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Construction of an Optimal Explicit Difference Formula in the Hilbert Space $W_2^{(3,2)}(0,1)$

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Abstract. In this paper, an optimal explicit difference formula for any integer k in the Hilbert space $W_2^{(3,2)}(0,1)$ is constructed. Here, an optimal function of the discrete argument is found, which is used to find the optimal error estimate for the explicit difference formula on the class $W_2^{(3,2)}(0,1)$. Furthermore, an optimal error estimate for the explicit difference formula on the class $W_2^{(3,2)}(0,1)$ is calculated. In addition, using the optimal function of the discrete argument, a new formula for the optimal error of explicit difference formulas on the classes $W_2^{(m,m-1)}(0,1)$ is obtained.

1. Introduction

The problem of solving ordinary differential equations is more complex than the problem of calculating single integrals, and accordingly, the proportion of problems that can be integrated explicitly is significantly smaller here.

When speaking of explicit integrability, we mean that the solution can be calculated using a finite number of elementary operations. Thus, the class of explicitly integrable problems has come to include problems whose solutions are expressed through special functions. However, even this broader class represents a relatively small fraction of the problems presented for solution. A significant expansion of the class of actually solvable differential equations, and consequently an expansion of the scope of computational mathematics, occurred with the development of numerical methods and the active use of computers.

One of the simplest methods for solving the Cauchy problem in its description is based on the use of the Taylor formula.

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Let it be required to find a solution to the differential equation

$$y' = f(x, y) \tag{1.1}$$

on the segment $[x_0,\ x_0 + X]$ with the initial condition

$$y\left(x_0\right) = y_0 \tag{1.2}$$

$f(x, y)$ is an analytic function.

For an approximate solution of the Cauchy problem (1.1)-(1.2) there are various methods, for example, the Euler method, the Adams methods, the Adams-Bashforth methods, the Adams-Moulton methods, Runge-Kutta methods, etc. (see [1]).

The numerical solution of ordinary differential equations remains a major challenge even today. Therefore, many researchers are currently working on this problem. In the article [2], S. Mehrkanoon, Z. A. Majid, M. Suleiman, K. I. Othman, and Z. B. Brahim proposed a new approach to solving a system of first-order ordinary differential equations using three points and three steps. In the works [3–5], the authors presented new semi-explicit and semi-implicit predictor-corrector methods. They studied the numerical stability of these methods by constructing stability regions and clearly showed that semi-explicit methods have higher numerical stability than conventional predictor-corrector algorithms.

In the article [6] Osama Y.Ababneh presented new numerical methods for solving ordinary differential equations, both in linear and nonlinear cases. In the work [7] Adekoya M. Odunayo and Z.O. Ogunwobi balanced the Adams-Bashforth-Multon and Milne-Simpson methods for a second-order differential equation. In [8], Sajal K. Kar used the Adams-Bashforth scheme to create a new time-difference predictor-corrector scheme. In [9], Emil Vitásek studied methods of arbitrarily high orders of accuracy for solving abstract ordinary differential equations.

In this paper, we construct optimal difference formulas for the approximate solution of the Cauchy problem in specific function spaces, and also investigate the properties of these methods and their error estimates. The next section presents an analytical expression for the squared norm of the error functional and a system for finding optimal coefficients of difference formulas.

2. Statement of the problem of optimization of difference formulas

Let us consider the problem of constructing difference formulas (see [10])

$$\sum_{\beta=0}^{k} C_\beta \varphi\left(h\beta\right) - h\sum_{\beta=0}^{k-1} C_\beta^{(1)}\varphi'\left(h\beta\right) \cong 0\,. \tag{2.1}$$

Here C_β and $C_\beta^{(1)}$ are the coefficients of the difference formulas. The class of problems under consideration is determined by the definition of the class $W_2^{(m,m-1)}\left(0,1\right)$, i.e., we will consider the functions $\varphi\left(x\right)$ from the class of real functions in the Hilbert space $W_2^{(m,m-1)}\left(0,1\right)$. The scalar product in this space is defined by the formula

$$\langle\varphi,\psi\rangle=\int_0^1\left(\varphi^{(m)}(x)+\varphi^{(m-1)}(x)\right)\left(\psi^{(m)}(x)+\psi^{(m-1)}(x)\right)dx.$$

Let the space $W_2^{(m,m-1)}(0,1)$ be embedded in the space $\mathbf{C}(0,1)$ of continuous functions, then the error functional of difference formulas will be linear [11]

$$\ell(\varphi)=(\ell,\varphi)=\sum_{\beta=0}^{k}C_\beta\varphi(h\beta)-h\sum_{\beta=0}^{k-1}C_\beta^{(1)}\varphi'(h\beta)$$

$$=\int_{-\infty}^{\infty}\left[\sum_{\beta=0}^{k}C_\beta\delta(x-h\beta)+h\sum_{\beta=0}^{k-1}C_\beta^{(1)}\delta'(x-h\beta)\right]\varphi(x)\,dx. \tag{2.2}$$

The quantity (2.2) is called the error of the difference formulas. The error of the difference formulas on classes $W_2^{(m,m-1)}(0,1)$ is called the quantity

$$\ell\left(W_2^{(m,m-1)}\right)=\sup_{\varphi\in W_2^{(m,m-1)}}\left|\ell(\varphi)\right|.$$

The lower bound

$$\mathring{\ell}\left(W_2^{(m,m-1)}\right)=\inf_{C_\beta,C_\beta^{(1)}}\left\|\ell\left(W_2^{(m,m-1)}\right)\right\|$$

is called the error of the optimal difference formula on the class under consideration. If there exists a difference formula for which

$$\ell\left(W_2^{(m,m-1)}\right)=\mathring{\ell}\left(W_2^{(m,m-1)}\right),$$

then such a difference formula is called the *optimal difference formula* on the class under consideration. And the coefficients $\mathring{C}_\beta$ and $\mathring{C}_\beta^{(1)}$ are called the *optimal coefficients* of the difference formulas.

The problem of constructing an optimal difference formula in a functional setting consists of minimizing the error of difference formulas on classes $W_2^{(m,m-1)}(0,1)$ of functions. Since Hilbert space $W_2^{(m,m-1)}(0,1)$, then this error on classes coincides with the norm of the error functional (2.2) of difference formulas. To find the error of difference formulas on classes, we will use the extremal function [11].

Definition 2.1. *If the equality holds*

$$(\ell,\psi_\ell)=\left\|\ell\left|W_2^{(m,m-1)*}\right\|\cdot\left\|\psi_\ell\left|W_2^{(m,m-1)}\right\|\right.\right.$$

and $\psi_\ell\in W_2^{(m,m-1)}(0,1)$*, is then* ψ_ℓ *called the extremal function of the functional* ℓ*.*

In the Hilbert space $W_2^{(m,m-1)}(0,1)$, the norm of a function $\varphi(x)$ is found by the equality

$$\left\|\varphi|W_2^{(m,m-1)}(0,1)\right\|=\left\{\int_0^1\left(\varphi^{(m)}(x)+\varphi^{(m-1)}(x)\right)^2dx\right\}^{1/2}.$$

So, as the error functional of difference formulas

$$\ell(x)=\sum_{\beta=0}^{k} C_\beta \delta\left(x-h\beta\right)+h\sum_{\beta=0}^{k-1} C_\beta^{(1)}\delta'\left(x-h\beta\right)$$

is defined and bounded in the Hilbert space $W_2^{(m,m-1)}\left(0,1\right)$, then

$$(\ell, x^\alpha)=0,\ \alpha=0,1,...,m-2,$$

$$(\ell, e^{-x})=0.$$

Theorem 2.1. *The maximizing element, i.e. the extremal function of the difference formulas* (2.1) *in the Hilbert space* $W_2^{(m,m-1)}\left(0,1\right)$ *is given by the equality*

$$\psi_\ell(x)=(-1)^m\ell(x)*G_m(x)+P_{m-2}(x)+de^{-x},$$

where

$$G_m(x)=\frac{\mathrm{sign}x}{2}\left(\frac{e^x-e^{-x}}{2}-\sum_{j=1}^{m-1}\frac{x^{2j-1}}{(2j-1)!}\right)$$

is the Green's function or fundamental solution to the following equation

$$G_m^{(2m)}(x)-G_m^{(2m-2)}(x)=\delta(x),$$

$d=const$*,* $sgnx=\begin{cases}1,\ x>0,\\ 0,\ x=0,\\ -1,\ x<0,\end{cases}$ $P_{m-2}\left(x\right)$*- some unknown polynomial of degree* $m-2$*.*

Theorem 2.2. *The error of the difference formulas* (2.1) *on classes* $W_2^{(m,m-1)}\left(0,1\right)$ *(i.e., the square of the norm of the error functional of difference formulas) is equal to*

$$\left\|\ell|W_2^{(m,m-1)*}\right\|^2=(-1)^m\left[\sum_{\gamma=0}^{k}C_\gamma\sum_{\beta=0}^{k}C_\beta G_m\left(h\gamma-h\beta\right)\right.$$

$$\left.-2h\sum_{\gamma=0}^{k-1}C_\gamma^{(1)}\sum_{\beta=0}^{k}C_\beta G'_m\left(h\gamma-h\beta\right)-h^2\sum_{\gamma=0}^{k-1}C_\gamma^{(1)}\sum_{\beta=0}^{k-1}C_\beta^{(1)}G''_m\left(h\gamma-h\beta\right)\right],$$

where $G'_m(x)$ *and* $G''_m(x)$ *are given by the equalities*

$$G'_m(x)=\frac{\mathrm{sign}x}{2}\left(\frac{e^x+e^{-x}}{2}-\sum_{j=1}^{m-1}\frac{x^{2j-2}}{(2j-2)!}\right),$$

$$G''_m(x)=\frac{\mathrm{sign}x}{2}\left(\frac{e^x-e^{-x}}{2}-\sum_{j=1}^{m-2}\frac{x^{2j-1}}{(2j-1)!}\right).$$

The proofs of these theorems follow from the works [12,13,20].

The problem of constructing difference formulas close to this work can be seen in the following works [14–19].

Let us now apply the method of undetermined Lagrange multipliers to find the minimum error of the difference formulas on the classes $W_2^{(m,m-1)}(0,1)$, i.e., norms. For this, we consider the auxiliary function [20]

$$\Psi\left(C, C^{(1)}, \breve{\ }\right) = \left\|\ell|W_2^{(m,m-1)*}\right\|^2 - 2(-1)^m \sum_{\alpha=0}^{m-2} \lambda_\alpha (\ell, x^\alpha) - 2(-1)^m \lambda_{m-1} (\ell, e^{-x}).$$

Here $C = (C_0, C_1, ..., C_k)$, $C^{(1)} = \left(C_0^{(1)}, C_1^{(1)}, ..., C_{k-1}^{(1)}\right)$, $\breve{\ } = (\lambda_0, \lambda_1, ..., \lambda_{m-1})$.

Now we calculate all partial derivatives with respect to $C_\beta^{(1)}$, $\beta = 0, 1, ..., k-1$ and λ_α, $\alpha = 0, 1, ..., m-1$ from the function $\Psi\left(C, C^{(1)}, \breve{\ }\right)$. Next, equating them to zero, we have

$$\frac{\partial \Psi}{\partial C_\beta^{(1)}} = 0, \quad \beta = 0, 1, ..., k-1,$$

$$\frac{\partial \Psi}{\partial \lambda_\alpha} = 0, \quad \alpha = 0, 1, ..., m-1.$$

These equalities give us a system of linear equations for finding the optimal coefficients $\mathring{C}_\beta^{(1)}$ and $\mathring{\lambda}_\alpha$, ($\alpha = 1, 2, ..., m-1$).

$$h \sum_{\gamma=0}^{k-1} \mathring{C}_\gamma^{(1)} G_m''(h\beta - h\gamma) + \mathring{P}_{m-3}(h\beta) + \mathring{\lambda}_{m-1} e^{-h\beta} = f_m(h\beta), \ \beta = 0, 1, ..., k, \tag{2.3}$$

$$h \sum_{\gamma=0}^{k-1} \mathring{C}_\gamma^{(1)} (h\gamma)^{\alpha-1} = g_\alpha, \quad \alpha = 1, 2, ..., m-2, \tag{2.4}$$

$$h \sum_{\gamma=0}^{k-1} \mathring{C}_\gamma^{(1)} e^{-h\gamma} = g_k. \tag{2.5}$$

Here $h = \frac{1}{N}$, $N = 1, 2...$,

$$G_m'(h\beta - h\gamma) = \frac{\operatorname{sign}(h\beta - h\gamma)}{2} \left(\frac{e^{h\beta - h\gamma} + e^{h\gamma - h\beta}}{2} - \sum_{j=1}^{m-1} \frac{(h\beta - h\gamma)^{2j-2}}{(2j-2)!} \right),$$

$$f_m(h\beta) = -\sum_{\gamma=0}^{k} C_\gamma G_m'(h\beta - h\gamma), \beta = 0, 1, ..., k,$$

$$G_m''(h\beta - h\gamma) = \frac{\operatorname{sign}(h\beta - h\gamma)}{2} \left(\frac{e^{h\beta - h\gamma} - e^{h\gamma - h\beta}}{2} - \sum_{j=1}^{m-2} \frac{(h\beta - h\gamma)^{2j-1}}{(2j-1)!} \right),$$

C_β- given coefficients,

$$g_\alpha = \frac{1}{\alpha}\sum_{\gamma=0}^{k} C_\gamma (h\gamma)^\alpha, \quad \alpha = 1,2,...,m-2$$

and

$$g_k = -\sum_{\gamma=0}^{k} C_\gamma e^{-h\gamma}.$$

For the system (2.3)-(2.5) under consideration, the unknowns are the optimal coefficients $\overset{\circ}{C}{}_\beta^{(1)}$ of difference formulas of the form (2.1) and the coefficients $\overset{\circ}{\lambda}_\alpha$, ($\alpha = 1,2,...,m-1$) of the optimal function $\overset{\circ}{P}_{m-3}(h\beta) + \overset{\circ}{\lambda}_{m-1}e^{-h\beta}$ of the discrete argument.

