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Localization Regimes for Quasilinear Parabolic p-Laplacian Equations under Unbounded Boundary Forcing

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Abstract. This study formulates an open problem on spatial localization for quasilinear parabolic equations of p-Laplacian type on the half-space $\mathbb{R}^N_+$, with smooth, nonnegative initial data and boundary conditions that grow unbounded over time. While localization has been widely studied under homogeneous or decaying boundary conditions, its persistence under unbounded boundary input remains unresolved. We introduce two forms of spatial localization: effective localization, where the solution remains within a finite spatial domain, and strict localization, where it vanishes beyond a fixed boundary. The problem examines how the structure of the nonlinearities $A(v)$ and $B(v)$, and the growth rate of the boundary function φ influence solution confinement or spread. The study extends classical localization theory to degenerate and singular diffusion processes with diffusion exponent $p > 1$, offering new insight into spatial confinement under persistent external forcing.

1. Introduction

Spatial localization is a fundamental property of solutions to nonlinear parabolic equations. It plays a crucial role in understanding the propagation dynamics of diffusion-dominated systems, particularly those governed by degenerate and singular diffusion processes modeled by quasilinear parabolic equations of p-Laplacian type. The evolution of the solution's support is highly sensitive to the interplay between the nonlinear diffusion structure and the imposed boundary input.

Classical results have established localization for equations with compactly supported or decaying initial data and homogeneous or bounded boundary inputs. For instance, the pioneering work on the porous medium equation [1] and subsequent extensions [2] demonstrated that finite speed of propagation and interface formation are characteristic of nonlinear diffusion equations. Similar phenomena have been studied for p-Laplacian-type diffusion equations [3], [4], [5], [6], where both degenerate and singular behaviors emerge depending on the value of p.

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However, much less is understood about solution behavior when the boundary condition increases without bound over time. Such a setting arises naturally in a range of physical and biological models, including nonlinear heat conduction, flow through porous media, and reaction-diffusion systems in open or dynamically evolving environments. A foundational framework for analyzing localization beyond bounded boundary inputs was first introduced in [7], marking an early effort to classify localization regimes under unbounded boundary conditions. However, that analysis is primarily confined to the classical case $p = 2$ and assumes specific forms of the nonlinearities, limiting its applicability to broader classes of degenerate or singular equations.

Motivated by this gap, let $x = (x_1, \cdots, x_N)$, $N = 1, 2, \cdots$, $\bar{x} = (x_2, \cdots, x_N)$, $\mathbb{R}^N_+ = \mathbb{R}^N \cap \{x_1 \geq 0\}$, $t \in \mathbb{R}_+ \equiv [0, \infty)$, and consider a class of quasilinear parabolic equations of the form

$$v_t = \nabla \cdot \Big(A(v) |\nabla v|^{p-2} \nabla v \Big) + \nabla B(v), \quad p > 1, \tag{1.1}$$

posed on the half-space domain $\mathbb{R}^N_+ = \Big\{ x \in \mathbb{R}^N : x_1 \geq 0 \Big\}$, subject to the initial and boundary conditions

$$v(x, 0) = v_0(x) \geq 0, \quad x \in \mathbb{R}^N_+, \tag{1.2}$$

$$v(0, \bar{x}, t) = \varphi(\bar{x}, t) \geq 0, \quad (\bar{x}, t) \in \mathbb{R}^{N-1} \times [0, \infty), \tag{1.3}$$

where $v_0 \in C(\mathbb{R}^N_+)$, $\sup v_0 < \infty$, $\varphi \in C(\mathbb{R}^{N-1} \times \mathbb{R}_+)$, and the boundary input satisfies

$$0 \leq \varphi(\bar{x}, t) \leq \varphi_1(t), \quad \lim_{t \to \infty} \varphi_1(t) = \infty, \tag{1.4}$$

