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TVaR-Based Capital Allocation under Liouville Copulas

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Abstract. This paper provides explicit closed-form expressions for key tail risk measures, namely the Tail Value-at-Risk (TVaR) and TVaR-based capital allocation, in a multivariate risk framework governed by Liouville distributions. Introduced by McNeil and Nešlehová (2010), Liouville copulas offer a flexible and tractable class of models for capturing asymmetric and non-exchangeable dependencies. We derive analytical expressions for the distribution and survival functions of aggregate risks under various parametric specifications, including Clayton-Liouville and generalized Clayton-Liouville models. The conditions under which TVaR is finite are discussed in relation to the existence of moments. Numerical illustrations highlight the impact of dependence parameters and generator shapes on aggregate tail risk and its decomposition, demonstrating the practical relevance of Liouville-based models for capital modeling and solvency assessment.

1. Introduction

Risk management remains a cornerstone of both the finance and insurance sectors, presenting complex challenges that require robust and economically meaningful mathematical frameworks. Central to this field is the concept of *risk measures*, which have long been instrumental in actuarial science and quantitative finance. In actuarial practice, these measures underpin the pricing of insurance and reinsurance contracts, the setting of regulatory capital requirements to ensure solvency, and the allocation of capital across diversified portfolios within insurance groups.

Among various risk measures, *Value at Risk (VaR)* is perhaps the most widely adopted by practitioners. Defined as the quantile of a risk distribution, VaR represents the threshold loss not exceeded with a specified confidence level $1-\alpha$. Despite its prevalence, VaR fails to satisfy the coherence criteria established by Artzner et al. [1], particularly due to its lack of subadditivity, which can underestimate risk in diversified portfolios.

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An alternative, more theoretically robust measure is the *Conditional Tail Expectation (CTE)*, also referred to as *Tail Value at Risk (TVaR)*. CTE captures the expected loss conditional on losses exceeding the VaR threshold, thus providing a fuller depiction of tail risk.

For a given confidence level $\kappa \in (0,1)$, the Value-at-Risk of the total loss S_d is defined as:

$$VaR_\kappa(S_d) = \inf\left\{x \in \mathbb{R} : F_{S_d}(x) \geq \kappa\right\}.$$

For continuous distributions, the Conditional Tail Expectation coincides with the Tail Value-at-Risk and is defined as

$$TVaR_\kappa(S_d) = \mathbb{E}\left[S_d \mid S_d > VaR_\kappa(S_d)\right]$$

It also admits the equivalent integral representation

$$\text{TVaR}_\kappa(S_d) = \frac{1}{1-\kappa}\int_\kappa^1 \text{VaR}_u(S_d)\,du.$$

Both expressions are used interchangeably.
CTE is a *coherent risk measure* in this setting, which makes it particularly attractive for capital adequacy and allocation applications Dhaene et al. [6], Furman and Zitikis [12], Furman and Landsman [10].

Explicit expressions for CTE have been derived for various distribution families: elliptical distributions Landsman and Valdez [14], phase-type distributions Cai and Li [3], and the Farlie-Gumbel-Morgenstern copula family Bargès et al. [2]. This breadth of literature underscores the continued academic interest in developing tractable and interpretable formulations of tail-based risk metrics.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ denote a random vector of d non-negative continuous risks, where each X_i corresponds to an individual claim or loss. The *aggregate risk* is given by $S_d = \sum_{i=1}^d X_i$. In capital allocation problems, one seeks to decompose the total risk into marginal contributions. Under a TVaR-based allocation rule, the *marginal contribution* of X_i to S_d is defined as:

$$TVaR_\kappa(X_i; S_d) = \mathbb{E}\left[X_i \mid S_d > VaR_\kappa(S_d)\right],$$

which satisfies the *additivity property*:

$$TVaR_\kappa(S_d) = \sum_{i=1}^{d} TVaR_\kappa(X_i; S_d).$$

This property enables exact allocation of the total capital among the individual risks.

Modeling dependence among risks is essential, particularly in settings where joint tail behavior has significant implications for systemic risk. Copula-based models allow for flexible and accurate representation of dependencies. Numerous studies have investigated CTE in multivariate settings: elliptical distributions Landsman and Valdez [14], phase-type Cai and Li [3], gamma Furman and Landsman [9], Tweedie Furman and Landsman [11], Pareto Chiragiev and Landsman [4], and mixtures with Archimedean copulas Sarabia et al. [22], Marri and Moutanabbir [15].

Beyond distributional approaches, several authors have investigated capital allocation using axiomatic or alternative risk-based frameworks. Maume-Deschamps et al. [16] analyzed the coherence of capital allocation rules derived from the minimization of multivariate risk indicators, emphasizing their relevance under Solvency II's ORSA requirements. Said [21] studied capital allocation using the Euler principle with expectiles as risk measures, focusing on how dependence structures influence marginal risk contributions across various actuarial models.

This paper contributes to the existing literature by deriving explicit expressions for both $TVaR_\kappa(S_d)$ and its component contributions $TVaR_\kappa(X_i; S_d)$ under *multivariate Liouville distributions*, a class of models introduced in McNeil and Nešlehová [19]. Liouville copulas are particularly suited to modeling *non-exchangeable* dependencies, a feature that enhances their relevance for applications in capital allocation. We complement the theoretical findings with *numerical illustrations*, highlighting the influence of dependence structures on TVaR and its allocations.

The remainder of this paper is structured as follows. Section 2 introduces multivariate Liouville distributions. Section 3 derives closed-form expressions for TVaR and TVaR-based allocations. In Section 4, we provide specific expressions for the aggregated distribution. Section 5 presents numerical examples to demonstrate the practical implications of our results.

Notation. Throughout the paper, vectors in $\mathbb{R}^d$ are denoted in boldface, and $\mathbf{1}_d$ represents the vector $(1, \ldots, 1)$ in $\mathbb{R}^d$.

2. Multivariate Liouville distributions

In this section, we briefly recall the definition and properties of multivariate Liouville distributions, which form the foundational framework of the risk models considered in this paper.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ be a random vector of d continuous non-negative random variables with joint survival function $\bar{H}$ and marginal survival functions $\bar{H}_i$, for $i = 1, \ldots, d$. The vector $\mathbf{X}$, defined on $\mathbb{R}_+^d = [0, \infty)^d$, is said to follow a *Liouville distribution* if it admits the following stochastic representation:

$$\mathbf{X} = R\mathbf{D}, \tag{2.1}$$

where R is a strictly positive random variable with distribution function F_R, independent of the d-dimensional random vector $\mathbf{D} = (D_1, \ldots, D_d)$, which follows a Dirichlet distribution with parameters $(\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$. The vector $\mathbf{D}$ takes values in the unit simplex:

$$\mathbb{S}_d = \left\{(x_1, \ldots, x_d) \in [0,1]^d : x_1 + \cdots + x_d = 1\right\}.$$

Properties of the Dirichlet and Liouville distributions can be found in Devroye [5] and Fang et al. [7]. It is well known that the Dirichlet vector $\mathbf{D}$ can be represented as:

$$\mathbf{D} \stackrel{d}{=} \left(\frac{Z_1}{\sum_{i=1}^d Z_i}, \ldots, \frac{Z_d}{\sum_{i=1}^d Z_i}\right),$$

where the Z_i are independent Gamma random variables with shape parameters α_i and unit scale, i.e., $Z_i \sim \text{Gamma}(\alpha_i, 1)$ for $i = 1, \ldots, d$.