3. Optimal coefficients of an explicit difference formula in Hilbert space $W_2^{(3,2)}(0,1)$

In this section, we will find the analytical form of the optimal coefficients of the explicit Adams-type difference formula in the Hilbert space $W_2^{(3,2)}(0,1)$. So, the following theorems will be proved here.

Theorem 3.1. *Optimal coefficients of the explicit difference formula of Adams type*

$$\varphi(hk) - \varphi(hk-h) \cong h\sum_{\beta=0}^{k-1} \overset{\circ}{C}{}_\beta^{(1)}\ \varphi'(h\beta) \tag{3.1}$$

are expressed by equalities

$$\overset{\circ}{C}{}_0^{(1)} = \frac{\lambda^k\left(e^h - he^h - 1\right)\left(\lambda^2 - 1\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}, \tag{3.2}$$

$$\overset{\circ}{C}{}_\beta^{(1)} = \frac{\left(e^h - he^h - 1\right)(1-\lambda)\left[\lambda^{k+\beta}\left(e^h-\lambda\right) + \lambda^{k-\beta}\left(\lambda e^h - 1\right)\right]}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)},\ \ \beta = 1,\ 2,\ \ldots,\ k-2, \tag{3.3}$$

$$\overset{\circ}{C}{}_{k-1}^{(1)} = \frac{\left(\lambda^{2k}-\lambda^2\right)\left(he^{2h} - e^h + 1\right) + \left(\lambda^{2k-1}-\lambda^3\right)\left(e^{2h} - he^{2h} - e^h\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}, \tag{3.4}$$

here λ is the root of a polynomial $P_2(x)$ of degree 2, i.e. the root of a polynomial

$$P_2(x) = \left(1 - e^{2h} + 2he^h\right)x^2 - 2\left(1 - e^{2h} + he^{2h} + h\right)x + \left(1 - e^{2h} + 2he^h\right)$$

with a modulus less than one, i.e.

$$\lambda = \frac{h\left(e^{2h}+1\right) - e^{2h} + 1 - \left(e^h - 1\right)\sqrt{h^2\left(e^h+1\right)^2 + 2h\left(1-e^{2h}\right)}}{1 - e^{2h} + 2he^h}.$$

It is obvious that the optimal explicit difference formula of the Adams type in space $W_2^{(3,2)}(0,1)$ has the form

$$\varphi_{n+k} = \varphi_{n+k-1} + h\sum_{\beta=0}^{k-1} \overset{\circ}{C}{}_\beta^{(1)}\ \varphi'_{n+\beta}, \quad n = 0,1,...,N-k, \quad k \geq 1.$$

To prove theorem 3.1, we first prove the following auxiliary lemma.

Lemma 3.1. *The following formulas are valid*

$$K_1 = \sum_{\gamma=1}^{\beta} \lambda^{\gamma} \cdot \gamma = \frac{\lambda - \lambda^{\beta+2} - (\beta+1)\,\lambda^{\beta+1}\,(1-\lambda)}{(1-\lambda)^2};$$

$$K_2 = \sum_{\gamma=1}^{\beta} \lambda^{-\gamma} \cdot \gamma = \frac{\lambda - \lambda^{-\beta} + (\beta+1)\,\lambda^{-\beta}\,(1-\lambda)}{(1-\lambda)^2};$$

$$K_3 = \sum_{\gamma=1}^{k-2} \lambda^{\gamma} \cdot \gamma = \frac{\lambda - \lambda^{k} - (k-1)\,\lambda^{k-1}\,(1-\lambda)}{(1-\lambda)^2};$$

$$K_4 = \sum_{\gamma=1}^{k-2} \lambda^{-\gamma} \cdot \gamma = \frac{\lambda - \lambda^{-k+2} + (k-1)\,\lambda^{-k+2}\,(1-\lambda)}{(1-\lambda)^2}.$$

Proof of Lemma 3.1. To prove this lemma, we use the following well-known formulas from [21].

$$\sum_{\gamma=0}^{n-1} q^{\gamma}\gamma^{p} = \frac{1}{1-q}\sum_{i=0}^{p}\left(\frac{q}{1-q}\right)^{i}\Delta^{i}0^{p} - \frac{q^{n}}{1-q}\sum_{i=0}^{p}\left(\frac{q}{1-q}\right)^{i}\Delta^{i}\gamma^{p}\Big|_{\gamma=n}, \tag{3.5}$$

where $\Delta^{i}\gamma^{p}$ is a finite difference of order i from γ^{p},

$$\Delta^{i}0^{p} = \Delta^{i}\gamma^{p}\Big|_{\gamma=0}, \quad \Delta^{i}\gamma^{p} = \sum_{\mu=0}^{k}\begin{pmatrix} k \\ p \end{pmatrix}\gamma^{p-\mu}\Delta^{i}0^{\mu}. \tag{3.6}$$

In formula (3.5) instead of q substituting q^{-1} we obtain the following summation formula

$$\sum_{\gamma=0}^{n-1} q^{-\gamma}\gamma^{p} = \frac{q}{q-1}\sum_{i=0}^{p}\left(\frac{1}{q-1}\right)^{i}\Delta^{i}0^{p} - \frac{q^{-n+1}}{q-1}\sum_{i=0}^{p}\left(\frac{1}{q-1}\right)^{i}\Delta^{i}\gamma^{p}\Big|_{\gamma=n}.$$

To prove Lemma 3.1, we will use formula (3.5) and (3.6) to calculate the following sum

$$K_1 = \sum_{\gamma=1}^{\beta}\lambda^{\gamma}\cdot\gamma = \frac{1}{1-\lambda}\sum_{i=0}^{1}\left(\frac{\lambda}{1-\lambda}\right)^{i}\Delta^{i}0^{1} - \frac{\lambda^{\beta+1}}{1-\lambda}\sum_{i=0}^{1}\left(\frac{\lambda}{1-\lambda}\right)^{i}\Delta^{i}\,\gamma^{1}\Big|_{\gamma=\beta+1}$$

$$= \frac{1}{1-\lambda}\left(1\cdot\Delta^{0}0^{1} + \frac{\lambda}{1-\lambda}\cdot\Delta^{1}0^{1}\right) - \frac{\lambda^{\beta+1}}{1-\lambda}\left(1\cdot\Delta^{0}\,\gamma^{1}\Big|_{\gamma=\beta+1} + \frac{\lambda}{1-\lambda}\cdot\Delta^{1}\,\gamma^{1}\Big|_{\gamma=\beta+1}\right).$$

Because

$$\Delta^{0}0^{1} = \gamma^{1}\Big|_{\gamma=0} = 0, \quad \Delta^{1}0^{1} = [\gamma+1-\gamma]\Big|_{\gamma=0} = 1,$$

$$\Delta^{0}\gamma^{1} = \gamma^{1}\Big|_{\gamma=\beta+1} = \beta+1, \quad \Delta^{1}\gamma^{1} = \left[\gamma^{1}-\gamma^{1}+1\right]\Big|_{\gamma=\beta+1} = 1.$$

Then K_1 it takes the form

$$K_1 = \frac{\lambda}{(1-\lambda)^2} - \frac{\lambda^{\beta+1}\,(\beta+1)}{1-\lambda} - \frac{\lambda^{\beta+2}}{(1-\lambda)^2} = \frac{\lambda - \lambda^{\beta+1}\,(1-\lambda)\,(\beta+1) - \lambda^{\beta+2}}{(1-\lambda)^2}.$$

Now we denote by K_2 the following sum

$$K_2 = \sum_{\gamma=1}^{\beta}\lambda^{-\gamma}\cdot\gamma = \frac{\lambda}{\lambda-1}\sum_{i=0}^{1}\left(\frac{1}{\lambda-1}\right)^{i}\Delta^{i}0^{1} - \frac{\lambda^{-\beta}}{\lambda-1}\sum_{i=0}^{1}\left(\frac{1}{\lambda-1}\right)^{i}\Delta^{i}\,\gamma^{1}\Big|_{\gamma=\beta+1}$$

$$=\frac{\lambda}{\lambda-1}\left(1\cdot\Delta^0 0^1+\frac{1}{\lambda-1}\cdot\Delta^1 0^1\right)-\frac{\lambda^{-\beta}}{\lambda-1}\left(1\cdot\Delta^0\left.\gamma^1\right|_{\gamma=\beta+1}+\frac{1}{\lambda-1}\cdot\Delta^1\left.\gamma^1\right|_{\gamma=\beta+1}\right)$$

$$=\frac{\lambda}{(\lambda-1)^2}-\frac{\lambda^{-\beta}(\beta+1)}{\lambda-1}-\frac{\lambda^{-\beta}}{(\lambda-1)^2}=\frac{\lambda+\lambda^{-\beta}(1-\lambda)(\beta+1)-\lambda^{-\beta}}{(1-\lambda)^2}.$$

Now we denote by K_3 the following sum

$$K_3=\sum_{\gamma=1}^{k-2}\lambda^{\gamma}\cdot\gamma=\frac{1}{1-\lambda}\sum_{i=0}^{1}\left(\frac{\lambda}{1-\lambda}\right)^i\Delta^i 0^1-\frac{\lambda^{k-1}}{1-\lambda}\sum_{i=0}^{1}\left(\frac{\lambda}{1-\lambda}\right)^i\Delta^i\left.\gamma^1\right|_{\gamma=k-1}$$

$$=\frac{1}{1-\lambda}\left(1\cdot\Delta^0 0^1+\frac{\lambda}{1-\lambda}\cdot\Delta^1 0^1\right)-\frac{\lambda^{k-1}}{1-\lambda}\left(1\cdot\Delta^0\left.\gamma^1\right|_{\gamma=k-1}+\frac{\lambda}{1-\lambda}\cdot\Delta^1\left.\gamma^1\right|_{\gamma=k-1}\right).$$

Because

$$\Delta^0 0^1=\left.\gamma^1\right|_{\gamma=0}=0,\quad \Delta^1 0^1=\left.[\gamma+1-\gamma]\right|_{\gamma=0}=1,$$

$$\Delta^0\gamma^1=\left.\gamma^1\right|_{\gamma=k-1}=k-1,\quad \Delta^1\gamma^1=\left.\left[\gamma^1-\gamma^1+1\right]\right|_{\gamma=k-1}=1.$$

Then K_4 it takes the form

$$K_4=\sum_{\gamma=1}^{k-2}\lambda^{-\gamma}\cdot\gamma=\frac{\lambda}{\lambda-1}\sum_{i=0}^{1}\left(\frac{1}{\lambda-1}\right)^i\Delta^i 0^1-\frac{\lambda^{-k+2}}{\lambda-1}\sum_{i=0}^{1}\left(\frac{1}{\lambda-1}\right)^i\Delta^i\left.\gamma^1\right|_{\gamma=k-1}$$

$$=\frac{\lambda}{\lambda-1}\left(1\cdot\Delta^0 0^1+\frac{1}{\lambda-1}\cdot\Delta^1 0^1\right)-\frac{\lambda^{-k+2}}{\lambda-1}\left(1\cdot\Delta^0\left.\gamma^1\right|_{\gamma=k-1}+\frac{1}{\lambda-1}\cdot\Delta^1\left.\gamma^1\right|_{\gamma=k-1}\right)$$

$$=\frac{\lambda}{(\lambda-1)^2}-\frac{\lambda^{-k+2}(k-1)}{\lambda-1}-\frac{\lambda^{-k+2}}{(\lambda-1)^2}=\frac{\lambda+\lambda^{-k+2}(1-\lambda)(k-1)-\lambda^{-k+2}}{(1-\lambda)^2}.$$

Lemma 3.1 is proved completely.

To prove theorem 3.1, we first present auxiliary results.

