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On Elliptic Equations with General Robin Boundary Conditions in Hölder Spaces: Non Commutative Cases

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Abstract. In this paper, we study a class of second-order abstract differential equation problems of the elliptic type with operator coefficients with general Robin boundary conditions in a non-commutative setting, i.e the unbounded linear operator in the equation does not commute with the one that appears in the boundary conditions containing two spectral complex parameters. We study the case when the second member belongs to the Hölder space. We give necessary and sufficient conditions of compatibility to obtain a strict solution and also to ensure that the strict solution has the maximal regularity property. This paper is naturally the continuation of the ones studied in [16] and [10].

1. Introduction and hypotheses

Let us consider the second order abstract differential equation

$$u''(x) + Au(x) - \omega u(x) = f(x), \quad x \in \left]0,1\right[, \tag{1.1}$$

together with the abstract boundary conditions of Robin's type

$$\begin{cases} u'(0) - Hu(0) - \mu u(0) = d_0 \\ u(1) = u_1. \end{cases} \tag{1.2}$$

Here A and H are closed linear operators in a complex Banach space X; d_0 , u_1 are given elements in X; ω, μ are two complex parameters and the second member f belongs to $C([0,1];X)$. Here, we develop a different approach from those used so far.

We will seek for a strict solution u to (1.1)-(1.2), that is a function u such that:

$$\begin{cases} u \in C^2([0,1];X) \cap C([0,1];D(A)) \\ u(0) \in D(H), \end{cases}$$

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and which satisfies (1.1)-(1.2).

In general, the condition $f \in C([0,1];X)$ is not sufficient to provide a strict solution to (1.1)-(1.2), this is why we assume, in all the paper, that:

$$f \in C^{\theta}([0,1];X), \ 0 < \theta < 1.$$

The techniques used here are essentially based on the Dunford functional calculus and the methods applied in [5], [10] and [16].

In some problems governed by PDEs, there appears a spectral parameter both in the equation and in the boundary conditions. As such, we cite the problems related to thermal condition or to vibrating string, see [12], [19], [22], [24].

In [4], the authors have considered in a complex Banach space X, problem (1.1)-(1.2) where $\lambda = \omega$ is some positive spectral parameter and $\mu = 0$. For ω large enough, under some geometrical assumption on the space X and hypotheses on operators $A - \omega I$ and H including the fact that they commute in the resolvent sense, the authors have furnished necessary and sufficient conditions on the data d_0, u_1 to obtain the existence and the uniqueness of a solution to (1.1)-(1.2) verifying $u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A))$. Then, in [5], they studied problem (1.1)-(1.2) without the two spectral parameters λ and μ, $\lambda = \omega$ and $\mu = 0$, in a commutative framework and in a Hölder space. In [6], the authors have developed an interesting new approach in a non commutative framework, concerning some general Sturm-Liouville problems with the same Robin boundary condition in 0. Recently in [10], the authors studied the same problem (1.1)-(1.2) in a non-commutative setting, when $f \in L^p(0,1;X)$ with $1 < p < \infty$ and when X is an UMD space. They have considered two cases:

(1) $D(H) \subset D(A)$
(2) $D\left(\sqrt{-A}\right) \subset D(H)$, when $\sqrt{-A}$ is defined.

This second case has been studied in [16] with one spectral positive real parameter in the equation.

In all this work, we will use the following notation: for $\varphi \in (0,\pi)$, we set

$$S_{\varphi} = \left\{z \in \mathbb{C} \setminus \{0\} : \ \left|\arg(z)\right| \leq \varphi\right\} \cup \{0\}.$$

We suppose that

$$\left\{ \begin{array}{l} \exists \varphi_0 \in (0,\pi) : S_{\varphi_0} \subset \rho(A) \text{ and } \exists C_A > 0 : \\ \quad \forall \omega \in S_{\varphi_0}, \left\| (A - \omega I)^{-1} \right\|_{\mathcal{L}(X)} \leq \dfrac{C_A}{1 + |\omega|}, \end{array} \right. \tag{1.3}$$

($\rho(A)$ denotes the resolvent set of A; this assumption means exactly the ellipticity of our problem as in Krein [18]).

We will assume, moreover that

$$D(H) \subset D(A), \tag{1.4}$$

and

$$\begin{cases} \exists \varphi_1 \in (0,\pi)\,, \exists C_H > 0 : \\ S_{\varphi_1} \subset \rho(-H) \text{ and } \sup\limits_{\mu \in S_{\varphi_1}} \left(1+|\mu|\right)\left\|(H+\mu I)^{-1}\right\|_{\mathcal{L}(X)} \leq C_H. \end{cases} \tag{1.5}$$

In all this work, we set for

$$\begin{cases} A_\omega = A - \omega I, \quad \omega \in S_{\varphi_0} \\ H_\mu = H + \mu I, \quad \mu \in S_{\varphi_1}. \end{cases}$$

Remark 1.1. *It is well known that assumption (1.3) implies that the square roots $Q = -\sqrt{-A}$ and $Q_\omega = -\sqrt{-A+\omega I}$ are well defined and generate analytic semigroups not strongly continuous at zero, see Balakrishnan [2] for dense domains and Martinez [20] for non dense domains. Note that, $\overline{D(A)} = \overline{D(Q)}$.*

Due to (1.3), we recall that $D_Q(\theta;+\infty) = (D(Q);X)_{1-\theta;+\infty}$ is the real interpolation space described see [13].
We will also suppose that

$$(Q-H)^{-1}(D(Q)) \subset D\left(Q^2\right), \tag{1.6}$$

and

$$Q^2(Q-H)^{-1}\left(D_Q(1+\theta;+\infty)\right) \subset \overline{D(Q)}; \tag{1.7}$$

here, for the moment, $(Q-H)^{-1}$ is taken in the set sense of the term. We will see later that this

$$(Q-H)^{-1} \in \mathcal{L}(X).$$

We recall the notation

$$D_Q(1+\theta;+\infty) = \left\{\varphi \in D(Q) : \; Q\varphi \in D_Q(\theta;+\infty)\right\}.$$

For the maximum regularity we replace (1.7) by

$$Q^2(Q-H)^{-1}\left(D_Q(1+\theta;+\infty)\right) \subset D_Q(\theta;+\infty). \tag{1.8}$$

This paper is the natural continuation of [16], where the authors have studied problem (1.1)-(1.2) under (1.3), (1.5)~(1.8) and also

$$D(Q) \subset D(H),$$

instead of (1.4).

Remark 1.2. *From (1.3) we deduce that, there exists $\theta_0 \in\,]0,\pi/2[$ and $r_0 > 0$ such that the resolvent operator of A satisfies:*

$$\rho(A) \supset S_{\varphi_0,r_0} = \left\{z \in \mathbb{C}\setminus\{0\} : \; |\arg(z)| \leq \theta_0\right\} \cup \overline{B(0,r_0)},$$

where $\overline{B(0,r_0)}$ is the closed ball of radius r_0 and centered in 0.