As a consequence, each component $X_i = RD_i$ of $\mathbf{X}$ is a scale mixture of a Beta distribution. Specifically, $D_i \sim \mathcal{B}(\alpha_i, \alpha - \alpha_i)$, where $\alpha = \sum_{j=1}^{d} \alpha_j$.

According to Theorem 2 in McNeil and Nešlehová [19], the joint survival function of $\mathbf{X}$ can be expressed for all $(x_1, \dots, x_d) \in \mathbb{R}_+^d$ as:

$$\bar{H}(x_1, \dots, x_d) = \sum_{i_1=0}^{\alpha_1 - 1} \cdots \sum_{i_d=0}^{\alpha_d - 1} (-1)^{i_1 + \cdots + i_d} \frac{\psi^{(i_1 + \cdots + i_d)}(x_1 + \cdots + x_d)}{i_1! \cdots i_d!} \prod_{j=1}^{d} x_j^{i_j}, \tag{2.2}$$

where $\alpha = \sum_{j=1}^{d} \alpha_j$ and $\psi = \mathfrak{W}_\alpha F_R$ denotes the Williamson α-transform of the distribution function F_R of the radial variable R. The transform ψ is defined as:

$$\mathfrak{W}_\alpha F_R(x) = \int_{(x,\infty)} \left(1 - \frac{x}{t}\right)^{\alpha - 1} dF_R(t) = \mathbb{E}\left[\left(1 - \frac{x}{R}\right)_+^{\alpha - 1}\right], \quad x \in [0, \infty).$$

When ψ is α-times differentiable almost everywhere, the vector $\mathbf{X}$ admits a joint density given by:

$$h(x_1, \dots, x_d) = (-1)^\alpha \psi^{(\alpha)}(x_1 + \cdots + x_d) \prod_{i=1}^{d} \frac{x_i^{\alpha_i - 1}}{\Gamma(\alpha_i)}.$$

Furthermore, as shown in McNeil and Nešlehová [19], the transform $\mathfrak{W}_\alpha F_R$ uniquely determines the distribution function F_R, which can be recovered via the inverse Williamson α-transform. If ψ is α-times differentiable, then $F_R(x) = \mathfrak{W}_\alpha^{-1} \psi(x)$ has a density given by:

$$f_R(x) = \frac{(-1)^\alpha x^{\alpha - 1} \psi^{(\alpha)}(x)}{(\alpha - 1)!}, \quad x \in (0, \infty). \tag{2.3}$$

In particular, the marginal survival functions of $\mathbf{X}$ satisfy, for all $x > 0$ and $i \in \{1, \dots, d\}$:

$$\bar{H}_i(x) = \mathbb{P}(X_i > x) = \sum_{j=0}^{\alpha_i - 1} \frac{(-1)^j x^j \psi^{(j)}(x)}{j!} = 1 - \mathfrak{W}_{\alpha_i}^{-1} \psi(x). \tag{2.4}$$

By Sklar's theorem for survival functions (see, e.g., [23], Nelsen [20]), the survival copula associated with $\mathbf{X}$, denoted by C, is defined for all $(u_1, \dots, u_d) \in [0,1]^d$ by:

$$C(u_1, \dots, u_d) = \bar{H}\left(\bar{H}_1^{-1}(u_1), \dots, \bar{H}_d^{-1}(u_d)\right). \tag{2.5}$$

In certain cases, the copula density can be obtained explicitly as:

$$c(u_1, \dots, u_d) = \frac{h\left(\bar{H}_1^{-1}(u_1), \dots, \bar{H}_d^{-1}(u_d)\right)}{\prod_{i=1}^{d} h_i\left(\bar{H}_i^{-1}(u_i)\right)}. \tag{2.6}$$

Although Liouville copulas generally do not have closed-form expressions, it follows from equations (2.2) and (2.4) that the dependence structure encoded in the copula C is entirely characterized by two components: the Williamson α-transform ψ of the radial distribution F_R, and the Dirichlet shape parameters $(\alpha_1, \dots, \alpha_d)$.

Copulas were first introduced in insurance modeling by Frees and Valdez [8] as a powerful tool to separate the dependence structure from marginal distributions. For an overview of copula theory, we refer the reader to Nelsen [20]. Liouville copulas, introduced in McNeil and Nešlehová [19],

generalize the Archimedean class by allowing for non-exchangeable dependence structures, thus providing greater modeling flexibility.
In the special case where $\alpha_i = 1$ for all $i = 1, \ldots, d$, the survival function in (2.2) simplifies to:

$$\bar{H}(x_1, \ldots, x_d) = \psi(x_1 + \cdots + x_d), \quad \text{with} \quad \bar{H}_i(x) = \psi(x).$$

In this case, the associated copula is an Archimedean survival copula with generator ψ, and is given for all $(u_1, \ldots, u_d) \in [0,1]^d$ by:

$$C(u_1, \ldots, u_d) = \psi\left(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)\right).$$

However, as noted in McNeil and Nešlehová [19], when $(\alpha_1, \ldots, \alpha_d) \neq \mathbf{1}_d$, the survival copula of $\mathbf{X} = R\mathbf{D}$ is no longer Archimedean. The Liouville class thus includes Archimedean copulas as a particular case, but also extends beyond the exchangeable framework.
Moreover, as shown in McNeil and Nešlehová [18], there exists a one-to-one correspondence between the function ψ and the radial distribution F_R. Specifically, ψ must be α-monotone, with $\psi(1) = 0$ and $\psi(x) \to 0$ as $x \to \infty$.

The moments of $\mathbf{X}$ can be conveniently expressed in terms of the moments of the radial variable R. The following proposition provides closed-form expressions for the marginal variances and bivariate covariances:

Proposition 2.1. *Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a Liouville-distributed vector with parameters $(\alpha_1, \ldots, \alpha_d)$ and radial distribution R. Then, for all $i, j \in \{1, \ldots, d\}$:*

$$\mathrm{Var}(X_i) = \frac{\alpha_i}{\alpha^2(\alpha+1)}\left[\alpha(\alpha_i+1)\mathrm{Var}(R) + (\alpha - \alpha_i)(\mathbb{E}[R])^2\right], \tag{2.7}$$

$$\mathrm{Cov}(X_i, X_j) = \frac{\alpha_i \alpha_j}{\alpha^2(\alpha+1)}\left[\alpha \mathrm{Var}(R) - (\mathbb{E}[R])^2\right]. \tag{2.8}$$

A general expression for Kendall's tau for Liouville copulas was derived in McNeil and Nešlehová [19], based on the radial distribution R. In the bivariate case ($d = 2$), the dependence coefficient is given by:

$$\tau(\mathbf{X}) = \tau(C) = 4 \sum_{i_1=0}^{\alpha_1-1} \sum_{i_2=0}^{\alpha_2-1} (-1)^{i_1+i_2} \frac{B(\alpha_1+i_1, \alpha_2+i_2)\Gamma(\alpha)}{i_1!\, i_2!\, B(\alpha_1,\alpha_2)\, \Gamma(\alpha - i_1 - i_2)} \, \mathbb{E}\left[Y^{i_1+i_2}(1-Y)_+^{\alpha-i_1-i_2-1}\right] - 1,$$

where $Y = R/R^*$ and R^* is an independent copy of R.