In this case, the system (2.3)-(2.5) for finding the optimal coefficients of the explicit difference formula takes the form

$$h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}G_3''(h\beta-h\gamma)+de^{-h\beta}+p_0=-\sum_{\gamma=0}^{k}C_\gamma G_3'(h\beta-h\gamma),\quad \beta=0,1,...,k-1, \tag{3.7}$$

$$\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}=\sum_{\gamma=0}^{k}C_\gamma(\gamma)\ , \tag{3.8}$$

$$h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}e^{-h\gamma}=-\sum_{\gamma=0}^{k}C_\gamma e^{-h\gamma}\,. \tag{3.9}$$

Here $\overset{\circ}{C}{}_{\gamma}^{(1)}$ are the desired optimal coefficients of the difference formula, d andp_0 -unknowns. Since we are considering difference formulas of the Adams type [22,23], then $C_k=1,\quad C_{k-1}=-1,\quad C_{k-i}=0$, for $i=2,3,...,k$, and the functions $G_3''(h\beta-h\gamma)$ and $G_3'(h\beta-h\gamma)$ are defined by the equalities

$$G_3''(h\beta-h\gamma)=\frac{sign\,(h\beta-h\gamma)}{2}\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-h\beta+h\gamma\right),$$

$$G_3'\left(h\beta-h\gamma\right)=\frac{sign\left(h\beta-h\gamma\right)}{2}\left(\frac{e^{h\beta-h\gamma}+e^{h\gamma-h\beta}}{2}-\frac{\left(h\beta+h\gamma\right)^2}{2}-1\right).$$

Let us calculate the right-hand sides of equalities (3.7), (3.8) and (3.9). For this, taking into account that the difference formula under consideration is an Adams-type formula, i.e. $C_k=1,\ \ C_{k-1}=-1,\ \ C_{k-i}=0$, at $i=2,3,...,k$ we have

$$f_3(h\beta)=-\sum_{\gamma=0}^{k}C_\gamma G_3'\left(h\beta-h\gamma\right)=G_3'\left(hk-h\beta\right)-G_3'\left(hk-h\beta-h\right)$$

$$=\frac{1}{2}\left[\frac{e^{-h\beta}}{2}\left(e^{hk}-e^{hk-h}\right)+\frac{e^{h\beta}}{2}\left(e^{-hk}-e^{-hk+h}\right)+\left(h\beta\right)h+\frac{h^2}{2}\left(1-2k\right)\right],$$

$$\beta=0,1,...,k-1, \tag{3.10}$$

$$\sum_{\gamma=0}^{k}C_\gamma\left(\gamma\right)=k-k+1=1,$$

$$g_k=-\sum_{\gamma=0}^{k}C_\gamma e^{-h\gamma}=e^{-hk+h}-e^{-hk}.$$

Due to the above notations, we can rewrite the system (3.7), (3.8) and (3.9) in the form

$$h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_\gamma^{(1)}\,G_3''\left(h\beta-h\gamma\right)+\overset{\circ}{d}\,e^{-h\beta}+\overset{\circ}{p}_0=f_3(h\beta),\quad \beta=0,1,...,k-1, \tag{3.11}$$

$$\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_\gamma^{(1)}=1\,, \tag{3.12}$$

$$h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_\gamma^{(1)}\,e^{-h\gamma}=g_k. \tag{3.13}$$

In the system (3.11)-(3.13) the unknowns are the optimal coefficients $\overset{\circ}{C}{}_\gamma^{(1)},\ \ \gamma=0,1,...,k-1$ of the explicit difference formula and $\overset{\circ}{d}\,e^{-h\beta}+\overset{\circ}{p}_0$ is the optimal function of the discrete argument.

To solve this system, we will use the representation of the optimal coefficients $\overset{\circ}{C}{}_\beta^{(1)}$ in the case $m=3$, i.e. in space $W_2^{(3,2)}\left(0,1\right)$ they have the following form.

$$\overset{\circ}{C}{}_0^{(1)}=\frac{he^h-e^h+1}{he^h\left(1-e^{hk-h}\right)}+M\frac{\lambda^k\left(e^h-1\right)-\lambda^2\left(e^{hk-h}-1\right)+\lambda e^h\left(e^{hk-2h}-1\right)}{\left(1-e^{hk-h}\right)\left(e^h-\lambda\right)\left(1-\lambda\right)}$$

$$+N\frac{\lambda^{k+1}e^h\left(1-e^{hk-2h}\right)+\lambda^k\left(e^{hk-h}-1\right)-\lambda^2\left(e^h-1\right)}{\left(1-e^{hk-h}\right)\left(\lambda e^h-1\right)\left(1-\lambda\right)}, \tag{3.14}$$

$$\overset{\circ}{C}{}_\beta^{(1)}=M\lambda^\beta+N\lambda^{k-\beta},\quad \beta=1,2,...,k-2, \tag{3.15}$$

$$\overset{\circ}{C}{}_{k-1}^{(1)}=\frac{e^{h}-1-he^{hk}}{he^{h}\left(1-e^{hk-h}\right)}+M\frac{\lambda^{k}e^{h}\left(e^{hk-2h}-1\right)-\lambda^{k-1}e^{h}\left(e^{hk-h}-1\right)+\lambda e^{hk-h}\left(e^{h}-1\right)}{\left(1-e^{hk-h}\right)\left(e^{h}-\lambda\right)\left(1-\lambda\right)}$$
$$+N\frac{\lambda^{k+1}e^{hk-h}\left(1-e^{h}\right)+\lambda^{3}e^{h}\left(e^{hk-h}-1\right)-\lambda^{2}e^{h}\left(e^{hk-2h}-1\right)}{\left(1-e^{hk-h}\right)\left(\lambda e^{h}-1\right)\left(1-\lambda\right)}. \tag{3.16}$$

On the other hand, these representations of the coefficients $\overset{\circ}{C}{}_{0}^{(1)}$ and $\overset{\circ}{C}{}_{k-1}^{(1)}$ can be obtained from the system (3.12) and (3.13). Next, we determine the unknowns M and N from (3.11).
Let us denote by S sum of the type

$$S=h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}\,\frac{sign\left(h\beta-h\gamma\right)}{2}\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-h\beta+h\gamma\right). \tag{3.17}$$

Taking into account the definition of the function $sign(h\beta-h\gamma)$, S we divide the sum into two parts

$$S=h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}\,\frac{sign\left(h\beta-h\gamma\right)}{2}\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-h\beta+h\gamma\right)$$
$$=\frac{h}{2}\sum_{\gamma=0}^{\beta}\overset{\circ}{C}{}_{\gamma}^{(1)}\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-h\beta+h\gamma\right)-\frac{h}{2}\sum_{\gamma=\beta}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-h\beta+h\gamma\right).$$

Adding and subtracting the first sum to S we have

$$S=h\sum_{\gamma=0}^{\beta}\overset{\circ}{C}{}_{\gamma}^{(1)}\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-h\beta+h\gamma\right)-\frac{h}{2}\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-h\beta+h\gamma\right)$$
$$=S_1-S_2. \tag{3.18}$$

Let's look at the sum first S_1

$$S_1=h\sum_{\gamma=0}^{\beta}\overset{\circ}{C}{}_{\gamma}^{(1)}\left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2}-h\beta+h\gamma\right)=\frac{he^{h\beta}}{2}\sum_{\gamma=0}^{\beta}\overset{\circ}{C}{}_{\gamma}^{(1)}\,e^{-h\gamma}-\frac{he^{-h\beta}}{2}\sum_{\gamma=0}^{\beta}\overset{\circ}{C}{}_{\gamma}^{(1)}\,e^{h\gamma}$$
$$-\left(h\beta\right)h\sum_{\gamma=0}^{\beta}\overset{\circ}{C}{}_{\gamma}^{(1)}+h^{2}\sum_{\gamma=0}^{\beta}\overset{\circ}{C}{}_{\gamma}^{(1)}\cdot\gamma.$$

From here, using the representation (3.15) of the coefficients $\overset{\circ}{C}{}_{\gamma}^{(1)}$, we have

$$S_1=h\overset{\circ}{C}{}_{0}^{(1)}\left(\frac{e^{h\beta}}{2}-\frac{e^{-h\beta}}{2}-h\beta\right)+\frac{he^{h\beta}}{2}\sum_{\gamma=1}^{\beta}\left(M\lambda^{\gamma}+N\lambda^{k-\gamma}\right)\cdot e^{-h\gamma}$$
$$-\frac{he^{-h\beta}}{2}\sum_{\gamma=1}^{\beta}\left(M\lambda^{\gamma}+N\lambda^{k-\gamma}\right)\cdot e^{h\gamma}-\left(h\beta\right)h\sum_{\gamma=1}^{\beta}\left(M\lambda^{\gamma}+N\lambda^{k-\gamma}\right)$$
$$+h^{2}\sum_{\gamma=1}^{\beta}\left(M\lambda^{\gamma}+N\lambda^{k-\gamma}\right)\cdot\gamma. \tag{3.19}$$

Now we move on to simplifying the following sum

$$S_2 = \frac{h}{2}\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} \left(\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2} - h\beta + h\gamma\right) = \frac{he^{h\beta}}{4}\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} e^{-h\gamma} - \frac{he^{-h\beta}}{4}\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} e^{h\gamma}$$

$$-\frac{(h\beta)}{2}h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} + \frac{h^2}{2}\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} \cdot\gamma.$$

Taking into account (3.12) and (3.13), S_2 we rewrite the expression in the form

$$S_2 = \frac{e^{h\beta}}{4} \overset{\circ}{g}_k - \frac{he^{-h\beta}}{4}\overset{\circ}{C}{}_{0}^{(1)} - \frac{he^{-h\beta}}{4}\overset{\circ}{C}{}_{k-1}^{(1)} e^{hk-h} - \frac{he^{-h\beta}}{4}\sum_{\gamma=1}^{k-2} \overset{\circ}{C}{}_{\gamma}^{(1)} e^{h\gamma}$$

$$-\frac{(h\beta)}{2}h + \frac{h^2}{2}\overset{\circ}{C}{}_{k-1}^{(1)} \cdot (k-1) + \frac{h^2}{2}\sum_{\gamma=1}^{k-2} \overset{\circ}{C}{}_{\gamma}^{(1)} \cdot\gamma.$$

From here, using the representation (3.15) of the coefficients $\overset{\circ}{C}{}_{\gamma}^{(1)}$, we have

$$S_2 = \frac{e^{h\beta}}{4} \overset{\circ}{g}_k - \frac{he^{-h\beta}}{4}\left(\overset{\circ}{C}{}_{0}^{(1)} + \overset{\circ}{C}{}_{k-1}^{(1)} e^{hk-h}\right) - \frac{he^{-h\beta}}{4}\sum_{\gamma=1}^{k-2} \left(M\lambda^{\gamma} + N\lambda^{k-\gamma}\right) e^{h\gamma}$$

$$-\frac{(h\beta)}{2}h + \frac{h^2}{2}\overset{\circ}{C}{}_{k-1}^{(1)} \cdot (k-1) + \frac{h^2}{2}\sum_{\gamma=1}^{k-2} \left(M\lambda^{\gamma} + N\lambda^{k-\gamma}\right)\cdot\gamma. \tag{3.20}$$

Let us proceed to the proof of Theorem 3.1. From the first sum in (3.19) we denote the following sum

$$S_1^1 = \sum_{\gamma=1}^{\beta} \left(M\lambda^{\gamma} + N\lambda^{k-\gamma}\right)\cdot e^{-h\gamma}.$$

From here, using the formula for the sum of a geometric progression, we have

$$S_1^1 = \sum_{\gamma=1}^{\beta} \left(M\lambda^{\gamma} + N\lambda^{k-\gamma}\right)\cdot e^{-h\gamma} = M\sum_{\gamma=1}^{\beta}\left(\frac{\lambda}{e^h}\right)^{\gamma} + N\lambda^k\sum_{\gamma=1}^{\beta}\left(\frac{1}{\lambda e^h}\right)^{\gamma}$$

$$= M\frac{\lambda\left(e^{h\beta}-\lambda^{\beta}\right)}{\left(e^h-\lambda\right)e^{h\beta}} + N\frac{\lambda^{k-\beta}\left(\lambda^{\beta}e^{h\beta}-1\right)}{\left(\lambda e^h-1\right)e^{h\beta}}. \tag{3.21}$$

Let us denote by S_1^2 from the second sum in equality (3.19) the following

$$S_1^2 = \sum_{\gamma=1}^{\beta} \left(M\lambda^{\gamma} + N\lambda^{k-\gamma}\right)\cdot e^{h\gamma}.$$

We will calculate this sum in a similar way.

$$S_1^2 = \sum_{\gamma=1}^{\beta} \left(M\lambda^{\gamma} + N\lambda^{k-\gamma}\right)\cdot e^{h\gamma} = M\sum_{\gamma=1}^{\beta}\left(\lambda e^h\right)^{\gamma} + N\lambda^k\sum_{\gamma=1}^{\beta}\left(\frac{e^h}{\lambda}\right)^{\gamma}$$

$$= M\frac{\lambda e^{h}\left(\lambda^{\beta}e^{h\beta}-1\right)}{\lambda e^{h}-1}+N\frac{\lambda^{k-\beta}e^{h}\left(e^{h\beta}-\lambda^{\beta}\right)}{e^{h}-\lambda}. \tag{3.22}$$

Let us denote by S_1^3 from the third sum in equality (3.19) the following

$$S_1^3=\sum_{\gamma=1}^{\beta}\left(M\lambda^{\gamma}+N\lambda^{k-\gamma}\right).$$

Here we also have similarly

$$S_1^3=\sum_{\gamma=1}^{\beta}\left(M\lambda^{\gamma}+N\lambda^{k-\gamma}\right)=M\sum_{\gamma=1}^{\beta}\lambda^{\gamma}+N\lambda^{k}\sum_{\gamma=1}^{\beta}\lambda^{-\gamma}$$

$$=M\frac{\lambda\left(1-\lambda^{\beta}\right)}{1-\lambda}+N\frac{\lambda^{k-\beta}\left(1-\lambda^{\beta}\right)}{1-\lambda}. \tag{3.23}$$

Next, we denote by S_1^4 from the fourth sum in equality (3.19)

$$S_1^4=\sum_{\gamma=1}^{\beta}\left(M\lambda^{\gamma}+N\lambda^{k-\gamma}\right)\cdot\gamma=M\sum_{\gamma=1}^{\beta}\lambda^{\gamma}\cdot\gamma+N\lambda^{k}\sum_{\gamma=1}^{\beta}\lambda^{-\gamma}\cdot\gamma$$

$$=M\frac{\lambda-\lambda^{\beta+2}-(\beta+1)\,\lambda^{\beta+1}\,(1-\lambda)}{(1-\lambda)^{2}}+N\frac{\lambda^{k+1}-\lambda^{k-\beta}+(\beta+1)\,\lambda^{k-\beta}\,(1-\lambda)}{(1-\lambda)^{2}}. \tag{3.24}$$