Remark 1.3. *Assume (1.3) and (1.4). Let $\omega \in S_{\varphi_0}$, we know that there exists $L_\omega \in \mathcal{L}(X)$, such that*

$$Q = Q_\omega + L_\omega \text{ and } L_\omega Q^{-1} = Q^{-1} L_\omega. \tag{1.9}$$

See [15], Proposition 3.1.7, p. 65. Form (1.9) we deduce that L_ω is continuous from $D(Q)$ into $D(Q)$ by writing for all $\varphi \in D(Q)$

$$\begin{aligned} L_\omega \varphi &= L_\omega Q^{-1} Q \varphi \\ &= Q^{-1} L_\omega Q \varphi, \end{aligned}$$

which gives

$$\begin{aligned} \left\| L_\omega \varphi \right\|_{D(Q)} &= \left\| L_\omega \varphi \right\|_X + \left\| Q L_\omega \varphi \right\|_X \\ &= \left\| L_\omega \varphi \right\|_X + \left\| Q Q^{-1} L_\omega Q \varphi \right\|_X \\ &= \left\| L_\omega \varphi \right\|_X + \left\| L_\omega Q \varphi \right\|_X \\ &\leq \| L_\omega \|_X \left\| \varphi \right\|_X + \| L_\omega \|_X \left\| Q \varphi \right\|_X \\ &\leq \| L_\omega \|_X \left[\left\| \varphi \right\|_X + \left\| Q \varphi \right\|_X \right] \\ &\leq \| L_\omega \|_X \left\| \varphi \right\|_{D(Q)}, \end{aligned}$$

therefore L_ω is continuous from any interpolation space $D_Q(\theta; +\infty)$ into itself.

Remark 1.4. *Let $\omega \in S_{\varphi_0}$, we have*

$$D_{Q_\omega}(\theta; +\infty) = D_Q(\theta; +\infty), \;\; D_{Q_\omega}(1+\theta; +\infty) = D_Q(1+\theta; +\infty).$$

Moreover $D(Q_\omega) = D(Q)$ and $D\left(Q_\omega^2\right) = D\left(Q^2\right)$ with

$$Q^2 = Q_\omega^2 + 2 L_\omega Q_\omega + L_\omega^2,$$

on the other hand, we have

$$\begin{aligned} Q_\omega^{-2} &= Q^{-2} - (-A)^{-1} + (-A + \omega I)^{-1} \\ &= Q^{-2} - (-A)^{-1} (-A + \omega I)(-A + \omega I)^{-1} + (-A)^{-1}(-A)(-A + \omega I)^{-1} \\ &= Q^{-2} - \omega (-A)^{-1} (-A + \omega I)^{-1} \\ &= Q^{-2} - \omega Q^{-2} Q_\omega^{-2}. \end{aligned}$$

Remark 1.5. *Let $\omega \in S_{\varphi_0}$, $\mu \in S_{\varphi_1}$ and let us show that if $(Q - H)^{-1}$ exists and verifies*

$$(Q - H)^{-1} (D(Q)) \subset D\left(Q^2\right), \tag{1.10}$$

then

$$\left(Q_\omega - H_\mu\right)^{-1} (D(Q_\omega)) \subset D\left(Q_\omega^2\right),$$

when $\left(Q_\omega - H_\mu\right)^{-1}$ exists.

Proof. Assume (1.10) and set

$$\left(Q_\omega - H_\mu\right)^{-1}\left(Q^{-1}(\xi)\right) = \eta,$$

then

$$Q^{-1}(\xi) = \left(Q_\omega - H_\mu\right)(\eta) = (Q_\omega - Q)(\eta) + \left(H - H_\mu\right)(\eta) + (Q - H)(\eta);$$

We have seen that from (1.9)

$$Q = Q_\omega + L_\omega \text{ and } Q^{-1}L_\omega = L_\omega Q^{-1};$$

Now, set

$$H = H_\mu + T_\mu \text{ when } T_\mu = -\mu I;$$

Since $\eta \in D(Q)$, then there exists $\zeta \in X$ such that $\eta = Q^{-1}(\zeta)$, therefore

$$\begin{aligned} Q^{-1}(\xi) &= (Q_\omega - Q) Q^{-1}(\zeta) + \left(H - H_\mu\right) Q^{-1}(\zeta) + (Q - H)\eta \\ &= -Q^{-1}L_\omega(\zeta) + T_\mu Q^{-1}(\zeta) + (Q - H)\eta, \end{aligned}$$

and

$$\eta = (Q - H)^{-1}\left[Q^{-1}(\xi) + Q^{-1}L_\omega(\zeta) - T_\mu Q^{-1}(\zeta)\right] \in D\left(Q^2\right).$$

□

Remark 1.6. *Let $\omega \in S_{\varphi_0}$ and $\mu \in S_{\varphi_1}$. Now, as above, let us prove that if $(Q - H)^{-1}$exists and verifies the following properties*

$$\begin{cases} (Q - H)^{-1}(D(Q)) \subset D\left(Q^2\right) \\ \text{For some } \xi \in X, \quad (Q - H)^{-1}(\xi) \in D\left(Q^2\right), \end{cases} \tag{1.11}$$

then, when $\left(Q_\omega - H_\mu\right)^{-1}$ exists, we have also

$$\left(Q_\omega - H_\mu\right)^{-1}(\xi) \in D\left(Q^2\right).$$

Proof. In fact, set

$$\left(Q_\omega - H_\mu\right)^{-1}(\xi) = \eta,$$

so

$$\xi = \left(Q_\omega - H_\mu\right)(\eta) = (Q_\omega - Q)(\eta) + \left(H - H_\mu\right)(\eta) + (Q - H)(\eta),$$

where $\eta = Q^{-1}(\chi)$, $\chi \in X$. As above, we have

$$\xi = -Q^{-1}L_\omega(\chi) + T_\mu Q^{-1}(\chi) + (Q - H)(\eta),$$

from which it follows

$$(Q - H)^{-1}(\xi) = -(Q - H)^{-1}Q^{-1}L_\omega(\chi) + (Q - H)^{-1}T_\mu Q^{-1}(\chi) + \eta,$$

in virtue of (1.11), we obtain

$$\eta = (Q - H)^{-1}(\xi) + (Q - H)^{-1}Q^{-1}L_\omega(\chi) - (Q - H)^{-1}T_\mu Q^{-1}(\chi) \in D\left(Q^2\right).$$

□

Remark 1.7. *Let $\omega \in S_{\varphi_0}$ and $\mu \in S_{\varphi_1}$. Now, as above, assume that $(Q-H)^{-1}$exists and verifies the four following properties*

$$\begin{cases} i)\ (Q-H)^{-1}(D(Q)) \subset D\left(Q^2\right) \\ ii)\ Q^2(Q-H)^{-1}(D(Q)) \subset \overline{D(Q)} \\ iii)\ \text{For some } \xi \in X,\ \ (Q-H)^{-1}(\xi) \in D\left(Q^2\right) \\ iv)\ Q^2(Q-H)^{-1}(\xi) + f(0) \in \overline{D(Q)}, \end{cases} \tag{1.12}$$

then, when $\left(Q_\omega - H_\mu\right)^{-1}$ exists, we have also

$$\begin{cases} \left(Q_\omega - H_\mu\right)^{-1}(\xi) \in D\left(Q^2\right) \text{ and} \\ Q^2\left(Q_\omega - H_\mu\right)^{-1}(\xi) + f(0) \in \overline{D(Q)}. \end{cases}$$

Proof. In fact, by setting

$$\left(Q_\omega - H_\mu\right)^{-1}(\xi) = \eta,$$

and using the last calculus, we get

$$\eta = (Q-H)^{-1}(\xi) + (Q-H)^{-1} Q^{-1} L_\omega(\chi) - (Q-H)^{-1} T_\mu Q^{-1}(\chi),$$

so

$$\begin{array}{rcl} Q^2\left(Q_\omega - H_\mu\right)^{-1}(\xi) + f(0) & = & Q^2(\eta) + f(0) \\ & = & Q^2(Q-H)^{-1}(\xi) + Q^2(Q-H)^{-1} Q^{-1} L_\omega(\chi) \\ & - & Q^2(Q-H)^{-1} T_\mu Q^{-1}(\chi) + f(0) \in \overline{D(Q)}. \end{array}$$

□

The plan of the paper is as follows:
In the first section and after having presented our problem (1.1)-(1.2) and some works of authors on the same theme, we introduced our hypotheses (1.3)~(1.8). We then showed all the consequences of these hypotheses in from of remarks. In the next Section, we collect some useful basic lemmas. In Section 3, we give the representation formula of solution u of problem (1.1)-(1.2). Section 4 is devoted to the proof of the main result. Finally in section 5 we give some concrete examples of partial differential equations to which our theory applies.