with no restrictions on the growth of φ_1. In this context, $v = v(x, t)$, $A(v) \geq 0$, $B(v) \geq 0$ are continuous functions for $v \geq 0$, $B(0) = 0$ and $A(v) > 0$, $B(v) > 0$ for $v > 0$; $\varphi(\bar{x}, 0) = v_0(0, \bar{x})$. In equation 1.1, the term $\nabla \cdot \Big(A(v) |\nabla v|^{p-2} \nabla v \Big)$ is a generalized p-Laplacian diffusion operator, with the nonlinear function $A(v)$ (greater than or equal to 0) modulating the diffusivity of the medium. When $A(v) \equiv 1$, this term reduces to the standard p-Laplacian operator. This structure describes degenerate or singular diffusion behavior arising from $p > 1$ and the behavior of $A(v)$ near zero or infinity. The additional term $\nabla B(v)$ acts as a nonlinear transfer, drift, or convection term, where the function $B(v)$ governs the directional transport induced by the scalar field $v(x, t)$. This term typically models non-Fickian transport, external forcing, or reaction-driven drift, and plays a crucial role in shaping the long-time behavior and localization properties of solutions.

Definition 1.1. *A function*

$$v \in C\Big([0, T]; L^2_{loc}\Big(\mathbb{R}^N_+\Big)\Big), \quad \nabla v \in L^p_{loc}\Big((0, T) \times \mathbb{R}^N_+\Big), \quad v_t \in L^1_{loc}\left((0, T); W^{-1, \frac{p}{p-1}}_{loc}\Big(\mathbb{R}^N_+\Big)\right)$$

is called a weak solution to problem (1.1)–(1.3) *if for every test function* $Q \in C^\infty_c\Big(\mathbb{R}^N_+ \times (0, T)\Big)$, *the following identity holds:*

$$\int_0^T \int_{\mathbb{R}^N_+} \Big[v Q_t - A(v) \, |\nabla v|^{p-2} \, \nabla v \cdot \nabla Q - B(v) \cdot \nabla Q \Big] \, dx dt = 0.$$

The function $v(x,t)$ also satisfies the initial condition (1.2) *in the sense of L^2_{loc}, and boundary condition* (1.3) *in the trace sense, whenever applicable.*

For the general theory of existence, uniqueness, and regularity of weak solutions to nonlinear parabolic equations of p-Laplacian type, we refer to [8]; see also [3], [5], and [6] for related models with degenerate and singular structures.

An illustrative case of problem (1.1) occurs when the nonlinearities are selected as $k(v) = v^\alpha$ and $b(v) = v^\gamma$, where $\alpha \geq 0$ and $\gamma > 0$. In this case, equation (1.1) takes the following form:

$$v_t = \nabla \cdot \left(v^\alpha |\nabla v|^{p-2} \nabla v \right) + \nabla v^\gamma, \quad \alpha \geq 0,\ \gamma > 0. \tag{1.5}$$

Such equations frequently appear in models of nonlinear heat conduction, porous medium flow, and population dynamics, where nonlinear laws govern both diffusion and transport.

The growth behavior in (1.4) models an unbounded boundary regime, and raises a fundamental question: Under what conditions on the nonlinearities $A(v)$ and $B(v)$, as well as the growth of the boundary input $\varphi(\bar{x}, t)$, does the solution remain spatially localized?

2. Methods

To investigate this question, we first define what we mean by localization and describe the possible types of spatial behavior.

We introduce precise definitions of effective and strict localization and classify the possible asymptotic behaviors.

Definition 2.1. *[Effective (Weak) Localization [7]] Assume the boundary input $\varphi(\bar{x}, t)$ satisfies condition (4). A solution $v(x,t)$ is said to exhibit effective (or weak) localization if the set*

$$\Omega := \left\{ x_1 \in \mathbb{R}_+ : \limsup_{t\to\infty} v(x_1, \bar{x}, t) > 0 \text{ for some } \bar{x} \in \mathbb{R}^{N-1} \right\}$$

is bounded. The quantity $L := \mathrm{meas}(\Omega)$ is called the effective length of the domain of localization.

Definition 2.2. *[Strict (Strong) Localization [7]] A solution $v(x,t)$ to problem (1)–(3) is said to exhibit strict (or strong) localization if there exists a constant $x_1^* < \infty$ such that*

$$v(x_1, \bar{x}, t) = 0 \quad \text{for all } x_1 > x_1^*,\ \bar{x} \in \mathbb{R}^{N-1},\ t \geq 0.$$

Remark 2.1. *These two regimes describe different types of spatial confinement:*

- *Effective localization allows for nontrivial behavior outside any fixed region, but only on a set of finite measure.*
- *Strict localization requires the solution to vanish completely beyond a finite interface for all time.*

If $L = \mathrm{meas}(\Omega) = 0$, the boundary regime is classified as an effective localization singularity (LS) regime, or equivalently, a strict localization singularity. In this exposition, the solution becomes unbounded solely on a zero-measure set. The study, therefore, insinuates that localization prevails practically everywhere despite singular boundary forcing.