Taking into account equalities (3.21)-(3.24), we rewrite expression (3.19) in the following form

$$S_1=h\overset{\circ}{C}{}_0^{(1)}\left(\frac{e^{h\beta}}{2}-\frac{e^{-h\beta}}{2}-h\beta\right)+\frac{he^{h\beta}}{2}\left(M\frac{\lambda\left(e^{h\beta}-\lambda^{\beta}\right)}{\left(e^{h}-\lambda\right)e^{h\beta}}+N\frac{\lambda^{k-\beta}\left(\lambda^{\beta}e^{h\beta}-1\right)}{\left(\lambda e^{h}-1\right)e^{h\beta}}\right)$$

$$-\frac{he^{-h\beta}}{2}\left(M\frac{\lambda e^{h}\left(\lambda^{\beta}e^{h\beta}-1\right)}{\lambda e^{h}-1}+N\frac{\lambda^{k-\beta}e^{h}\left(e^{h\beta}-\lambda^{\beta}\right)}{e^{h}-\lambda}\right)$$

$$-(h\beta)\,h\left(M\frac{\lambda\left(1-\lambda^{\beta}\right)}{1-\lambda}+N\frac{\lambda^{k-\beta}\left(1-\lambda^{\beta}\right)}{1-\lambda}\right)$$

$$+h^{2}\left(M\frac{\lambda-\lambda^{\beta+2}-(\beta+1)\,\lambda^{\beta+1}\,(1-\lambda)}{(1-\lambda)^{2}}+N\frac{\lambda^{k+1}-\lambda^{k-\beta}+(\beta+1)\,\lambda^{k-\beta}\,(1-\lambda)}{(1-\lambda)^{2}}\right). \tag{3.25}$$

Then also denoting by S_2^1 from the first sum in (3.20), i.e.

$$S_2^1=\sum_{\gamma=1}^{k-2}\left(M\lambda^{\gamma}+N\lambda^{k-\gamma}\right)e^{h\gamma}=M\sum_{\gamma=1}^{k-2}\left(\lambda e^{h}\right)^{\gamma}+N\lambda^{k}\sum_{\gamma=1}^{k-2}\left(\frac{e^{h}}{\lambda}\right)^{\gamma}$$

$$=M\frac{\lambda e^{h}\left(\lambda^{k-2}e^{hk-2h}-1\right)}{\lambda e^{h}-1}+N\frac{\lambda^{2}e^{h}\left(e^{hk-2h}-\lambda^{k-2}\right)}{e^{h}-\lambda}. \tag{3.26}$$

Next, we denote by S_2^2 from the second sum in equality (3.20), i.e.

$$S_2^2=\sum_{\gamma=1}^{k-2}\left(M\lambda^{\gamma}+N\lambda^{k-\gamma}\right)\cdot\gamma=M\sum_{\gamma=1}^{k-2}\lambda^{\gamma}\cdot\gamma+N\lambda^{k}\sum_{\gamma=1}^{k-2}\lambda^{-\gamma}\cdot\gamma$$

$$= M\frac{\lambda-\lambda^{k}-(k-1)\lambda^{k-1}(1-\lambda)}{(1-\lambda)^{2}}+N\frac{\lambda^{k+1}-\lambda^{2}+(k-1)\lambda^{2}(1-\lambda)}{(1-\lambda)^{2}}. \tag{3.27}$$

Taking into account equalities (3.26) and (3.27), we write expression (3.20) in the following form

$$\begin{aligned} S_2 = \frac{e^{h\beta}}{4}g_k - \frac{h\overset{\circ}{e^{-h\beta}}}{4}\left(\overset{\circ}{C}{}_0^{(1)}+\overset{\circ}{C}{}_{k-1}^{(1)}e^{hk-h}\right)-\frac{(h\beta)}{2}h+\frac{h^2}{2}\overset{\circ}{C}{}_{k-1}^{(1)}\cdot(k-1) \\ -\frac{he^{-h\beta}}{4}\left(M\frac{\lambda e^{h}\left(\lambda^{k-2}e^{hk-2h}-1\right)}{\lambda e^{h}-1}+N\frac{\lambda^{2}e^{h}\left(e^{hk-2h}-\lambda^{k-2}\right)}{e^{h}-\lambda}\right) \\ +\frac{h^2}{2}\left(M\frac{\lambda-\lambda^{k}-(k-1)\lambda^{k-1}(1-\lambda)}{(1-\lambda)^{2}}+N\frac{\lambda^{k+1}-\lambda^{2}+(k-1)\lambda^{2}(1-\lambda)}{(1-\lambda)^{2}}\right). \end{aligned} \tag{3.28}$$

Using expressions (3.25) and (3.28) for the sum S determined by formula (3.18), we have

$$\begin{aligned} S = S_1 - S_2 = \frac{e^{h\beta}}{2}\left(h\overset{\circ}{C}{}_0^{(1)}+M\frac{h\lambda}{e^{h}-\lambda}+N\frac{h\lambda^{k}}{\lambda e^{h}-1}-\frac{g_k}{2}\right) \\ +\frac{e^{-h\beta}}{2}\left(\frac{he^{hk-h}}{2}\overset{\circ}{C}{}_{k-1}^{(1)}-\frac{h}{2}\overset{\circ}{C}{}_0^{(1)}+M\frac{h\lambda e^{h}\left(\lambda^{k-2}e^{hk-2h}+1\right)}{2(\lambda e^{h}-1)}+N\frac{h\lambda^{2}e^{h}\left(e^{hk-2h}+\lambda^{k-2}\right)}{2(e^{h}-\lambda)}\right) \\ +(h\beta)\left(\frac{h}{2}-h\overset{\circ}{C}{}_0^{(1)}+M\frac{h\lambda}{\lambda-1}+N\frac{h\lambda^{k}}{1-\lambda}\right) \\ +\frac{h^2}{2}\left(M\frac{\lambda-2\lambda^{\beta+1}+\lambda^{k}+(k-1)\lambda^{k-1}(1-\lambda)}{(1-\lambda)^{2}}\right. \\ \left.+N\frac{\lambda^{k+1}-2\lambda^{k-\beta+1}+\lambda^{2}-(k-1)\lambda^{2}(1-\lambda)}{(1-\lambda)^{2}}-(k-1)\overset{\circ}{C}{}_{k-1}^{(1)}\right) \\ +M\frac{h\lambda^{\beta+1}\left(1-e^{2h}\right)}{2(e^{h}-\lambda)(\lambda e^{h}-1)}+N\frac{h\lambda^{k-\beta+1}\left(1-e^{2h}\right)}{2(e^{h}-\lambda)(\lambda e^{h}-1)}. \end{aligned} \tag{3.29}$$

Then, by virtue of (3.17), (3.29), the left side of equality (3.11) takes the following form

$$\begin{aligned} S = \frac{e^{h\beta}}{2}\left(h\overset{\circ}{C}{}_0^{(1)}+M\frac{h\lambda}{e^{h}-\lambda}+N\frac{h\lambda^{k}}{\lambda e^{h}-1}-\frac{g_k}{2}\right) \\ +\frac{e^{-h\beta}}{2}\left(\frac{he^{hk-h}}{2}\overset{\circ}{C}{}_{k-1}^{(1)}-\frac{h}{2}\overset{\circ}{C}{}_0^{(1)}+M\frac{h\lambda e^{h}\left(\lambda^{k-2}e^{hk-2h}+1\right)}{2(\lambda e^{h}-1)}\right. \\ \left.+N\frac{h\lambda^{2}e^{h}\left(e^{hk-2h}+\lambda^{k-2}\right)}{2(e^{h}-\lambda)}\right)+(h\beta)\left(\frac{h}{2}-h\overset{\circ}{C}{}_0^{(1)}+M\frac{h\lambda}{\lambda-1}+N\frac{h\lambda^{k}}{1-\lambda}\right) \\ +\frac{h^2}{2}\left(M\frac{\lambda-2\lambda^{\beta+1}+\lambda^{k}+(k-1)\lambda^{k-1}(1-\lambda)}{(1-\lambda)^{2}}\right. \\ +N\frac{\lambda^{k+1}-2\lambda^{k-\beta+1}+\lambda^{2}-(k-1)\lambda^{2}(1-\lambda)}{(1-\lambda)^{2}}-(k-1)\overset{\circ}{C}{}_{k-1}^{(1)} \end{aligned}$$

$$
+M\frac{h\lambda^{\beta+1}\left(1-e^{2h}\right)}{\left(e^{h}-\lambda\right)\left(\lambda e^{h}-1\right)}+N\frac{h\lambda^{k-\beta+1}\left(1-e^{2h}\right)}{\left(e^{h}-\lambda\right)\left(\lambda e^{h}-1\right)}\Bigg). \tag{3.30}
$$

Due to (3.10) and (3.30), equality (3.11) takes the following form

$$
S=\frac{e^{h\beta}}{2}\left(h\overset{\circ}{C}{}_{0}^{(1)}+M\frac{h\lambda}{e^{h}-\lambda}+N\frac{h\lambda^{k}}{\lambda e^{h}-1}-\frac{g_k}{2}\right)
$$

$$
+\frac{e^{-h\beta}}{2}\Bigg(\frac{he^{hk-h}}{2}\overset{\circ}{C}{}_{k-1}^{(1)}-\frac{h}{2}\overset{\circ}{C}{}_{0}^{(1)}+M\frac{h\lambda e^{h}\left(\lambda^{k-2}e^{hk-2h}+1\right)}{2\left(\lambda e^{h}-1\right)}
$$

$$
+N\frac{h\lambda^{2}e^{h}\left(e^{hk-2h}+\lambda^{k-2}\right)}{2\left(e^{h}-\lambda\right)}+2d\Bigg)+(h\beta)\left(\frac{h}{2}-h\overset{\circ}{C}{}_{0}^{(1)}+M\frac{h\lambda}{\lambda-1}+N\frac{h\lambda^{k}}{1-\lambda}\right)
$$

$$
+h^{2}\Bigg(M\frac{\lambda-2\lambda^{\beta+1}+\lambda^{k}+(k-1)\lambda^{k-1}(1-\lambda)}{2(1-\lambda)^{2}}
$$

$$
+N\frac{\lambda^{k+1}-2\lambda^{k-\beta+1}+\lambda^{2}-(k-1)\lambda^{2}(1-\lambda)}{2(1-\lambda)^{2}}-\frac{k-1}{2}\overset{\circ}{C}{}_{k-1}^{(1)}+M\frac{h^{-1}\lambda^{\beta+1}\left(1-e^{2h}\right)}{2\left(e^{h}-\lambda\right)\left(\lambda e^{h}-1\right)}
$$

$$
+N\frac{h^{-1}\lambda^{k-\beta+1}\left(1-e^{2h}\right)}{2\left(e^{h}-\lambda\right)\left(\lambda e^{h}-1\right)}+h^{-2}p_0\Bigg)=f_3(h\beta),\qquad \beta=0,1,...,k-1.
$$

From here, equating the coefficients at $\frac{e^{h\beta}}{2}$, $(h\beta)$, $\frac{e^{-h\beta}}{2}$ and $\frac{h^2}{2}$ we obtain a system for finding unknowns M, N, d and p_0

$$
h\overset{\circ}{C}{}_{0}^{(1)}+M\frac{h\lambda}{e^{h}-\lambda}+N\frac{h\lambda^{k}}{\lambda e^{h}-1}-\frac{g_k}{2}=\frac{e^{-hk}-e^{-hk+h}}{2}, \tag{3.31}
$$

$$
\frac{h}{2}-h\overset{\circ}{C}{}_{0}^{(1)}+M\frac{h\lambda}{\lambda-1}+N\frac{h\lambda^{k}}{1-\lambda}=\frac{h}{2}, \tag{3.32}
$$

$$
\frac{he^{hk-h}}{2}\overset{\circ}{C}{}_{k-1}^{(1)}-\frac{h}{2}\overset{\circ}{C}{}_{0}^{(1)}+M\frac{h\lambda e^{h}\left(\lambda^{k-2}e^{hk-2h}+1\right)}{2\left(\lambda e^{h}-1\right)}
$$

$$
+N\frac{h\lambda^{2}e^{h}\left(e^{hk-2h}+\lambda^{k-2}\right)}{2\left(e^{h}-\lambda\right)}+2d=\frac{e^{hk}-e^{hk-h}}{2}, \tag{3.33}
$$

$$
M\frac{\lambda-2\lambda^{\beta+1}+\lambda^{k}+(k-1)\lambda^{k-1}(1-\lambda)}{2(1-\lambda)^{2}}+N\frac{\lambda^{k+1}-2\lambda^{k-\beta+1}+\lambda^{2}-(k-1)\lambda^{2}(1-\lambda)}{2(1-\lambda)^{2}}
$$

$$
-\frac{k-1}{2}\overset{\circ}{C}{}_{k-1}^{(1)}+M\frac{h^{-1}\lambda^{\beta+1}\left(1-e^{2h}\right)}{2\left(e^{h}-\lambda\right)\left(\lambda e^{h}-1\right)}+N\frac{h^{-1}\lambda^{k-\beta+1}\left(1-e^{2h}\right)}{2\left(e^{h}-\lambda\right)\left(\lambda e^{h}-1\right)}+h^{-2}p_0=\frac{1-2k}{2}. \tag{3.34}
$$

From (3.31) it follows that

$$
h\overset{\circ}{C}{}_{0}^{(1)}+M\frac{h\lambda}{e^{h}-\lambda}+N\frac{h\lambda^{k}}{\lambda e^{h}-1}=0. \tag{3.35}
$$