2. Technical lemmas

In this work all the constants in the estimates are independent of ω, μ. Due to [9], Lemma 2.6, statement b, p. 103, we have:

Lemma 2.1. *There exists a constant $M \geq 0$ independent of $\omega \in S_{\varphi_0}$, such that for any $\omega \in S_{\varphi_0}$, operators $I \pm e^{2Q_\omega}$ are invertible in $\mathcal{L}(X)$ and*

$$\forall \omega \in S_{\varphi_0}, \quad \left\| \left(I \pm e^{2Q_\omega}\right)^{-1} \right\|_{\mathcal{L}(X)} \leq M. \tag{2.1}$$

Proof. For the proof, see [10], Lemma 5.1. □

Lemma 2.2. *Assume that (1.3) holds. There exists a constants $C > 0$ such that, for all $\omega \in S_{\varphi_0}$ and $z \in S_{\varphi_0}$, we have*

$$\left\|(Q_\omega - zI)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{C}{\sqrt{1+|\omega|}+|z|}.$$

Proof. Although the proof is given in [3], Lemma 3.1, p. 67-68, we give a proof for the reader's convenience and completeness.

Let γ be a sectorial curve surrounding the spectrum of $-A_\omega$. Using S. G. Krein [18], p. 116-117 and for all $z \geq 0$, we have

$$\begin{aligned} (Q_\omega - zI)^{-1} &= -\left(\sqrt{-A_\omega} + zI\right)^{-1} \\ &= \frac{-1}{2\pi i}\int_\gamma \frac{(-A_\omega - \lambda I)^{-1}}{z+\sqrt{\lambda}}d\lambda \\ &= \frac{-1}{\pi}\int_0^{+\infty} \frac{\sqrt{s}\,(-A+\omega I + sI)^{-1}}{s+z^2}ds. \end{aligned}$$

From the hypothesis (1.3), we obtain

$$\begin{aligned} \left\|(Q_\omega - zI)^{-1}\right\|_{\mathcal{L}(X)} &\leq C\int_0^{+\infty}\frac{\sqrt{s}}{(1+|\omega|+s)\,(s+z^2)}ds \\ &\leq C\int_0^{+\infty}\frac{t^2}{(1+|\omega|+t^2)\,(t^2+z^2)}dt \\ &\leq \frac{C}{1+|\omega|-z^2}\left[\int_0^{+\infty}\frac{1+|\omega|}{1+|\omega|+t^2}dt - \int_0^{+\infty}\frac{z^2}{t^2+z^2}dt\right] \\ &\leq \frac{C}{\sqrt{1+|\omega|}+|z|}. \end{aligned}$$

□

Lemma 2.3. *For all $\omega \in S_{\varphi_0}$, $\mu \in S_{\varphi_1}$, we have $Q_\omega^2 H_\mu^{-1} \in \mathcal{L}(X)$; moreover there exists a constant $M > 0$ independent of ω and μ such that:*

$$\max\left\{\left\|HH_\mu^{-1}\right\|_{\mathcal{L}(X)},\ \left\|AH_\mu^{-1}\right\|_{\mathcal{L}(X)}\right\} \leq M, \tag{2.2}$$

$$\left\|Q_\omega^2 H_\mu^{-1}\right\|_{\mathcal{L}(X)} \leq M\frac{1+|\omega|+|\mu|}{1+|\mu|}, \tag{2.3}$$

and

$$\left\|Q_\omega H_\mu^{-1}\right\|_{\mathcal{L}(X)} \leq M\frac{1+|\omega|+|\mu|}{\left(1+|\mu|\right)\sqrt{1+|\omega|}}. \tag{2.4}$$

Proof. See [10], Lemma 5.2. □

We now introduce, for $r > 0$, the notation

$$\Omega_{\varphi_0,\varphi_1,r} = \left\{(\omega,\mu) \in S_{\varphi_0}\times S_{\varphi_1} : |\omega| \geq r \text{ and } \frac{|\mu|^2}{|\omega|} \geq r)\right\}.$$

Lemma 2.4. *Assume that (1.3)~(1.5) hold. Then there exists $\rho_0 > 0$ and a constant $C > 0$ independent of $\omega \in S_{\varphi_0}$ and $\mu \in S_{\varphi_1}$ such that the operator $Q_\omega \pm H_\mu$ is boundedly invertible, moreover*

$$\left\|(Q_\omega \pm H_\mu)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{C}{1+|\mu|}. \tag{2.5}$$

Proof. From (43), there exists $\rho_0 > 0$ such that for all $(\omega,\ \mu) \in \Omega_{\varphi_0,\varphi_1,\rho_0}$ we have

$$\left\|Q_\omega H_\mu^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{1}{2},$$

see [10], Lemma 5.3. Then from (1.5), that operator $Q_\omega \pm H_\mu$ is invertible and we have

$$\begin{aligned}\left\|\left(Q_\omega \pm H_\mu\right)^{-1}\right\|_{\mathcal{L}(X)} &= \left\|H_\mu^{-1}\left(Q_\omega H_\mu^{-1} \pm I\right)^{-1}\right\|_{\mathcal{L}(X)} \\ &\leq \frac{1}{1-\left\|Q_\omega H_\mu^{-1}\right\|_{\mathcal{L}(X)}}\left\|H_\mu^{-1}\right\|_{\mathcal{L}(X)} \\ &\leq \frac{C}{1+|\mu|}.\end{aligned}$$

(Note that, you can replace (2.5) by $\left\|(Q_\omega \pm H_\mu)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{C}{\sqrt{1+|\omega|}}$). □

Now, for $\omega \in S_{\varphi_0}$ and $\mu \in S_{\varphi_1}$, we define operator $\Lambda_{\omega,\mu}$ by:

$$\Lambda_{\omega,\mu} = Q_\omega - H_\mu + e^{2Q_\omega}\left(Q_\omega + H_\mu\right).$$

Note that, since $D\left(H_\mu\right) \subset D\left(Q_\omega^2\right)$, we have $D\left(\Lambda_{\omega,\mu}\right) = D\left(H_\mu\right) = D\left(H\right)$ and furnish results on $\Lambda_{\omega,\mu}$.

Lemma 2.5. (1) *There exist $r_0 > 0$ and $M > 0$ such that for all $(\omega,\mu) \in \Omega_{\varphi_0,\varphi_1,r}$, we have*

$$\begin{cases} 0 \in \rho\left(\left(I-e^{2Q_\omega}\right)^{-1}\left(I+e^{2Q_\omega}\right)Q_\omega H_\mu^{-1} - I\right) \\ \left\|\left[\left(I-e^{2Q_\omega}\right)^{-1}\left(I+e^{2Q_\omega}\right)Q_\omega H_\mu^{-1} - I\right]^{-1}\right\|_{\mathcal{L}(X)} \leq 2. \end{cases} \tag{2.6}$$

(2) *The operator $\Lambda_{\omega,\mu}$ is boundedly invertible and*

$$\left\|\Lambda_{\omega,\mu}^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M}{1+|\mu|}, \tag{2.7}$$

and

$$\left\|Q_\omega^2\Lambda_{\omega,\mu}^{-1}\right\|_{\mathcal{L}(X)} \leq M\frac{1+|\omega|+|\mu|}{1+|\mu|}, \tag{2.8}$$

Note that $Q_\omega^2\Lambda_{\omega,\mu}^{-1}$ has the same behavior as $Q_\omega H_\mu^{-1}$.