Compared to Archimedean copulas, Liouville copulas offer increased flexibility, particularly in modeling non-symmetric dependence structures. In the next section, we explore how such dependence influences risk aggregation and capital allocation based on the Tail Value-at-Risk.

3. TVaR and TVaR-Based Allocation under Liouville Copulas

In this section, we consider an individual risk model where dependence among risks is captured using a multivariate Liouville distribution. Let $(X_1, \ldots, X_d)$ be a random vector following the Liouville structure defined in (2.1). The aggregate claim amount is given by $S_d = X_1 + \cdots + X_d$,

with cumulative distribution function F_{S_d}. Since the Dirichlet weights satisfy $D_1 + \cdots + D_d = 1$, it follows that $S_d = R$.
Our objective is twofold: first, to derive a closed-form expression for the Tail Value-at-Risk (TVaR) of the aggregate risk; second, to determine the TVaR-based capital allocation to each individual risk component. Theorem 3.1 provides a general expression for $\text{TVaR}_\kappa(S_d)$ under Liouville dependence, extending known results for Archimedean copulas. Theorem 3.2 characterizes the corresponding TVaR-based allocation among the marginal risks.

Theorem 3.1. *Let $X_1, X_2, \ldots, X_d$ be dependent non-negative random variables joined by a Liouville copula, and let $S_d = \sum_{i=1}^d X_i$ denote the aggregate risk. Then, the Tail Value-at-Risk of S_d at level $\kappa \in (0,1)$ is given by:*

$$TVaR_\kappa(S_d) = \frac{\alpha}{1-\kappa} \int_{VaR_\kappa(S_d)}^{\infty} \psi(x)\, dx + \frac{\alpha}{1-\kappa} \sum_{i=1}^{\alpha} (-1)^{\alpha+i+1} \frac{1}{i!}\, VaR_\kappa^i(S_d)\, \psi^{(i-1)}\left(VaR_\kappa(S_d)\right), \tag{3.1}$$

where $\psi^{(k)}$ denotes the k-th derivative of the Williamson α-transform ψ.

Proof. Substituting the expression of the density $f_R(x)$ from (2.3) into the definition of $TVaR_\kappa(S_d)$, we obtain:

$$TVaR_\kappa(S_d) = \frac{1}{1-\kappa} \cdot \frac{(-1)^\alpha}{(\alpha-1)!} \int_{VaR_\kappa(S_d)}^{\infty} x^\alpha\, \psi^{(\alpha)}(x)\, dx.$$

An integration-by-parts formula yields the identity:

$$\int_{VaR_\kappa(S_d)}^{\infty} x^\alpha \psi^{(\alpha)}(x)\, dx = \sum_{i=1}^{\alpha} (-1)^{i+1} \frac{\alpha!}{i!}\, VaR_\kappa^i(S_d)\, \psi^{(i-1)}\left(VaR_\kappa(S_d)\right) + (-1)^\alpha \alpha! \int_{VaR_\kappa(S_d)}^{\infty} \psi(x)\, dx.$$

Combining the two expressions gives the result:

$$TVaR_\kappa(S_d) = \frac{1}{1-\kappa} \cdot \frac{(-1)^\alpha}{\Gamma(\alpha)} \sum_{i=1}^{\alpha} (-1)^{i+1} \frac{\alpha!}{i!}\, VaR_\kappa^i(S_d)\, \psi^{(i-1)}\left(VaR_\kappa(S_d)\right) + \frac{\alpha}{1-\kappa} \int_{VaR_\kappa(S_d)}^{\infty} \psi(x)\, dx.$$

□

Theorem 3.2. *Let $\mathbf{X} = R\mathbf{D}$ be a Liouville random vector with Dirichlet parameters $(\alpha_1, \ldots, \alpha_d)$ and a strictly positive radial variable R. Then, the TVaR-based capital allocation to the i-th risk component X_i at level $\kappa \in (0,1)$ is given by:*

$$TVaR_\kappa(X_i; S_d) = \frac{\alpha_i}{\alpha}\, TVaR_\kappa(S_d), \tag{3.2}$$

where $\alpha = \sum_{j=1}^d \alpha_j$.

Proof. The TVaR-based allocation to risk X_i can be written as:

$$TVaR_\kappa(X_i; S_d) = \frac{1}{1-\kappa}\, \mathbb{E}\left[X_i \cdot \mathbf{1}_{\{S_d > VaR_\kappa(S_d)\}}\right].$$

This expression can be rewritten using the joint density of (X_i, S_d):

$$TVaR_\kappa(X_i; S_d) = \frac{1}{1-\kappa} \int_{VaR_\kappa(S_d)}^{\infty} \int_0^s x\, f_{X_i,S_d}(x,s)\, dx\, ds.$$

Since $S_d = X_i + (S_d - X_i)$, the joint density admits the factorized form:

$$f_{X_i,S_d-X_i}(x,y) = (-1)^\alpha\, \psi^{(\alpha)}(x+y) \cdot \frac{x^{\alpha_i-1} y^{\alpha-\alpha_i-1}}{\Gamma(\alpha_i)\Gamma(\alpha-\alpha_i)}.$$

Substituting and integrating gives:

$$\begin{aligned} TVaR_\kappa(X_i; S_d) &= \frac{(-1)^\alpha}{\Gamma(\alpha_i)\Gamma(\alpha-\alpha_i)} \cdot \frac{1}{1-\kappa} \int_{VaR_\kappa(S_d)}^{\infty} \psi^{(\alpha)}(s) \int_0^s x^{\alpha_i}(s-x)^{\alpha-\alpha_i-1}\, dx\, ds \\ &= \frac{(-1)^\alpha \alpha_i}{\Gamma(\alpha+1)} \cdot \frac{1}{1-\kappa} \int_{VaR_\kappa(S_d)}^{\infty} s^\alpha\, \psi^{(\alpha)}(s)\, ds \\ &= \frac{\alpha_i}{\alpha}\, TVaR_\kappa(S_d), \end{aligned}$$

which completes the proof. □

Remark 3.1 (Alternate proof)**.** *Since $S_d = R$ and $\mathbb{E}[X_i \mid S = s] = \mathbb{E}[D_i]\, s = (\alpha_i/\alpha)\, s$ for Dirichlet D, integrating from* $\mathrm{VaR}_\kappa(S_d)$ *to ∞ yields*

$$\mathrm{TVaR}_\kappa(X_i; S_d) = \frac{\alpha_i}{\alpha}\, \mathrm{TVaR}_\kappa(S_d).$$

The result stated in Theorem 3.2 highlights the role of non-exchangeability in shaping the structure of TVaR-based capital allocation. It provides a clear interpretation of the Dirichlet parameters $(\alpha_1, \ldots, \alpha_d)$ as representing the relative contributions of the marginal risks to the aggregate risk. In this framework, each risk X_i receives a proportion α_i/α of the total allocated capital, where $\alpha = \sum_{j=1}^{d} \alpha_j$.

Beyond its theoretical appeal, the main advantage of the Liouville copula framework lies in its tractability. The availability of closed-form expressions for both the TVaR of the aggregate loss and its decomposition into marginal contributions enables the implementation of efficient numerical procedures. These results offer a valuable foundation for risk quantification and allocation in practical applications.