From (3.32) it follows that

$$M\frac{h\lambda}{\lambda-1}+N\frac{h\lambda^k}{1-\lambda}-h\overset{\circ}{C}{}_0^{(1)}=0. \tag{3.36}$$

From here, by virtue of (3.14), we obtain the following form of equality (3.35)

$$\frac{he^h-e^h+1}{e^h\left(1-e^{hk-h}\right)}+M\frac{h\left[\lambda^k\left(e^h-1\right)-\lambda^2\left(e^{hk-h}-1\right)+\lambda e^h\left(e^{hk-2h}-1\right)\right]}{\left(1-e^{hk-h}\right)\left(e^h-\lambda\right)(1-\lambda)}$$

$$+N\frac{h\left[\lambda^{k+1}e^h\left(1-e^{hk-2h}\right)+\lambda^k\left(e^{hk-h}-1\right)-\lambda^2\left(e^h-1\right)\right]}{\left(1-e^{hk-h}\right)\left(\lambda e^h-1\right)(1-\lambda)}+M\frac{h\lambda}{e^h-\lambda}+N\frac{h\lambda^k}{\lambda e^h-1}=0.$$

After canceling out similar terms, we have

$$M\frac{h\lambda\left(\lambda^{k-1}-1\right)\left(e^h-1\right)}{\left(e^h-\lambda\right)(1-\lambda)}+N\frac{h\lambda^2\left(\lambda^{k-1}-1\right)\left(e^h-1\right)}{\left(\lambda e^h-1\right)(1-\lambda)}=\frac{e^h-he^h-1}{e^h}. \tag{3.37}$$

Now, by virtue of (3.14), we obtain the following form of equality (3.36)

$$M\frac{h\lambda}{\lambda-1}+N\frac{h\lambda^k}{1-\lambda}-\frac{he^h-e^h+1}{e^h\left(1-e^{hk-h}\right)}$$

$$-M\frac{h\left[\lambda^k\left(e^h-1\right)-\lambda^2\left(e^{hk-h}-1\right)+\lambda e^h\left(e^{hk-2h}-1\right)\right]}{\left(1-e^{hk-h}\right)\left(e^h-\lambda\right)(1-\lambda)}$$

$$-N\frac{h\left[\lambda^{k+1}e^h\left(1-e^{hk-2h}\right)+\lambda^k\left(e^{hk-h}-1\right)-\lambda^2\left(e^h-1\right)\right]}{\left(1-e^{hk-h}\right)\left(\lambda e^h-1\right)(1-\lambda)}=0.$$

After canceling out similar terms, we have

$$M\frac{h\lambda\left(\lambda^{k-1}-e^{hk-h}\right)\left(e^h-1\right)}{\left(e^h-\lambda\right)(1-\lambda)}+N\frac{h\lambda^2\left(\lambda^{k-1}e^{hk-h}-1\right)\left(e^h-1\right)}{\left(\lambda e^h-1\right)(1-\lambda)}=\frac{e^h-he^h-1}{e^h}. \tag{3.38}$$

By virtue of (3.37), (3.38) we obtain a system for finding unknowns M and N

$$\begin{cases} M\frac{h\lambda\left(\lambda^{k-1}-1\right)\left(e^h-1\right)}{\left(e^h-\lambda\right)(1-\lambda)}+N\frac{h\lambda^2\left(\lambda^{k-1}-1\right)\left(e^h-1\right)}{\left(\lambda e^h-1\right)(1-\lambda)}=\frac{e^h-he^h-1}{e^h}, \\ M\frac{h\lambda\left(\lambda^{k-1}-e^{hk-h}\right)\left(e^h-1\right)}{\left(e^h-\lambda\right)(1-\lambda)}+N\frac{h\lambda^2\left(\lambda^{k-1}e^{hk-h}-1\right)\left(e^h-1\right)}{\left(\lambda e^h-1\right)(1-\lambda)}=\frac{e^h-he^h-1}{e^h}. \end{cases} \tag{3.39}$$

From system (3.39) we find the unknowns M and in a unique way N, i.e.

$$M=\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(e^h-\lambda\right)\lambda^k}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)};\ \ N=\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(\lambda e^h-1\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}. \tag{3.40}$$

Now let's simplify the coefficients $\overset{\circ}{C}{}_0^{(1)}, \overset{\circ}{C}{}_\beta^{(1)}$, $\beta=1,2,...,k-2$, and $\overset{\circ}{C}{}_{k-1}^{(1)}$ defined by formulas (3.14), (3.15), and (3.16). To do this, we take into account expression (3.40) for M, N and λ is the root of a polynomial $P_2(x)=\left(1-e^{2h}+2he^h\right)x^2-2\left(1-e^{2h}+he^{2h}+h\right)x+\left(1-e^{2h}+2he^h\right)$ of degree 2.
Let's look at this first

$$\overset{\circ}{C}{}_0^{(1)}=\frac{he^h-e^h+1}{he^h\left(1-e^{hk-h}\right)}+M\frac{\lambda^k\left(e^h-1\right)-\lambda^2\left(e^{hk-h}-1\right)+\lambda e^h\left(e^{hk-2h}-1\right)}{\left(1-e^{hk-h}\right)\left(e^h-\lambda\right)(1-\lambda)}$$

$$+N\frac{\lambda^{k+1}e^h\left(1-e^{hk-2h}\right)+\lambda^k\left(e^{hk-h}-1\right)-\lambda^2\left(e^h-1\right)}{\left(1-e^{hk-h}\right)\left(\lambda e^h-1\right)(1-\lambda)}.$$

Taking into account equalities (3.40) we obtain

$$\overset{\circ}{C}{}^{(1)}_0=\frac{he^h-e^h+1}{he^h\left(1-e^{hk-h}\right)}$$

$$+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(e^h-\lambda\right)\lambda^k}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\cdot\frac{\lambda^k\left(e^h-1\right)-\lambda^2\left(e^{hk-h}-1\right)+\lambda e^h\left(e^{hk-2h}-1\right)}{\left(1-e^{hk-h}\right)\left(e^h-\lambda\right)(1-\lambda)}$$

$$+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(\lambda e^h-1\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\cdot\frac{\lambda^{k+1}e^h\left(1-e^{hk-2h}\right)+\lambda^k\left(e^{hk-h}-1\right)-\lambda^2\left(e^h-1\right)}{\left(1-e^{hk-h}\right)\left(\lambda e^h-1\right)(1-\lambda)}.$$

So, we finally have

$$\overset{\circ}{C}{}^{(1)}_0=\frac{\lambda^k\left(e^h-he^h-1\right)\left(\lambda^2-1\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}.$$

Now let's look at

$$\overset{\circ}{C}{}^{(1)}_{k-1}=\frac{e^h-1-he^{hk}}{he^h\left(1-e^{hk-h}\right)}+M\frac{\lambda^k e^h\left(e^{hk-2h}-1\right)-\lambda^{k-1}e^h\left(e^{hk-h}-1\right)+\lambda e^{hk-h}\left(e^h-1\right)}{\left(1-e^{hk-h}\right)\left(e^h-\lambda\right)(1-\lambda)}$$

$$+N\frac{\lambda^{k+1}e^{hk-h}\left(1-e^h\right)+\lambda^3e^h\left(e^{hk-h}-1\right)-\lambda^2e^h\left(e^{hk-2h}-1\right)}{\left(1-e^{hk-h}\right)\left(\lambda e^h-1\right)(1-\lambda)}$$

$$=\frac{e^h-1-he^{hk}}{he^h\left(1-e^{hk-h}\right)}+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(e^h-\lambda\right)\lambda^k}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}$$

$$\times\frac{\lambda^k e^h\left(e^{hk-2h}-1\right)-\lambda^{k-1}e^h\left(e^{hk-h}-1\right)+\lambda e^{hk-h}\left(e^h-1\right)}{\left(1-e^{hk-h}\right)\left(e^h-\lambda\right)(1-\lambda)}$$

$$+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(\lambda e^h-1\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\cdot\frac{\lambda^{k+1}e^{hk-h}\left(1-e^h\right)+\lambda^3e^h\left(e^{hk-h}-1\right)-\lambda^2e^h\left(e^{hk-2h}-1\right)}{\left(1-e^{hk-h}\right)\left(\lambda e^h-1\right)(1-\lambda)}$$

$$=\frac{\left(\lambda^{2k}-\lambda^2\right)\left(he^{2h}-e^h+1\right)+\left(\lambda^{2k-1}-\lambda^3\right)\left(e^{2h}-he^{2h}-e^h\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}.$$

Means,

$$\overset{\circ}{C}{}^{(1)}_{k-1}=\frac{\left(\lambda^{2k}-\lambda^2\right)\left(he^{2h}-e^h+1\right)+\left(\lambda^{2k-1}-\lambda^3\right)\left(e^{2h}-he^{2h}-e^h\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}.$$

In this case

$$\overset{\circ}{C}{}^{(1)}_\beta=M\lambda^\beta+N\lambda^{k-\beta}$$

$$=\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(e^h-\lambda\right)\lambda^k}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\cdot\lambda^\beta+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(\lambda e^h-1\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\cdot\lambda^{k-\beta}$$

$$=\frac{\left(e^h-he^h-1\right)(1-\lambda)\left[\lambda^{k+\beta}\left(e^h-\lambda\right)+\lambda^{k-\beta}\left(\lambda e^h-1\right)\right]}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)},\quad \beta=1,2,\ldots,k-2.$$

Means,

$$\overset{\circ}{C}{}^{(1)}_{\beta}=\frac{\left(e^{h}-he^{h}-1\right)(1-\lambda)\left[\lambda^{k+\beta}\left(e^{h}-\lambda\right)+\lambda^{k-\beta}\left(\lambda e^{h}-1\right)\right]}{he^{h}\left(e^{h}-1\right)\left(\lambda^{2k}-\lambda^{2}\right)},\ \ \beta=1,2,\ldots,k-2.$$

Theorem 3.1 is completely proven.

4. Optimal function of a discrete argument

In this section we find the unknown optimal function $\overset{\circ}{F}(h\beta)$ of the discrete argument, which plays an important role in finding the optimal error estimate of an explicit difference formula of the form (3.1), i.e. calculating the square of the norm of the error functional of the optimal explicit difference formula of the Adams type in the Hilbert space $W_2^{(3,2)}(0,1)$.

The main result of this section is the following theorem.

Theorem 4.1. *The optimal function $\overset{\circ}{F}(h\beta)=\overset{\circ}{d}\,e^{-h\beta}+\overset{\circ}{p}_0$ of the discrete argument is expressed by the equality*

$$\overset{\circ}{F}(h\beta)=\left(\frac{e^{hk-h}\left(e^{h}-h-1\right)}{4}+\frac{e^{hk-2h}\left(e^{h}-he^{h}-1\right)\left[\left(e^{h}-1\right)\left(\lambda^{2k-1}-\lambda^{2k-2}e^{h}\right)-\lambda e^{h}+1\right]}{4\left(e^{h}-1\right)\left(\lambda^{2k-2}-1\right)\left(\lambda e^{h}-1\right)}\right.$$
$$\left.+\frac{e^{hk-2h}\left(e^{h}-he^{h}-1\right)\left(\lambda e^{2h}-\lambda e^{h}-\lambda+1\right)}{4\left(e^{h}-1\right)\left(\lambda^{2k-2}-1\right)\left(e^{h}-\lambda\right)}-\frac{\lambda^{k-1}\left(e^{h}-he^{h}-1\right)\left(1-\lambda^{2}\right)\left(e^{2h}-1\right)}{4e^{h}\left(\lambda^{2k-2}-1\right)\left(e^{h}-\lambda\right)\left(\lambda e^{h}-1\right)}\right)\cdot e^{-h\beta}$$
$$-h^{2}\left(\frac{1}{4}+\frac{\left(e^{h}-he^{h}-1\right)\left(1+\lambda^{k-1}\right)\left(\lambda^{k-1}e^{h}-\lambda^{k}+\lambda e^{h}-1\right)}{2he^{h}\left(e^{h}-1\right)\left(\lambda^{2k-2}-1\right)(1-\lambda)}\right).$$

Proof of Theorem 4.1. From equality (3.33) it follows that

$$\overset{\circ}{d}=\frac{e^{hk}-e^{hk-h}}{4}-\frac{he^{hk-h}}{4}\overset{\circ}{C}{}^{(1)}_{k-1}+\frac{h}{4}\overset{\circ}{C}{}^{(1)}_{0}-M\frac{h\lambda e^{h}\left(\lambda^{k-2}e^{hk-2h}+1\right)}{4\left(\lambda e^{h}-1\right)}$$
$$-N\frac{h\lambda^{2}e^{h}\left(e^{hk-2h}+\lambda^{k-2}\right)}{2\left(e^{h}-\lambda\right)}.$$