Proof. See [10], Lemma 5.3. □

Lemma 2.6. *Assume (1.3)~(1.6). Fix $(\omega_1,\mu_1) \in \Omega_{\varphi_0,\varphi_1,r}$. Then there exists $C > 0$ such that for all $(\omega,\mu) \in \Omega_{\varphi_0,\varphi_1,r}$, we have*

(1) $Q_\omega^2\left(Q_\omega-H_\mu\right)^{-1}Q_\omega^{-1}\in\mathcal{L}(X)$, *with*

$$\left\|Q_\omega^2\left(Q_\omega-H_\mu\right)^{-1}Q_\omega^{-1}\right\|_{\mathcal{L}(X)}\leq C.$$

(2) *There exists* $W_{\omega,\mu}\in\mathcal{L}(X)$ *such that*

$$\Lambda_{\omega,\mu}^{-1}=\left(Q_\omega-H_\mu\right)^{-1}\left(I-e^{2Q_\omega}W_{\omega,\mu}\right),$$

with

$$\left\|W_{\omega,\mu}\right\|_{\mathcal{L}(X)}\leq C\ \text{ and }\ \left\|Q_\omega^2\Lambda_{\omega,\mu}^{-1}Q_\omega^{-1}\right\|_{\mathcal{L}(X)}\leq C.$$

Proof. Let $(\omega,\mu)\in\Omega_{\varphi_0,\varphi_1,r}$.

(1) Due to Lemma 2.3 and Lemma 2.4 and by using the same techniques as in [10], Lemma 5.3, we obtain that

$$\left\|Q_\omega^2\left(Q_\omega-H_\mu\right)^{-1}Q_\omega^{-1}\right\|_{\mathcal{L}(X)}=\left\|Q_\omega^2H_\mu^{-1}\left(Q_\omega H_\mu^{-1}-I\right)^{-1}Q_\omega^{-1}\right\|_{\mathcal{L}(X)}\leq C.$$

(2) For the second assertion, see [10], Lemma 6.4. Statement 3.

□

3. Representation formula of the solution

In order to solve problem (1.1)-(1.2), we use the well know Krein's reduction order method, see [18]. Under assumptions (1.3)~(1.5), suppose that the problem has a strict solution u, that is, $u\in C^2([0,1],X)\cap C([0,1],D(A))$ such that $u(0)\in D(H)$ and (1.1)-(1.2) are satisfied. Then u admits a representation formula given in [16]

$$u(x)=S(x)\mu_0+S(1-x)\mu_1+I(x)+J(x),\quad x\in[0,1],\tag{3.1}$$

where

$$\begin{cases} I(x)=\dfrac{1}{2}Q_\omega^{-1}\int_0^x e^{(x-s)Q_\omega}f(s)\,ds,\\[2mm] J(x)=\dfrac{1}{2}Q_\omega^{-1}\int_x^1 e^{(s-x)Q_\omega}f(s)\,ds,\\[2mm] S(x)=T_\omega\left(e^{xQ_\omega}-e^{(1-x)Q_\omega}e^{Q_\omega}\right),\end{cases}\tag{3.2}$$

with $T_\omega=\left(I-e^{2Q_\omega}\right)^{-1}$ and

$$\begin{cases}\mu_1=u_1-I(1)\\ \mu_0=\Lambda_{\omega,\mu}^{-1}\beta_0-J(0)\\ \beta_0=\left(I-e^{2Q_\omega}\right)d_0+2Q_\omega e^{Q_\omega}\mu_1+2Q_\omega J(0).\end{cases}\tag{3.3}$$

We have $T_\omega = I + e^{2Q_\omega} T_\omega$. In view to use the h ölderianity assumption on f, we need to adapt the previous representation, then there exist $\tau_0, \varphi_0, \varphi_1 \in X$ (clearly computable) such that

$$\begin{aligned}
u(x) &= \left(I + e^{2Q_\omega} T_\omega\right)\left(e^{xQ_\omega} - e^{(1-x)Q_\omega} e^{Q_\omega}\right)\mu_0 \\
&\quad + \left(I + e^{2Q_\omega} T_\omega\right)\left(e^{(1-x)Q_\omega} - e^{xQ_\omega} e^{Q_\omega}\right)(u_1 - I(1)) \\
&\quad + I(x) + J(x) \\
&= e^{xQ_\omega}\mu_0 + e^{(1-x)Q_\omega}(u_1 - I(1)) + I(x) + J(x) + e^{xQ_\omega} e^{Q_\omega}\varphi_0 + e^{(1-x)Q_\omega} e^{Q_\omega}\varphi_1 \\
&= e^{xQ_\omega}\left(\Lambda_{\omega,\mu}^{-1}[d_0 + 2Q_\omega J(0)] - J(0)\right) + I(x) \\
&\quad + e^{(1-x)Q_\omega}(u_1 - I(1)) + J(x) \\
&\quad + e^{xQ_\omega}\Lambda_{\omega,\mu}^{-1} e^{Q_\omega}\tau_0 + e^{xQ_\omega} e^{Q_\omega}\varphi_0 + e^{(1-x)Q_\omega} e^{Q_\omega}\varphi_1.
\end{aligned}$$

Moreover $\Lambda_{\omega,\mu}^{-1} = \left(Q_\omega - H_\mu\right)^{-1} - \left(Q_\omega - H_\mu\right)^{-1} e^{2Q_\omega} W_{\omega,\mu}$, so there exist τ_1, ψ_0, ψ_1 in X (clearly computable) such that

$$\begin{aligned}
u(x) &= e^{xQ_\omega}\left[\left(Q_\omega - H_\mu\right)^{-1}(d_0 + 2Q_\omega J(0)) - J(0)\right] + I(x) \\
&\quad + e^{(1-x)Q_\omega}(u_1 - I(1)) + J(x) \\
&\quad + e^{xQ_\omega}\left(Q_\omega - H_\mu\right)^{-1} e^{Q_\omega}\tau_1 \\
&\quad + e^{xQ_\omega} e^{Q_\omega}\psi_0 + e^{(1-x)Q_\omega} e^{Q_\omega}\psi_1.
\end{aligned} \tag{3.4}$$