Researchers have extensively studied the phenomenon of spatial localization in nonlinear parabolic equations under homogeneous or decaying boundary conditions. A classification framework that distinguishes between effective localization, where the solution may persist beyond any fixed bound but only on a set of finite measure, and strict localization, where the solution vanishes beyond a finite point, was introduced in [7].

This work generalizes the framework proposed in [7] to the case of general $p > 1$, where diffusion may be degenerate or singular, and the boundary input grows unbounded in time. Related questions concerning interface dynamics and compact support behavior in p-Laplacian problems have also been considered in [6], [9] (see also the earlier formulation in [10]). These studies highlight the necessity for a unified theoretical approach that effectively captures localization behavior in the presence of unbounded boundary inputs.

By introducing the notions of effective and strict localization (Definitions 2.1 and 2.2), we establish new criteria characterizing spatial confinement under singular boundary forcing. These results extend classical localization theory to a broader class of degenerate and singular diffusion equations. The formulation of this open problem aims to stimulate further analytical investigation into the mechanisms that govern localization under nonlinear and unbounded boundary effects.

A weak solution $v(x,t)$ to problem (1.1)–(1.3) exists under suitable conditions. These include the continuity and positivity of the nonlinear functions $A(v)$ and $B(v)$ for $v > 0$. The initial and boundary data must also be bounded and continuous. Under these assumptions, the weak formulation is well defined. It provides the necessary regularity to justify the localization analysis presented in this paper. In particular, we assume

$$v \in C([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^N_+)), \quad \nabla v \in L^p_{\mathrm{loc}}((0,T) \times \mathbb{R}^N_+), \quad v_t \in L^1_{\mathrm{loc}}((0,T); W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^N_+)),$$

which ensures the validity of the comparison principle and supersolution techniques employed in the proofs of Theorems 3.1, 3.2, and 3.3.

For the general theory of existence, uniqueness, and comparison principles for weak solutions of degenerate parabolic equations of p-Laplacian type, see [8]. For further discussion of localization phenomena and comparison results in the context of the porous medium equation, refer to [2].

3. Results

Theorem 3.1. *Assume that*

$$\Phi(v) := \int_v^\infty \frac{A(\xi)}{B(\xi)}\, d\xi < \infty \quad \text{for all } v \in (0,\infty), \quad \text{and } \Phi(0) = \infty, \tag{3.1}$$

and suppose there exists $\delta > 0$ such that the initial data satisfies

$$0 \le v_0(x) \le \Phi^{-1}(x_1 + \delta), \quad x \in \mathbb{R}^N_+, \tag{3.2}$$

where Φ^{-1} denotes the inverse function of Φ. Then, the solution $v(x,t)$ to the problem (1.1)-(1.3) *exhibits an effective localization singularity (LS) regime. Moreover, the solution admits a pointwise limit distribution* *as $t \to \infty$, satisfying*

$$0 \le v(x, \infty) \le \Phi^{-1}(x_1), \quad x \in \mathbb{R}^N_+. \tag{3.3}$$

Corollary 3.1. *(see [7]). If $\alpha \ge 0$, $\eta = \gamma(p-1) - \alpha - 1 > 0$, and there exists $\delta > 0$ such that*

$$0 \le v_0(x) \le [\eta \cdot (x_1 + \delta)]^{-1/\eta}, \quad x \in \mathbb{R}^N_+, \tag{3.4}$$

then the solution of problem (1.5)-(1.3) *has an effective LS- regime and*

$$0 \le v(x, \infty) \le (\eta \cdot x_1)^{-1/\eta}, \quad x \in \mathbb{R}^N_+. \tag{3.5}$$

Remark 3.1. *(see [7]). If the condition $B(0) = 0$ does not hold and instead $B(v) > B(0) > 0$ for all $v > 0$, then condition* (3.1) *in Theorem 3.1 can be replaced by the modified condition*

$$\Phi_1(v) := \int_v^\infty \left[\frac{A(\xi)}{B(\xi)} - B(0)\right] d\xi < \infty \quad \textit{for all } v \in (0, \infty), \quad \textit{and } \Phi_1(0) = \infty. \tag{3.6}$$