From here, using formulas (3.2), (3.4) and (3.40), we have

$$\overset{\circ}{d}=\frac{e^{hk}-e^{hk-h}}{4}-\frac{he^{hk-h}}{4}\cdot\frac{\left(\lambda^{2k}-\lambda^{2}\right)\left(he^{2h}-e^{h}+1\right)+\left(\lambda^{2k-1}-\lambda^{3}\right)\left(e^{2h}-he^{2h}-e^{h}\right)}{he^{h}\left(e^{h}-1\right)\left(\lambda^{2k}-\lambda^{2}\right)}$$
$$+\frac{h}{4}\cdot\frac{\lambda^{k}\left(e^{h}-he^{h}-1\right)\left(\lambda^{2}-1\right)}{he^{h}\left(e^{h}-1\right)\left(\lambda^{2k}-\lambda^{2}\right)}$$
$$-\frac{\left(e^{h}-he^{h}-1\right)(1-\lambda)\left(e^{h}-\lambda\right)\lambda^{k}}{he^{h}\left(e^{h}-1\right)\left(\lambda^{2k}-\lambda^{2}\right)}\cdot\frac{h\lambda e^{h}\left(\lambda^{k-2}e^{hk-2h}+1\right)}{4\left(\lambda e^{h}-1\right)}$$
$$-\frac{\left(e^{h}-he^{h}-1\right)(1-\lambda)\left(\lambda e^{h}-1\right)}{he^{h}\left(e^{h}-1\right)\left(\lambda^{2k}-\lambda^{2}\right)}\cdot\frac{h\lambda^{2}e^{h}\left(e^{hk-2h}+\lambda^{k-2}\right)}{2\left(e^{h}-\lambda\right)}$$

$$= \frac{e^{hk-h}\left(e^h-h-1\right)}{4}+\frac{e^{hk-2h}\left(e^h-he^h-1\right)\left[\left(e^h-1\right)\left(\lambda^{2k-1}-\lambda^{2k-2}e^h\right)-\lambda e^h+1\right]}{4\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)\left(\lambda e^h-1\right)}$$

$$+\frac{e^{hk-2h}\left(e^h-he^h-1\right)\left(\lambda e^{2h}-\lambda e^h-\lambda+1\right)}{4\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)\left(e^h-\lambda\right)}-\frac{\lambda^{k-1}\left(e^h-he^h-1\right)\left(1-\lambda^2\right)\left(e^{2h}-1\right)}{4e^h\left(\lambda^{2k-2}-1\right)\left(e^h-\lambda\right)\left(\lambda e^h-1\right)}.$$

So, we have found one of the coefficients of the optimal function $\overset{\circ}{F}(h\beta)$ of the discrete argument

$$\overset{\circ}{d} = \frac{e^{hk-h}\left(e^h-h-1\right)}{4}+\frac{e^{hk-2h}\left(e^h-he^h-1\right)\left[\left(e^h-1\right)\left(\lambda^{2k-1}-\lambda^{2k-2}e^h\right)-\lambda e^h+1\right]}{4\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)\left(\lambda e^h-1\right)}$$

$$+\frac{e^{hk-2h}\left(e^h-he^h-1\right)\left(\lambda e^{2h}-\lambda e^h-\lambda+1\right)}{4\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)\left(e^h-\lambda\right)}-\frac{\lambda^{k-1}\left(e^h-he^h-1\right)\left(1-\lambda^2\right)\left(e^{2h}-1\right)}{4e^h\left(\lambda^{2k-2}-1\right)\left(e^h-\lambda\right)\left(\lambda e^h-1\right)}.$$

Let's move on to finding the second coefficient of the optimal function $\overset{\circ}{F}(h\beta)$ of the discrete argument.

From equalities (3.34) we find

$$\overset{\circ}{p}_0 = -h^2\left[\frac{2k-1}{4}-\frac{k-1}{2}\overset{\circ}{C}{}^{(1)}_{k-1}+M\frac{\lambda-2\lambda^{\beta+1}+\lambda^k+(k-1)\lambda^{k-1}(1-\lambda)}{2(1-\lambda)^2}\right.$$

$$+N\frac{\lambda^{k+1}-2\lambda^{k-\beta+1}+\lambda^2-(k-1)\lambda^2(1-\lambda)}{2(1-\lambda)^2}$$

$$\left.+M\frac{h^{-1}\lambda^{\beta+1}\left(1-e^{2h}\right)}{2\left(e^h-\lambda\right)\left(\lambda e^h-1\right)}+N\frac{h^{-1}\lambda^{k-\beta+1}\left(1-e^{2h}\right)}{2\left(e^h-\lambda\right)\left(\lambda e^h-1\right)}\right].$$

From here, using formulas (3.4), (3.40), after some simplifications, we obtain

$$\overset{\circ}{p}_0 = -h^2\left[\frac{2k-1}{4}-\frac{k-1}{2}\cdot\frac{\left(\lambda^{2k}-\lambda^2\right)\left(he^{2h}-e^h+1\right)+\left(\lambda^{2k-1}-\lambda^3\right)\left(e^{2h}-he^{2h}-e^h\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\right.$$

$$+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(e^h-\lambda\right)\lambda^k}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\cdot\frac{\lambda-2\lambda^{\beta+1}+\lambda^k+(k-1)\lambda^{k-1}(1-\lambda)}{2(1-\lambda)^2}$$

$$+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(\lambda e^h-1\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\cdot\frac{\lambda^{k+1}-2\lambda^{k-\beta+1}+\lambda^2-(k-1)\lambda^2(1-\lambda)}{2(1-\lambda)^2}$$

$$+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(e^h-\lambda\right)\lambda^k}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\cdot\frac{h^{-1}\lambda^{\beta+1}\left(1-e^{2h}\right)}{2\left(e^h-\lambda\right)\left(\lambda e^h-1\right)}$$

$$\left.+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(\lambda e^h-1\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\cdot\frac{h^{-1}\lambda^{k-\beta+1}\left(1-e^{2h}\right)}{2\left(e^h-\lambda\right)\left(\lambda e^h-1\right)}\right]$$

$$= -h^2\left[\frac{2k-1}{4}+\frac{\lambda^{k+1}\left(e^h-he^h-1\right)\left(1+\lambda^{k-1}\right)\left(e^h-\lambda\right)}{2he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)(1-\lambda)}-\frac{\lambda^{2k}\left(e^h-he^h-1\right)(k-1)}{2he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}\right.$$

$$+\frac{\lambda^2\left(e^h-he^h-1\right)\left(\lambda e^h-1\right)\left(1+\lambda^{k-1}\right)}{2he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)(1-\lambda)}+\frac{\lambda^2\left(e^h-he^h-1\right)(k-1)}{2he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}+\frac{\left(e^h-he^h-1\right)(k-1)}{2he^h\left(e^h-1\right)}$$

$$+\frac{\lambda^{k+\beta+1}\left(e^h-he^h-1\right)\left[\left(1+2he^h-e^{2h}\right)\lambda^2-2\left(he^{2h}+h-e^{2h}+1\right)\lambda+\left(1+2he^h-e^{2h}\right)\right]}{2h^2e^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)\left(\lambda e^h-1\right)(1-\lambda)}$$

$$+\left.\frac{\lambda^{k-\beta+1}\left(e^h-he^h-1\right)\left[\left(1+2he^h-e^{2h}\right)\lambda^2-2\left(he^{2h}+h-e^{2h}+1\right)\lambda+\left(1+2he^h-e^{2h}\right)\right]}{2h^2e^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)\left(e^h-\lambda\right)(1-\lambda)}\right], \tag{4.1}$$

here λ is the root of a polynomial $P_2(x) = \left(1-e^{2h}+2he^h\right)x^2-2\left(he^{2h}+h-e^{2h}+1\right)x+\left(1-e^{2h}+2he^h\right)$ of degree 2, i.e.

$$\left(1+2he^h-e^{2h}\right)\lambda^2-2\left(he^{2h}+h-e^{2h}+1\right)\lambda+\left(1+2he^h-e^{2h}\right)=0. \tag{4.2}$$

Due to equality (4.2), expression (4.1) is reduced to the form

$$\overset{\circ}{p}_0=-h^2\left[\frac{2k-1}{4}+\frac{\lambda^{k+1}\left(e^h-he^h-1\right)\left(1+\lambda^{k-1}\right)\left(e^h-\lambda\right)}{2he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)(1-\lambda)}\right.$$

$$-\frac{\lambda^{2k}\left(e^h-he^h-1\right)(k-1)}{2he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}+\frac{\lambda^2\left(e^h-he^h-1\right)\left(\lambda e^h-1\right)\left(1+\lambda^{k-1}\right)}{2he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)(1-\lambda)}$$

$$\left.+\frac{\lambda^2\left(e^h-he^h-1\right)(k-1)}{2he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}+\frac{\left(e^h-he^h-1\right)(k-1)}{2he^h\left(e^h-1\right)}\right]$$

$$=-h^2\left[\frac{1}{4}+\frac{\left(e^h-he^h-1\right)\left(1+\lambda^{k-1}\right)\left(\lambda^{k-1}e^h-\lambda^k+\lambda e^h-1\right)}{2he^h\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)(1-\lambda)}\right]$$

So, we have found the second coefficient of the optimal function $\overset{\circ}{F}(h\beta)$ of the discrete argument

$$\overset{\circ}{p}_0=-h^2\left[\frac{1}{4}+\frac{\left(e^h-he^h-1\right)\left(1+\lambda^{k-1}\right)\left(\lambda^{k-1}e^h-\lambda^k+\lambda e^h-1\right)}{2he^h\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)(1-\lambda)}\right].$$

Theorem 4.1 is completely proven.

5. Optimal error estimate of the difference formula on the class $W_2^{(3,2)}(0,1)$

In this section, we will calculate the optimal error estimate for an explicit difference formula of the form (3.1) on the class $W_2^{(3,2)}(0,1)$. In addition, using the optimal function discrete argument, a new formula will be obtained for the optimal error estimate of the explicit difference formula of the form (3.1) on the class $W_2^{(3,2)}(0,1)$.

Theorem 5.1. *Optimal error estimate for explicit difference formulas Adams type on classes $W_2^{(m,m-1)}(0,1)$, for any $m\geq 2$ is given by the formula*

$$\left\|\ell\left|W_2^{(m,m-1)*}\right.\right\|^2=(-1)^m\left\{\frac{e^{-h}-e^h}{2}+\sum_{j=1}^{m-1}\frac{h^{2j-1}}{(2j-1)!}+h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_\gamma^{(1)}\left[\overset{\circ}{d}\,e^{-h\gamma}+\overset{\circ}{P}_{m-3}(h\gamma)+f_m(h\gamma)\right]\right\},$$

here are $\overset{\circ}{C}{}_{\beta}^{(1)}$ *−the optimal coefficients of the explicit difference formulas,* $\overset{\circ}{d}\, e^{-h\beta} + \overset{\circ}{P}_{m-3}(h\beta)$ *is the optimal function discrete argument,*

$$f_m(h\beta) = \frac{1}{2}\left(\left(1-e^h\right)\left(\frac{e^{h\beta-hk}}{2} - \frac{e^{hk-h\beta-h}}{2}\right) - \sum_{j=2}^{m-1}\left(\frac{(h\beta-hk)^{2j-2}}{(2j-2)!} - \frac{(h\beta-hk+h)^{2j-2}}{(2j-2)!}\right)\right).$$

Theorem 5.2. *Optimal error estimate of the explicit difference formula* (3.1) *Adams type in the class* $W_2^{(3,2)}(0,1)$, *is expressed by the formula*

$$\left\|\overset{\circ}{\ell}\,|W_2^{(3,2)*}(0,1)\right\|^2 = \frac{\left(e^h - he^h - 1\right)}{2e^{2h}\left(\lambda^{2k-2}-1\right)\left(\lambda e^h - 1\right)\left(e^h-\lambda\right)}\left[2\left(e^{2h}-e^h+2\right)\left(\lambda^{2k-2}-1\right)\lambda e^h\right.$$
$$\left.+\left(e^h-3\right)\left(\lambda^{2k-4}-1\right)\lambda^2 e^{2h} - \left(2e^{2h}-e^h+1\right)\left(\lambda^{2k}-1\right)\right] + \frac{h\left(e^h-1\right)}{2e^h}$$
$$+\frac{h^2\left(e^h-he^h-1\right)\left(\lambda^{2k-2}e^h - \lambda^{2k-1} + \lambda e^h - 1\right)}{e^h\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)(1-\lambda)} + \frac{h^3}{3}.$$

In Theorem 2.2, an expression is obtained for the error of explicit difference formulas of general form on classes $W_2^{(m,m-1)}(0,1)$. For explicit difference formulas, this expression takes the following form

$$\left\|\ell|W_2^{(m,m-1)*}\right\|^2 = (-1)^m\left[\sum_{\gamma=0}^{k} C_\gamma \sum_{\beta=0}^{k} C_\beta G_m\left(h\gamma-h\beta\right)\right.$$
$$\left.-2h\sum_{\gamma=0}^{k-1} C_\gamma^{(1)} \sum_{\beta=0}^{k} C_\beta G_m'\left(h\gamma-h\beta\right) - h^2\sum_{\gamma=0}^{k-1} C_\gamma^{(1)} \sum_{\beta=0}^{k-1} C_\beta^{(1)} G_m''\left(h\gamma-h\beta\right)\right].$$

Minimizing this norm with respect to the coefficients $C_\beta^{(1)}$ subject to the conditions

$$h\sum_{\gamma=0}^{k-1} C_\gamma^{(1)}(h\gamma)^{\alpha-1} = \frac{h^\alpha}{\alpha}\left(k^\alpha - (k-1)^\alpha\right),\ \ \alpha = 1,2,...,m-2, \tag{5.1}$$

$$h\sum_{\gamma=0}^{k-1} C_\gamma^{(1)} e^{-h\gamma} = e^{-hk+h} - e^{-hk} \tag{5.2}$$

a system for finding the optimal coefficients $\overset{\circ}{C}{}_{\beta}^{(1)}$ of explicit difference formulas and the optimal function was obtained discrete argument

$$h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} \frac{sign(h\beta-h\gamma)}{2}\left[\frac{e^{h\beta-h\gamma}-e^{h\gamma-h\beta}}{2} - \sum_{j=1}^{m-2}\frac{(h\beta-h\gamma)^{2j-1}}{(2j-1)!}\right] + \overset{\circ}{d}\, e^{-h\beta}$$
$$+\overset{\circ}{\underset{m-3}{P}}(h\beta) = f_m(h\beta), \beta = 0,1,...,k-1, \tag{5.3}$$

here

$$f_m(h\beta) = -\sum_{\gamma=0}^{k} C_\gamma \frac{sign(h\beta-h\gamma)}{2}\left[\frac{e^{h\beta-h\gamma}+e^{h\gamma-h\beta}}{2} - \sum_{j=1}^{m-1}\frac{(h\beta-h\gamma)^{2j-2}}{(2j-2)!}\right],$$

$\overset{\circ}{C}{}_{\beta}^{(1)}$- the desired optimal coefficients of difference formulas, $\overset{\circ}{d}\, e^{-h\beta} + \overset{\circ}{P}_{m-3}(h\beta)$- the desired optimal function of the discrete argument.