Now, we write

$$\begin{aligned}
J(0) &= \frac{1}{2} Q_\omega^{-1} \int_0^1 e^{sQ_\omega}(f(s) - f(0))\, ds + \frac{1}{2} Q_\omega^{-1} \int_0^1 e^{sQ_\omega} f(0)\, ds \\
&= \frac{1}{2} Q_\omega^{-1} \int_0^1 e^{sQ_\omega}(f(s) - f(0))\, ds + \frac{1}{2} Q_\omega^{-2} e^{Q_\omega} f(0) - \frac{1}{2} Q_\omega^{-2} f(0),
\end{aligned}$$

and

$$\begin{aligned}
I(1) &= \frac{1}{2} Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega}(f(s) - f(1))\, ds + \frac{1}{2} Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega} f(1)\, ds \\
&= \frac{1}{2} Q_\omega^{-1} \int_0^1 e^{(1-s)Q_\omega}(f(s) - f(1))\, ds - \frac{1}{2} Q_\omega^{-2} f(1) + \frac{1}{2} Q_\omega^{-2} e^{Q_\omega} f(1),
\end{aligned}$$

so from (3.4), we can define $\tau_2, \psi_2, \psi_3 \in X$ such that

$$\begin{aligned}
u(x) &= e^{xQ_\omega}\left[\left(Q_\omega - H_\mu\right)^{-1}[d_0 - Q_\omega^{-1} f(0)] + \frac{1}{2} Q_\omega^{-2} f(0)\right] + I(x) \\
&\quad + e^{(1-x)Q_\omega}\left(u_1 + \frac{1}{2} Q_\omega^{-2} f(1)\right) + J(x) \\
&\quad + R_1(x) + R_2(x) + R_3(x),
\end{aligned} \tag{3.5}$$

with

$$\begin{cases} R_1(x) = e^{xQ_\omega}\left(Q_\omega - H_\mu\right)^{-1}\left(e^{Q_\omega}\tau_2 + \int_0^1 e^{sQ_\omega}\left(f(s) - f(0)\right) ds\right) \\ R_2(x) = -\frac{1}{2}e^{xQ_\omega}Q_\omega^{-1}\int_0^1 e^{sQ_\omega}\left(f(s) - f(0)\right) ds \\ -\frac{1}{2}e^{(1-x)Q_\omega}Q_\omega^{-1}\int_0^1 e^{(1-s)Q_\omega}\left(f(s) - f(1)\right) ds \\ R_3(x) = e^{xQ_\omega}e^{Q_\omega}\psi_2 + e^{(1-x)Q_\omega}e^{Q_\omega}\psi_3. \end{cases} \tag{3.6}$$

i): Note that, in (3.5) the first term gives the behavior of u near 0, the second term concerns the behavior of u near one 1 and $R_1(x)$, $R_2(x)$, $R_3(x)$ are regular terms (see Propositions 4.4 and 4.5 below).

ii): Note that we did not need a commutativity hypothesis between H_μ and Q_ω in the above calculus.

iii): In virtue of the commutativity assumption supposed in ([5]), which given

$$\Lambda_{\omega,\mu}^{-1}e^{Q_\omega} = e^{Q_\omega}\Lambda_{\omega,\mu}^{-1}, \tag{3.7}$$

the term $R_1(x)$ is regular.

In this work, we cannot use (3.7) but the regularity of $R_1(x)$ using (1.6) and (1.7).

4. Regularity of the solution

We start the section by proving some technical lemmas useful for the analysis of the terms of the solution.

Proposition 4.1. (1) *Let $\varphi \in X$ and $\theta \in]0,1[$. Then the two following assertions are equivalent.*
  (a) $e^{\cdot Q}\varphi \in C([0,1];X)$.
  (b) $\varphi \in \overline{D(Q)}$.
(2) *Let $\theta \in]0,1[$, $g \in C^\theta([0,1];X)$ and $\varphi \in X$. Set*

$$v(x) = e^{xQ}\varphi + \int_0^x e^{(x-s)Q}g(s)\,ds.$$

Then the two following assertions are equivalent.
  (a) $v \in C^1([0,1];X) \cap C([0,1];D(Q))$.
  (b) $\varphi \in D(Q)$ *and* $g(0) + Q\varphi \in \overline{D(Q)}$.

Considering the well known real interpolation space

$$(D(Q), X)_{1-\theta,\infty} = (X, D(Q))_{\theta,\infty} = D_Q(\theta;+\infty),$$

(see H. Triebel see [23] p. 25 and 76), we have also:

Proposition 4.2. (1) *Let $\varphi \in X$ and $\theta \in]0,1[$. Then the two following assertions are equivalent.*
  (a) $e^{\cdot Q}\varphi \in C^\theta([0,1];X)$.
  (b) $\varphi \in D_Q(\theta;+\infty)$.

(2) *Let* $\theta \in \left]0,1\right[$ *and* $g \in C^{\theta}\left([0,1];X\right)$. *Set*

$$v(x) = \int_0^x e^{(x-s)Q}\left(g(s) - g(0)\right) ds.$$

Then $v \in C^{1,\theta}\left([0,1];X\right) \cap C^{\theta}\left([0,1];D(Q)\right).$

(3) *Let* $g \in C\left([0,1];X\right)$ *and* $\varphi \in X$. *Set*

$$w(x) = e^{xQ}\varphi + \int_0^x e^{(x-s)Q} g(s)\, ds.$$

Then the two following assertions are equivalent.

(a) $w \in C^{1,\theta}\left([0,1];X\right) \cap C^{\theta}\left([0,1];D(Q)\right).$

(b) $g \in C^{\theta}\left([0,1];X\right), \varphi \in D(Q)$ *and* $g(0) + Q\varphi \in D_Q(\theta;+\infty).$

(4) *Let* $g \in C^{\theta}\left([0,1];X\right)$. *Then*

$$Q\int_0^1 e^{sQ}\left(f(s) - f(0)\right) ds \in \left(D(Q), X\right)_{1-\theta,\infty}.$$

Proof. Statement 2 is obtained by applying the Da Prato-Grisvard sum theory [8]. Statement 3 which improves Statement 2 is due to E. Sinestrari [21], see also G. Da Prato [7]. We can find in D. Guidetti [14] a simple proof of these results (see Corollary 2.1 and Theorem 2.4, p. 136). □

notation: Let g and h be two given X-valued functions defined on $[0,1]$ and $\theta \in \left]0,1\right[$. We write

$$g \simeq_\theta h,$$

if

$$g - h \in C^{\theta}\left([0,1];X\right).$$

As a consequence of Proposition 4.2 we get (see [11] Proposition 8, p. 976):

Proposition 4.3. *Let* $g \in C^{\theta}\left([0,1];X\right)$ *with* $\theta \in \left]0,1\right[$ *and* $\varphi \in D(Q)$. *Set*

$$w(x) = e^{xQ}\varphi + \int_0^x e^{(x-s)Q} g(s)\, ds.$$

Then

$$Qw(\cdot) \simeq_\theta e^{\cdot Q}\left(Q\varphi + g(0)\right).$$

Proposition 4.4. *Assume (1.3) ~(1.7) . Let* $f \in C^{\theta}([0,1];X)$, *with* $0 < \theta < 1$ *and* $d_0, u_1 \in X$. *Then for* $\omega \in S_{\varphi_0}$ *and* $\mu \in S_{\varphi_1}$*:*

(1) $Q_\omega^2 R_3 \in C^{\infty}([0,1];X)$,
(2) $R_1 \in C^2([0,1];X)$ *and* $Q_\omega^2 R_1 \in C([0,1];X)$,
(3) $R_2 \in C^2([0,1];X)$ *and* $Q_\omega^2 R_2 \in C([0,1];X)$,
(4) $Q^2 R_1, Q^2 R_2, Q^2 R_3 \in C([0,1];X)$.

Proof. (1) For any $\psi \in X$ and $n \in \mathbb{N}^*$, we have

$$e^{Q_\omega}\psi \in \mathcal{L}\left(X, D(Q^n)\right),$$

so

$$e^{\cdot Q_\omega}e^{Q_\omega}\psi \in C^\infty([0,1];X),$$

applying A_ω, we obtain

$$A_\omega e^{\cdot Q_\omega}e^{Q_\omega}\psi = -e^{\cdot Q_\omega}Q_\omega^2 e^{Q_\omega}\psi \in C^\infty([0,1];X).$$

Using (3.6), we have for $x \in]0,1[$

$$A_\omega R_3\left(x\right) = e^{xQ_\omega}e^{Q_\omega}\varphi + e^{(1-x)Q_\omega}e^{Q_\omega}\xi,$$

it is clear that

$$A_\omega R_3\left(x\right) \in C^\infty([0,1];X),$$

which gives the result.