4. Models Derived from the Liouville Copula

A natural approach to constructing d-dimensional Liouville copulas involves selecting a generator ψ that satisfies α-monotonicity. In this section, we examine several dependence structures obtained from distinct choices of such generators (see McNeil and Nešlehová [18], [19] for further details). Building on the theoretical results established in Section 3, we derive explicit expressions for both the Tail Value-at-Risk of the aggregate risk, $TVaR_\kappa(S_d)$, and its component contributions $TVaR_\kappa(X_i; S_d)$ under each specification.

4.1. **Clayton–Liouville Copulas.** According to McNeil and Nešlehová [19], the copula associated with a Liouville-distributed vector $\mathbf{X}$ is called a *Clayton–Liouville copula* when the Williamson α-transform of the radial distribution F_R is given by:

$$\psi_\theta(x) = \mathfrak{W}_\alpha F_R(x) = \{\max(0, 1+\theta x)\}^{-1/\theta}, \quad \theta \geq -1/(\alpha-1).$$

In this work, we focus on the case where $\theta > 0$, corresponding to positive dependence.
The k-th derivative of the generator ψ_θ takes the following form:

$$\psi_\theta^{(k)}(x) = \begin{cases} (-1)^k c(\theta,k)(1+\theta x)^{-(1/\theta+k)} & \text{if } 1+\theta x > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $c(\theta,k) = \theta^{k-1}\Gamma(1/\theta+k)/\Gamma(1/\theta+1)$.
Substituting (4.1) into the general expressions for the marginal and joint survival functions (see Section 2), we obtain:

$$\bar{H}_i(x) = \psi_\theta(x) \sum_{j=0}^{\alpha_i-1} \frac{c(\theta,j)}{j!} \left(\frac{x}{1+\theta x}\right)^j, \quad x \in \mathbb{R}_+, \tag{4.2}$$

and

$$\bar{H}(x_1,\ldots,x_d) = \sum_{i_1=0}^{\alpha_1-1} \cdots \sum_{i_d=0}^{\alpha_d-1} \frac{c(\theta, \sum_{j=1}^d i_j)}{i_1! \cdots i_d!} \left(\max\left(1+\theta\sum_{j=1}^d x_j, 0\right)\right)^{-1/\theta-\sum_{j=1}^d i_j} \prod_{j=1}^d x_j^{i_j}. \tag{4.3}$$

Remark 4.1. *Setting* $\alpha_1 = \cdots = \alpha_d = 1$ *in* (4.3) *yields the multivariate Pareto survival function:*

$$\bar{H}(x_1,\ldots,x_d) = \left(\max\left(1+\theta\sum_{j=1}^d x_j, 0\right)\right)^{-1/\theta}.$$

The corresponding survival marginals are:

$$\bar{H}_i(x) = (\max(0, 1+\theta x))^{-1/\theta}.$$

In this special case, the density and tail risk measures of the aggregate risk S_d *coincide with the results derived by Sarabia et al. [22].*

Theorem 4.1. *Let* $S_d = X_1 + \cdots + X_d$ *denote the aggregate of* d *dependent random variables with joint distribution defined by a Clayton–Liouville copula. Then, the probability density function of* S_d *is given by:*

$$f_{S_d}(x) = \frac{\theta^\alpha}{B(\alpha, 1/\theta)} x^{\alpha-1}(1+\theta x)^{-(\alpha+1/\theta)}, \quad x > 0, \tag{4.4}$$

where $\theta > 0$, $\alpha = \sum_{i=1}^d \alpha_i$, *and* $B(\cdot,\cdot)$ *denotes the Beta function.*

Proof. Substituting the k-th derivative of ψ_θ given in (4.1) into the general formula (2.3) with $k = \alpha$, we obtain:

$$f_{S_d}(x) = \frac{(-1)^\alpha x^{\alpha-1}\psi^{(\alpha)}(x)}{(\alpha-1)!} = \frac{\theta^{\alpha-1}}{(\alpha-1)!} \cdot \frac{\Gamma(1/\theta+\alpha)}{\Gamma(1/\theta+1)} x^{\alpha-1}(1+\theta x)^{-(\alpha+1/\theta)}.$$

Using the identity $B(\alpha, 1/\theta) = \Gamma(\alpha)\Gamma(1/\theta)/\Gamma(\alpha+1/\theta)$ leads to the stated result. □

Remark 4.2. *The density in* (4.4) *corresponds to a Beta distribution of the second kind, denoted* $S_d \sim \mathfrak{B}2(1/\theta, \alpha, 1/\theta)$*; see McDonald and Xu [17]. Its cumulative distribution function can be expressed as:*

$$F_{S_d}(x) = IB\left(\frac{x}{x+1/\theta}; \alpha, 1/\theta\right),$$

where $IB(x; p, q)$ *denotes the incomplete Beta ratio function:*

$$IB(x; p, q) = \frac{\int_0^x t^{p-1}(1-t)^{q-1}dt}{B(p, q)}.$$

The k-th moment of S_d *is given by:*

$$\mathbb{E}\left[S_d^k\right] = \frac{\Gamma(\alpha+k)\Gamma(1/\theta-k)}{\theta^k\,\Gamma(\alpha)\Gamma(1/\theta)}, \quad \text{for } -\alpha < k < 1/\theta.$$

Theorem 4.2. *Let* $X_1, \ldots, X_d$ *be dependent random variables with a joint distribution given by a Clayton–Liouville copula, and let* $S_d = \sum_{i=1}^d X_i$ *denote the aggregate risk. Then, the Tail Value-at-Risk of* S_d *at level* $\kappa \in (0, 1)$ *is given by:*

$$TVaR_\kappa(S_d) = \mathbb{E}\Big[S_d\Big] \cdot \frac{1 - IB\left(\frac{VaR_\kappa(S_d)}{VaR_\kappa(S_d)+1/\theta}; \alpha+1, 1/\theta-1\right)}{1-\kappa}, \tag{4.5}$$

where $IB(x; p, q)$ *denotes the incomplete Beta ratio function defined as:*

$$IB(x; p, q) = \frac{\int_0^x t^{p-1}(1-t)^{q-1}dt}{B(p, q)},$$

and $B(p, q)$ *is the Beta function.*

Proof. Using the density function of S_d given in (4.4), the TVaR is computed as:

$$TVaR_\kappa(S_d) = \frac{1}{1-\kappa}\int_{VaR_\kappa(S_d)}^{\infty} x\, f_{S_d}(x)\, dx.$$

From the known moment expression $\mathbb{E}[S_d] = \frac{\alpha}{\theta(1/\theta-1)}$ and the fact that S_d follows a second-kind Beta distribution, it follows (see McDonald and Xu [17]) that:

$$TVaR_\kappa(S_d) = \mathbb{E}\Big[S_d\Big] \cdot \frac{1 - IB\left(\frac{VaR_\kappa(S_d)}{VaR_\kappa(S_d)+1/\theta}; \alpha+1, 1/\theta-1\right)}{1-\kappa}.$$

This completes the proof. □

Remark 4.3. *When all Dirichlet parameters are equal to one, i.e.,* $\alpha_1 = \cdots = \alpha_d = 1$*, formula* (4.5) *reduces to the TVaR expression obtained by Sarabia et al. [22] for the multivariate Pareto model associated with Archimedean copulas.*

4.2. **Generalized Clayton–Liouville Copulas.** We now consider a Liouville-distributed random vector with integer parameters $\alpha_1, \ldots, \alpha_d$ and a radial distribution F_R whose Williamson α-transform takes the form:

$$\psi_{a,b}(x) = \mathfrak{W}_\alpha F_R(x) = \left(1 + \frac{x}{b}\right)_+^{-a},$$

where $a \geq 1 - \alpha$ and $\alpha = \sum_{j=1}^d \alpha_j$. The derivatives of this generalized Clayton generator are given by:

$$\psi_{a,b}^{(k)}(x) = \begin{cases} (-1)^k \, c(a,b,k) \left(1 + \frac{x}{b}\right)^{-(a+k)} & \text{if } a \geq 1 - \alpha, \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

where $c(a,b,k) = \dfrac{b^{-k}\Gamma(a+k)}{\Gamma(a)}$.

