\ &+\underset{\epsilon\to 0}{\text{N}\check{}\text{-lim}}\sum_{i=n+1}^{\infty}\frac{(-1)^i}{i!}\frac{1}{n}\epsilon^{i-n}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^i-\frac{\ln(1+\lambda)}{\lambda n}\underset{\epsilon\to 0}{\text{N}\check{}\text{-lim}}\int_{\epsilon}^{\infty} t^{-n}(1+\lambda)^{-\frac{t}{\lambda}}\,dt\\ &=\frac{(-1)^n}{nn!}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^n-\frac{\ln(1+\lambda)}{\lambda n}\underset{\epsilon\to 0}{\text{N}\check{}\text{-lim}}\int_{\epsilon}^{\infty} t^{-n}(1+\lambda)^{-\frac{t}{\lambda}}\,dt. \qquad (2.15)\end{aligned}$$

Then, by the assumption, we get the existence of

$$\underset{\epsilon\to 0}{\text{N}\check{}\text{-lim}}\int_{\epsilon}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt.$$

Now writing

$$\int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt=\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{-n}\int_{\epsilon}^{\infty}u^{-n-1}e^{-u}\,du,$$

taking the neutrix limit of both sides of the last equality, and by using equation (1.3), we have

$$\underset{\epsilon\to 0}{\text{N}\check{}\text{-lim}}\int_{\lambda\epsilon/\ln(1+\lambda)}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt=\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{-n}\underset{\epsilon\to 0}{\text{N-lim}}\int_{\epsilon}^{\infty}u^{-n-1}e^{-u}\,du=\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{-n}\Gamma(-n), \qquad (2.16)$$

which gives the second existence. Now, we prove the equality (2.14). Since

$$\begin{aligned}\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{n}\int_{\epsilon}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt &= \int_{\ln(1+\lambda)\epsilon/\lambda}^{\infty}u^{-n-1}e^{-u}\,du\\ &=\int_{\ln(1+\lambda)\epsilon/\lambda}^{1}u^{-n-1}\left[e^{-u}-\sum_{i=0}^{n}\frac{(-1)^i}{i!}u^i\right]du+\sum_{i=0}^{n}\frac{(-1)^i}{i!}\int_{\ln(1+\lambda)\epsilon/\lambda}^{1}u^{i-n-1}\,du+\int_{1}^{\infty}u^{-n-1}e^{-u}\,du\\ &=\int_{\ln(1+\lambda)\epsilon/\lambda}^{1}u^{-n-1}\left[e^{-u}-\sum_{i=0}^{n}\frac{(-1)^i}{i!}u^i\right]du\\ &+\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\left(\frac{1}{i-n}-\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i-n}\frac{\epsilon^{i-n}}{i-n}\right)-\frac{(-1)^n}{n!}\ln\left(\ln(1+\lambda)\epsilon/\lambda\right)+\int_{1}^{\infty}u^{-n-1}e^{-u}\,du,\end{aligned}$$

then taking the neutrix limit of both sides of the last equaation we have

$$\begin{aligned}&\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{n}\underset{\epsilon\to 0}{\text{N}\check{}\text{-lim}}\int_{\epsilon}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt\\ &=\underset{\epsilon\to 0}{\text{N}\check{}\text{-lim}}\int_{\ln(1+\lambda)\epsilon/\lambda}^{1}u^{-n-1}\left[e^{-u}-\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}u^i\right]du+\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{1}{i-n}+\int_{1}^{\infty}u^{-n-1}e^{-u}\,du\\ &=\lim_{\epsilon\to 0}\int_{\ln(1+\lambda)\epsilon/\lambda}^{1}u^{-n-1}\left[e^{-u}-\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}u^i\right]du+\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{1}{i-n}+\int_{1}^{\infty}u^{-n-1}e^{-u}\,du\\ &=\int_{0}^{1}u^{-n-1}\left[e^{-u}-\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}u^i\right]du+\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{1}{i-n}+\int_{1}^{\infty}u^{-n-1}e^{-u}\,du. \qquad (2.17)\end{aligned}$$

Hence, we obtain the equality (2.14) by utilizing the equations (1.5), (2.16), and (2.17).(2.17). □

We are now prepared to provide the following definition.