(2) Now, we have for all $x \in [0,1]$

$$\begin{aligned} R_1\left(x\right) &= e^{xQ_\omega}\left(Q_\omega - H_\mu\right)^{-1}\left(e^{Q_\omega}\tau_2 + \int_0^1 e^{sQ}(f(s)-f(0))ds\right) \\ &= e^{xQ_\omega}\left(Q_\omega - H_\mu\right)^{-1}e^{Q_\omega}\tau_2 \\ &\quad + e^{xQ_\omega}\left(Q_\omega - H_\mu\right)^{-1}\int_0^1 e^{sQ}(f(s)-f(0))ds, \end{aligned}$$

then

$$\begin{aligned} Q_\omega^2 R_1\left(x\right) &= e^{xQ_\omega}Q_\omega^2\left(Q_\omega - H_\mu\right)^{-1}Q_\omega^{-1}\left(Q_\omega e^{Q_\omega}\tau_2\right) \\ &\quad + e^{xQ_\omega}Q_\omega^2\left(Q_\omega - H_\mu\right)^{-1}Q_\omega^{-1}\left(Q_\omega\int_0^1 e^{sQ}(f(s)-f(0))ds\right). \end{aligned}$$

The term

$$\left(Q_\omega\int_0^1 ...ds\right) \in \left(D\left(Q\right), X\right)_{1-\theta,\infty} \subset \overline{D\left(Q\right)},$$

see Proposition 4.1. Then the first therm and the second are continuous on $[0,1]$, in virtue of (1.6) and (1.7).

(3) From Proposition 4.1, we have

$$\left\{\begin{array}{l} Q_\omega\int_0^1 e^{sQ_\omega}(f(s)-f(0))ds \in \left(D\left(Q\right), X\right)_{1-\theta,\infty} \subset \overline{D\left(Q\right)}, \\ Q_\omega\int_0^1 e^{(1-s)Q_\omega}(f(s)-f(1))ds \in \left(D\left(Q\right), X\right)_{1-\theta,\infty} \subset \overline{D\left(Q\right)}, \end{array}\right.$$

then, from Proposition 4.1, statement 1), $Q_\omega^2 R_2 \in C([0,1];X)$, since

$$\begin{aligned} Q_\omega^2 R_2(x) &= -\frac{1}{2}e^{xQ_\omega}Q_\omega\int_0^1 e^{sQ_\omega}(f(s)-f(0))ds \\ &\quad -\frac{1}{2}e^{(1-x)Q_\omega}Q_\omega\int_0^1 e^{(1-s)Q_\omega}(f(s)-f(1))ds. \end{aligned}$$

(4) For $j \in \{1,2,3\}$, write $Q^2R_j = Q^2Q_\omega^{-2}Q_\omega^2R_j$ with $Q^2Q_\omega^{-2} \in \mathcal{L}(X)$.

□

Proposition 4.5. *Assume (1.3) ~(1.8). Let $f \in C^\theta([0,1];X)$, with $0 < \theta < 1$, and d_0, $u_1 \in X$. Then for $\omega \in S_{\varphi_0}$ and $\mu \in S_{\varphi_1}$:*

(1) $Q_\omega^2R_3 \in C^\infty([0,1];X)$,
(2) $R_1 \in C^2([0,1];X)$ *and* $Q_\omega^2R_1 \in C^\theta([0,1];X)$,
(3) $R_2 \in C^2([0,1];X)$ *and* $Q_\omega^2R_2 \in C^\theta([0,1];X)$,
(4) $Q^2R_1, Q^2R_2, Q^2R_3 \in C^\theta([0,1];X)$.

Proof. The proof is similar to that of Proposition 4.4, we simply replace hypothesis (1.7) by (1.8), $\overline{D(Q)}$ by $D_Q(\theta;+\infty)$ and $C([0,1];X)$ by $C^\theta([0,1];X)$. □

5. Main result

According to the above study on the regularity of the solution, we are ready to state our main result on the existence and the uniqueness of the strict solution of problem (1.1)-(1.2).

Theorem 5.1. *Assume (1.3)~(1.8). Let $f \in C^\theta([0,1];X)$, with $0 < \theta < 1$ and $d_0, u_1 \in X$. Then, for any $\omega \in S_{\varphi_0}$ and $\mu \in S_{\varphi_1}$:*

(1) *Problem (1.1)-(1.2). has a unique strict solution u if and only if*

$$\begin{cases} i)\ \left(Q_\omega - H_\mu\right)^{-1}[d_0 - Q_\omega^{-1}f(0)] \in D\left(Q^2\right) \\ ii)\ Q_\omega^2\left(Q_\omega - H_\mu\right)^{-1}[d_0 - Q_\omega^{-1}f(0)] + f(0) \in \overline{D(Q)} \\ iii)\ u_1 \in D\left(Q^2\right) \\ iv)\ Q_\omega^2u_1 + f(1) \in \overline{D(Q)}. \end{cases} \tag{5.1}$$

(2) *Problem (1.1)-(1.2) has a unique strict solution u satisfying the maximal regularity property u'', $A_\omega u \in C^\theta([0,1];X)$ if and only if*

$$\begin{cases} i)\ \left(Q_\omega - H_\mu\right)^{-1}[d_0 - Q_\omega^{-1}f(0)] \in D\left(Q^2\right) \\ ii)\ Q_\omega^2\left(Q_\omega - H_\mu\right)^{-1}[d_0 - Q_\omega^{-1}f(0)] + f(0) \in D_Q(\theta,+\infty) \\ iii)\ u_1 \in D\left(Q^2\right) \\ iv)\ Q_\omega^2u_1 + f(1) \in D_Q(\theta,+\infty). \end{cases} \tag{5.2}$$

Proof. For (1), we first suppose that (5.1) is satisfied and we have to prove that u given by (3.5) is the strict solution u of problem (1.1)-(1.2). But to prove that

$$u \in C^2([0,1];X) \cap C([0,1];D(A)),$$

it is enough to show that $Q_\omega^2u \in C([0,1];X)$.
Using the Proposition 4.4, we have

$$Q_\omega^2R_i \in C([0,1];X),\ i = 1,2 \text{ and } Q_\omega^2R_3 \in C^\infty([0,1];X),$$

so we deduce

$$Q_\omega^2 \widetilde{u} = Q_\omega^2 u - Q_\omega^2 R_1 - Q_\omega^2 R_2 - Q_\omega^2 R_3 \in C\left([0,1];X\right),$$

where

$$\begin{aligned} \widetilde{u} &= e^{xQ_\omega}\left[\left(Q_\omega - H_\mu\right)^{-1}\left[d_0 - Q_\omega^{-1} f(0)\right] + \frac{1}{2} Q_\omega^{-2} f(0)\right] + I(x) \\ &\quad + e^{(1-x)Q_\omega}\left(u_1 + \frac{1}{2} Q_\omega^{-2} f(1)\right) + J(x). \end{aligned}$$

Then it is now known from Proposition 4.1, by statement 1, that this term is continuous on $[0,1]$ if and only if

$$\begin{cases} Q_\omega^2 \left(Q_\omega - H_\mu\right)^{-1}\left[d_0 - Q_\omega^{-1} f(0)\right] + f(0) \in \overline{D(Q)} \\ Q_\omega^2 u_1 + f(1) \in \overline{D(Q)}. \end{cases}$$