Substituting (4.6) into the general formulas (2.2) and (2.4), we obtain closed-form expressions for the survival function of the random vector $\mathbf{X}$ and its marginal survival functions:

$$\bar{H}(x_1, \ldots, x_d) = \psi_{a,b}\left(\sum_{j=1}^d x_j\right) \sum_{i_1=0}^{\alpha_1-1} \cdots \sum_{i_d=0}^{\alpha_d-1} \frac{b^{-\sum_{j=1}^d i_j} \Gamma\left(a + \sum_{j=1}^d i_j\right)}{\Gamma(a)\, i_1! \cdots i_d!} \prod_{j=1}^d x_j^{i_j},$$

$$\bar{H}_i(x) = \psi_{a,b}(x) \sum_{j=0}^{\alpha_i-1} \frac{c(a,b,j)}{j!} \left(\frac{x}{1+b}\right)^j, \quad x \in \mathbb{R}_+.$$

Remark 4.4. *If we set $\alpha_1 = \cdots = \alpha_d = 1$, the survival function reduces to the multivariate Pareto form:*

$$\bar{H}(x_1, \ldots, x_d) = \left(1 + \sum_{j=1}^d \frac{x_j}{b}\right)^{-a}, \quad x_1, \ldots, x_d > 0,$$

with univariate survival margins:

$$\bar{H}_i(x) = \left(1 + \frac{x}{b}\right)^{-a}, \quad x > 0.$$

The probability density function of the aggregate risk $S_d = \sum_{i=1}^d X_i$, as well as its tail risk measures such as the TVaR, can then be derived explicitly. See Sarabia et al. [22] for related results.

Theorem 4.3. *Let $S_d = X_1 + X_2 + \cdots + X_d$ be the sum of d dependent random variables governed by a multivariate Generalized Clayton–Liouville copula. Then, the probability density function (pdf) of the aggregate risk S_d is given by:*

$$f_{S_d}(x) = \frac{b^{-\alpha}}{B(\alpha,a)} x^{\alpha-1} \left(1 + \frac{x}{b}\right)^{-(a+\alpha)}, \quad x > 0, \tag{4.7}$$

where $B(\alpha, a)$ denotes the Beta function.

Proof. From expression (2.3) and the k-th derivative of the generator given in (4.6), we obtain:

$$f_{S_d}(x) = \frac{(-1)^\alpha x^{\alpha-1} \psi^{(\alpha)}(x)}{(\alpha-1)!} = \frac{b^{-\alpha}}{B(\alpha,a)} x^{\alpha-1} \left(1 + \frac{x}{b}\right)^{-(a+\alpha)}.$$

□

Remark 4.5. *The density in* (4.7) *corresponds to a Beta distribution of the second kind, denoted* $S_d \sim \mathfrak{B}2(\alpha, a, b)$*; see McDonald and Xu [17] and Kleiber and Kotz [13]. The cumulative distribution function (cdf) of* S_d *is given by:*

$$F_{S_d}(x) = IB\left(\frac{x}{x+b}; \alpha, a\right),$$

where $IB(x; p, q)$ *denotes the incomplete Beta ratio function:*

$$IB(x; p, q) = \frac{\int_0^x t^{p-1}(1-t)^{q-1}\, dt}{B(p,q)}.$$

The k-th moment of S_d *exists for* $k < a$ *and is given by:*

$$\mathbb{E}\left[S_d^k\right] = b^k \frac{\Gamma(\alpha+k)\,\Gamma(a-k)}{\Gamma(\alpha)\,\Gamma(a)}.$$

Theorem 4.4. *Let* $X_1, X_2, \ldots, X_d$ *be dependent random variables joined by a Generalized Clayton–Liouville copula. Then, the Tail Value-at-Risk (TVaR) of the aggregate risk* $S_d = \sum_{i=1}^d X_i$ *at level* $\kappa \in (0,1)$ *is given by:*

$$TVaR_\kappa(S_d) = \frac{b\alpha}{a-1} \cdot \frac{1 - IB\left(\frac{VaR_\kappa(S_d)}{VaR_\kappa(S_d)+b}; \alpha+1, a-1\right)}{1-\kappa}, \quad \text{for } a > 1, \tag{4.8}$$

where $IB(x; p, q)$ *denotes the incomplete Beta ratio function:*

$$IB(x; p, q) = \frac{\int_0^x t^{p-1}(1-t)^{q-1} dt}{B(p,q)},$$

and $B(p,q)$ *is the Beta function.*

Proof. Using the expression of the pdf in (4.7) and recalling that $\mathbb{E}\left[S_d\right] = \frac{b\alpha}{a-1}$ for $a > 1$, we substitute into the definition of TVaR:

$$TVaR_\kappa(S_d) = \mathbb{E}\left[S_d\right] \cdot \frac{1 - IB\left(\frac{VaR_\kappa(S_d)}{VaR_\kappa(S_d)+b}; \alpha+1, a-1\right)}{1-\kappa}.$$

□

Remark 4.6. *In the case where* $\alpha_1 = \cdots = \alpha_d = 1$*, the model reduces to a multivariate Pareto distribution with survival function:*

$$\bar{H}(x_1, \ldots, x_d) = \left(1 + \sum_{j=1}^d \frac{x_j}{b}\right)^{-a}, \quad x_1, \ldots, x_d > 0,$$

and univariate survival margins given by:

$$\bar{H}(x) = \left(1 + \frac{x}{b}\right)^{-a}, \quad x > 0.$$

In this setting, the pdf and TVaR of S_d *correspond to the expressions derived in Sarabia et al. [22].*

4.3. **Gamma-Liouville Copulas.** We consider a Liouville-distributed random vector with integer parameters $\alpha_1, \ldots, \alpha_d$ and a radial component R with distribution function F_R, whose Williamson α-transform is given by:

$$\psi_{a,b}(x) = \mathfrak{W}_\alpha F_R(x) = \frac{\Gamma(a, bx)}{\Gamma(a)}, \quad \text{with} \quad a < 1, \tag{4.9}$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t}\, dt$ denotes the upper incomplete Gamma function.
The k-th derivative of the generator $\psi_{a,b}$ can be computed using Leibniz's rule and is given by:

$$\psi_{a,b}^{(k)}(x) = \sum_{i=0}^{k-1} \frac{b^a}{\Gamma(a-i)} \binom{k-1}{i} (-1)^{k-i} b^{k-i-1} e^{-bx} x^{a-i-1}, \quad k \geq 1. \tag{4.10}$$