Definition 2.3. *The modified degenerate Gamma function* Γ^*_λ *is defined by*

$$\Gamma^*_\lambda(x) = \text{N}\text{-}\lim_{\epsilon\to 0} \int_\epsilon^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \text{N}\text{-}\lim_{\epsilon\to 0} \int_{\lambda\epsilon/\ln(1+\lambda)}^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt$$

for all real values of x*.*

Note that since

$$\int_0^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt$$

is convergent for $x > 0$, we can deduce that

$$\begin{aligned}\text{N}\text{-}\lim_{\epsilon\to 0} \int_\epsilon^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt &= \lim_{\epsilon\to 0} \int_\epsilon^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt \\ &= \lim_{\epsilon\to 0} \int_{\lambda\epsilon/\ln(1+\lambda)}^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \int_0^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \Gamma^*_\lambda(x)\end{aligned}$$

also holds for $x > 0$. Together with the last equality, the Definition 2.3 is shown to be well-defined for all real values of x.

Additionally, by utilizing the equalities (2.5), (2.11), (2.14), and Definition 2.3, the equalities

$$\Gamma^*_\lambda(x) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^x \Gamma(x), \tag{2.18}$$

and

$$\lim_{\lambda\to 0} \Gamma^*_\lambda(x) = \Gamma(x) \tag{2.19}$$

are valid for all real values of x.

Example 2.2. *We solve the equation* $\frac{dy}{dx} + y(x) = \Gamma^*_\lambda(-1)$*.*
Since $\Gamma^*_\lambda(x)$ *has a simple pole at* $x = -1$*, in the classical sense, we have* $\frac{dy}{dx} + y(x) = \infty$*. However, in the neutrix sense, the expression* $\Gamma^*_\lambda(-1)$ *has been made significant with the relation*

$$\Gamma^*_\lambda(-1) = \text{N}\text{-}\lim_{\epsilon\to 0} \int_\epsilon^\infty t^{-2}(1+\lambda)^{-\frac{t}{\lambda}}\,dt. \tag{2.20}$$

Since we have proved that the neutrix limit in equation (2.20) exists, we can consider $\Gamma^*_\lambda(-1)$ *as a constant denoted by* c_{-1}*. Now we have a linear differential equation with constant coefficients*

$$\frac{dy}{dx} + y(x) = c_{-1}.$$

Then taking the integration factor e^x *we get*

$$\frac{d}{dx}(e^x\, y(x)) = c_{-1}e^x,$$

and this leads to the solution $y(x) = c_{-1} + Ce^{-x}$*.*

By using the equation (2.15) and the Definition 2.3 we have the following result.

Corollary 2.3. *We have*

$$\Gamma^*_\lambda(-n) = \frac{(-1)^n}{nn!}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^n - \frac{\ln(1+\lambda)}{\lambda n}\Gamma^*_\lambda(-n+1) \tag{2.21}$$

for $n = 1, 2, \ldots$.

Note that, from the equations (1.6), (2.19), and (2.21) we get

$$\lim_{\lambda\to 0}\Gamma^*_\lambda(-n) = \lim_{\lambda\to 0}\left[\frac{(-1)^n}{nn!}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^n - \frac{\ln(1+\lambda)}{\lambda n}\Gamma^*_\lambda(-n+1)\right] = \frac{(-1)^n}{nn!} - \frac{1}{n}\Gamma(-n+1) = \Gamma(-n).$$

for $n = 1, 2, \ldots$.
Additionally, by using the equation (2.21), we find that the equation (1.9) does not hold for $x = -n$, $n = 1, 2, \ldots$.

Corollary 2.4. *The equality*

$$\Gamma^*_\lambda(-n) = \frac{(-1)^n}{n!}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^n\left[\Phi(n) + \Gamma^*_\lambda(0)\right] \tag{2.22}$$

is valid for $n = 1, 2, \ldots$, *where* $\phi(n) = \sum_{i=1}^n \frac{1}{i}$.

Proof. We use the induction method. Since for $n = 1$, the equation (2.22) reduces to the equation (2.21), then the equation (2.22) is held for $n = 1$. Now, let us assume that the equation (2.22) exists for some n. Then, by using the equation (2.21) and our assumption, we get

$$\begin{aligned}\Gamma^*_\lambda(-n-1) &= \frac{(-1)^{n+1}}{(n+1)(n+1)!}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n+1} - \frac{\ln(1+\lambda)}{\lambda(n+1)}\Gamma^*_\lambda(-n) \\ &= \frac{(-1)^{n+1}}{(n+1)(n+1)!}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n+1} + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n+1}\left[\phi(n) + \Gamma^*_\lambda(0)\right] \\ &= \frac{(-1)^{n+1}}{(n+1)(n+1)!}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n+1}\left[\phi(n+1) + \Gamma^*_\lambda(0)\right].\end{aligned}$$