Since we are considering a difference formula of the Adams type, then $C_k = 1,\ C_{k-1} = -1,\ C_{k-i} = 0$ at $i = 0, 1, ..., k-2$ in this case $f_m(h\beta)$, it takes the following form

$$f_m(h\beta) = \frac{1}{2}\left(\left(1 - e^h\right)\left(\frac{e^{h\beta - hk}}{2} - \frac{e^{hk - h\beta - h}}{2}\right) - \sum_{j=2}^{m-1}\left(\frac{(h\beta - hk)^{2j-2}}{(2j-2)!} - \frac{(h\beta - hk + h)^{2j-2}}{(2j-2)!}\right)\right). \tag{5.4}$$

Proof of Theorem 5.1. To do this, using the value $C_\beta,\ \beta = 0, 1, ..., k$, we simplify the following double sum

$$\sum_{\gamma=0}^{k}\sum_{\beta=0}^{k} C_\gamma C_\beta \frac{sign(h\beta - h\gamma)}{2}\left[\frac{e^{h\beta - h\gamma} - e^{h\gamma - h\beta}}{2} - \sum_{j=1}^{m-1}\frac{(h\beta - h\gamma)^{2j-1}}{(2j-1)!}\right]$$

$$= \sum_{\gamma=0}^{k} C_\gamma\left[\frac{sign(hk - h\gamma)}{2}\left[\frac{e^{hk - h\gamma} - e^{h\gamma - hk}}{2} - \sum_{j=1}^{m-1}\frac{(hk - h\gamma)^{2j-1}}{(2j-1)!}\right]\right.$$

$$\left. - \frac{sign(hk - h - h\gamma)}{2}\left[\frac{e^{hk - h - h\gamma} - e^{h\gamma - hk + h}}{2} - \sum_{j=1}^{m-1}\frac{(hk - h - h\gamma)^{2j-1}}{(2j-1)!}\right]\right]$$

$$= \frac{e^{-h} - e^h}{2} - \sum_{j=1}^{m-1}\frac{(-h)^{2j-1}}{(2j-1)!} = \frac{e^{-h} - e^h}{2} + \sum_{j=1}^{m-1}\frac{h^{2j-1}}{(2j-1)!}. \tag{5.5}$$

Taking into account equalities (5.3), (5.5) and expressions (5.1), (5.2), after some simplifications for the error of difference formulas on classes $W_2^{(m,m-1)}(0,1)$ next

$$\left\|\ell\left|W_2^{(m,m-1)*}\right\|^2 = (-1)^m\left\{\frac{e^{-h} - e^h}{2} + \sum_{j=1}^{m-1}\frac{h^{2j-1}}{(2j-1)!} + 2h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)} f_m(h\gamma)\right.$$

$$\left. -h^2\sum_{\gamma=0}^{k-1}\sum_{\beta=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}\overset{\circ}{C}{}_{\beta}^{(1)}\frac{sign(h\beta - h\gamma)}{2}\left[\frac{e^{h\beta - h\gamma} - e^{h\gamma - h\beta}}{2} - \sum_{j=1}^{m-2}\frac{(h\beta - h\gamma)^{2j-1}}{(2j-1)!}\right]\right\}$$

$$= (-1)^m\left\{\frac{e^{-h} - e^h}{2} + \sum_{j=1}^{m-1}\frac{h^{2j-1}}{(2j-1)!} + h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}\left[\overset{\circ}{d}\, e^{-h\gamma} + \overset{\circ}{P}_{m-3}(h\gamma) + f_m(h\gamma)\right]\right\}.$$

From here we obtain the statement of Theorem 4.1, i.e.

$$\left\|\ell\left|W_2^{(m,m-1)*}\right\|^2 = (-1)^m\left\{\frac{e^{-h} - e^h}{2} + \sum_{j=1}^{m-1}\frac{h^{2j-1}}{(2j-1)!} + h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}\left[\overset{\circ}{d}\, e^{-h\gamma} + \overset{\circ}{P}_{m-3}(h\gamma) + f_m(h\gamma)\right]\right\}. \tag{5.6}$$

Theorem 5.1 is proven.

Now we move on to the proof of Theorem 5.2. In the case $m=3$ of formula (5.6) it follows that the error of the explicit difference formula on the class $W_2^{(3,2)}(0,1)$ takes the following form

$$\left\|\overset{\circ}{\ell}|W_2^{(3,2)*}(0,1)\right\|^2=\frac{e^h-e^{-h}}{2}-h-\frac{h^3}{6}-h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}\left[\overset{\circ}{d}\,e^{-h\gamma}+\overset{\circ}{p}_0+f_3(h\gamma)\right]. \tag{5.7}$$

Here $\overset{\circ}{C}{}_{\beta}^{(1)}$, $\beta=0,1,\dots,k-1$ are the optimal coefficients of the explicit difference formula in space $W_2^{(3,2)}(0,1)$ and they are given in Theorem 3.1, i.e.

$$\overset{\circ}{C}{}_{0}^{(1)}=\frac{\lambda^k\left(e^h-he^h-1\right)\left(\lambda^2-1\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)},$$

$$\overset{\circ}{C}{}_{\beta}^{(1)}=\frac{\left(e^h-he^h-1\right)(1-\lambda)\left[\lambda^{k+\beta}\left(e^h-\lambda\right)+\lambda^{k-\beta}\left(\lambda e^h-1\right)\right]}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)},\ \ \beta=1,\,2,\,\dots,\,k-2,$$

$$\overset{\circ}{C}{}_{k-1}^{(1)}=\frac{\left(\lambda^{2k}-\lambda^2\right)\left(he^{2h}-e^h+1\right)+\left(\lambda^{2k-1}-\lambda^3\right)\left(e^{2h}-he^{2h}-e^h\right)}{he^h\left(e^h-1\right)\left(\lambda^{2k}-\lambda^2\right)}.$$

The optimal function $\overset{\circ}{F}(h\beta)=\overset{\circ}{d}\,e^{-h\beta}+\overset{\circ}{p}_0$ of the discrete argument is given by the formula

$$\begin{aligned}\overset{\circ}{F}(h\beta)=&\left(\frac{e^{hk-h}\left(e^h-h-1\right)}{4}+\frac{e^{hk-2h}\left(e^h-he^h-1\right)\left[\left(e^h-1\right)\left(\lambda^{2k-1}-\lambda^{2k-2}e^h\right)-\lambda e^h+1\right]}{4\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)\left(\lambda e^h-1\right)}\right.\\&\left.+\frac{e^{hk-2h}\left(e^h-he^h-1\right)\left(\lambda e^{2h}-\lambda e^h-\lambda+1\right)}{4\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)\left(e^h-\lambda\right)}-\frac{\lambda^{k-1}\left(e^h-he^h-1\right)\left(1-\lambda^2\right)\left(e^{2h}-1\right)}{4e^h\left(\lambda^{2k-2}-1\right)\left(e^h-\lambda\right)\left(\lambda e^h-1\right)}\right)\cdot e^{-h\beta}\\&-h^2\left(\frac{1}{4}+\frac{\left(e^h-he^h-1\right)\left(1+\lambda^{k-1}\right)\left(\lambda^{k-1}e^h-\lambda^k+\lambda e^h-1\right)}{2he^h\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)(1-\lambda)}\right).\end{aligned} \tag{5.8}$$

In the case $m=3$ of formula (5.4) for the function $f_3(h\beta)$ we have the following

$$\begin{aligned}f_3(h\beta)&=\frac{1}{2}\left[\left(1-e^h\right)\left(\frac{e^{h\beta-hk}}{2}-\frac{e^{hk-h\beta-h}}{2}\right)-\sum_{j=2}^{2}\left(\frac{(h\beta-hk)^{2j-2}}{(2j-2)!}-\frac{(h\beta-hk+h)^{2j-2}}{(2j-2)!}\right)\right]\\&=\frac{1}{2}\left[\frac{e^{-h\beta}}{2}\left(e^{hk}-e^{hk-h}\right)+\frac{e^{h\beta}}{2}\left(e^{-hk}-e^{-hk+h}\right)+(h\beta)\,h+\frac{h^2}{2}\,(1-2k)\right]\end{aligned} \tag{5.9}$$

By virtue of (5.8) and (5.9) we have

$$\begin{aligned}\overset{\circ}{F}(h\beta)+f_3(h\beta)&=\overset{\circ}{d}\,e^{-h\beta}+\overset{\circ}{p}_0+f_3(h\beta)=\overset{\circ}{d}\,e^{-h\beta}+\overset{\circ}{p}_0\\&+\frac{1}{2}\left[\frac{e^{-h\beta}}{2}\left(e^{hk}-e^{hk-h}\right)+\frac{e^{h\beta}}{2}\left(e^{-hk}-e^{-hk+h}\right)+(h\beta)\,h+\frac{h^2}{2}\,(1-2k)\right]\\&=\left(\overset{\circ}{d}+\frac{e^{hk}-e^{hk-h}}{4}\right)e^{-h\beta}+\overset{\circ}{p}_0+\frac{e^{-hk}-e^{-hk+h}}{4}e^{h\beta}+\frac{h^2}{2}\beta+\frac{h^2\,(1-2k)}{4}.\end{aligned} \tag{5.10}$$

It is known that

$$h\sum_{\gamma=0}^{k-1}\overset{\circ}{C}{}_{\gamma}^{(1)}\,e^{-h\gamma}=e^{-hk+h}-e^{-hk}, \tag{5.11}$$

$$\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} = 1. \tag{5.12}$$

Using equalities (5.10)-(5.12) we calculate the following sum

$$h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} \left(\overset{\circ}{d}\, e^{-h\gamma} + \overset{\circ}{p}_0 + f_3(h\gamma)\right) = \frac{e^h-1}{e^{hk}}\overset{\circ}{d} + h\overset{\circ}{p}_0 + \frac{\left(e^h-1\right)^2}{4e^h}$$

$$+\frac{h^3\,(1-2k)}{4} + \frac{\left(1-e^h\right)h}{4e^{hk}}\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} e^{h\gamma} + \frac{h^3}{2}\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)}\gamma\,. \tag{5.13}$$

Using the formulas for the sum of a geometric progression and (3.5), we obtain

$$\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} e^{h\gamma} = \frac{\left(e^h-he^h-1\right)(1-\lambda)\left(e^h-\lambda\right)\left(1-\lambda^{k-2}e^{hk-2h}\right)\lambda^{k-1}}{h\,(e^h-1)\,(\lambda^{2k-2}-1)\,(1-\lambda e^h)}$$

$$+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(\lambda e^h-1\right)\left(e^{hk-2h}-\lambda^{k-2}\right)}{h\,(e^h-1)\,(\lambda^{2k-2}-1)\,(e^h-\lambda)}, \tag{5.14}$$

$$\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)}\gamma = \frac{\left(e^h-he^h-1\right)(1-\lambda)\left(e^h-\lambda\right)\left[1-\lambda^{k-1}-(k-1)\,\lambda^{k-2}\,(1-\lambda)\right]\lambda^{k-1}}{he^h\,(e^h-1)\,(\lambda^{2k-2}-1)\,(1-\lambda)^2}$$

$$+\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(\lambda e^h-1\right)\left[\lambda^{k-1}-1+(k-1)\,(1-\lambda)\right]}{he^h\,(e^h-1)\,(\lambda^{2k-2}-1)\,(1-\lambda)^2}. \tag{5.15}$$