As a result, the survival margins of the Liouville vector $\mathbf{X}$ take the form:

$$\begin{aligned}\bar{H}_i(x) &= \psi_{a,b}(x) + \sum_{j=1}^{\alpha_i-1} \sum_{\ell=1}^{j} \frac{(-1)^{j-\ell}}{j!} \frac{1}{\Gamma(a-j+\ell)} \binom{j-1}{j-\ell} b^{a+\ell-1} e^{-bx} x^{a+\ell-1} \\ &= \psi_{a,b}(x) + \sum_{\ell=1}^{\alpha_i-1} c(a, b, \ell)\, e^{-bx} x^{a+\ell-1},\end{aligned}$$

where the coefficients $c(a, b, \ell)$ are defined by:

$$c(a, b, \ell) = \sum_{j=\ell}^{\alpha_i-1} \frac{(-1)^{j-\ell}}{j!} \cdot \frac{1}{\Gamma(a-j+\ell)} \binom{j-1}{j-\ell} b^{a+\ell-1}.$$

Theorem 4.5. *Let $S_d = X_1 + X_2 + \cdots + X_d$ be the sum of d dependent random variables whose joint distribution is defined by a Gamma-Liouville copula. Then, the probability density function (pdf) of the aggregate variable S_d is given by:*

$$f_{S_d}(x) = \sum_{i=0}^{\alpha-1} c(i, a, b)\, e^{-bx} x^{\alpha+a-i-2}, \tag{4.11}$$

where the coefficients $c(i, a, b)$ are defined by:

$$c(i, a, b) = \frac{1}{(\alpha-1)!} \cdot \frac{1}{\Gamma(a-i)} \binom{\alpha-1}{i} (-1)^i b^{\alpha+a-i-1}.$$

Proof. Substituting the generator $\psi_{a,b}$ from (4.9) into the general expression for the density of S_d from (2.3), we get:

$$\begin{aligned}f_{S_d}(x) &= \frac{1}{(\alpha-1)!} \sum_{i=0}^{\alpha-1} \frac{1}{\Gamma(a-i)} \binom{\alpha-1}{i} (-1)^i b^{\alpha+a-i-1} e^{-bx} x^{\alpha+a-i-2} \\ &= \sum_{i=0}^{\alpha-1} c(i, a, b)\, e^{-bx} x^{\alpha+a-i-2}.\end{aligned}$$

□

Remark 4.7. *The density function $f_{S_d}(x)$ can be expressed as a mixture of Gamma densities:*

$$f_{S_d}(x) = \sum_{i=0}^{\alpha-1} d(i,a,b)\, f_{Z_i}(x),$$

where $Z_i \sim Gamma(\alpha + a - i - 1, b)$ and

$$d(i,a,b) = \frac{1}{(\alpha-1)!} \cdot \frac{\Gamma(\alpha + a - i - 1)}{\Gamma(a-i)} \binom{\alpha-1}{i} (-1)^i.$$

It follows that the cumulative distribution function (cdf) of S_d is given by:

$$F_{S_d}(x) = \sum_{i=0}^{\alpha-1} d(i,a,b)\, F_{Z_i}(x).$$

Theorem 4.6. *Let $X_1, X_2, \ldots, X_d$ be dependent random variables joined by a Gamma-Liouville copula. Then, the Tail Value-at-Risk of the aggregate risk $S_d = \sum_{i=1}^{d} X_i$ at level $\kappa \in (0,1)$ is given by:*

$$TVaR_\kappa(S_d) = \frac{1}{1-\kappa} \sum_{i=0}^{\alpha-1} d(i,a,b)\, \mathbb{E}\Big[Z_i\Big] \bar{F}_{Y_i}(VaR_\kappa(S_d))\,.$$

where $Y_i \sim Gamma(\alpha + a - i, b)$ and $Z_i \sim Gamma(\alpha + a - i - 1, b)$.

Proof. Substituting the density expression (4.11) into the definition of the Tail Value-at-Risk, we obtain:

$$\begin{aligned} TVaR_\kappa(S_d) &= \frac{1}{1-\kappa} \int_{VaR_\kappa(S_d)}^{\infty} x\, f_{S_d}(x)\, dx \\ &= \frac{1}{1-\kappa} \sum_{i=0}^{\alpha-1} d(i,a,b) \int_{VaR_\kappa(S_d)}^{\infty} x\, f_{Z_i}(x)\, dx \\ &= \frac{1}{1-\kappa} \sum_{i=0}^{\alpha-1} d(i,a,b)\, \mathbb{E}\Big[Z_i\Big] \bar{F}_{Y_i}\left(VaR_\kappa(S_d)\right), \end{aligned}$$

where Y_i is a Gamma variable with shape parameter $\alpha + a - i$ and scale $1/b$. The result follows. □

For all the models studied in this section, the composition of the TVaR-based allocation can be readily derived by applying Theorem 3.2 to the explicit expressions obtained for the aggregate TVaR. This section has illustrated the analytical tractability of the Liouville copula framework through several parametric examples, each providing closed-form formulas for both the density and the risk measures of interest. These results reinforce the practical relevance of the theoretical findings established in Section 3, particularly in the context of risk aggregation and capital allocation. In addition to their interpretability, the models presented here facilitate efficient numerical implementation for actuarial and financial applications.

5. Numerical illustrations

This section provides numerical illustrations to complement the analytical results derived in Section 4. We focus on two bivariate models derived from the Liouville copula: the Clayton–Liouville and the Generalized Clayton–Liouville models. These illustrations aim to shed light on how tail risk measures such as *VaR* and *TVaR*, as well as their allocation-based decompositions, respond to changes in model parameters and dependence structures.
In each case, we fix the Dirichlet parameters (α_1, α_2) to reflect an asymmetric risk profile and vary the dependence parameters to examine their impact on risk aggregation. For each model, we report the values of *VaR*, *TVaR*, and component *TVaR*-based allocations at different confidence levels, complemented by graphical representations to highlight sensitivity with respect to the parameters.

Example 5.1 (Bivariate Clayton–Liouville Model)**.** *In this example, we examine a bivariate Clayton–Liouville model with fixed Dirichlet parameters* $\alpha_1 = 1$ *and* $\alpha_2 = 2$*. As shown in Section 4.1, the bivariate survival function in this case is given by:*

$$\bar{H}(x_1, x_2) = \psi_\theta(x_1 + x_2)\left[1 + \frac{x_2}{1 + \theta(x_1 + x_2)}\right],$$

where the survival margins are defined as

$$\bar{H}_1(x) = \psi_\theta(x) = (1 + \theta x)^{-1/\theta}, \quad \bar{H}_2(x) = \psi_\theta(x)\left[1 + \frac{x}{1 + \theta x}\right].$$

We compute the Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), and TVaR-based allocations for X_1 *and* X_2 *at different confidence levels* κ*. Numerical results for* $\theta = 0.1$ *and* $\theta = 0.49$ *are presented in Tables 1 and 2, corresponding to weak and moderate dependence levels, respectively.*

Table 1. $\rho(X_1, X_2) = 0.1348$

$\theta = 0.1$	$\kappa = 0.5$	0.75	0.95	0.99	0.995
$VaR_\kappa(S_d)$	2.77	4.31	7.80	11.61	13.42
$TVaR_\kappa(S_d)$	4.99	6.52	10.22	14.38	16.36
$TVaR_\kappa(X_1; S)$	1.66	2.17	3.41	4.79	5.45
$TVaR_\kappa(X_2; S)$	3.33	4.35	6.81	9.59	10.91