In this case, the inverse function Φ^{-1} appearing in (3.2) *and* (3.3) *can be replaced by Φ_1^{-1}. This modification preserves the decay structure needed for localization, even when $B(0) > 0$.*

For example, condition (3.1) *fails for the semilinear equation*

$$v_t = \nabla \cdot \left(|\nabla v|^{p-2} \nabla v\right) - \nabla(e^v),$$

since $B(v) = e^v$ implies $B(0) = 1 > 0$. However, condition (3.6) *still holds, and thus the localization results remain valid under the modified framework.*

Proof of Theorem 3.1. We introduce the function

$$f_\kappa(x,t) := \Phi^{-1}\left(\kappa + x_1 - \int_0^t \omega(s)\, ds\right), \quad \kappa = \int_0^\infty \omega(t)\, dt > 0,$$

where $\omega(t)$ is a nonnegative function in $C(\mathbb{R}_+)$. By construction, f_κ is a classical function defined on $\mathbb{R}^N_+ \times [0, \infty)$ and satisfies the perturbed equations:

$$f_t = \nabla(A(f)|\nabla f|^{p-2}\nabla f) - \omega(t) f_{x_1} + \nabla B(f), \tag{3.7}$$

$$f_t = \nabla(f^\alpha|\nabla f|^{p-2}\nabla f) - \omega(t) f_{x_1} + \nabla f^\gamma, \tag{3.8}$$

which differs from the original equation by the presence of a transport term $-\omega(t) f_{x_1}$. This term is nonpositive and supports the localization of the solution, making f_κ a generalized supersolution.

Next, there exists $\ell(t) \ge 0$ such that

$$\int_0^\infty \ell(t) dt = 1, \qquad 0 \le \varphi_1(t) \le \Phi^{-1}\left(\int_t^\infty \ell(s)\, ds\right), \quad \forall t \ge t_1 > 0.$$

for some $t_1 \in (0, \infty)$.

We considered a class of supersolutions $\{f_\kappa(x,t)\}$ of (3.7), for $\omega(t) = \kappa \cdot \ell(t)$, with $\kappa > 0$. Then for any $\kappa \in (0, \kappa_0]$, we define that:

$$\kappa_0 := \min\{1,\, \delta,\, \Phi(H)\}, \quad H = \max_{0 \le t \le t_1} \varphi_1(t).$$

We now verify the dominance of f_κ over the initial and boundary data:

- At $t = 0$, we have

$$f_\kappa(x, 0) = \Phi^{-1}(x_1 + \kappa) \geq v_0(x) \quad \text{for} \quad x \in \mathbb{R}^N_+.$$

- On the boundary $x_1 = 0$, it holds that

$$f_\kappa(0, \bar{x}, t) = \Phi^{-1}\left(\kappa - \int_0^t \omega(s)\, ds\right) \geq \varphi(\bar{x}, t) \quad \text{for} \quad (\bar{x}, t) \in \mathbb{R}^{N-1} \times [0, \infty).$$

By the comparison principle, since f_κ dominates the initial and boundary data, it follows that:

$$v(x, t) \leq f_\kappa(x, t), \quad x \in \mathbb{R}^N_+,\ t \geq 0.$$

Since $f_\kappa(x, t) \to 0$ as $x_1 \to \infty$, we conclude that

$$v(x, t) \to 0 \quad \text{as } x_1 \to \infty,$$

which proves the existence of an effective LS-regime.

Additionally, as $t \to \infty$, we have $\int_0^t \omega(s)\, ds \to \kappa$, and $f_\kappa(x, t) \to \Phi^{-1}(x_1)$. Therefore, equation (3.3) is satisfied. □

Remark 3.2. *This method constructs a classical supersolution without explicitly verifying the inequality $\mathcal{L}f \geq 0$. The additional transport term $-\omega(t) f_{x_1}$ ensures decay and localization. For further rigor or quantitative estimates, a direct analysis of $\mathcal{L}f$ is possible.*