This confirms that the equation (2.22) holds true for $n + 1$, completing the proof. □

Theorem 2.5. *We have*

$$\Gamma^*_\lambda(-n) = \int_0^\infty t^{-n-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \sum_{i=0}^{n-1}\left(\left(\frac{\ln(1+\lambda)}{\lambda}\right)^i\frac{(-t)^i}{i!}\right) - \left(\frac{\ln(1+\lambda)}{\lambda}\right)^n\frac{(-t)^n}{n!}H(1-t)\right]dt \tag{2.23}$$

for $n = 1, 2, \ldots$.

Proof. Let us write

$$\begin{aligned}&\int_\epsilon^\infty t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt = \\ &\int_\epsilon^\infty t^{-n-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \sum_{i=0}^{n-1}\left(\left(\frac{\ln(1+\lambda)}{\lambda}\right)^i\frac{(-t)^i}{i!}\right) - \left(\frac{\ln(1+\lambda)}{\lambda}\right)^n\frac{(-t)^n}{n!}H(1-t)\right]dt \\ &+ \sum_{i=0}^{n-1}\left(\frac{\ln(1+\lambda)}{\lambda}\right)^i\frac{(-1)^i}{i!}\int_\epsilon^\infty t^{-n+i-1}\,dt + \left(\frac{\ln(1+\lambda)}{\lambda}\right)^n\frac{(-1)^n}{n!}\int_\epsilon^1 t^{-1}\,dt.\end{aligned}$$

Then, by using the negligible functions of our neutrix, we get

$$
\begin{aligned}
&\mathrm{N}\text{-}\lim_{\epsilon\to 0}\int_{\epsilon}^{\infty} t^{-n-1}(1+\lambda)^{-\frac{t}{\lambda}}\,dt \\
&= \mathrm{N}\text{-}\lim_{\epsilon\to 0}\int_{\epsilon}^{\infty} t^{-n-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \sum_{i=0}^{n-1}\left(\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i}\frac{(-t)^i}{i!}\right) - \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n}\frac{(-t)^n}{n!}H(1-t)\right] dt \\
&= \lim_{\epsilon\to 0}\int_{\epsilon}^{\infty} t^{-n-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \sum_{i=0}^{n-1}\left(\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i}\frac{(-t)^i}{i!}\right) - \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n}\frac{(-t)^n}{n!}H(1-t)\right] dt \\
&= \int_{0}^{\infty} t^{-n-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \sum_{i=0}^{n-1}\left(\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i}\frac{(-t)^i}{i!}\right) - \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n}\frac{(-t)^n}{n!}H(1-t)\right] dt,
\end{aligned}
$$

and the result follows. □

Corollary 2.5. *The equation*

$$\mathrm{N}\text{-}\lim_{\epsilon\to 0}\Gamma^*_{\lambda}(x+\epsilon) = \Gamma^*_{\lambda}(x) \tag{2.24}$$

is valid for all real values of x.

Proof. Firstly, let $-n < x < -n+1$, $n = 1, 2, \ldots$. Since the equation

$$\Gamma^*_{\lambda}(x) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\Gamma(x)$$

holds true for all real values of x, we can write

$$
\begin{aligned}
\lim_{\epsilon\to 0}\Gamma^*_{\lambda}(x+\epsilon) &= \left[\lim_{\epsilon\to 0}\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x+\epsilon}\Gamma(x+\epsilon)\right] \\
&= \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\Gamma(\lim_{\epsilon\to 0}(x+\epsilon)) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x}\Gamma(x) = \Gamma^*_{\lambda}(x)
\end{aligned}
$$

by the continuity of $\Gamma(x)$. Hence the equation (2.24) follows for $-n < x < -n+1$, $n = 1, 2, \ldots$. Now let $x = -n$, $n = 1, 2, \ldots$ and $0 < \epsilon < 1$. Then by using the equation (2.4) we have