Table 2. $\rho(X_1, X_2) = 0.5677$

$\theta = 0.49$	$\kappa = 0.5$	0.75	0.95	0.99	0.995
$VaR_\kappa(S_d)$	3.17	6.17	18.09	44.02	63.22
$TVaR_\kappa(S_d)$	10.06	15.68	38.83	89.61	127.24
$TVaR_\kappa(X_1; S)$	3.35	5.23	12.94	29.87	42.41
$TVaR_\kappa(X_2; S)$	6.71	10.45	25.89	59.74	84.82

These results show that both VaR, TVaR, and the absolute contributions to TVaR increase with the level of dependence, as captured by the parameter θ*. However, the allocation proportions* $TVaR_\kappa(X_i; S)/TVaR_\kappa(S)$ *remain constant, as they are governed by the Dirichlet structure of the model.*

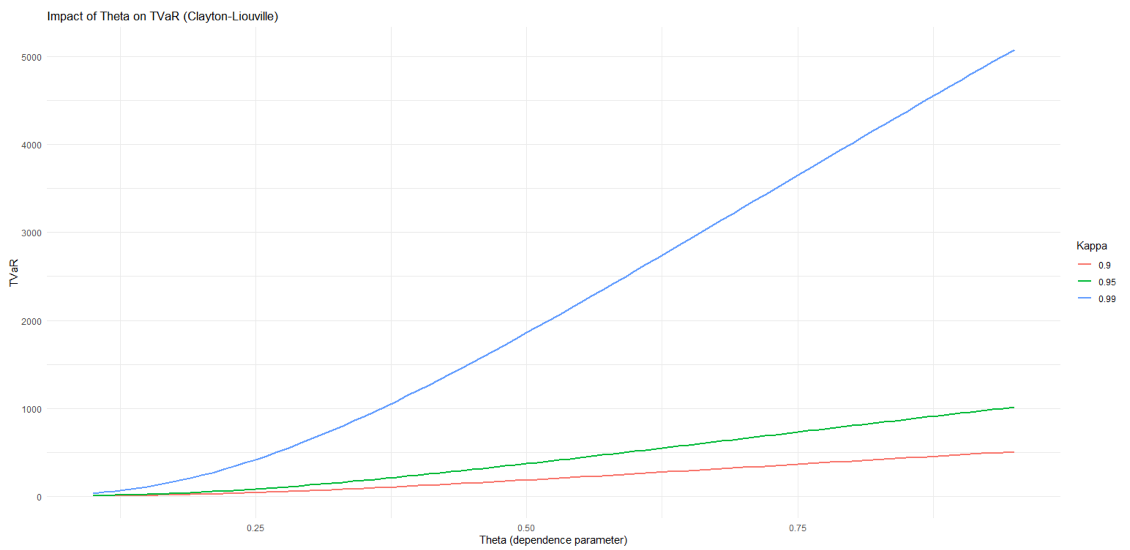

Figure 1. Impact of θ on $TVaR_\kappa(S_d)$ for different values of κ in the Clayton–Liouville model.

Figure 1 illustrates how the TVaR of the aggregate risk increases with the dependence parameter θ*. The convexity of the curves, especially for high values of* κ*, emphasizes how stronger tail dependence leads to disproportionately larger extreme losses.*

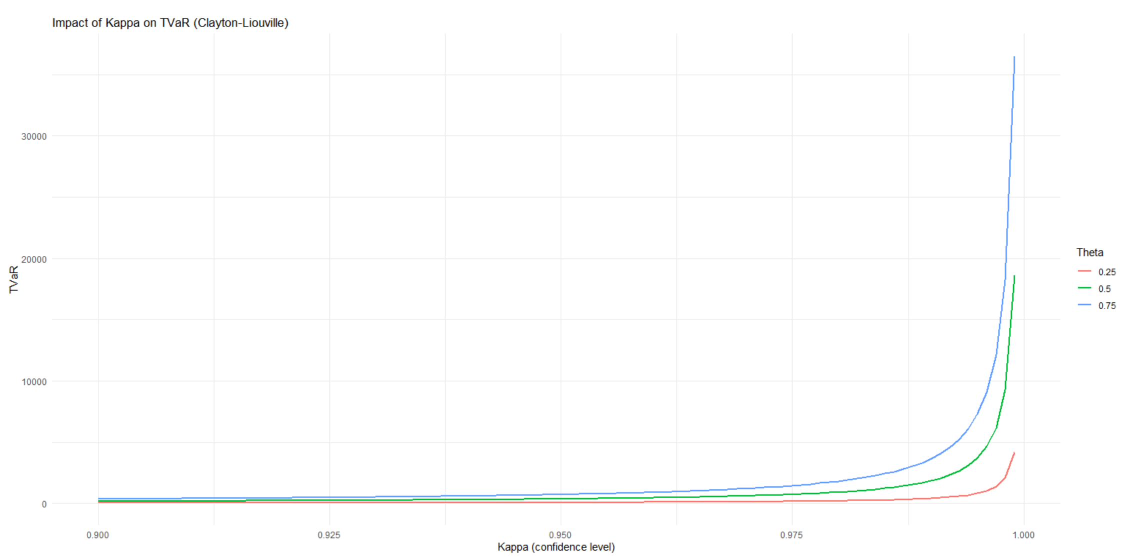

Figure 2. Impact of κ on $TVaR_\kappa(S_d)$ for different values of θ in the Clayton–Liouville model.

Figure 2 shows the sharp increase of $TVaR_\kappa(S_d)$ as the confidence level κ approaches 1, reflecting the accumulation of risk in the extreme tails. This increase is significantly more pronounced for higher values of θ, underscoring the need to account for dependence in solvency assessments and capital allocations.

Example 5.2. *(Bivariate Generalized Clayton–Liouville Model)*

We now turn to the Generalized Clayton–Liouville copula model with parameters $\alpha_1 = 1$, $\alpha_2 = 2$, and generator parameters $a > 1$ and $b > 0$. As shown in Section 4.2, the bivariate survival function under this model is given by:

$$\bar{H}(x_1, x_2) = \psi_{a,b}(x_1 + x_2)\left[1 + \frac{ax_2}{1 + b(x_1 + x_2)}\right],$$

with marginal survival functions:

$$\bar{H}_1(x) = \psi_{a,b}(x) = \left(1 + \frac{x}{b}\right)^{-a}, \quad \bar{H}_2(x) = \psi_{a,b}(x)\left[1 + \frac{ax}{1 + bx}\right].$$

We compute the Value-at-Risk, Tail Value-at-Risk, and TVaR-based allocation components for X_1 and X_2 at different confidence levels κ, using two sets of parameter values corresponding to distinct correlation levels. The numerical results are summarized in Tables 3 and 4.