Conversely, we suppose that (5.1) is satisfied and show that the problem (1.1)-(1.2) has a strict solution u. From Proposition 4.1, we have

$$\begin{cases} e^{xQ_\omega}\left[Q_\omega^2\left(Q_\omega - H_\mu\right)^{-1}\left[d_0 - Q_\omega^{-1} f(0)\right] + f(0)\right] \in C\left([0,1];X\right) \\ e^{(1-x)Q_\omega}\left[Q_\omega^2 u_1 + f(1)\right] \in C\left([0,1];X\right). \end{cases}$$

Which implies that $Q_\omega^2 \widetilde{u}$ in $C([0,1];X)$. So

$$A_\omega u \in C\left([0,1];X\right),$$

then we deduce

$$u \in C\left([0,1];D(A)\right).$$

Since $f \in C^\theta([0,1];X)$, so

$$u'' = f - A_\omega u \in C\left([0,1];X\right),$$

this means

$$u \in C^2\left([0,1];X\right) \cap C\left([0,1];D(A)\right).$$

Moreover, we have according to (3.3) $u(1) = \mu_1 + I(1) = u_1,$ and taking into account the fact that

$$\begin{cases} u(0) = \mu_0 + J(0) = \Lambda_{\omega,\mu}^{-1}\beta_0 \in D(H) \\ H_\mu u(0) = T_\omega Q_\omega\left(I + e^{2Q_\omega}\right)\Lambda_{\omega,\mu}^{-1}\beta_0 - T_\omega \beta_0, \end{cases}$$

where

$$\begin{cases} \mu_0 = \Lambda_{\omega,\mu}^{-1}\beta_0 - J(0) \\ \beta_0 = \left(I - e^{2Q_\omega}\right) d_0 + 2Q_\omega e^{Q_\omega}\mu_1 + 2Q_\omega J(0) \in X. \end{cases}$$

And also, we have

$$\begin{aligned} u'(0) &= Q_\omega T_\omega\left(I + e^{2Q_\omega}\right)\mu_0 - 2Q_\omega T_\omega e^{Q_\omega}\mu_1 - Q_\omega J(0) \\ &= Q_\omega T_\omega\left(I + e^{2Q_\omega}\right)\Lambda_{\omega,\mu}^{-1}\beta_0 - Q_\omega T_\omega\left(I + e^{2Q_\omega}\right) J(0) - 2Q_\omega T_\omega e^{Q_\omega}\mu_1 - Q_\omega J(0), \end{aligned}$$

so

$$\begin{aligned} u'(0)-Hu(0)-\mu u(0) &= T_\omega\beta_0 - Q_\omega T_\omega\left(I+e^{2Q_\omega}\right)J(0) - 2Q_\omega T_\omega e^{Q_\omega}\mu_1 - Q_\omega J(0) \\ &= d_0 + 2T_\omega Q_\omega e^{Q_\omega}\mu_1 + 2T_\omega Q_\omega J(0) - Q_\omega T_\omega\left(I+e^{2Q_\omega}\right)J(0) \\ &\quad -2Q_\omega T_\omega e^{Q_\omega}\mu_1 - Q_\omega J(0) \\ &= d_0. \end{aligned}$$

Finally (5.1) is satisfied.
For (2), it is enough to replace the condition (5.1) by (5.2),then we use the Proposition 4.2. □

Corollary 5.1. *Assume (1.3) ~(1.8), let* $f \in C^\theta([0,1];X)$ *, with* $0<\theta<1$ *and* $d_0, u_1 \in X$*. If* $u_1 \in D(A)$*,* $d_0 \in D(Q)$ *and* Qd_0*,* $f(0), Q^2u_1 + f(1) \in D_Q(\theta,+\infty)$*, then problem (1.1)-(1.2) has a unique strict solution u satisfying maximal regularity property* u''*,* $A_\omega u \in C^\theta([0,1];X)$ *.*

6. Application

Let us illustrate our abstract results by the following problem. Set

$$\Omega = (0,1)\times(0,1),$$

and consider:

$$(P)\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) - \dfrac{\partial^4 u}{\partial y^4}(x,y) - \omega u(x,y) = f(x,y), & (x,y)\in\Omega \\ \dfrac{\partial u}{\partial x}(0,y) + a\dfrac{\partial^4 u}{\partial y^4}(0,y) - \mu u(0,y) = d_0(y), & y\in(0,1) \\ u(1,y) = u_1(y), & y\in(0,1) \\ u(x,0) = u(x,1) = \dfrac{\partial^2 u}{\partial y^2}(x,0) = \dfrac{\partial^2 u}{\partial y^2}(x,1) = 0, & x\in(0,1), \end{cases}$$

where $\omega \geq 0$, $\mu \geq 0$, $a>0$ and d_0, u_1 are some given functions and
$f \in C^\theta([0,1];L^p(0,1))$, $0<\theta<1$, $1<p<+\infty$.
Now, in the Banach space $X = L^p(0,1)$, we define:

$$\begin{cases} D(A) = \left\{\psi \in W^{4,p}(0,1) : \psi(0) = \psi(1) = \psi''(0) = \psi''(1) = 0\right\} \\ A\psi(y) = -\psi^{(4)}(y), \end{cases}$$

and

$$\begin{cases} D(H) = \left\{\psi \in W^{4,p}(0,1) : \psi(0) = \psi(1) = \psi''(0) = \psi''(1) = 0\right\} \\ H\psi(y) = -a\psi^{(4)}(y), \end{cases}$$

Then, it is not difficult to see that our problem (P) writes, in X, in the abstract following form

$$\begin{cases} u''(x) + Au(x) - \omega u(x) = f(x), \quad x \in]0,1[\\ u'(0) - Hu(0) - \mu u(0) = d_0, \\ u(1) = u_1. \end{cases}$$

Let us verify our assumptions (1.3)~(1.8). First we have

$$D(H) = D(A),$$

on the other hand A verifies (1.3) and $-H$ verifies (1.5) in

$$S_{\varphi_0} = S_{\varphi_1} = S_{\pi-\varepsilon_0},$$

for any $\varepsilon_0 \in \left]0, \pi\right[$. From this we know that operator $-\sqrt{-A} = Q$ generates an analytic semigroup strongly continuous at zero, see Balakrishnan [2] and from [1] we have

$$D(Q) = \left\{\psi \in W^{2,p}(0,1) : \psi(0) = \psi(1) = 0\right\}.$$

It is not difficult to prove that Q, $Q-H$ and H are boundedly invertible. Therefore for any $\psi \in D(H)$

$$\begin{aligned} Q\psi &= \frac{1}{\pi}\int_0^{+\infty} \frac{(A-\lambda I)^{-1}}{\sqrt{\lambda}}(-A)\psi d\lambda \\ &= \frac{1}{\pi}\int_0^{+\infty} \frac{(A-\lambda I)^{-1}}{a\sqrt{\lambda}} H\psi d\lambda, \end{aligned}$$

see [2]. By virtue of (1.3), we have

$$\begin{aligned} \left\|Q\psi\right\| &\leq k\left\|H\psi\right\|\int_0^{+\infty} \frac{d\lambda}{\sqrt{\lambda}(1+\lambda)} \\ &\leq \frac{k}{\pi}\left\|H\psi\right\|\int_0^{+\infty} \frac{d\lambda}{\sqrt{\lambda}(1+\lambda)} \\ &\leq k\left\|H\psi\right\|. \end{aligned}$$

So we can choose two positive constants k_1 and k_2 with k_1 very large as possible and $k_2 < 1$ such that

$$\left\|Q\psi\right\| \leq k_1\left\|\psi\right\| + k_2\left\|H\psi\right\|,$$

by virtue of (1.4) and (1.5) from where the linear operator $Q-H$ is closed and invertible, see [17], Theorem 1.1, p. 190.