Due to (5.14) and (5.15), equality (5.13) can be reduced to the form

$$h\sum_{\gamma=0}^{k-1} \overset{\circ}{C}{}_{\gamma}^{(1)} \left(\overset{\circ}{d}\, e^{-h\gamma} + \overset{\circ}{p}_0 + f_3(h\gamma)\right) = \frac{\left(e^h-1\right)\left(e^h-h-1\right)}{4e^h}$$

$$+\frac{\left(e^h-he^h-1\right)\left[\left(e^h-1\right)\left(\lambda^{2k-1}-\lambda^{2k-2}e^h\right)-\lambda e^h+1\right]}{4e^{2h}\,(\lambda^{2k-2}-1)\,(\lambda e^h-1)}$$

$$+\frac{\left(e^h-he^h-1\right)\left(\lambda e^{2h}-\lambda e^h-\lambda+1\right)}{4e^{2h}\,(\lambda^{2k-2}-1)\,(e^h-\lambda)} - \frac{\left(e^h-he^h-1\right)\left(1-\lambda^2\right)\left(e^{3h}-e^{2h}-e^h+1\right)\lambda^{k-1}}{4e^{hk+h}\,(\lambda^{2k-2}-1)\,(e^h-\lambda)\,(\lambda e^h-1)}$$

$$-\frac{h^3}{4} - \frac{h^2\left(e^h-he^h-1\right)\left(1+\lambda^{k-1}\right)\left(\lambda^{k-1}e^h-\lambda^k+\lambda e^h-1\right)}{2e^h\,(e^h-1)\,(\lambda^{2k-2}-1)\,(1-\lambda)} + \frac{\left(e^h-1\right)^2}{4e^h} + \frac{h^3\,(1-2k)}{4}$$

$$-\frac{\left(e^h-he^h-1\right)(1-\lambda)\left(e^h-\lambda\right)\left(1-\lambda^{k-2}e^{hk-2h}\right)\lambda^{k-1}}{4e^{hk}\,(\lambda^{2k-2}-1)\,(1-\lambda e^h)} - \frac{\left(e^h-he^h-1\right)(1-\lambda)\left(\lambda e^h-1\right)\left(e^{hk-2h}-\lambda^{k-2}\right)}{4e^{hk}\,(\lambda^{2k-2}-1)\,(e^h-\lambda)}$$

$$-\frac{h^2\left(e^h-he^h-1\right)(1-\lambda)\left(e^h-\lambda\right)\left[1-\lambda^{k-1}-(k-1)\,\lambda^{k-2}\,(1-\lambda)\right]\lambda^{k-1}}{2e^h\,(e^h-1)\,(\lambda^{2k-2}-1)\,(1-\lambda)^2}$$

$$-\frac{h^2\left(e^h-he^h-1\right)(1-\lambda)\left(\lambda e^h-1\right)\left[\lambda^{k-1}-1+(k-1)\,(1-\lambda)\right]}{2e^h\,(e^h-1)\,(\lambda^{2k-2}-1)\,(1-\lambda)^2}$$

$$= \frac{\left(e^h-1\right)\left(e^h-h-1\right)}{2e^h} - \frac{h^3}{2} - \frac{h^2\left(e^h-he^h-1\right)\left(\lambda^{2k-2}e^h-\lambda^{2k-1}+\lambda e^h-1\right)}{e^h\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)\left(1-\lambda\right)}$$
$$-\frac{\left(e^h-he^h-1\right)\left(\lambda^{2k-3}e^h-\lambda^{2k-2}-\lambda e^h+1\right)}{4e^{2h}\left(\lambda^{2k-2}-1\right)}$$
$$-\frac{\left(e^h-he^h-1\right)\left(1-\lambda\right)\left(\lambda^{2k-3}e^{2h}+\lambda^{2k-1}-2\lambda^{2k-2}e^h-2\lambda e^h+\lambda^2e^{2h}+1\right)}{4e^{2h}\left(\lambda^{2k-2}-1\right)\left(e^h-\lambda\right)\left(\lambda e^h-1\right)}$$
$$-\frac{\left(e^h-he^h-1\right)\left(\left(1-e^h\right)\left(\lambda-e^h\right)\lambda^{2k-2}+\lambda e^h-1\right)}{4e^{2h}\left(\lambda^{2k-2}-1\right)\left(\lambda e^h-1\right)} - \frac{\left(e^h-he^h-1\right)\left(\lambda e^h-\lambda e^{2h}+\lambda-1\right)}{4e^{2h}\left(\lambda^{2k-2}-1\right)\left(e^h-\lambda\right)}. \tag{5.16}$$

Substituting (5.16) into formula (5.7), we obtain

$$\left\|\overset{\circ}{\ell}|W_2^{(3,2)*}(0,1)\right\|^2 = \frac{e^h-e^{-h}}{2} - h - \frac{h^3}{6} - \frac{\left(e^h-1\right)\left(e^h-h-1\right)}{2e^h} + \frac{h^3}{2}$$
$$+\frac{h^2\left(e^h-he^h-1\right)\left(\lambda^{2k-2}e^h-\lambda^{2k-1}+\lambda e^h-1\right)}{e^h\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)\left(1-\lambda\right)} + \frac{\left(e^h-he^h-1\right)\left(\lambda^{2k-3}e^h-\lambda^{2k-2}-\lambda e^h+1\right)}{4e^{2h}\left(\lambda^{2k-2}-1\right)}$$
$$+\frac{\left(e^h-he^h-1\right)\left(1-\lambda\right)\left(\lambda^{2k-3}e^{2h}+\lambda^{2k-1}-2\lambda^{2k-2}e^h-2\lambda e^h+\lambda^2e^{2h}+1\right)}{4e^{2h}\left(\lambda^{2k-2}-1\right)\left(e^h-\lambda\right)\left(\lambda e^h-1\right)}$$
$$+\frac{\left(e^h-he^h-1\right)\left(\left(1-e^h\right)\left(\lambda-e^h\right)\lambda^{2k-2}+\lambda e^h-1\right)}{4e^{2h}\left(\lambda^{2k-2}-1\right)\left(\lambda e^h-1\right)} + \frac{\left(e^h-he^h-1\right)\left(\lambda e^h-\lambda e^{2h}+\lambda-1\right)}{4e^{2h}\left(\lambda^{2k-2}-1\right)\left(e^h-\lambda\right)}.$$

From here, after some simplifications, we obtain

$$\left\|\overset{\circ}{\ell}|W_2^{(3,2)*}(0,1)\right\|^2 = \frac{\left(e^h-he^h-1\right)}{2e^{2h}\left(\lambda^{2k-2}-1\right)\left(\lambda e^h-1\right)\left(e^h-\lambda\right)}\Big[2\left(e^{2h}-e^h+2\right)\left(\lambda^{2k-2}-1\right)\lambda e^h$$
$$+\left(e^h-3\right)\left(\lambda^{2k-4}-1\right)\lambda^2e^{2h}-\left(2e^{2h}-e^h+1\right)\left(\lambda^{2k}-1\right)\Big] + \frac{h\left(e^h-1\right)}{2e^h}$$
$$+\frac{h^2\left(e^h-he^h-1\right)\left(\lambda^{2k-2}e^h-\lambda^{2k-1}+\lambda e^h-1\right)}{e^h\left(e^h-1\right)\left(\lambda^{2k-2}-1\right)\left(1-\lambda\right)} + \frac{h^3}{3}.$$

Theorem 5.2 is completely proven.

Since $\lambda = \frac{h\left(e^{2h}+1\right)-e^{2h}+1-\left(e^h-1\right)\sqrt{h^2\left(e^h+1\right)^2+2h\left(1-e^{2h}\right)}}{1-e^{2h}+2he^h}$, then for large $k$values λ^k close to zero, i.e. for $k\to\infty$, $\lambda^k\to 0$, therefore we have:

$$\left\|\overset{\circ}{\ell}|W_2^{(3,2)*}(0,1)\right\|^2 = \frac{h^3}{3} - \frac{h^2\left(e^h-he^h-1\right)\left(\lambda e^h-1\right)}{e^h\left(e^h-1\right)\left(1-\lambda\right)} + \frac{h\left(e^h-1\right)}{2e^h}$$
$$-\frac{\left(e^h-he^h-1\right)\left[\left(2e^{2h}-e^h+1\right)-2\left(e^{2h}-e^h+2\right)\lambda e^h-\left(e^h-3\right)\lambda^2e^{2h}\right]}{2e^{2h}\left(\lambda e^h-1\right)\left(e^h-\lambda\right)}.$$

Expansion of the square norm of the optimal difference formula into a Taylor series:

$$\left\|\overset{\circ}{\ell}|W_2^{(3,2)*}(0,1)\right\|^2 = \frac{h^5}{30\left(\lambda-1\right)^2}\cdot\left[7\lambda+8+\frac{31\lambda^{10}-1}{4\left(\lambda^8+\lambda^6+\lambda^4+\lambda^2+1\right)}\right]$$

$$-\frac{h^6}{144}\cdot\frac{16\lambda^3\left(\lambda^2-\lambda+1\right)-17\lambda^2\left(\lambda^4+1\right)+8\lambda\left(\lambda^6+1\right)-35\left(\lambda^8+1\right)}{\left(\lambda^4+\lambda^2+1\right)\left(\lambda-1\right)^4}+O\left(h^7\right).$$

Conclusion

Optimal explicit difference formula for any integer k in the Hilbert space $W_2^{(3,2)}(0,1)$ was constructed. Here, an optimal function of the discrete argument was found, which was used to find the optimal error estimate for an explicit difference formula on the class $W_2^{(3,2)}(0,1)$. Further, an optimal error estimate for an explicit difference formula on the class $W_2^{(3,2)}(0,1)$ was calculated. In addition, using the optimal function of the discrete argument, a new formula for the optimal error of explicit difference formulas on classes $W_2^{(m,m-1)}(0,1)$ was obtained.

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References

[1] R.L. Burden, J.D. Faires, A.M. Burden, Numerical Analysis, Cengage Learning, (2016).

[2] S. Mehrkanoon, Z.A. Majid, M. Suleiman, K.I. Othman, Z.B. Ibrahim, 3-Point Implicit Block Multistep Method for the Solution of First Order ODEs, Bull. Malays. Math. Sci. Soc. (2) 35 (2012), 547–555.

[3] L. Beuken, O. Cheffert, A. Tutueva, D. Butusov, V. Legat, Numerical Stability and Performance of Semi-Explicit and Semi-Implicit Predictor?Corrector Methods, Mathematics 10 (2022), 2015. https://doi.org/10.3390/math10122015.

[4] A. Tutueva, D. Butusov, Stability Analysis and Optimization of Semi-Explicit Predictor?Corrector Methods, Mathematics 9 (2021), 2463. https://doi.org/10.3390/math9192463.

[5] N. Hoang, R. Sidje, On the Equivalence of the Continuous Adams?Bashforth Method and Nordsieck?s Technique for Changing the Step Size, Appl. Math. Lett. 26 (2013), 725–728. https://doi.org/10.1016/j.aml.2013.02.001.

[6] O.Y. Ababneh, New Numerical Methods for Solving Differential Equations, J. Adv. Math. 16 (2019), 8384–8390. https://doi.org/10.24297/jam.v16i0.8280.

[7] M.A. Odunayo, Z. Ogunwobi, Comparison of Adams-Bashforth-Moulton Method and Milne-Simpson Method on Second Order Ordinary Differential Equation, Turk. J. Anal. Number Theory 9 (2021), 1–8. https://doi.org/10.12691/tjant-9-1-1.

[8] S.K. Kar, An Explicit Time-Difference Scheme with an Adams?Bashforth Predictor and a Trapezoidal Corrector, Mon. Weather. Rev. 140 (2012), 307–322. https://doi.org/10.1175/mwr-d-10-05066.1.

[9] E. Vitasek, Approximate Solutions of Abstract Differential Equations, Appl. Math. 52 (2007), 171–183. https://doi.org/10.1007/s10492-007-0008-3.

[10] I. Babushka, E. Vitasek, M. Prager, Numerical Processes in Differential Equations, Mir, 1969.

[11] S.L. Sobolev, Introduction to the Theory of Cubature Formulas, Nauka, (1974).

[12] K.M. Shadimetov, A.R. Hayotov, Construction of Interpolation Splines Minimizing Semi-Norm in $W_2^{(m,m-1)}(0,1)$ Space, BIT Numer. Math. 53 (2013), 545–563. https://doi.org/10.1007/s10543-012-0407-z.

[13] K.M. Shadimetov, A.R. Hayotov, Optimal Quadrature Formulas in the Sense of Sard in $W_2^{(m,m-1)}$ Space, Calcolo 51 (2013), 211–243. https://doi.org/10.1007/s10092-013-0076-6.

[14] K. Shadimetov, A. Hayotov, R. Karimov, Optimization of Explicit Difference Methods in the Hilbert Space $W_2^{(2,1)}$, AIP Conf. Proc. 2997 (2023), 020054. https://doi.org/10.1063/5.0144805.

[15] K.M. Shadimetov, R.S. Karimov, Optimization of Adams-Type Difference Formulas in Hilbert Space $W_2^{(2,1)}(0,1)$, J. Comput. Anal. Appl. 32 (2024), 300–319.

[16] K.M. Shadimetov, R.S. Karimov, Optimal Coefficients of an Implicit Difference Formula in the Hilbert Space, AIP Conf. Proc. 3045 (2024), 060030. https://doi.org/10.1063/5.0200065.

[17] G. Ismatullaev, R. Mirzakabilov, R. Karimov, Cubature Formulas in the Parabolic Domain, AIP Conf. Proc. 3045 (2024), 060040. https://doi.org/10.1063/5.0200066.

[18] K.M. Shadimetov, R.N. Mirzakabilov, Optimal Difference Formulas in the Sobolev Space, J. Math. Sci. 278 (2024), 712–721. https://doi.org/10.1007/s10958-024-06951-2.

[19] K.M. Shadimetov, S. Shonazarov, On an Implicit Optimal Difference Formula, Uzb. Math. J. 68 (2025), 128–136. https://doi.org/10.29229/uzmj.2024-4-15.

[20] K.M. Shadimetov, R.S. Karimov, Wiener-Hopf Type System for Finding Optimal Coefficients of Difference Formulas in the Hilbert Space, Probl. Comput. Appl. Math. 59 (2024), 74–84.

[21] R.W. Hamming, Numerical Methods for Scientists and Engineers, Dover Publications, (2012).

[22] G. Dahlquist, Convergence and Stability in the Numerical Integration of Ordinary Differential Equations, Math. Scand. 4 (1956), 33–53. https://www.jstor.org/stable/24490010.

[23] G. Dahlquist, Stability and Error Bounds in the Numerical Integration of Ordinary Differential Equations, Thesis, University of Stockholm, 1958.