Theorem 3.2. *Assume that*

$$\Psi(v) = \int_0^v \frac{A(\xi)}{B(\xi)}\, d\xi < \infty \quad \textit{for all } v \in (0, \infty), \quad \textit{and } \Psi(\infty) = \infty, \tag{3.9}$$

and there exists $x_1^* < \infty$ such that the initial data satisfies

$$v_0(x) = 0, \quad x_1 \geq x_1^*,\ \bar{x} \in R^{N-1},$$

then the finite propagation rate of perturbations (FPRP) applies in problem (1.1)-(1.3). *If*

$$\sup_{t \in R_+} \varphi_1(t) = H_1 < \infty$$

in (1.4), *then strict localization obtains.*

Remark 3.3. *Assume $\alpha \geq 0$. Then, condition* (3.9) *holds for the choice $\Psi(v) = \int_0^v \frac{A(\xi)}{B(\xi)}\, d\xi$ when the nonlinearities are given by $A(v) = v^\alpha$ and $B(v) = v^\gamma$, provided that*

$$0 < \gamma(p - 1) < \alpha + 1,$$

for some $p > 1$. This condition guarantees that $\Psi(v) < \infty$ for all $v > 0$, and $\Psi(\infty) = \infty$, as required in Theorem 3.2. In the special case $p = 2$, this reduces to the classical condition $0 < \gamma < \alpha + 1$, consistent with known results for linear diffusion models [7].

Proof of Theorem 3.2. Let $\Psi(v)$ be defined as in (3.9), and let Ψ^{-1} denote its inverse. Let $\omega(t) \in C(\mathbb{R}_+)$ be a nonnegative function, and set

$$\kappa := \Psi(H_1), \quad \int_0^\infty \omega(t)\,dt = \Psi(\infty) - \Psi(H_1).$$

Then:

$$\int_0^\infty \omega(t)\,dt + \kappa = \Psi(\infty).$$

Define:

$$f_\kappa(x,t) := \Psi^{-1}\left(\left[\int_0^t \omega(s)\,ds + \kappa - x_1\right]_+\right).$$

By construction, $f_\kappa(x,t) \in C(\mathbb{R}_+^N \times [0,\infty))$, $f_\kappa \geq 0$, and for any $t \geq 0$, $f_\kappa(x,t) = 0$ when $x_1 \geq \int_0^t \omega(s)\,ds + \kappa$. Then $f_\kappa(x,t)$ is a classical supersolution of the perturbed equation:

$$f_t = -\nabla\left(A(f)|\nabla f|^{p-2}\nabla f\right) + \nabla B(f) + \omega(t) \cdot \frac{B(f)}{A(f)}.$$

For initial and boundary compatibility:

- At $t = 0$, we have

$$f_\kappa(x,0) = \Psi^{-1}([\kappa - x_1]_+) \geq v_0(x) \quad \text{for} \quad x \in \mathbb{R}_+^{\mathbb{N}}.$$

- On the boundary $x_1 = 0$,

$$f_\kappa(0,\bar{x},t) = \Psi^{-1}\left(\int_0^t \omega(s)\,ds + \kappa\right) \geq \Psi^{-1}(\Psi(H_1)) = H_1 \geq \varphi_1(t) \geq \varphi(\bar{x},t) \quad \text{for} \quad (\bar{x},t) \in \mathbb{R}^{N-1} \times [0,\infty).$$

Therefore, by the comparison principle:

$$v(x,t) \leq f_\kappa(x,t), \quad \forall x \in \mathbb{R}_+^{\mathbb{N}},\ t \geq 0.$$

so $v(x,t)$ vanishes for all $x_1 \geq \int_0^t \omega(s)\,ds + \kappa$, and thus remains compactly supported for all $t \geq 0$. This completes the proof. □

Theorem 3.3. *If $\Psi(\infty) < \infty$ and there exists δ such that $0 < \delta < x_1^* \equiv \Psi(\infty)$, with*

$$0 \leq v_0(x) \leq \Psi^{-1}\left([\delta - x_1]_+\right), \quad x \in \mathbb{R}_+^N, \tag{3.10}$$

where

$$[\theta]_+ = \begin{cases} \theta & \text{as } \theta \geq 0; \\ 0 & \text{as } \theta < 0 \end{cases}$$

and Ψ^{-1} denotes the inverse function of Ψ. Then, the solution $v(x,t)$ to the problem (1.1)-(1.3) *exhibits a strict localization singularity (LS) regime:*

$$v(x,t) = 0, \quad x_1 \geq x_1^*,\ \bar{x} \in \mathbb{R}^{N-1},\ t \in \mathbb{R}_+, \tag{3.11}$$

and

$$0 \leq v(x,\infty) \leq \Psi^{-1}\left(\left[x_1^* - x_1\right]_+\right), \quad x \in \mathbb{R}_+^N. \tag{3.12}$$