The hypotheses $(Q-H)^{-1}(D(Q)) \subset D\left(Q^2\right)$ and $Q^2(Q-H)^{-1}\left(D_Q(1+\theta;+\infty)\right) \subset \overline{D(Q)}$ are checked automatically since $D\left(Q^2\right) = D(H) \subset D(Q)$ and $\overline{D(Q)} = \overline{D(A)} = X = L^p(0,1)$., therefore, all our assumptions are satisfied

Now we must assume verify the following conditions of u_1 and d_0.

(1) The condition $u_1 \in D\left(Q^2\right)$.

This, with respect to the variable y, translates into

$$u_1 \in W^{2,p}(0,1) : u_1(0) = u_1(1) = 0.$$

(2) The condition $Q^2 u_1 + f(1) \in \overline{D(Q)} = L^p(0,1)$.

It means that

$$y \mapsto u_1''(y) + f(1,y) \in L^p(0,1).$$

This is automatically verified, because $u_1'' \in L^p(0,1)$ and

$$f \in C^\theta([0,1]; L^p(0,1)),$$

which implies $y \mapsto f(1,y) = f(1)(y) \in L^p(0,1)$.

(3) It is clear that the condition $Q(Q-H)^{-1}\left(d_0 - Q^{-1}f(0)\right) \in D(Q)$.

(4) The condition $Q^2(Q-H)^{-1}\left(d_0 - Q^{-1}f(0)\right) + f(0) \in \overline{D(Q)} = L^p(0,1)$.

It becomes $d_0 \in D(Q)$.

Therefore, Theorem 5.1 for the existence and the uniqueness of the strict solution applies.

We do the same for the maximal regularity.

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References

[1] P. Auscher, A. McIntosh, A. Nahmod, The Square Root Problem of Kato in One Dimension, and First Order Elliptic Systems, Ind. Univ. Math. J. 46 (1997), 659–695. https://doi.org/10.1512/iumj.1997.46.1423.

[2] A. Balakrishnan, Fractional Powers of Closed Operators and the Semigroups Generated by Them, Pac. J. Math. 10 (1960), 419–437. https://doi.org/10.2140/pjm.1960.10.419.

[3] M. Cheggag, Problèmes de Sturm-liouville Abstraits pour une Équation Differentielle Abstraite Complète Elliptique du Second Ordre dans Divers Espaces, Thesis, Université Ahmed Ben Bella-Oran 1, 2008.

[4] M. Cheggag, A. Favini, R. Labbas, S. Maingot, A. Medeghri, Sturm-Liouville Problems for an Abstract Differential Equation of Elliptic Type in UMD Spaces, Differ. Integral Equ. 21 (2008), 981–1000. https://doi.org/10.57262/die/1356038596.

[5] M. Cheggag, A. Favini, R. Labbas, S. Maingot, A. Medeghri, Abstract Differential Equations of Elliptic Type with General Robin Boundary Conditions in Hölder Spaces, Appl. Anal. 91 (2012), 1453–1475. https://doi.org/10.1080/00036811.2011.635653.

[6] M. Cheggag, A. Favini, R. Labbas, S. Maingot, K. Ould Melha, New Results on Complete Elliptic Equations with Robin Boundary Coefficient-Operator Conditions in Non Commutative Case, Bull. South Ural. State Univ. Ser. Math. Model. Program. Comput. Softw. 10 (2017), 70–96. https://doi.org/10.14529/mmp170105.

[7] G. Da Prato, Abstract Differential Equations, Maximal Regularity, and Linearization, in: Proceedings of Symposia in Pure Mathematics, American Mathematical Society, Providence, Rhode Island, 1986, pp. 359–370. https://doi.org/10.1090/pspum/045.1/843572.

[8] G. Da Prato, P. Grisvard, Sommes d'Opérateurs Linéaires et Equations Différentielles Opérationnelles, J. Math. Pures Appl. 54 (1975), 305–387. https://cir.nii.ac.jp/crid/1371132820324105090.

[9] G. Dore, S. Yakubov, Semigroup Estimates and Noncoercive Boundary Value Problems, Semigroup Forum 60 (2000), 93–121. https://doi.org/10.1007/pl00020982.

[10] A. Favini, R. Labbas, S. Maingot, A. Thorel, Elliptic Differential-Operators with an Abstract Robin Boundary Condition and Two Spectral Parameters, Results Math. 80 (2025), 188. https://doi.org/10.1007/s00025-025-02507-1.

[11] A. Favini, R. Labbas, S. Maingot, H. Tanabe, A. Yagi, Necessary and Sufficient Conditions for Maximal Regularity in the Study of Elliptic Differential Equations in Hölder Spaces, Discret. Contin. Dyn. Syst. 22 (2008), 973–987. https://doi.org/10.3934/dcds.2008.22.973.

[12] C.T. Fulton, Two-Point Boundary Value Problems with Eigenvalue Parameter Contained in the Boundary Conditions, Proc. R. Soc. Edinb.: Sect. A Math. 77 (1977), 293–308. https://doi.org/10.1017/s030821050002521x.

[13] P. Grisvard, Spazi di Tracce e Applicazioni, Rend. Mat. (4), 5 (1972), 657–729.

[14] D. Guidetti, An Introduction to Maximal Regularity for Parabolic Problems and Interpolation Theory With Application to an Inverse Problem, International Minicourse-Workshop, Interplay Between (C0)-Semigroups and PDEs: Theory and Applications, pp. 113-146, 2003.

[15] M. Haase, The Functional Calculus for Sectorial Operators, Birkhäuser Basel, 2006. https://doi.org/10.1007/3-7643-7698-8.

[16] R. Haoua, R. Labbas, S. Maingot, A. Medeghri, New Results on Abstract Elliptic Problems with General Robin Boundary Conditions in Hölder Spaces: Non Commutative Cases, Boll. Un Mat. Ital. 15 (2022), 413–436. https://doi.org/10.1007/s40574-021-00311-7.

[17] T. Kato, Perturbation Theory for Linear Operators, Springer, Berlin, 1980. https://doi.org/10.1007/978-3-642-53393-8.

[18] S.G. Kreui, Linear Differential Equations in Banach Space, American Mathematical Society, (2011).

[19] A.V. Lykov, Y.A. Mikhailov, Theory of Heat and Mass Transfer, Jerusalem, (1965).

[20] C. Martinez, M. Sanz, The Theory of Fractional Powers of Operators, North-Holland, 2001. https://doi.org/10.1016/S0304-0208(00)X8025-5.

[21] E. Sinestrari, On the Abstract Cauchy Problem of Parabolic Type in Spaces of Continuous Functions, J. Math. Anal. Appl. 107 (1985), 16–66. https://doi.org/10.1016/0022-247x(85)90353-1.

[22] A.N. Tikhonov, A.A. Samarskii, Equations Of Mathematical Physics, Dover Publications, New York, (1963).

[23] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North Holland, 1978.

[24] N.N. Voitovich, B.Z. Katsenelbaum, A.N. Sivov, Generalized Method of Eigene-Vibration in the Theory of Diffraction, Nakua, Moscow, 1997.