Table 3. $\rho(X_1, X_2) = 0.014$

$a = 100, b = 10$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$VaR_\kappa(S_d)$	0.2683	0.3958	0.6432	0.8680	0.9618
$TVaR_\kappa(S_d)$	0.4380	0.5492	0.7826	1.0022	1.0950
$TVaR_\kappa(X_1; S)$	0.1460	0.1831	0.2609	0.3341	0.3650
$TVaR_\kappa(X_2; S)$	0.2920	0.3662	0.5218	0.6681	0.7300

Table 4. $\rho(X_1, X_2) = 0.554$

$a = 2.1, b = 10$	$\kappa = 0.5$	$\kappa = 0.75$	$\kappa = 0.95$	$\kappa = 0.99$	$\kappa = 0.995$
$VaR_\kappa(S_d)$	15.0361	29.0234	83.5468	199.5321	284.1308
$TVaR_\kappa(S_d)$	46.4265	71.9220	175.0281	396.1255	557.5598
$TVaR_\kappa(X_1; S)$	15.4755	23.9740	58.3427	132.0418	185.8533
$TVaR_\kappa(X_2; S)$	30.9510	47.9480	116.6854	264.0837	371.7065

To illustrate the effect of the generator parameters on tail risk, we present two additional simulations. In the first figure, we fix $\kappa = 0.95$ and evaluate how TVaR changes with parameter a for different values of b. The results, shown in Figure 3, confirm a sharp decrease in TVaR as a increases, indicating that the tails become lighter. Moreover, for a given a, increasing b results in a vertical shift of the TVaR, highlighting the multiplicative role of b in scaling the distribution.

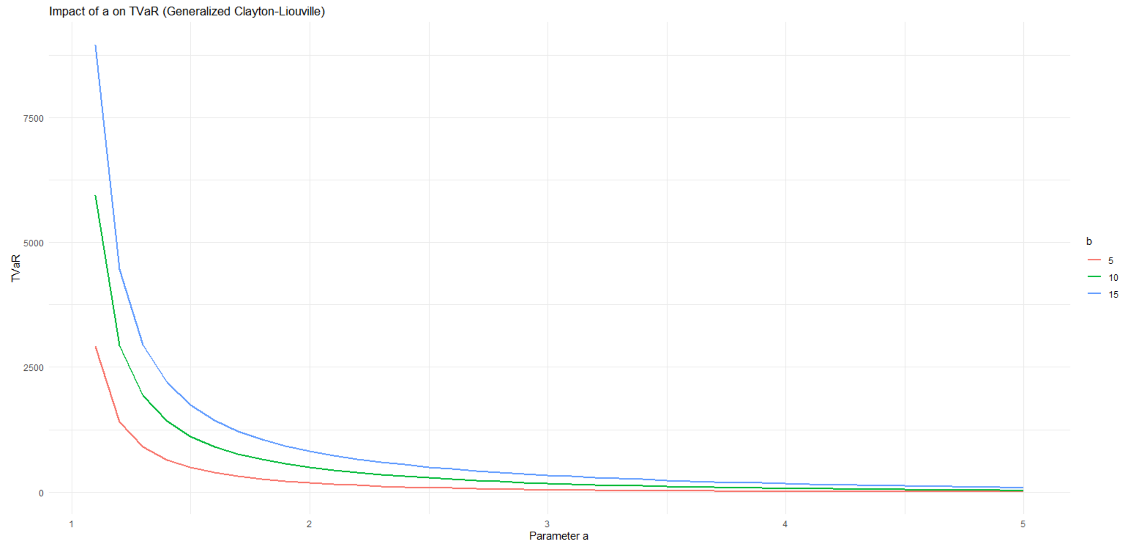

Figure 3. Impact of parameter a on $TVaR$ for different values of b (fixed $\kappa = 0.95$).

We then examine the impact of varying b for fixed values of a. The curves in Figure 4 demonstrate the expected monotonic increase of $TVaR$ *with respect to b, but with a speed that strongly depends on the value of a. When a is small, the slope is steep, reflecting the greater sensitivity of tail risk to b in the heavy-tailed regime.*

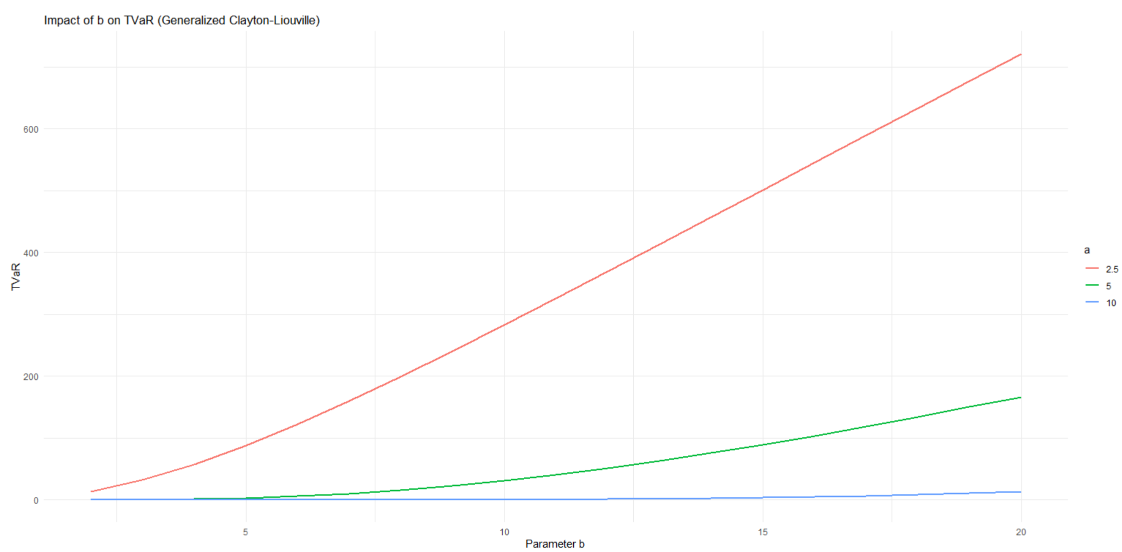

Figure 4. Impact of parameter b on $TVaR$ for different values of a (fixed $\kappa = 0.95$).

These numerical illustrations highlight how both the dependence structure and the generator parameters shape the behavior of tail risk measures and their allocation. In both models examined, VaR and $TVaR$ increase with stronger dependence, as measured by the correlation $\rho(X_1, X_2)$. The composition of the TVaR-based allocation reflects the asymmetry induced by the Dirichlet parameters. Moreover, the generator parameters, namely θ in the Clayton model and (a, b) in the generalized model, provide interpretable levers to control tail heaviness and adjust sensitivity to extreme risks. These findings support the relevance of Liouville-based copulas for capital modeling, solvency assessment, and risk aggregation in actuarial and financial applications.

Conclusion

This paper has explored the potential of Liouville copulas as a flexible class of multivariate dependence models, particularly suited for capturing asymmetric risk structures beyond the scope of exchangeable Archimedean copulas. We have emphasized their usefulness in the context of risk aggregation and capital allocation, focusing on their analytical tractability and their ability to model non-exchangeable dependencies.

We derived closed-form expressions for key tail risk measures, namely the Value-at-Risk and the Tail Value-at-Risk, for several parametric families within the Liouville framework. Particular attention was given to the decomposition of TVaR-based allocations, which naturally reflect the asymmetry of the Dirichlet components embedded in the construction. Numerical illustrations confirmed and complemented the theoretical results, highlighting the role of generator parameters in modulating tail heaviness and dependence strength.

Overall, the Liouville copula approach offers a promising avenue for constructing interpretable and computationally efficient risk models with direct applications in actuarial capital modeling, solvency analysis and stress testing. Future research may extend this framework to higher dimensions, explore statistical inference techniques, evaluate empirical performance on real insurance portfolios and investigate the extension of the results to settings with arbitrary marginal distributions, as recommended by the reviewers.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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