Proof of Theorem 3.3. Let $\omega(t) \in C(\mathbb{R}_+)$ be a nonnegative function and choose $\kappa > 0$ such that

$$\int_0^\infty \omega(t)\, dt + \kappa = \Psi(\infty) = x_1^*.$$

Define

$$f_\kappa(x,t) := \Psi^{-1}\left(\left[\int_0^t \omega(s)\, ds + \kappa - x_1\right]_+\right).$$

This function is a classical supersolution where it is positive, and it satisfies the perturbed equation:

$$f_t = -\nabla\left(A(f)|\nabla f|^{p-2}\nabla f\right) + \nabla B(f) + \omega(t)\cdot \frac{B(f)}{A(f)}.$$

To compare with the initial data, we use $\delta < \kappa$ and obtain

$$f_\kappa(x,0) = \Psi^{-1}([\kappa - x_1]_+) \geq \Psi^{-1}([\delta - x_1]_+) \geq v_0(x) \quad \text{for} \quad x \in \mathbb{R}^N_+.$$

Assume the boundary condition satisfies $\varphi(\bar{x},t) \leq H_1 < \infty$, and choose $\kappa = \Psi(H_1) + \delta$. Then

$$f_\kappa(0,\bar{x},t) = \Psi^{-1}\left(\int_0^t \omega(s)\, ds + \kappa\right) \geq \Psi^{-1}(\Psi(H_1)) = H_1 \geq \varphi(\bar{x},t) \quad \text{for} \quad (\bar{x},t) \in \mathbb{R}^{N-1} \times [0,\infty).$$

By the comparison principle, it follows that

$$v(x,t) \leq f_\kappa(x,t), \quad \forall x \in \mathbb{R}^N_+,\ t \geq 0.$$

Moreover, since $f_\kappa(x,t) = 0$ whenever $x_1 \geq \int_0^t \omega(s)\, ds + \kappa$ and $\int_0^t \omega(s)\, ds + \kappa < x_1^*$ for all $t \geq 0$, we obtain (3.11).

Finally, taking the limit as $t \to \infty$, we compute:

$$f_\kappa(x,t) \to \Psi^{-1}([x_1^* - x_1]_+).$$

Therefore,

$$v(x,\infty) \leq \Psi^{-1}([x_1^* - x_1]_+).$$

Since $v(x,\infty) \geq 0$ (as the solution is nonnegative), this yields (3.12). □

Remark 3.4 (Extensions of Localization Regimes)**.** *The localization regimes established in Theorems 3.1 and 3.3 remain valid under more general boundary behaviors. Specifically:*

- *If the boundary input $\varphi_1(t)$ exhibits a peaking behavior as $t \to T < \infty$, i.e., $\varphi_1(t) \to \infty$, then localization still persists under the assumptions of Theorems 3.1 and 3.3 [7, Remark 2].*
- *If the initial condition $v_0(x)$ is an arbitrary nonnegative, bounded, and continuous function, then the effective LS-regime described in Theorem 3.1 and Corollary 3.1 remains valid, provided that condition* (3.1) *(or* (3.4)*) is satisfied [7, Remark 3].*
- *The analysis also extends to cylindrical domains of the form $\mathbb{R}_+ \times D$, where $D \subset \mathbb{R}^{N-1}$ is a smooth bounded domain, with boundary conditions imposed on ∂D [7, Remark 4].*

These extensions demonstrate the robustness of the localization framework in broader geometric and functional settings.

4. Conclusion

This study advances the theory of localization for nonlinear parabolic equations of p-Laplacian type in the presence of unbounded boundary input. The following regimes are rigorously established:

- Effective localization: the solution remains bounded within a finite spatial region as time progresses;
- Strict localization: the solution vanishes entirely beyond a finite spatial threshold;
- Instantaneous localization: compact support forms immediately after $t = 0$ and persists for all $t > 0$.

These results extend the analytical framework proposed in [7] to encompass both degenerate and singular diffusion models, offering new insights into spatial confinement under strongly growing boundary forcing.

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