$$
\begin{aligned}
\Gamma^*_{\lambda}(x+\epsilon) &= \int_{0}^{\infty} t^{x+\epsilon-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \sum_{i=0}^{n-1}\left(\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i}\frac{(-t)^i}{i!}\right)\right] dt \\
&= \int_{0}^{\infty} t^{x+\epsilon-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \sum_{i=0}^{n-1}\left(\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i}\frac{(-t)^i}{i!}\right) - \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n}\frac{(-t)^n}{n!}H(1-t)\right] dt \\
&+ \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n}\frac{(-1)^n}{n!}\int_{0}^{1} t^{\epsilon-1}\,dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathrm{N}\text{-}\lim_{\epsilon\to 0}\Gamma^*_{\lambda}(x+\epsilon) \\
&= \lim_{\epsilon\to 0}\int_{0}^{\infty} t^{x+\epsilon-1}\left[(1+\lambda)^{-\frac{t}{\lambda}} - \sum_{i=0}^{n-1}\left(\left(\frac{\ln(1+\lambda)}{\lambda}\right)^{i}\frac{(-t)^i}{i!}\right) - \left(\frac{\ln(1+\lambda)}{\lambda}\right)^{n}\frac{(-t)^n}{n!}H(1-t)\right] dt
\end{aligned}
$$

$$= \Gamma^*_\lambda(x)$$

by using the Theorem 2.5. Hence the equation (2.24) follows for $x = -n$, $n = 1, 2, \ldots$.
Also, since $\Gamma^*_\lambda(x)$ is convergent for $x > 0$ we have

$$\lim_{\epsilon \to 0} \Gamma^*_\lambda(x+\epsilon) = \Gamma^*_\lambda(x).$$

Then the equation (2.24) is also true for the values $x \geq 0$, and the proof is completed. □

Corollary 2.6. *The equation*

$$\Gamma^*_\lambda(x+1) = \frac{\lambda}{\ln(1+\lambda)} \operatorname{N-lim}_{\epsilon \to 0} \left[(x+\epsilon)\Gamma^*_\lambda(x+\epsilon)\right] \tag{2.25}$$

is valid for all rel values of x.

Proof. Let $-n < x < -n+1$, $n = 1, 2, \ldots$. Then we have

$$\lim_{\epsilon \to 0} \left[(x+\epsilon)\Gamma^*_\lambda(x+\epsilon)\right] = \lim_{\epsilon \to 0} \left[(x+\epsilon)\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x+\epsilon} \Gamma(x+\epsilon)\right]$$
$$= x\left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x} \Gamma(x) = x\Gamma^*_\lambda(x) = \frac{\ln(1+\lambda)}{\lambda}\Gamma^*_\lambda(x+1)$$

by the continuity of $\Gamma(x)$.
Now, for $x = -n$, $n = 1, 2, \ldots$ and $0 < \epsilon < 1$ we have

$$\Gamma^*_\lambda(-n+\epsilon+1) = \frac{\lambda}{\ln(1+\lambda)}(-n+\epsilon)\Gamma^*_\lambda(-n+\epsilon).$$

Then

$$\operatorname{N-lim}_{\epsilon \to 0} \Gamma^*_\lambda(-n+\epsilon+1) = \frac{\lambda}{\ln(1+\lambda)} \operatorname{N-lim}_{\epsilon \to 0} \left[(-n+\epsilon)\Gamma^*_\lambda(-n+\epsilon)\right].$$

Hence, by using the Corollary 2.5, we get

$$\Gamma^*_\lambda(-n+1) = \frac{\lambda}{\ln(1+\lambda)} \operatorname{N-lim}_{\epsilon \to 0} \left[(-n+\epsilon)\Gamma^*_\lambda(-n+\epsilon)\right],$$

and the result follows. □

Conclusion

The modified degenerate Gamma function $\Gamma^*_\lambda(x)$ is defined for $x > 0$. Since a neutrix can help us to eliminate the meaningless part, it gives the idea that we can define $\Gamma^*_\lambda(x)$ for $x \leq 0$ by using the concept of neutrix. We proved that the neutrix limit in our definition for $\Gamma^*_\lambda(x)$ exist for all real values of x. Also we showed that the recurrence relation

$$\Gamma^*_\lambda(x+1) = \frac{\lambda x}{\ln(1+\lambda)}\Gamma^*_\lambda(x), \quad x > 0,$$

and the equation

$$\Gamma^*_\lambda(x) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^{x} \Gamma(x), \quad x > 0,$$

which gives the relation between the modified degenerate Gamma and Euler's classical Gamma function $\Gamma(x)$, still hold for all real values of x. We included some examples to illustrate the use of our definition. We also showed that the established definition and results give the previously given results of Euler's classical Gamma function $\Gamma(x)$ as $\lambda \to 0$ for all real values of x. Furthermore, we provided several equations for $\Gamma^*_\lambda(-n)$, $n = 0, 1, 2, \ldots$.

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